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Spectral Theory of the Dirac Operator

W. Desmond Evans

1. Introduction

The object of this paper is to investigate the spectrum of the Dirac operator T for a particle in an electro-magnetic field. Our main concern is with the case when the magnetic field has vector potential $Q=h(-x_2, x_1, 0)$ (h constant) and the electric field has scalar potential $V=V(\rho)$ where $\rho=(x_1^2+x_2^2)^{\frac{1}{2}}$, (x_1, x_2, x_3) being Cartesian co-ordinates. In an unpublished note [7] Titchmarsh studied a linked pair of ordinary differential equations in the variable ρ which is related to T with the latter vector and scalar potentials by separation of variables in cylindrical co-ordinates (ρ, θ, z). He proved that if $V(\rho)=\rho v(\rho)$ and either $v(\rho)\to\infty$ as $\rho\to\infty$ or $v(\rho)=k$ with $|k|>|h|$, then the spectrum consists of $(-\infty, \infty)$ and is continuous. However, quite apart from the difficulty of relating the spectral notions in Titchmarsh's work to the standard definitions of operator theory it is not clear how the nature of the spectrum of T can be derived from a knowledge of the spectrum of the operator T_n+V say, considered by Titchmarsh, although one would expect the spectrum to consist of $(-\infty, \infty)$ in the above case. In § 2 below, in view of this latter difficulty, we consider T directly, our aim being to obtain criteria on Q and V for the spectrum of T to consist of $(-\infty, \infty)$. In fact, our results are concerned with general vector and scalar potentials and our criteria contain only as a special case those of Titchmarsh given above. The method which we use is based on that of Eastham [1] and this leads to a further generalisation of Titchmarsh's result in that the conditions are only imposed in a certain sequence of cubes.

In the remainder of the paper we are primarily concerned with the operator T_n+V discussed in [7] with $V=V(\rho)$ satisfying $\rho^{-1}V(\rho)\to 0$ as $\rho\to\infty$. Under this condition Titchmarsh proved that the spectrum is discrete. This is a consequence of our result that V represents a relatively compact perturbation of the operator T_n whose spectrum is known to be discrete. Our approach is based on the method used by Jörgens in [4]. The result we obtain is a generalisation of that of Titchmarsh in that it does not impose the smoothness conditions on V which are necessary in Titchmarsh's work. In the case $V=0$, and in this case only, we are able to determine the spectrum of the Dirac operator T.

2. Conditions for the Spectrum to consist of $(-\infty, \infty)$

The Dirac operator for an electron in an electro-magnetic field with vector potential $Q=(Q_1, Q_2, Q_3)$ and scalar potential V is of the form

$$\tau=\sum_{j=1}^{3}\alpha_j(i\partial_j-Q_j)+\alpha_4+VI$$

in suitable units, where $\partial_j = \partial/\partial x_j$. The Dirac matrices α_j $(j=1,2,3,4)$ are Hermitian and satisfy

$$\alpha_i \alpha_j = -\alpha_j \alpha_i \quad (i \neq j), \qquad \alpha_i^2 = I \tag{2.1}$$

where I is the unit matrix. We choose for the α_j

$$\alpha_1 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}, \qquad \alpha_2 = \begin{pmatrix} & & & -i \\ & & i & \\ & -i & & \\ i & & & \end{pmatrix},$$

$$\alpha_3 = \begin{pmatrix} & & 1 & \\ & & & -1 \\ 1 & & & \\ & -1 & & \end{pmatrix}, \qquad \alpha_4 = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}. \tag{2.2}$$

τ acts on C^4-valued functions, where C^4 is 4-dimensional complex space, and we shall use the following notation for the inner product and norm in C^4:

$$u \cdot v = \sum_{i=1}^{4} u_i \bar{v}_i, \qquad |u|^2 = \sum_{i=1}^{4} |u_i|^2,$$

where $u=(u_1, u_2, u_3, u_4)$, $v=(v_1, v_2, v_3, v_4)$. It is useful to note that from the above properties of the α_i we have

$$\alpha_i u \cdot v = u \cdot \alpha_i v, \qquad i=1,2,3,4 \tag{2.3}$$

and

$$\left| \sum_{i=1}^{4} \alpha_i f_i u \right|^2 = \left(\sum_{i=1}^{4} f_i^2 \right) |u|^2 \tag{2.4}$$

for real numbers f_1, f_2, f_3, f_4.

Let $\mathscr{H}$ denote the Hilbert space of measurable C^4-valued functions u which satisfy

$$\|u\|^2 = \int_{R^3} |u(x)|^2 \, dx < \infty$$

with inner-product

$$(u, v) = \int_{R^3} u(x) \cdot v(x) \, dx.$$

Let $\mathscr{D}$ be the subset of $\mathscr{H}$ consisting of all C^4-valued functions $\varphi \in \mathscr{H}$ which satisfy:

i) φ has continuous first order derivatives,

ii) φ has compact support,

iii) $\tau \varphi \in \mathscr{H}$.

We assume throughout that V and the components of Q are real-valued and also that V and $\left(\sum_{i=1}^{3} Q_i^2\right)^{\frac{1}{2}}$ are in $L^2_{\mathrm{loc}}(R^3)$ – the space of functions which are locally square integrable in Euclidean 3-space R^3. The operator T defined on $\mathscr{D}$ by $T\varphi = \tau\varphi$ is then a symmetric operator in $\mathscr{H}$. We also remark that as in [2] it can be shown that T is in fact essentially self-adjoint if V and $\left(\sum_{i=1}^{3} Q_i^2\right)^{\frac{1}{2}}$ are in the Stummel space $\mathscr{S}_{\alpha\,\mathrm{loc}}$ – the set of functions f for which

$$\int_{|y-x|\leqq 1} \frac{|f|^2}{|y-x|^{1+\alpha}}\, dy$$

is locally bounded for some α, $0<\alpha\leqq 1$. In what follows we only require that T has a self-adjoint extension and we shall denote any such extension by $\tilde{T}$.

Our aim in this section is to obtain conditions on Q and V under which the spectrum of $\tilde{T}$ consists of the whole real line. The method we shall use is based on that developed by Eastham in [1] and depends upon showing that for any real number λ there exists a sequence of vectors $\varphi^{(m)}\in\mathscr{D}$ which are such that $\|(\tilde{T}-\lambda)\,\varphi^{(m)}\| \to 0$ as $m\to\infty$. It is convenient and helpful at this stage to describe the ideas behind the construction of these functions $\varphi^{(m)}$.

The construction involves a sequence of cubes A_m with side $2a_m$ where $a_m\to\infty$ as $m\to\infty$ and the A_m have their edges parallel to the co-ordinate axes. Let u_m be a continuously differentiable function of one variable which satisfies

$$u_m(t)=\begin{cases}1, & (|t|\leqq a_m-1)\\ 0, & (|t|\geqq a_m),\end{cases}$$

$$0\leqq u_m(t)\leqq 1 \qquad (-\infty<t<\infty)$$

and put

$$h_m(x)=\prod_{j=1}^{3} u_m(x_j-c_{jm})$$

where (c_{1m}, c_{2m}, c_{3m}) is the centre of A_m. Let $F(x,\lambda,m)$ be a function which is defined in A_m, has continuous first order partial derivatives and is real-valued for $m>m_0$. Let $g(x,\lambda,m)$ be a C^4-valued function defined in A_m which has continuous first order partial derivatives. We also define in A_m a vector $q_m=(q_{1m}, q_{2m}, q_{3m})$ and function v_m which approximate to Q and V respectively in a certain sense. The functions v_m are assumed to be continuously differentiable and do not vanish in A_m for $m>m_0$. We then define

$$\varphi^{(m)}(x)=\varphi^{(m)}(x,\lambda)=k_m \exp\bigl(-i\,F(x,\lambda,m)\bigr)\, v_m^{-1}(x)\, h_m(x)\, g(x,\lambda,m) \qquad (2.5)$$

in A_m and $\varphi^{(m)}(x)=0$ otherwise, where k_m is the positive constant making $\|\varphi^{(m)}\|=1$. Clearly $\varphi^{(m)}\in\mathscr{D}$ and hence $\varphi^{(m)}$ lies in the domain of $\tilde{T}$. Also, writing

$f_j = \partial_j F$ we have

$$
\begin{aligned}
(\tilde{T}-\lambda)\,\varphi^{(m)} &= (\tau-\lambda)\,\varphi^{(m)} \\
&= \left\{\sum_{j=1}^{3}\alpha_j(i\,\partial_j - q_{jm}) + \alpha_4 + (v_m-\lambda)\,I\right\}\varphi^{(m)} \\
&\quad + \sum_{j=1}^{3}\alpha_j(q_{jm}-Q_j)\,\varphi^{(m)} + (V-v_m)\,\varphi^{(m)} \\
&= \left\{\sum_{j=1}^{3}\alpha_j(f_j - q_{jm}) + \alpha_4 + (v_m-\lambda)\,I\right\}\varphi^{(m)} \qquad (2.6)\\
&\quad + i\,k_m\sum_{j=1}^{3}\alpha_j\{\partial_j(v_m^{-1}h_m)\,g + v_m^{-1}h_m\,\partial_j g\}\exp(-iF) \\
&\quad + \sum_{j=1}^{3}\alpha_j(q_{jm}-Q_j)\,\varphi^{(m)} + (V-v_m)\,\varphi^{(m)}.
\end{aligned}
$$

We choose F and g such that

$$\left\{\sum_{j=1}^{3}\alpha_j(f_j - q_{jm}) + \alpha_4 + (v_m-\lambda)\right\} g = 0$$

so that the first term in (2.6) vanishes. Writing

$$\beta_j = f_j - q_{jm} \qquad (j=1,2,3),$$
$$\beta_4 = 1$$

this latter equation becomes

$$\sum_{j=1}^{4}\alpha_j\beta_j\,g = (\lambda - v_m)\,g. \qquad (2.7)$$

Now, the matrix $\sum_{j=1}^{4}\alpha_j\beta_j$ is readily shown to have eigenvalues $\pm|\beta|$ where

$$|\beta| = \left(\sum_{j=1}^{4}\beta_j^2\right)^{\frac{1}{2}}$$

and so (2.7) will be satisfied if

$$|\beta| = \lambda - v_m \qquad (2.8)$$

and

$$g = \begin{pmatrix} |\beta| - \beta_4 \\ 0 \\ \beta_3 \\ \beta_1 + i\beta_2 \end{pmatrix} \qquad (2.9)$$

this being an eigenvector of $\sum_{j=1}^{4} \alpha_j \beta_j$ corresponding to the eigenvalue $|\beta|$. We will show that under suitable conditions on q_m and v_m it is possible to obtain from (2.8) and (2.9) a function F and vector g which are such that the vector $\varphi^{(m)}$ defined by (2.5) satisfies $\|(\tilde{T}-\lambda)\varphi^{(m)}\| \to 0$ as $m \to \infty$.

Theorem 1. *Let $\{B_m\}$ be a sequence of cubes of side $2b_m$ which are such that $b_m \to \infty$ as $m \to \infty$ and B_m does not contain the origin for large enough m. Suppose that there are functions v_m, q_m defined in B_m which satisfy:*

i) *v_m and q_m are real-valued functions of $r=|x|$ with continuous first derivatives,*

ii) $\min_{x \in B_m} |v_m(r)| \to \infty$ *as* $m \to \infty$,

iii) $\int_{B_m} \left\{ \frac{v_m'(r)}{v_m(r)} \right\}^2 dx = o(b_m^3)$,

iv) $\int_{B_m} (V(x)-v_m(r))^2 \, dx = o(b_m^3)$,

v) $\int_{B_m} \sum_{j=1}^{3} (Q_j(x)-\partial_j q_m(r))^2 \, dx = o(b_m^3)$.

Then the spectrum of $\tilde{T}$ consists of $(-\infty, \infty)$.

For a discussion of the conditions imposed above on V and v_m we refer the reader to [1] where examples are given of functions satisfying a similar set of conditions. We note that the only condition imposed on Q is (v) and this requires Q to be close to a gradient vector in B_m for large m. If Q is itself a gradient vector in R^3 then, by the gauge transformation, there is no magnetic field and this theorem then is merely Theorem 13 of [3].

Proof. We begin by defining a suitable sequence of cubes A_m lying in B_m and these are chosen as in [1]. Let the centre of B_m be at a distance R from the origin and let Σ_m be the region consisting of those points of B_m which lie on $|x|=R$. Let Ω_m be the region consisting of those points of B_m which lie on a line joining the origin to a point of Σ_m. Then, it can be shown that Ω_m always contains a fixed portion of B_m and we can choose a cube A_m lying in Ω_m with its edges parallel to the co-ordinate axes and having side $2a_m$, where

$$a_m = k\, b_m \tag{2.10}$$

for some constant k. In addition we can also choose A_m to lie in $|x| \geqq R$.

In (2.6) we choose $q_{jm} = \partial_j q_m$ and so, on taking F in (2.5) to be a function of r only, we can write (2.8) as

$$\begin{aligned}(\lambda - v_m(r))^2 = |\beta|^2 &= \sum_{j=1}^{3} \{\partial_j (F(r)-q_m(r))\}^2 + 1 \\ &= \left\{ \frac{\partial}{\partial r} (F(r)-q_m(r)) \right\}^2 + 1.\end{aligned}$$

Let

$$F(r)=F(r,\lambda,m)=q_m(r)+\int_R^r\{(\lambda-v_m)^2-1\}^{\frac{1}{2}}\,dt. \tag{2.11}$$

The choice of A_m above ensures that F is defined on A_m and by (ii) for sufficiently large, $m>m_0$ say, F is real-valued and has continuous first derivatives.

With this F and g given by (2.9) we thus have from (2.6), for $m>m_0$,

$$\begin{aligned}\|(\tilde{T}-\lambda)\,\varphi^{(m)}\| &\leqq k_m\left\|\sum_{j=1}^{3}\alpha_j\{\partial_j(v_m^{-1}h_m)\,g+v_m^{-1}h_m\,\partial_j g\}\right\|\\ &\quad+\left\|\sum_{j=1}^{3}\alpha_j(\partial_j q_m-Q_j)\,\varphi^{(m)}\right\|+\|(V-v_m)\,\varphi^{(m)}\|\\ &\leqq k_m\left\|\sum_{j=1}^{3}\alpha_j\,\partial_j(v_m^{-1}h_m)\,g\right\|+k_m\left\|\sum_{j=1}^{3}\alpha_j v_m^{-1}h_m\,\partial_j g\right\|\\ &\quad+\left\|\sum_{j=1}^{3}\alpha_j(\partial_j q_m-Q_j)\,\varphi^{(m)}\right\|+\|(V-v_m)\,\varphi^{(m)}\|\\ &=I_1+I_2+I_3+I_4\end{aligned}$$

say. Now, from (2.9) and (2.8), we have

$$|g|^2=2|\beta|^2-2|\beta|\,\beta_4=2(\lambda-v_m)(\lambda-v_m-1).$$

Hence, as $m\to\infty$, we have from (ii)

$$\begin{aligned}\|\varphi^{(m)}\|^2&=k_m^2\int_{A_m}v_m^{-2}h_m^2|g|^2\,dx\\ &\sim 2k_m^2\int_{A_m}\left\{\prod_{j=1}^{3}u_m(x_j-c_{jm})\right\}^2dx\\ &=2k_m^2\prod_{j=1}^{3}\int_{|x_j-c_{jm}|\leqq a_m}u_m^2(x_j-c_{jm})\,dx_j\\ &\sim 16k_m^2\,a_m^3\end{aligned}$$

so that

$$k_m\sim\tfrac{1}{4}a_m^{-\frac{3}{2}}. \tag{2.12}$$

Also

$$\begin{aligned}|\varphi^{(m)}(x)|^2&\leqq k_m^2\,v_m^{-2}(r)\,|g|^2\\ &\sim 2k_m^2.\end{aligned} \tag{2.13}$$

In view of (2.4) we have

$$I_1^2 = k_m^2 \left\| \sum_{j=1}^{3} \alpha_j \partial_j (v_m^{-1} h_m)\, g \right\|^2 = k_m^2 \int_{A_m} \sum_{j=1}^{3} (\partial_j (v_m^{-1} h_m))^2 \, |g|^2 \, dx$$

$$= k_m^2 \int_{A_m} |g|^2 \sum_{j=1}^{3} \{(\partial_j v_m^{-1})\, h_m + v_m^{-1} \partial_j h_m\}^2 \, dx$$

$$\leqq 2 k_m^2 \int_{A_m} |g|^2 \left\{ \sum_{j=1}^{3} (\partial_j v_m^{-1})^2 + v_m^{-2} \sum_{j=1}^{3} (\partial_j h_m)^2 \right\} dx$$

$$= 2 k_m^2 \int_{A_m} |g|^2 \left\{ \left(\frac{\partial}{\partial r} v_m^{-1} \right)^2 + v_m^{-2} |\operatorname{grad} h_m|^2 \right\} dx$$

$$= O\left(k_m^2 \int_{A_m} |v_m|^2 \frac{(v_m')^2}{v_m^4} \, dx + k_m^2 \int_{A_m} \frac{v_m^2}{v_m^2} |\operatorname{grad} h_m|^2 \, dx \right)$$

$$= O\left(k_m^2 \int_{B_m} \left(\frac{v_m'}{v_m} \right)^2 dx + k_m^2 \int_{A_m} |\operatorname{grad} h_m|^2 \, dx \right).$$

The second integral is $O(k_m^2 a_m^2)$ since $u_m(x_j - c_{jm}) = 1$ in $|x_j - c_{jm}| \leqq a_m - 1$ and so, we get from (iii), (2.10) and (2.12) that $I_1 = o(1)$.

$$I_2^2 = k_m^2 \left\| \sum_{j=1}^{3} v_m^{-1} h_m \alpha_j \partial_j g \right\|^2$$

$$\leqq k_m^2 \int_{A_m} v_m^{-2} \left| \sum_{j=1}^{3} \alpha_j \partial_j g \right|^2 dx.$$

From (2.3) and (2.2)

$$\left| \sum_{j=1}^{3} \alpha_j \partial_j g \right|^2 = \sum_{j=1}^{3} |\partial_j g|^2 + 2 \operatorname{Re} \{ \partial_1 g \cdot \alpha_1 \alpha_2 \partial_2 g + \partial_1 g \cdot \alpha_1 \alpha_3 \partial_3 g + \partial_2 g \cdot \alpha_2 \alpha_3 \partial_3 g \}$$

$$= \sum_{j=1}^{3} |\partial_j g|^2 + 2 \{ \partial_1 \beta_1 \partial_2 \beta_2 + \partial_1 \beta_1 \partial_3 \beta_3 + \partial_2 \beta_2 \partial_3 \beta_3 \}$$

$$- 2 \{ \partial_1 \beta_2 \partial_2 \beta_1 + \partial_1 \beta_3 \partial_3 \beta_1 + \partial_2 \beta_3 \partial_3 \beta_2 \} \tag{2.14}$$

$$= \sum_{j=1}^{3} (\partial_j |\beta|)^2 + \left(\sum_{j=1}^{3} \partial_j \beta_j \right)^2$$

$$+ (\partial_1 \beta_3)^2 + (\partial_2 \beta_3)^2 + (\partial_2 \beta_1)^2 + (\partial_3 \beta_1)^2 + (\partial_1 \beta_2)^2 + (\partial_3 \beta_2)^2$$

$$- 2 \{ (\partial_1 \beta_2)(\partial_2 \beta_1) + (\partial_1 \beta_3)(\partial_3 \beta_1) + (\partial_2 \beta_3)(\partial_3 \beta_2) \}.$$

As $\beta_j=\partial_j(F-q_m)$ and $|\beta|=\lambda-v_m$ where F, q_m and v_m are functions of r only, this gives

$$\left|\sum_{j=1}^{3}\alpha_j\partial_j g\right|^2=\left(\frac{\partial}{\partial r}|\beta|\right)^2+(\nabla^2(F-q_m))^2$$
$$=\left(\frac{\partial}{\partial r}|\beta|\right)^2+\left\{\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)(F-q_m)\right\}^2$$
$$=(v_m')^2+\left\{\frac{-(\lambda-v_m)v_m'}{[(\lambda-v_m)^2-1]^{\frac{1}{2}}}+\frac{2}{r}[(\lambda-v_m)^2-1]^{\frac{1}{2}}\right\}^2.$$

Hence, as $m\to\infty$,

$$I_2^2=O\left(k_m^2\int_{A_m}\left\{\left(\frac{v_m'}{v_m}\right)^2+\frac{1}{r^2}\right\}dx\right).$$

For m sufficiently large B_m does not contain the origin and so $r>b_m$ for x in A_m. Consequently $I_2=o(1)$ by (iii), (2.10) and (2.12).

$$I_3^2=\left\|\sum_{j=1}^{3}\alpha_j(\partial_j q_m-Q_j)\varphi^{(m)}\right\|^2$$
$$=\int_{A_m}\sum_{j=1}^{3}(\partial_j q_m-Q_j)^2|\varphi^{(m)}|^2\,dx,$$

by (2.4),

$$=O\left(k_m^2\int_{A_m}\sum_{j=1}^{3}(\partial_j q_m-Q_j)^2\,dx\right),$$

by (2.13),

$$=o(1)$$

from (v), (2.10) and (2.12). Finally

$$I_4^2=\|(V-v_m)\varphi^{(m)}\|^2$$
$$=O\left(k_m^2\int_{A_m}(V-v_m)^2\,dx\right)=o(1)$$

from (iv). Thus $\|(\tilde{T}-\lambda)\varphi^{(m)}\|\to 0$ as $m\to\infty$ as required.

Theorem 2. *Let $\{B_m'\}$ be a sequence of cubes of side $2b_m$ which are such that $b_m\to\infty$ as $m\to\infty$ and B_m' does not intersect the axis $p=0$ for m large enough. Let $q(x)=h(-x_2,x_1,0)$ for some constant h in the union of the B_m' and suppose that there is a function v_m defined in B_m' which satisfies:*

i) *v_m is a real-valued function of $\rho=(x_1^2+x_2^2)^{\frac{1}{2}}$ with a continuous first derivative,*

ii) $\min_{x\in B_m'}\dfrac{(v_m^2-h^2\rho^2)}{|v_m|}\to\infty$ *as* $m\to\infty$,

iii) $\int\limits_{B'_m} \frac{(v'_m)^2}{(v_m^2 - h^2\rho^2)}\, dx = o(b_m^3)$,

iv) $\int\limits_{B'_m} (V(x) - v_m(\rho))^2\, dx = o(b_m^3)$.

Then, if

v) $\int\limits_{B'_m} \sum_{j=1}^{3} (Q_j(x) - q_j(x))^2\, dx = o(b_m^3)$

the spectrum of $\tilde{T}$ *consists of* $(-\infty, \infty)$.

Proof. As in the proof of Theorem 1 our first task is to obtain a suitable sequence of cubes A_m. Let the centre of B'_m be at a distance T from the x_3-axis and let Σ'_m be the intersection of B'_m with $\rho = T$. Let Ω'_m consist of those points of B'_m which lie on a line $\theta = \text{constant}$, $z = \text{constant}$, perpendicular to the x_3-axis and passing through a point of Σ'_m. Then Ω'_m contains a fixed portion of B'_m and it is possible to choose a cube A_m lying in the part of Ω'_m for which $\rho \geqq T$, with its edges parallel to the co-ordinate axes and having side $2a_m$ where

$$a_m = k\, b_m \tag{2.15}$$

for some constant k.

We now have ($q_m = q$)

$$\beta_1 = \partial_1 F + h\, x_2, \quad \beta_2 = \partial_2 F - h\, x_1, \quad \beta_3 = \partial_3 F, \quad \beta_4 = 1,$$

and if we take F to be a function of $\rho = (x_1^2 + x_2^2)^{\frac{1}{2}}$ only then (2.8) becomes

$$\begin{aligned}(\lambda - v_m)^2 &= |\beta|^2 \\ &= \sum_{j=1}^{2} (\partial_j F)^2 + 2h(x_2\, \partial_1 F - x_1\, \partial_2 F) + h^2\rho^2 + 1 \\ &= \left(\frac{\partial}{\partial\rho} F\right)^2 + h^2\rho^2 + 1\end{aligned}$$

in cylindrical co-ordinates $x_1 = \rho\cos\theta$, $x_2 = \rho\sin\theta$. Let

$$F(\rho) = F(\rho, \lambda, m) = \int_T^{\rho} \{(\lambda - v_m)^2 - h^2 t^2 - 1\}^{\frac{1}{2}}\, dt. \tag{2.16}$$

This is defined on A_m and, by (ii), for m sufficiently large, $m > m_0$ say, F is real and has continuous first derivatives.

Proceeding as in the proof of Theorem 1, we have from (2.6), with g defined by (2.9) and $m > m_0$,

$$\begin{aligned}\|(\tilde{T} - \lambda)\, \varphi^{(m)}\| &\leqq k_m \left\| \sum_{j=1}^{3} \alpha_j \partial_j (v_m^{-1} h_m)\, g \right\| + k_m \left\| \sum_{j=1}^{3} \alpha_j v_m^{-1} h_m \partial_j g \right\| \\ &\quad + \left\| \sum_{j=1}^{3} \alpha_j (q_j - Q_j)\, \varphi^{(m)} \right\| + \|(V - v_m)\, \varphi^{(m)}\| \\ &= I_1 + I_2 + I_3 + I_4\end{aligned}$$

say. Note that (ii) implies that v_m does not have a zero in A_m for m sufficiently large, $m > m_0$ say, indeed $\min_{x \in B'_m} |v_m(\rho)| \to \infty$ as $m \to \infty$. Also the estimates (2.12) and (2.13) again hold.

As before, we have on using (2.4)

$$\begin{aligned} I_1^2 &\leqq 2k_m^2 \int_{A_m} |g|^2 \left\{ \sum_{j=1}^{3} (\partial_j v_m^{-1})^2 + v_m^{-2} \sum_{j=1}^{3} (\partial_j h_m)^2 \right\} dx \\ &= 2k_m^2 \int_{A_m} |g|^2 \left\{ \left(\frac{\partial}{\partial \rho} v_m^{-1} \right)^2 + v_m^{-2} |\operatorname{grad} h_m|^2 \right\} dx \\ &= O\left(k_m^2 \int_{A_m} \left(\frac{v'_m(\rho)}{v_m(\rho)} \right)^2 dx + k_m^2 \int_{A_m} |\operatorname{grad} h_m|^2 \, dx \right) \\ &= O\left(k_m^2 \int_{A_m} \frac{(v'_m(\rho))^2}{v_m^2(\rho) - h^2 \rho^2} \, dx \right) + O(k_m^2 a_m^2) \\ &= o(1) \end{aligned}$$

by (iii), (2.12) and (2.15).

$$I_2^2 \leqq k_m^2 \int_{A_m} v_m^{-2} \left| \sum_{j=1}^{3} \alpha_j \partial_j g \right|^2 dx.$$

From (2.14), it can be shown that since $|\beta| = \lambda - v_m$ is a function of ρ only

$$\begin{aligned} \left| \sum_{j=1}^{3} \alpha_j \partial_j g \right|^2 &= \left(\frac{\partial}{\partial \rho} |\beta| \right)^2 + (V^2 F)^2 + 4h^2 \\ &= \left(\frac{\partial}{\partial \rho} |\beta| \right)^2 + \left\{ \left(\frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} \right) F \right\}^2 + 4h^2 \\ &= (v'_m)^2 + \left\{ \frac{-(\lambda - v_m) v'_m - 2h^2 \rho}{[(\lambda - v_m)^2 - h^2 \rho^2 - 1]^{\frac{1}{2}}} + \frac{1}{\rho} [(\lambda - v_m)^2 - h^2 \rho^2 - 1]^{\frac{1}{2}} \right\}^2 + 4h^2 \\ &= O\left((v'_m)^2 + \frac{v_m^2 (v'_m)^2 + \rho^2}{v_m^2 - h^2 \rho^2} + \frac{v_m^2 - h^2 \rho^2}{\rho^2} + 1 \right). \end{aligned}$$

Thus

$$I_2^2 = O\left(k_m^2 \int_{A_m} \left\{ \left(\frac{v'_m}{v_m} \right)^2 + \frac{(v'_m)^2}{v_m^2 - h^2 \rho^2} + \frac{\rho^2}{v_m^2 (v_m^2 - h^2 \rho^2)} + \frac{1}{\rho^2} + \frac{1}{v_m^2} \right\} dx \right).$$

From (ii), for all $x \in A_m$ and m sufficiently large,

$$v_m^2 - h^2 \rho^2 > |v_m| > |h| \rho.$$

This gives

$$I_2^2 = O\left(k_m^2 \int_{A_m} \left\{ \frac{(v'_m)^2}{v_m^2 - h^2 \rho^2} + \frac{1}{\rho} + \frac{1}{\rho^2} \right\} dx \right).$$

As B'_m does not intersect $\rho=0$ for m sufficiently large $\rho>b_m$ for x in A_m. Hence from (iii), (2.12) and (2.15) $I_2=o(1)$. It follows from (iv) and (v), exactly as in the proof of Theorem 1, that I_3 and I_4 are $o(1)$ as $m\to\infty$ and so

$$\|(\tilde{T}-\lambda)\,\varphi^{(m)}\|\to 0$$

as $m\to\infty$. The theorem is therefore proved.

Let us finally in this section consider examples of functions V and vectors Q for which the conditions of the last theorem are satisfied. We concentrate on the kind of scalar potential V investigated by Titchmarsh in [7] (see § 1).

Let A be a region in R^3 which contains a sequence of cubes B'_m with side $2b_m$, where $b_m\to\infty$ as $m\to\infty$ and having the property that for any $x\in B'_m$, $\rho=(x_1^2+x_2^2)^{\frac{1}{2}}\to\infty$ if and only if $m\to\infty$. A could be, for instance, any regular semi-infinite cone with vertex on the x_3-axis and whose axis is perpendicular to the x_3-axis, in which case B'_m could be taken as the cube inscribed in A with centre on the axis of A at a distance m from the vertex.

Corollary 3. *Suppose* $V(x)=\rho\,v(\rho)+w(x)$ *in* A *(arbitrary otherwise) where* $|w(x)|\to 0$ *as* $\rho\to\infty$ *in* A *and* v *is a function with a continuous first derivative and which satisfies any one of the following conditions:*

i) $|v(\rho)|\to k$, *where* $|h|<k\leqq\infty$, *and* $v'(\rho)=o\big(v(\rho)\big)$ *as* $\rho\to\infty$.

ii) $v(\rho)=h\left(1+\dfrac{u(\rho)}{\rho}\right)$ *where* $u(\rho)\to+\infty$, $\dfrac{u(\rho)}{\rho}\to 0$ *and* $u'(\rho)=O\big(u(\rho)\big)$ *as* $\rho\to\infty$.

Let $Q(x)=h(-x_2,x_1,0)+\big(q_1(x),q_2(x),q_3(x)\big)$ *in* A *where* $\sum_{j=1}^{3}q_j^2(x)\to 0$ *as* $\rho\to\infty$ *in* A.

Then the spectrum of $\tilde{T}$ *consists of* $(-\infty,\infty)$.

This Corollary follows easily from Theorem 2 with $v_m=\rho\,v$.

3. Separation of Variables

In Corollary 3 above we have a description of the spectrum of $\tilde{T}$ when $Q=h(-x_2,x_1,0)$, $V=V(\rho)$ and $V(\rho)/\rho$ is restricted from being too small as $\rho\to\infty$. We now investigate the case when $V(\rho)/\rho\to 0$ as $\rho\to\infty$. With such Q and V

$$\tau=\sum_{j=1}^{3}\alpha_j(i\,\partial_j-Q_j)+\alpha_4+VI$$

is separable in cylindrical co-ordinates and we shall mainly concern ourselves with the ρ equation which results from this separation. Our investigations are incomplete in the sense that we are unable to derive from our results the spectrum of $\tilde{T}$, except in the case $V=0$.

Let $Q(x)=h(-x_2,x_1,0)$, $V=V(\rho)$, and put $x_1=\rho\cos\theta$, $x_2=\rho\sin\theta$, $x_3=z$ so that

$$\partial_1=\cos\theta\,\partial_\rho-\frac{\sin\theta}{\rho}\,\partial_\theta,\qquad \partial_2=\sin\theta\,\partial_\rho+\frac{\cos\theta}{\rho}\,\partial_\theta$$

where we have written ∂_ρ, ∂_θ, for $\partial/\partial\rho$ and $\partial/\partial\theta$ respectively. Let

$$\psi(x)=\rho^{-\frac{1}{2}}\begin{pmatrix}\varphi_1(\rho)\,w_1(z)\,e^{in\theta}\\ -i\,\varphi_2(\rho)\,w_2(z)\,e^{i(n+1)\theta}\\ -\varphi_3(\rho)\,w_2(z)\,e^{in\theta}\\ -i\,\varphi_4(\rho)\,w_1(z)\,e^{i(n+1)\theta}\end{pmatrix}. \tag{3.1}$$

Then, we get, with $q(n,\rho)=\dfrac{n+\frac{1}{2}}{\rho}+h\rho$,

$$\tau\psi(x)= \tag{3.2}$$
$$\rho^{-\frac{1}{2}}\begin{pmatrix}\{[(V-1)\varphi_1+(\partial_\rho+q(n,\rho))\varphi_4]\,w_1-i\varphi_3\,\partial_z w_2\}\,e^{in\theta}\\ \{-[(-V+1)\varphi_2+(-\partial_\rho+q(n,\rho))\varphi_3]\,w_2-i\varphi_4\,\partial_z w_1\}\,\bullet\,-i\,e^{i(n+1)\theta}\\ \{[(-V-1)\varphi_3+(\partial_\rho+q(n,\rho))\varphi_2]\,w_2+i\varphi_1\,\partial_z w_1\}\,e^{in\theta}\\ \{[(V+1)\varphi_4+(-\partial_\rho+q(n,\rho))\varphi_1]\,w_1-i\varphi_2\,\partial_z w_2\}\,\bullet\,-i\,e^{i(n+1)\theta}\end{pmatrix}.$$

In the case $V=0$ we put $\varphi_3=\varphi_1$, $\varphi_4=\varphi_2$ so that separation of variables gives rise to the linked pair of equations

$$-\varphi_1+[\partial_\rho+q(n,\rho)]\,\varphi_2=\lambda\varphi_1,$$
$$\varphi_2+[-\partial_\rho+q(n,\rho)]\,\varphi_1=\lambda\varphi_2$$

or, in matrix form,

$$(\tau_n-\lambda)\,\varphi=0$$

where

$$\tau_n=\beta_1\partial_\rho+\left[\frac{n+\frac{1}{2}}{\rho}+h\rho\right]\beta_2+\beta_3,$$
$$\beta_1=\begin{pmatrix}&1\\-1&\end{pmatrix},\quad \beta_2=\begin{pmatrix}1&\\&1\end{pmatrix},\quad \beta_3=\begin{pmatrix}-1&\\&1\end{pmatrix} \tag{3.3}$$

and $\varphi=(\varphi_1,\varphi_2)$. Note that the matrices β_i satisfy

$$\beta_i\beta_j=-\beta_j\beta_i\quad(i\neq j),\qquad -\beta_1^2=\beta_2^2=\beta_3^2=I. \tag{3.4}$$

For a general $V=V(\rho)$ we shall consider the equations

$$(\tau_n+V-\lambda)\,\varphi=0.$$

Such equations with V and $-V$ do arise from the separation of variables if for instance ψ is assumed to be independent of z or if either $\varphi_2=\varphi_3=0$ or $\varphi_1=\varphi_4=0$. In general, separation of variables leads to a system of four equations in the variable ρ. However a study of these equations has not been able to help us in our quest for the spectrum of $\tilde{T}$. In view of this and the fact that this system of four equations can be treated in a similar way to the equations $(\tau_n+V-\lambda)\,\varphi=0$ we shall confine our attention to the latter set of equations.

Another reason for doing this is that the equations $(\tau_n + V - \lambda)\varphi = 0$ are the ones considered by Titchmarsh in [7] and one of our aims is to generalise his results.

4. The Case $V=0$

Let $\mathscr{H}_\rho$ be the space of all measurable C^2-valued functions u which are such that

$$\|u\|^2 = \int_0^\infty |u|^2 \, d\rho < \infty, \qquad |u|^2 = |u_1|^2 + |u_2|^2,$$

with inner product

$$(u, v) = \int_0^\infty u \cdot v \, d\rho, \qquad u \cdot v = u_1 \bar{v}_1 + u_2 \bar{v}_2,$$

where $u=(u_1, u_2)$, $v=(v_1, v_2)$.

Let T_n be the operator defined by $T_n\varphi = \tau_n\varphi$ on the set $\mathscr{D}(T_n)$ of functions $\varphi \in \mathscr{H}_\rho$ which satisfy the conditions:

i) φ has a continuous first derivative and its support is a compact subset of $[0, \infty)$,

ii) $\varphi_1(\rho) = O(\rho^{|n|+\frac{1}{2}})$, $\varphi_2(\rho) = O(\rho^{|n+1|+\frac{1}{2}})$ as $\rho \to 0$,

iii) $\tau_n\varphi \in \mathscr{H}_\rho$.

It can easily be shown as in ([8], § 4) that the equation $(\tau_n - \lambda)u = 0$ has a solution φ which satisfies (ii) above. Moreover, up to linear dependence, φ is the only solution which is square integrable at $\rho = 0$. For, if ψ is another such solution, we have for all n and $\rho \to 0$, since the bilinear form $\varphi_1\psi_2 - \psi_1\varphi_2$ is independent of ρ,

$$\begin{aligned}
\rho(\varphi_1\psi_2 - \psi_1\varphi_2) &= \int_0^\rho (\varphi_1\psi_2 - \psi_1\varphi_2)\, dt \\
&= O\left(\int_0^\rho t^{\frac{1}{2}} |\psi| \, dt\right) \\
&= O\left(\int_0^\rho t \, dt \int_0^\rho |\psi|^2 \, dt\right)^{\frac{1}{2}} \\
&= O\left(\rho \left(\int_0^\rho |\psi|^2 \, dt\right)^{\frac{1}{2}}\right) \\
&= o(\rho).
\end{aligned}$$

Hence $\varphi_1\psi_2 - \psi_1\varphi_2 = 0$ for all values of n and so ψ and φ are linearly dependent. τ_n is easily shown to be in the limit point case at infinity and it follows that T_n is essentially self-adjoint (see [2], § 2). The unique self-adjoint extension $\tilde{T}_n$ of T_n is the maximal operator generated by τ_n in $\mathscr{H}_\rho$, namely, the operator defined by $\tilde{T}_n\varphi = \tau_n\varphi$ on the set of functions $\varphi \in \mathscr{H}_\rho$ which are absolutely continuous on any compact subset of $(0, \infty)$ and satisfy $\tau_n\varphi \in \mathscr{H}_\rho$.

In [7] it is shown that the spectrum of $\tilde{T}_n$ is discrete, consisting of eigenvalues

$$\lambda_{\pm m}^{(n)} = \pm\{1+4|h|\, m+2(|h\, n|+h\, n)\}^{\frac{1}{2}} \tag{4.1}$$

$m=1, 2, 3, \ldots$ with eigenvectors

$$\begin{aligned}\varphi_1^{(n,\pm m)}(\rho) &= \rho^{-\frac{1}{2}} W_{m+\frac{1}{2}|n|+\frac{1}{2},\frac{1}{2}|n|}(h\rho^2)\\ \varphi_2^{(n,\pm m)}(\rho) &= \frac{2h^{\frac{1}{2}}}{(\lambda_{\pm m}^{(n)}-1)}\rho^{-\frac{1}{2}} W_{m+\frac{1}{2}|n|+1,\frac{1}{2}|n+1|}(h\rho^2)\end{aligned} \tag{4.2}$$

when $h>0$, and

$$\begin{aligned}\varphi_1^{(n,\pm m)}(\rho) &= \rho^{-\frac{1}{2}} W_{m+\frac{1}{2}|n+1|+1,\frac{1}{2}|n|}(|h|\rho^2)\\ \varphi_2^{(n,\pm m)}(\rho) &= \frac{2|h|^{\frac{1}{2}}}{(\lambda_{\pm m}^{(n)}-1)}\rho^{-\frac{1}{2}} W_{m+\frac{1}{2}|n+1|+\frac{1}{2},\frac{1}{2}|n+1|}(|h|\rho^2)\end{aligned} \tag{4.3}$$

when $h<0$, W being the Whittaker function. We note from (4.1) that, for all m, n, $\lambda_m^{(n)} \geqq (1+4|h|)^{\frac{1}{2}}$, this lower bound being attained when $m=1$, $n=0$.

In order to derive the spectrum of $\tilde{T}$ it is necessary to investigate the relationship which exists between $\tilde{T}$ and those operators determined by τ in the Hilbert spaces $\mathscr{H}_\rho$ and $\mathscr{H}_z$, the latter being the space of measurable C^2-valued functions $w=(w_1, w_2)$ of z, $-\infty<z<\infty$ for which

$$\|w\|_z^2 = \int_{-\infty}^{\infty} |w|^2\, dz < \infty.$$

For similar considerations see [6], chapters 1 and 3. We shall denote the inner products of $\mathscr{H}_\rho$ and $\mathscr{H}_z$ by $(\cdot,\cdot)_\rho$, $(\cdot,\cdot)_z$ respectively and also write $(\cdot,\cdot)_\theta$ for the inner product of $L^2(-\pi, \pi)$.

Let $f_1^{(n)}(\theta) = (2\pi)^{-\frac{1}{2}} e^{in\theta}$, $f_2^{(n)}(\theta) = -i\, e^{i\theta} f_1^{(n)}(\theta)$ and $f^{(n)} = (f_1^{(n)}, f_2^{(n)})$, $n=0, \pm 1, \ldots$. If $\Omega=(\Omega_1, \Omega_2, \Omega_3, \Omega_4)$, $\varphi=(\varphi_1, \varphi_2)$, $w=(w_1, w_2)$ are vector-valued functions of (ρ, z), ρ and z respectively we define:

$$\begin{aligned}\Omega f^{(n)} = f^{(n)}\Omega &= (\Omega_1 f_1^{(n)}, \Omega_2 f_2^{(n)}, \Omega_3 f_1^{(n)}, \Omega_4 f_2^{(n)}),\\ \varphi w = w\varphi &= \rho^{-\frac{1}{2}}(\varphi_1 w_1, \varphi_2 w_2, -\varphi_1 w_2, \varphi_2 w_1).\end{aligned} \tag{4.4}$$

Note that if $\varphi\in\mathscr{H}_\rho$ and $w\in\mathscr{H}_z$ then $\psi=\varphi w f^{(n)}\in\mathscr{H}$ and if $\psi'=\varphi' w' f^{(m)}$, then

$$(\psi, \psi') = (\varphi, \varphi')_\rho (w, w')_z (f_1^{(n)}, f_1^{(m)})_\theta \tag{4.5}$$

where, of course, $(f_1^{(n)}, f_1^{(m)})_\theta = \delta_{n,m}$.

Let S be the operator $i\beta_2\partial_z$ on the linear manifold $\mathscr{D}(S)$ of functions in $\mathscr{H}_z$ whose components belong to $C_0^\infty(-\infty, \infty)$. Let $\mathscr{U}$ be the linear manifold in $\mathscr{H}$ which is spanned by $\psi^{(n,m,k)} = \varphi^{(n,m)} w^{(k)} f^{(n)}$, where $\{\varphi^{(n,m)} | m=\pm 1, \pm 2, \ldots\}$ is the complete orthonormal set of eigenvectors of $\tilde{T}_n$ exhibited above ((4.2), (4.3)) and $\{w^{(k)} | k=1, 2, \ldots\}$ is a sequence which spans $\mathscr{D}(S)$ (a Hamel basis for $\mathscr{D}(S)$). Denoting by P the operator defined on $\mathscr{U}$ by $P\psi^{(n,m,k)} = \tau\psi^{(n,m,k)}$ and by linearity, we have the following lemma.

Lemma 4. *The operator P is essentially self-adjoint.*

Proof. We need to prove first that $\mathscr{U}$ is dense in $\mathscr{H}$ and, in fact, once this has been established the remainder presents no problem. For this, we are required to show that for any $F\in\mathscr{H}$ and $\varepsilon>0$, there exist positive integers M, N, K and complex numbers $\alpha_{n,m,k}$ such that

$$\left\| F - \sum_{\substack{0\leqq|n|\leqq N\\ 1\leqq|m|\leqq M\\ 1\leqq k\leqq K}} \alpha_{n,m,k}\psi^{(n,m,k)} \right\| < \varepsilon.$$

It is clearly sufficient to prove this for any F which is continuous and has compact support. Let the support of F be contained in the cylinder $C=\{x|0\leqq\rho\leqq R, 0\leqq|z|\leqq T, -\pi\leqq\theta\leqq\pi\}$. If $\Omega_i^{(n)}(\rho,z)$ are the Fourier co-efficients

$$\Omega_i^{(n)}(\rho,z)=\int_{-\pi}^{\pi} F_i(x)\,\bar{f}_i^{(n)}(\theta)\,d\theta$$

where $f_3^{(n)}=f_1^{(n)}$, $f_4^{(n)}=f_2^{(n)}$, then there exists an integer N such that for all (ρ,z) in $[0,R]\times[-T,T]$ and $i=1,2,3,4$,

$$\|F_i(\rho,\cdot,z)-\sum_{0\leqq|n|\leqq N}\Omega_i^{(n)}(\rho,z)\,f_i^{(n)}\|_\theta^2<\frac{\varepsilon^2}{36R^2T}$$

and consequently

$$\|F-\sum_{0\leqq|n|\leqq N}\Omega^{(n)}f^{(n)}\|^2<\frac{\varepsilon^2}{9}. \tag{4.6}$$

Let $\Gamma_1^{(n)}=(\Omega_1^{(n)},\Omega_4^{(n)})$ and $\Gamma_2^{(n)}=(-\Omega_3^{(n)},\Omega_2^{(n)})$. Then, for each z in $[-T,T]$, $\rho^{\frac{1}{2}}\Gamma_1^{(n)}$ and $\rho^{\frac{1}{2}}\Gamma_2^{(n)}\in\mathscr{H}_\rho$, being in fact continuous functions of (ρ,z) with support in $[0,R]\times[-T,T]$. Putting

$$c_i^{(n,m)}(z)=(\rho^{\frac{1}{2}}\Gamma_i^{(n)}(\cdot,z),\varphi^{(n,m)})_\rho$$

we have, since $\{\varphi^{(n,m)}\}$ is a complete orthonormal set for each n, that there exists an integer M such that for $0\leqq|n|\leqq N$ and $0\leqq|z|\leqq T$,

$$\|\rho^{\frac{1}{2}}\Gamma_i^{(n)}(\cdot,z)-\sum_{1\leqq|m|\leqq M}c_i^{(n,m)}(z)\,\varphi^{(n,m)}\|_\rho^2<\frac{\varepsilon^2}{36T(2N+1)}$$

$i=1,2$. Consequently,

$$\sum_{\substack{0\leqq|n|\leqq N\\ i=1,2}}\int_{-\infty}^{\infty}\int_0^\infty|\Gamma_i^{(n)}(\rho,z)-\sum_{1\leqq|m|\leqq M}c_i^{(n,m)}(z)\,\rho^{-\frac{1}{2}}\varphi^{(n,m)}(\rho)|^2\,\rho\,d\rho\,dz<\frac{\varepsilon^2}{9}. \tag{4.7}$$

Finally, $c^{(n,m)}=(c_1^{(n,m)},c_2^{(n,m)})\in\mathscr{H}_z$ and so, since the span of $\{w^{(k)}\}$ is dense in $\mathscr{H}_z$ there exists an integer K and complex numbers $\alpha_{n,m,k}$ such that, for $0\leqq|n|\leqq N$,

$1 \leqq |m| \leqq M$,

$$\left\| c^{(n,m)} - \sum_{1 \leqq k \leqq K} \alpha_{n,m,k} w^{(k)} \right\|_z^2 < \frac{\varepsilon^2}{9(2N+1)(2M)}. \tag{4.8}$$

Hence, with the understanding that $1 \leqq |m| \leqq M$, $0 \leqq |n| \leqq N$, $1 \leqq k \leqq K$, in all the sums that appear, we have from (4.6), (4.7), (4.8) and the orthonormality of $\{f^{(n)}\}$ and $\{\varphi^{(n,m)}\}$ for each n,

$$\begin{aligned}
&\left\| F - \sum_{n,m,k} \alpha_{n,m,k} \varphi^{(n,m)} w^{(k)} f^{(n)} \right\| \\
&\quad \leqq \left\| F - \sum_n \Omega^{(n)} f^{(n)} \right\| + \left\| \sum_n \left(\Omega^{(n)} - \sum_m c^{(n,m)} \varphi^{(n,m)} \right) f^{(n)} \right\| \\
&\qquad + \left\| \sum_n \sum_m \left(c^{(n,m)} - \sum_k \alpha_{n,m,k} w^{(k)} \right) \varphi^{(n,m)} f^{(n)} \right\| \\
&\quad = \left\| F - \sum_n \Omega^{(n)} f^{(n)} \right\| \\
&\qquad + \left\{ \sum_{n=1,2} \int_{-\infty}^{\infty} \int_0^{\infty} \left| \Gamma_i^{(n)}(\rho, z) - \sum_m c_i^{(n,m)}(z) \rho^{-\frac{1}{2}} \varphi^{(n,m)}(\rho) \right|^2 \rho \, d\rho \, dz \right\}^{\frac{1}{2}} \\
&\qquad + \left\{ \sum_{n,m} \left\| c^{(n,m)} - \sum_k \alpha_{n,m,k} w^{(k)} \right\|_z^2 \right\}^{\frac{1}{2}} \\
&\quad < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\end{aligned}$$

as we set out to prove.

It remains for us to prove that P is symmetric and essentially self-adjoint. From (3.2) ($V=0$)

$$\begin{aligned}
P\psi^{(n,m,k)} &= \rho^{-\frac{1}{2}} \begin{pmatrix} [\lambda_m^{(n)} w_1 - i \partial_z w_2] \varphi_1^{(n,m)} f_1^{(n)} \\ [-\lambda_m^{(n)} w_2 - i \partial_z w_1] \varphi_2^{(n,m)} f_2^{(n)} \\ [\lambda_m^{(n)} w_2 + i \partial_z w_1] \varphi_1^{(n,m)} f_1^{(n)} \\ [\lambda_m^{(n)} w_1 - i \partial_z w_2] \varphi_2^{(n,m)} f_2^{(n)} \end{pmatrix} \\
&= \varphi^{(n,m)} (S^{(n,m)} w^{(k)}) f^{(n)}
\end{aligned}$$

in the notation of (4.4), where $S^{(n,m)}$ is the symmetric operator $-\lambda_m^{(n)} \beta_3 - S$. Hence, from (4.5)

$$\begin{aligned}
(P\psi^{(n,m,k)}, \psi^{(n',m',k')}) &= (\varphi^{(n,m)}, \varphi^{(n',m')})_\rho (S^{(n,m)} w^{(k)}, w^{(k')})_z (f_1^{(n)}, f_1^{(n')})_\theta \\
&= (S^{(n,m)} w^{(k)}, w^{(k')})_z \delta_{n,n'} \delta_{m,m'} = (S^{(n',m')} w^{(k)}, w^{(k')})_z \delta_{nn'} \delta_{mm'} \\
&= (w^{(k)}, S^{(n',m')} w^{(k')})_z \delta_{nn'} \delta_{mm'} = (\psi^{(n,m,k)}, P\psi^{(n',m',k')})
\end{aligned}$$

and, by linearity, P is symmetric on $\mathscr{U}$. In order to prove that P is essentially self-adjoint it is sufficient to show that $(P-\lambda)\mathscr{U}$ is dense in $\mathscr{H}$ for any non-real λ. For any $w \in \mathscr{D}(S)$,

$$(P-\lambda) \varphi^{(n,m)} w f^{(n)} = \varphi^{(n,m)} (S^{(n,m)} - \lambda) w f^{(n)}.$$

But, $S^{(n,m)}$ is known to be essentially self-adjoint in $\mathscr{H}_z$ for each n, m (since S is) and so $(S^{(n,m)}-\lambda)\,\mathscr{D}(S)$ is dense in $\mathscr{H}_z$. It therefore follows, as for $\mathscr{U}$ above, that $(P-\lambda)\,\mathscr{U}$ is dense in $\mathscr{H}$ and the lemma is proved.

Since $V=0$ in τ, T is essentially self-adjoint and its unique self-adjoint extension $\tilde{T}$ is the closure of T, this being in fact the operator defined by $\tilde{T}u=\tau u$ in the distribution sense on $\mathscr{D}(\tilde{T})=\{u|u\in\mathscr{H}\cap\mathscr{H}_{1\,\mathrm{loc}}, \tau u\in\mathscr{H}\}$ (see § 2 and [2], § 3). All the vectors ψ in $\mathscr{U}$ above have continuous first derivatives and also satisfy $\psi\in\mathscr{H}$, $\tau\psi\in\mathscr{H}$. Thus $\mathscr{U}\subset\mathscr{D}(\tilde{T})$, $P\subset\tilde{T}$ and since P is essentially self-adjoint, $\tilde{T}=\bar{P}$ the closure of P.

At last we are in a position to establish the spectrum of $\tilde{T}$. We make use of the criterion that λ is in the spectrum $\sigma(\tilde{T})$ of $\tilde{T}$ if and only if, given $\varepsilon>0$, there exists a $\varphi\in\mathscr{D}(\tilde{T})$ such that $\|(\tilde{T}-\lambda)\,\varphi\|<\varepsilon\,\|\varphi\|$; or equivalently, since $\tilde{T}=\bar{P}$, if and only if there exists a $\psi\in\mathscr{U}$ such that $\|(P-\lambda)\,\psi\|<\varepsilon\,\|\psi\|$. For $w\in\mathscr{D}(S)$

$$\begin{aligned}\|(P-\lambda)\,\varphi^{(n,m)}\,w f^{(n)}\| &= \|\varphi^{(n,m)}(S^{(n,m)}-\lambda)\,w f^{(n)}\| \\ &= \|\varphi^{(n,m)}\|_\rho\,\|(S^{(n,m)}-\lambda)\,w\|_z\end{aligned} \tag{4.9}$$

and this implies that $\lambda\in\sigma(\tilde{T})$ if $\lambda\in\sigma(S^{(n,m)})$ for any n, m. Since

$$(S^{(n,m)})^2=\{(\lambda_m^{(n)})^2-\partial_z^2\}\,I$$

and $-\partial_z^2$ has spectrum $[0,\infty)$, it follows that $\sigma\{(S^{(n,m)})^2\}=\big[(\lambda_m^{(n)})^2,\infty\big)$. But, if $\lambda\in\sigma(S^{(n,m)})$ then so does $-\lambda$, for if $\|(S^{(n,m)}-\lambda)\,w\|<\varepsilon\,\|w\|$ then also, in view of (3.4),

$$\begin{aligned}\|(S^{(n,m)}+\lambda)\,\beta_1 w\| &= \|\beta_1(-S^{(n,m)}+\lambda)\,w\| = \|(S^{(n,m)}-\lambda)\,w\| \\ &< \varepsilon\,\|w\| = \varepsilon\,\|\beta_1 w\|.\end{aligned}$$

Hence

$$\sigma(S^{(n,m)})=\{\lambda\,|\,|\lambda|\geqq|\lambda_m^{(n)}|\}$$

and

$$\sigma(\tilde{T})\supset\{\lambda\,|\,|\lambda|\geqq\min_{n,m}|\lambda_m^{(n)}|=(1+4|h|)^{\frac{1}{2}}\}.$$

Conversely, if $\lambda\in\sigma(\tilde{T})$ there exists a finite linear combination $\psi=\sum\varphi^{(n,m)}\,w f^{(n)}$, $w\in\mathscr{D}(S)$, such that

$$\begin{aligned}\varepsilon^2\,\|\psi\|^2 &> \|(P-\lambda)\,\psi\|^2 \\ &= \|\sum\varphi^{(n,m)}(S^{(n,m)}-\lambda)\,w f^{(n)}\|^2 \\ &= \sum\|\varphi^{(n,m)}\|_\rho^2\,\|(S^{(n,m)}-\lambda)\,w\|_z^2.\end{aligned} \tag{4.10}$$

If $|\lambda|<\min_{m,n}|\lambda_m^{(n)}|$, then $\|(S^{(n,m)}-\lambda)\,w\|_z>\varepsilon\,\|w\|_z$ for all $w\in\mathscr{D}(S)$ and all m, n, if ε is sufficiently small, since $(-|\lambda_m^{(n)}|, |\lambda_m^{(n)}|)$ is in the resolvent set of $S^{(n,m)}$. This leads to a contradiction in (4.10). We have therefore proved.

Theorem 5. $\sigma(\tilde{T})=\{\lambda\,|\,|\lambda|\geqq(1+4|h|)^{\frac{1}{2}}\}$.

5. Relatively Compact Perturbations of $\tilde{T}_n$

Let V be the self-adjoint operator defined by multiplication by $V(\rho)$ in $\mathscr{H}_\rho$. If $V \in L^2_{\mathrm{loc}}[0, \infty)$, the operator $T_n + V$ is symmetric on $\mathscr{D}(T_n)$ (see § 4) for each n. We shall prove the following:

Theorem 6. *V is a relatively compact perturbation of $\tilde{T}_n$ if $V(\rho)$ satisfies the conditions:*

i) $V \in L^2_{\mathrm{loc}}(0, \infty)$,

ii) $V(\rho)/\rho \to 0$ *as* $\rho \to \infty$,

iii) *if* $n \neq 0, -1$, $\rho V(\rho) \to 0$ *as* $\rho \to 0$,

iv) *if* $n = 0, -1$, $\rho^{\frac{1+\alpha}{2}} V(\rho) \to 0$ *as* $\rho \to 0$ *for some* α *satisfying* $0 \leqq \alpha < 1$.

Under these conditions on V, $T_n + V$ is essentially self-adjoint and its unique self-adjoint extension has a wholly discrete spectrum.

The proof will follow from a series of lemmas given below, and is based on the method used by Jorgens in [4]. For the notion of relative compactness and its consequences we refer the reader to [4] and [5].

The first step of the proof is to obtain an integral representation for any $\varphi \in \mathscr{D}(T_n)$. Let $g_\delta(t, s)$ be the function defined on $[0, \infty) \times [0, \infty)$ by

$$g_\delta(t, s) = \begin{cases} 1 - \dfrac{t-s}{\delta} & (t - \delta \leqq s \leqq t), \\ -1 + \dfrac{s-t}{\delta} & (t < s \leqq t + \delta), \\ 0 & (|t-s| \geqq \delta). \end{cases}$$

Then, for $\varphi \in \mathscr{D}(T_n)$ we obtain

$$\begin{aligned} \int_0^\infty g_\delta(t, s)\, \varphi'(s)\, ds &= \int_{|t-s| \leqq \delta} g_\delta(t, s)\, \varphi'(s)\, ds \\ &= 2\varphi(t) - \int_{|t-s| \leqq \delta} h_\delta(t, s)\, \varphi(s)\, ds \end{aligned}$$

where

$$h_\delta(t, s) = \begin{cases} \dfrac{1}{\delta} & (|t-s| \leqq \delta), \\ 0 & \text{otherwise}. \end{cases}$$

Therefore, since

$$T_n \varphi = \beta_1 \varphi' + \left[\frac{n + \frac{1}{2}}{\rho} + h\rho \right] \beta_2 \varphi + \beta_3 \varphi \quad \text{and} \quad \beta_1^2 = -I,$$

we get

$$\begin{aligned} \varphi(t) &= -\tfrac{1}{2} \int_{|t-s| \leqq \delta} g_\delta(t, s)\, \beta_1 (\beta_1 \varphi'(s))\, ds + \tfrac{1}{2} \int_{|t-s| \leqq \delta} h_\delta(t, s)\, \varphi(s)\, ds \\ &= \int_{|t-s| \leqq \delta} G_\delta(t, s)\, T_n \varphi(s)\, ds + \int_{|t-s| \leqq \delta} H_\delta(t, s)\, \varphi(s)\, ds \end{aligned} \tag{5.1}$$

where

$$G_\delta(t,s) = -\tfrac{1}{2}\, g_\delta(t,s)\,\beta_1,$$
$$H_\delta(t,s) = \tfrac{1}{2}\, h_\delta(t,s)\, I + \tfrac{1}{2}\, g_\delta(t,s)\left\{\left[\frac{n+\frac{1}{2}}{s} + h\, s\right]\beta_2 + \beta_3\right\}.$$

Lemma 7. *Let Ω be any open interval (a, b) with $a>0$, and let $\Omega_\delta=(a-\delta, b+\delta)$. Then, for $V\in L^2_{\mathrm{loc}}(0,\infty)$, $\varphi\in\mathscr{D}(T_n)$ and $\delta<\frac{1}{2}a$*

$$\|V\varphi\|^2_\Omega \leqq c_1\,\delta\,\|T_n\varphi\|^2_{\Omega_\delta} + \frac{c_2}{\delta}\,\|\varphi\|^2_{\Omega_\delta}$$

where c_1 and c_2 are constants which depend only on n and Ω.

Here $\|\cdot\|_\Omega$ denotes the $\mathscr{H}_\rho$ norm with the integration over Ω. Throughout this section the $\mathscr{H}_\rho$ norm itself will be written $\|\cdot\|$, there being no possibility of confusion with the $\mathscr{H}$ norm.

Proof. For $t\in\Omega$, $s\in\Omega_\delta$ and $\delta<\frac{1}{2}a$ we have for any $\psi=(\psi_1,\psi_2)$

$$\begin{aligned} &|G_\delta(t,s)\,\psi(s)| \leqq \tfrac{1}{2}\,|\psi(s)|,\\ &|H_\delta(t,s)\,\psi(s)| \leqq \frac{1}{2}\left\{\frac{1}{\delta} + \frac{|n+1|}{s} + |h|\,s + 1\right\}|\psi(s)| \\ &\qquad\qquad < \frac{c}{\delta}\,|\psi(s)| \end{aligned} \tag{5.2}$$

where c is a constant depending only on Ω and n. Hence, from (5.1)

$$\begin{aligned} |\varphi(t)|^2 &\leqq 2\Big|\int_{|t-s|\leqq\delta} G_\delta(t,s)\,T_n\varphi(s)\,ds\Big|^2 + 2\Big|\int_{|t-s|\leqq\delta} H_\delta(t,s)\,\varphi(s)\,ds\Big|^2\\ &\leqq 4\delta\int_{|t-s|\leqq\delta}|G_\delta(t,s)\,T_n\varphi(s)|^2\,ds + 4\delta\int_{|t-s|\leqq\delta}|H_\delta(t,s)\,\varphi(s)|^2\,ds\\ &\leqq \delta\int_{|t-s|\leqq\delta}|T_n\varphi|^2\,ds + \frac{4c^2}{\delta}\int_{|t-s|\leqq\delta}|\varphi|^2\,ds. \end{aligned}$$

Consequently

$$\begin{aligned} \int_\Omega V^2(t)\,|\varphi(t)|^2\,dt &\leqq \delta\int_\Omega V^2(t)\Big\{\int_{|t-s|\leqq\delta}|T_n\varphi|^2\,ds\Big\}\,dt + \frac{4c^2}{\delta}\int_\Omega V^2(t)\Big\{\int_{|t-s|\leqq\delta}|\varphi|^2\,ds\Big\}\,dt\\ &\leqq \delta\int_{\Omega_\delta}|T_n\varphi|^2\,N_V(s)\,ds + \frac{4c^2}{\delta}\int_{\Omega_\delta}|\varphi|^2\,N_V(s)\,ds \end{aligned}$$

where $N_V(s)=\int_{|t-s|\leqq\delta} V^2(t)\,dt$. Since $V\in L^2_{\mathrm{loc}}(0,\infty)$, N_V is bounded on $\Omega_{\frac{1}{2}a}\supset\Omega_\delta$ and hence the lemma follows.

Lemma 8. *For $\varphi \in \mathscr{D}(T_n)$ there exist positive constants c_1, c_2 depending only on n such that*

i) $\|t\varphi\|^2 \leqq c_1 \|T_n \varphi\|^2 + c_2 \|\varphi\|^2$.

ii) $\|t^{-1}\varphi\|^2 \leqq c_1 \|T_n \varphi\|^2 + c_2 \|\varphi\|^2$ *if* $n \neq 0, -1$.

iii) $\|t^{-\frac{1+\alpha}{2}} \varphi\|^2 \leqq c_1 \|T_n \varphi\|^2 + c_2 \|\varphi\|^2$ *if* $n = 0, -1$, *where* α *is any real number satisfying* $0 \leqq \alpha < 1$.

Proof. Since τ_n is real it is clearly sufficient to prove the lemma for real φ.

i) $$h \|t\varphi\|^2 = (ht\beta_2 \varphi, t\beta_2 \varphi)$$
$$= \left(\left[T_n - \beta_1 \partial_t - \frac{n+\frac{1}{2}}{t} \beta_2 - \beta_3 \right] \varphi, t\beta_2 \varphi \right)$$
$$= (T_n \varphi, t\beta_2 \varphi) - (\beta_1 \varphi', t\beta_2 \varphi) - (n+\tfrac{1}{2}) \|\varphi\|^2$$

since $\beta_3 \varphi \cdot \beta_2 \varphi = 0$. Now, for any positive real number μ,

$$|(T_n \varphi, t\beta_2 \varphi)| \leqq \|T_n \varphi\| \, \|t\beta_2 \varphi\| = \|T_n \varphi\| \, \|t\varphi\|$$
$$\leqq \tfrac{1}{2} \left(\mu^2 \|T_n \varphi\|^2 + \frac{1}{\mu^2} \|t\varphi\|^2 \right).$$

Also,

$$-(\beta_1 \varphi', t\beta_2 \varphi) = -\int_0^\infty t(\varphi_2 \varphi_2' - \varphi_1 \varphi_1') \, dt$$
$$= -\tfrac{1}{2} \int_0^\infty t\{(\varphi_2^2)' - (\varphi_1^2)'\} \, dt = \tfrac{1}{2} \int_0^\infty (\varphi_2^2 - \varphi_1^2) \, dt \leqq \tfrac{1}{2} \|\varphi\|^2$$

since φ has compact support and $\varphi_i(t) = o(1)$ as $t \to 0$. Hence

$$|h| \, \|t\varphi\|^2 \leqq \tfrac{1}{2} \left(\mu^2 \|T_n \varphi\|^2 + \frac{1}{\mu^2} \|t\varphi\|^2 \right) + \tfrac{1}{2} \|\varphi\|^2 + |n+\tfrac{1}{2}| \, \|\varphi\|^2$$

giving

$$\left(|h| - \frac{1}{2\mu^2} \right) \|t\varphi\|^2 \leqq \tfrac{1}{2} \mu^2 \|T_n \varphi\|^2 + (\tfrac{1}{2} + |n+\tfrac{1}{2}|) \|\varphi\|^2.$$

The assertion follows on choosing $\dfrac{1}{\mu^2} < 2|h|$.

ii) $$(n+\tfrac{1}{2}) \|t^{-1}\varphi\|^2 = \left(\frac{n+\frac{1}{2}}{t} \beta_2 \varphi, \beta_2 \frac{\varphi}{t} \right)$$
$$= (T_n \varphi, t^{-1}\beta_2 \varphi) - (\beta_1 \varphi', t^{-1}\beta_2 \varphi) - (t\beta_2 \varphi, t^{-1}\beta_2 \varphi).$$
$$(\beta_1 \varphi', t^{-1}\beta_2 \varphi) = \tfrac{1}{2} \int_0^\infty t^{-1}\{(\varphi_2^2)' - (\varphi_1^2)'\} \, dt = \tfrac{1}{2} \int_0^\infty t^{-2}(\varphi_2^2 - \varphi_1^2) \, dt$$
$$\leqq \tfrac{1}{2} \|t^{-1}\varphi\|^2$$

since, for $n \neq 0, -1$, $\varphi_i(t) = O(t^{\frac{3}{2}})$ as $t \to 0$. The remainder of the proof of (ii) follows as for (i) above.

iii) We shall prove this for $n=0$, the proof for $n=-1$ being similar.

$$\tfrac{1}{2}\|t^{-\frac{1+\alpha}{2}}\varphi\|^2 = \left(\frac{1}{2t}\beta_2\varphi, \frac{1}{t^\alpha}\beta_2\varphi\right)$$

$$= (T_n\varphi, t^{-\alpha}\beta_2\varphi) - (\beta_1\varphi', t^{-\alpha}\beta_2\varphi) - (ht\beta_2\varphi, t^{-\alpha}\beta_2\varphi).$$

$$|(T_n\varphi, t^{-\alpha}\beta_2\varphi)| = \left|\int_0^1 \frac{(T_n\varphi\cdot\beta_2\varphi)}{t^{\frac{1+\alpha}{2}}}\, t^{\frac{1-\alpha}{2}}\, dt + \int_1^\infty (T_n\varphi\cdot\beta_2\varphi)\, t^{-\alpha}\, dt\right|$$

$$\leqq \int_0^1 |T_n\varphi|\,|\varphi|\, t^{-\frac{1+\alpha}{2}}\, dt + \int_1^\infty |T_n\varphi|\,|\varphi|\, dt$$

$$\leqq \|T_n\varphi\|\,\|t^{-\frac{1+\alpha}{2}}\varphi\| + \|T_n\varphi\|\,\|\varphi\|$$

$$\leqq \tfrac{1}{2}\left\{\mu^2\|T_n\varphi\|^2 + \frac{1}{\mu^2}\|t^{-\frac{1+\alpha}{2}}\varphi\|^2\right\} + \tfrac{1}{2}\{\|T_n\varphi\|^2 + \|\varphi\|^2\}$$

for arbitrary $\mu \neq 0$.

$$(\beta_1\varphi', t^{-\alpha}\beta_2\varphi) = \tfrac{1}{2}\int_0^\infty t^{-\alpha}\{(\varphi_2^2)' - (\varphi_1^2)'\}\, dt = \frac{\alpha}{2}\int_0^\infty t^{-1-\alpha}(\varphi_2^2 - \varphi_1^2)\, dt$$

$$\leqq \frac{\alpha}{2}\|t^{-\frac{1+\alpha}{2}}\varphi\|^2$$

as long as $\alpha < 1$, since $\varphi_1(t) = O(t^{\frac{1}{2}})$, $\varphi_2(t) = O(t^{\frac{3}{2}})$ at $t=0$.

$$(t\beta_2\varphi, t^{-\alpha}\beta_2\varphi) = (t\varphi, t^{-\alpha}\varphi) \leqq \int_0^1 |\varphi|^2\, dt + \int_1^\infty t^2\,|\varphi|^2\, dt$$

$$\leqq \|\varphi\|^2 + \|t\varphi\|^2$$

$$\leqq c_1\|T_n\varphi\|^2 + c_2\|\varphi\|^2$$

from (i). The result now follows by a suitable choice of μ.

Proof of Theorem 6. Let $\Omega = (a, b)$, $a > 0$, where a, b are such that, given $\varepsilon > 0$,

$$|t^{-1}V(t)|^2 < \frac{\varepsilon}{3c_1} \qquad (b \leqq t < \infty),$$

$$|tV(t)|^2 < \frac{\varepsilon}{3c_1} \qquad (0 < t \leqq a,\ n \neq 0, -1),$$

$$|t^{\frac{1+\alpha}{2}}V(t)|^2 < \frac{\varepsilon}{3c_1} \qquad (0 < t \leqq a,\ n = 0, -1),$$

where c_1 is the constant in Lemma 8. From Lemma 8 we therefore have, for any $\varphi \in \mathscr{D}(T_n)$,

$$\int_b^\infty |V\varphi|^2\,dt = \int_b^\infty |t^{-1}\,V(t)\,t\,\varphi|^2\,dt$$

$$< \frac{\varepsilon}{3c_1}\,\|t\varphi\|^2 \tag{5.3}$$

$$< \frac{\varepsilon}{3}\left\{\|T_n\varphi\|^2 + \frac{c_2}{c_1}\,\|\varphi\|^2\right\}$$

and similarly

$$\int_0^a |V\varphi|^2\,dt < \frac{\varepsilon}{3}\left\{\|T_n\varphi\|^2 + \frac{c_2}{c_1}\,\|\varphi\|^2\right\}. \tag{5.4}$$

If in Lemma 7 we choose $\delta < \min\left(\frac{a}{2}, \frac{\varepsilon}{3c_1}\right)$ we get

$$\int_a^b |V\varphi|^2\,dt < \frac{\varepsilon}{3}\,\|T_n\varphi\|^2_{\Omega_\delta} + \frac{c_2}{\delta}\,\|\varphi\|^2_{\Omega_\delta}. \tag{5.5}$$

It therefore follows that

$$\|V\varphi\|^2 < \varepsilon\,\|T_n\varphi\|^2 + c\,\|\varphi\|^2$$

and since ε is arbitrary small ($\varepsilon < 1$), we have that V is a weak perturbation of T_n. In particular this implies that $T_n + V$ is essentially self-adjoint and its unique self-adjoint extension is $\tilde{T}_n + V$ on the same domain as $\tilde{T}_n$.

In order to prove that V is in fact $\tilde{T}_n$-compact it is sufficient to show that if $\{\varphi_m\} \subset \mathscr{D}(T_n)$ is such that φ_m and $T_n\varphi_m$ converge weakly to zero then $V\varphi_m$ converges strongly to zero (see (4) Lemma 1.3). Thus suppose $\varphi_m \rightharpoonup 0$, $T_n\varphi_m \rightharpoonup 0$. The sequences $\{\varphi_m\}$, $\{T_n\varphi_m\}$ are therefore bounded, and in view of (5.3), (5.4) and (5.5) above we can write, given $\varepsilon' > 0$,

$$\|V\varphi_m\|^2 < \frac{\varepsilon'}{2} + c\int_{\Omega_\delta} |\varphi_m|^2\,dt \tag{5.6}$$

where c depends only on Ω_δ. Let G_δ, H_δ be the integral operators with kernels $G_\delta(t, s)$, $H_\delta(t, s)$ respectively so that by (5.1), for any $\varphi \in \mathscr{D}(T_n)$

$$\varphi = G_\delta(T_n\varphi) + H_\delta\varphi. \tag{5.7}$$

For δ sufficiently small the kernels $G_\delta(t, s)$, $H_\delta(t, s)$ are bounded for $t \in \Omega_\delta$, $s \in \Omega_{2\delta}$ (see (5.2)) and so the operators G_δ, H_δ are compact (indeed Hilbert-Schmidt) operators mapping $\mathscr{H}_\rho(\Omega_{2\delta})$ into $\mathscr{H}_\rho(\Omega_\delta)$. Since $\varphi_m \rightharpoonup 0$ and $T_n\varphi_m \rightharpoonup 0$ in $\mathscr{H}_\rho$ and hence in $\mathscr{H}_\rho(\Omega_{2\delta})$ it therefore follows from (5.7) that $\varphi_m \to 0$ in $\mathscr{H}_\rho(\Omega_\delta)$. Hence, there exists a positive integer m_0 such that for $m > m_0$

$$\int_{\Omega_\delta} |\varphi_m|^2\,dt < \frac{\varepsilon'}{2c}$$

and so, from (5.6),

$$\|V\varphi_m\|^2 < \varepsilon'$$

whenever $m > m_0$. V is therefore $\tilde{T}_n$-compact. Since we have already shown that $\tilde{T}_n$ has a discrete spectrum and relatively compact perturbations preserve the essential spectrum, it follows that the self-adjoint extension $\tilde{T}_n + V$ of $T_n + V$ has a discrete spectrum.

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Relative Gleichverteilung in lokalkompakten Räumen

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§1. Einleitung

Liegt ein kompakter Raum X mit abzählbarer Umgebungsbasis vor, ist λ ein normiertes Maß auf X, dann heißt eine Folge $x_1, x_2, \ldots$ von Elementen aus X λ-gleichverteilt, wenn für alle Elemente A einer gewissen Klasse von Teilmengen von X

$$\lim_{N\to\infty}\frac{1}{N}\sum_{\substack{x_n\in A\\ n\leqq N}} 1=\lambda(A) \tag{1}$$

gilt (Definition von Hlawka [10] für das arithmetische Mittel). Durch Spezialisierung ergibt sich daraus die klassische Definition von H. Weyl. Das Wesentliche an der Gleichverteilung in diesem Sinn ist das Vorhandensein einer Verteilungswahrscheinlichkeit, also eines normierten Maßes.

In dieser Arbeit soll nun die Definition (1) erweitert werden: Aus (1) ergibt sich sofort für je zwei Teilmengen A, B mit $\lambda(B)>0$:

$$\lim_{N\to\infty}\frac{\sum\limits_{\substack{x_n\in A\\ n\leqq N}} 1}{\sum\limits_{\substack{x_n\in B\\ n\leqq N}} 1}=\frac{\lambda(A)}{\lambda(B)}. \tag{2}$$

Wir werden das als Definition verwenden und nennen eine Folge $x_1, x_2, \ldots$ von Elementen aus X λ-relativ gleichverteilt, wenn (2) für je zwei Elemente A, B (mit $\lambda(B)>0$) einer gewissen Klasse von Teilmengen von X gilt. Diese Definition umfaßt die Definition (1), hat aber den Vorteil, auch für lokalkompakte Räume X und nichtnormierte Maße λ anwendbar zu sein. Auf wen eine Definition dieser Art zurückgeht, konnte ich nicht feststellen; Untersuchungen in dieser Richtung finden sich in den Arbeiten [13, 1, 11]. Diese Arbeiten bildeten auch den Ausgangspunkt zur vorliegenden Untersuchung.

Eine andere Erweiterung der Definition (1) gab Helmberg in [7]; wir werden die Beziehung zwischen den verschiedenen Definitionen in § 3 (und § 7) untersuchen. In § 4 bringen wir einige Kriterien zur relativen Gleichverteilung. Die Existenz von relativ gleichverteilten Punktfolgen zu vorgegebenem Maß in beliebigen lokalkompakten Räumen mit abzählbarer Umgebungsbasis zeigen wir in § 5; der nächste Paragraph enthält einen genaueren Existenzsatz für Folgen im Produkt eines kompakten und eines lokalkompakten Raumes. Das asymptotische Verhalten von $\sum 1$ (summiert wird über alle $n\leqq N$, für die $x_n\in A$ liegt) für $N\to\infty$ wird in § 7 untersucht. Es wird sich hier ein signifi-

kanter Unterschied zwischen endlichen und unendlichen Maßen einerseits, zwischen Maßen mit kompaktem und Maßen mit nichtkompaktem Träger anderseits ergeben. Schließlich enthält § 8 einen Approximationssatz, der als Spezialfall einen Umordnungssatz enthält (jede in X dichte Folge $x_1, x_2, \ldots$ kann zu einer λ-relativ gleichverteilten Folge umgeordnet werden).

§ 2. Einige Definitionen und Hilfssätze

X bezeichne stets einen lokalkompakten Hausdorffschen Raum mit abzählbarer Umgebungsbasis. Maße auf X bezeichnen wir mit kleinen griechischen Buchstaben $\lambda, \mu, \ldots$ und meinen damit stets, wenn nicht anders vermerkt, positive Radonmaße auf X (im Sinne von Bourbaki).

Sei λ ein Maß auf X. Eine Teilmenge $A \subset X$ heißt *λ-Stetigkeitsmenge* (im folgenden kurz λ-Menge genannt), wenn A relativ kompakt ist und der Rand von A, Rd A, das λ-Maß Null hat ($\lambda(\mathrm{Rd}\, A)=0$).

Einige Eigenschaften von X (vgl. [14]):

E 1: X ist abzählbar im Unendlichen; d.h., es gibt eine Folge $(Q_n)_{n=1}^{\infty}$ von Teilmengen von X mit: Q_n offen, $\bar{Q}_n$ = kompakt,

$$\bar{Q}_n \subset Q_{n+1} \quad (n=1,2,\ldots) \quad \text{und} \quad \bigcup_{n=1}^{\infty} Q_n = X.$$

E 2: X ist parakompakt und es gilt genauer: Zu jeder offenen Überdeckung von X gibt es eine feinere lokalendliche offene Überdeckung von X, die aus abzählbar vielen relativ kompakten Mengen besteht.

E 3: X ist normal.

E 4: X ist metrisierbar. d bezeichne im folgenden stets eine mit der Topologie von X verträgliche Metrik.

Ist $A \subset X$, und $r>0$, dann sei

$$V_r(A)=\{x \in X \mid d(x, A)<r\} \quad \text{und} \quad W_r(A)=\{x \in X \mid d(x, A) \leqq r\}.$$

$\#_{n \leqq N} p(n)$ bedeutet die Anzahl der natürlichen Zahlen $n \leqq N$ mit der Eigenschaft p.

Für alle ohne Literaturangabe verwendeten Sätze aus der Theorie der Gleichverteilung vergleiche man [3].

Wir leiten jetzt einige Hilfssätze ab, die wir im folgenden immer wieder verwenden werden. (Diese Sätze sind sicher wohlbekannt, doch konnte ich keine geeignete Referenz finden.)

Hilfssatz 2.1. *Sei λ ein Maß auf X, $A \subset X$ eine relativ kompakte Teilmenge von X und $\varepsilon>0$. Dann gibt es stets eine offene λ-Menge $B \subset X$ mit*

$$A \subset B \subset V_{\varepsilon}(A).$$

Beweis. Nach Voraussetzung ist $\bar{A}$ kompakt. Dann gibt es nach [4] (3.18.2) ein $r>0$, so daß $V_r(\bar{A})$ relativ kompakt ist. Sei $s=\min(r/2, \varepsilon/2)$. Dann ist auch

$V_s(\bar{A})$ relativ kompakt und

$$A \subset \bar{A} \subset V_s(\bar{A}) \subset V_\varepsilon(A).$$

Nach E 3 ist X normal, $\bar{A}$ und $X \setminus V_s(\bar{A})$ sind disjunkte, abgeschlossene Mengen in X; es gibt daher eine stetige Funktion

$$f: X \to [0, 1]$$

mit $f|\bar{A} = 1$ und $f|(X \setminus V_s(\bar{A})) = 0$.

Für $0 < \sigma < 1$ definieren wir nun offene Umgebungen U_σ von $\bar{A}$ durch:

$$U_\sigma := \{x | f(x) > \sigma\}.$$

Dann gilt natürlich $\bar{A} \subset U_\sigma \subset V_s(\bar{A})$. Für den Rand von U_σ haben wir

$$\mathrm{Rd}\, U_\sigma \subset \{x | f(x) = \sigma\} = A_\sigma.$$

Nun gilt $A_\sigma \cap A_\tau = \emptyset$ für $\sigma \neq \tau$ und $A_\sigma \subset V_s(\bar{A})$ für $0 < \sigma < 1$. Daraus folgt aber

$$\lambda(A_\sigma) = 0$$

für alle $\sigma \in (0, 1)$ bis auf höchstens abzählbar viele, denn sonst wäre ja $\lambda(V_s(\bar{A})) = \infty$, was einen Widerspruch zu $V_s(\bar{A})$ relativ kompakt bedeutet (vgl. [6], Anmerkung zu Satz 3).

Sei nun $\sigma \in (0, 1)$, so daß $\lambda(\mathrm{Rd}\, U_\sigma) = 0$ ist. Wir setzen $B = U_\sigma$. Dann ist $A \subset B$, B ist eine offene λ-Menge und

$$B = U_\sigma \subset V_s(\bar{A}) \subset V_\varepsilon(A). \quad \text{W.z.b.w.}$$

Hilfssatz 2.2. *Sei $(A_i)_{i=1}^\infty$ ein lokalendliches System von Teilmengen von X, $(B_i)_{i=1}^\infty$ ein System von Teilmengen von X mit $A_i \subset B_i \subset V_{\varepsilon_i}(A_i)$ $(i = 1, 2, \ldots)$, wobei $\lim_{i \to \infty} \varepsilon_i = 0$. Dann ist auch das System (B_i) lokalendlich.*

Beweis. Indirekt. Angenommen, das System (B_i) ist nicht lokalendlich, dann gibt es also ein $x \in X$, so daß für alle Umgebungen U_x von x gilt: $U_x \cap B_i \neq \emptyset$ für unendlich viele i; aber es gibt eine Umgebung U_x' von x mit $U_x' \cap A_i \neq \emptyset$ für nur endlich viele i. Dann können wir ein $\delta > 0$ finden, so daß

$$V_\delta(x) \subset U_x \cap U_x',$$
$$V_\delta(x) \cap A_i \neq \emptyset \quad \text{nur für } i = i_1, \ldots, i_k,$$
$$V_\delta(x) \cap B_i \neq \emptyset \quad \text{für } i = i_1', i_2', \ldots.$$

Sei nun

$$d\Big(x, \bigcup_{i \neq i_1, \ldots, i_k} A_i\Big) = \alpha (> 0).$$

Wegen

$$V_\delta(x) \cap \bigcup_{i \neq i_1, \ldots, i_k} A_i = \emptyset \quad \text{gilt: } \alpha \geqq \delta.$$

Wegen $\lim \varepsilon_i = 0$ gibt es eine natürliche Zahl n_0 $(> \max(i_1, \ldots, i_k))$ mit $\varepsilon_n < \delta/2$ für $n \geqq n_0$.

Sei nun $n \geqq n_0$ und $y \in V_{\delta/4}(x) \cap B_n$. Dann gilt

$$d(x, y) < \delta/4, \quad d(y, A_n) \leqq \varepsilon_n, \quad d(x, A_n) \geqq \alpha \geqq \delta.$$

Nun ist aber

$$\delta \leqq d(x, A_n) \leqq d(x, y) + d(y, A_n) < \frac{\delta}{4} + \varepsilon_n < \frac{\delta}{4} + \frac{\delta}{2} = \frac{3\delta}{4}.$$

Das ist unmöglich; also ist

$$V_{\delta/4}(x) \cap B_n = \emptyset \quad \text{für } n \geqq n_0,$$

oder

$$V_{\delta/4}(x) \cap B_n \neq \emptyset \quad \text{höchstens für } n = 1, \ldots, n_0.$$

Das heißt aber, das System (B_i) ist lokalendlich. W.z.b.w.

Hilfssatz 2.3. *Sei λ ein Maß auf X. Dann gibt es ein System $(X_i)_{i=1}^{\infty}$ von Teilmengen von X mit:*

1) *alle X_i sind kompakt,*
2) *alle X_i sind λ-Mengen,*
3) $\bigcup_{i=1}^{\infty} X_i = X$,
4) $\lambda(X_i \cap X_k) = 0$ *für alle* $i \neq k$,
5) $(X_i)_{i=1}^{\infty}$ *ist lokalendlich.*

Bemerkung. Aus den Eigenschaften 3) und 4) folgt: Es gibt mindestens ein i, so daß $\lambda(X_i) > 0$. Wir werden im folgenden stets $\lambda(X_1) > 0$ voraussetzen.

Beweis. Nach E 2 gibt es eine abzählbare, lokalendliche Überdeckung $(A_i)_{i=1}^{\infty}$ von X, wo die A_i offen und relativ kompakt sind. Sei $(\varepsilon_i)_{i=1}^{\infty}$ eine Folge positiver, reeller Zahlen mit $\lim_{i \to \infty} \varepsilon_i = 0$. Dann gibt es nach Hilfssatz 2.1 ein abzählbares System $(B_i)_{i=1}^{\infty}$ mit

$$A_i \subset B_i \subset V_{\varepsilon_i}(A_i)$$

und alle B_i sind offene λ-Mengen. Nach Hilfssatz 2.2 ist das System (B_i) lokalendlich und es gilt weiter:

$$\bigcup_{i=1}^{\infty} B_i = X.$$

Wir setzen nun

$$X_1 = \bar{B}_1, \quad X_k = \bar{B}_k \setminus \bigcup_{i=1}^{k-1} B_i \quad (k = 2, 3, \ldots).$$

Wir zeigen, daß das System $(X_i)_{i=1}^{\infty}$ alle im Hilfssatz verlangten Eigenschaften erfüllt:

1) $\bar{B}_k$ ist kompakt, alle B_i sind offen, daher sind auch alle X_k kompakt.
2) Alle B_i sind λ-Mengen, daher ist auch $\bigcup_{i=1}^{k-1} B_i$ eine λ-Menge, denn es ist

$$\lambda\left(\mathrm{Rd} \bigcup_{i=1}^{k-1} B_i\right) = \lambda\left(\mathrm{Rd}\left(X \setminus \bigcup_{i=1}^{k-1} B_i\right)\right) \leqq \lambda\left(\bigcup_{i=1}^{k-1} \mathrm{Rd}\, B_i\right) = 0.$$

Nach 1) sind alle X_i kompakt und es gilt

$$\lambda(\mathrm{Rd}\, X_k)=\lambda\left(\mathrm{Rd}\left(\bar{B}_k\cap\left(X\setminus\bigcup_{i=1}^{k-1}B_i\right)\right)\right)\leqq\lambda(\mathrm{Rd}\,\bar{B}_k)+\lambda(\mathrm{Rd}(X\setminus\bigcup B_i))=0,$$

denn $\mathrm{Rd}\,\bar{B}_k=\mathrm{Rd}\,B_k$, da B_k offen.

3) Wir haben

$$X=\bigcup B_i=\bigcup_{i=1}^{\infty}X_i.$$

4) Es gilt für $k>i$:

$$X_i\cap X_k=\bar{B}_i\cap\left(X\setminus\bigcup_{l=1}^{i-1}B_l\right)\cap\bar{B}_k\cap\left(X\setminus\bigcup_{l=1}^{k-1}B_l\right)\subset\bar{B}_i\cap(X\setminus B_i)=\mathrm{Rd}\,B_i$$

und $\lambda(\mathrm{Rd}\,B_i)=0$ nach 2).

5) Das System (B_i) ist lokalendlich, daher auch nach Hilfssatz 2.2 das System $(\bar{B}_i)$ und daher auch das System (X_i) (denn $X_i\subset\bar{B}_i$). W.z.b.w.

Streicht man von der im Hilfssatz 2.3 konstruierten Folge (X_i) alle X_i mit $\lambda(X_i)=0$, so ergibt sich der

Hilfssatz 2.3a. *In Hilfssatz* 2.3 *kann die Eigenschaft* 3) *durch* 3a)

$$\lambda\left(X\setminus\bigcup_{i=1}^{\infty}X_i\right)=0\quad\textit{und}\quad\lambda(X_i)>0\qquad(i=1,2,\ldots)$$

ersetzt werden.

Ersetzen wir das in Hilfssatz 2.3 konstruierte System (X_i) durch das System $(Y_i)_{i=1}^{\infty}$ mit

$$Y_1=X_1,\qquad Y_k=\left(X_k\setminus\bigcup_{i=1}^{k-1}Y_i\right),$$

so gilt der

Hilfssatz 2.3b. *Sei λ ein Maß auf X. Dann gibt es ein System $(Y_i)_{i=1}^{\infty}$ von Teilmengen von X mit*

1) *alle Y_i sind λ-Mengen,*
2) $\bigcup_{i=1}^{\infty}Y_i=X$,
3) $Y_i\cap Y_k=\emptyset$ *für alle* $i\neq k$,
4) (Y_i) *ist lokalendlich,*
5) $\lambda(Y_1)>0$.

Und nun

Hilfssatz 2.4. *Eine relativ kompakte Menge C hat höchstens mit endlich vielen Mengen eines lokalendlichen Mengensystems $(A_i)_{i\in I}$ einen nichtleeren Durchschnitt.*

Beweis. Nach Voraussetzung gibt es zu jedem $x \in \bar{C}$ eine offene Umgebung U_x, so daß $U_x \cap A_i \neq \emptyset$ für nur endlich viele i. Da

$$C \subset \bar{C} \subset \bigcup_{k=1}^{l} U_{x_k},$$

so ist alles bewiesen. W.z.b.w.

§ 3. Relative Gleichverteilung

Definition 1. Sei λ ein Maß auf X, $(x_n)_{n=1}^{\infty}$ eine Folge von Elementen aus X. Die Folge (x_n) heißt *λ-relativ gleichverteilt* (λ-rel.glv.), wenn für alle λ-Mengen $A, B \subset X$ mit $\lambda(B) > 0$ gilt:

$$\lim_{N \to \infty} \frac{\sum_{n \leqq N} \chi_A(x_n)}{\sum_{n \leqq N} \chi_B(x_n)} \text{ exist.} = \frac{\lambda(A)}{\lambda(B)} \tag{3}$$

(χ_A = charakteristische Funktion von A).

Wir vermerken gleich: Ist X kompakt, λ ein normiertes Maß auf X, so ist $B = X$ zulässig und (3) ist die übliche Definition für λ-Gleichverteilung ([10]). In diesem Fall ist also λ-Gleichverteilung (λ-Glv.) und λ-rel.Glv. dasselbe. Durch weitere Spezialisierung ($X = [0, 1)$, λ = Lebesguesches Maß) erhält man die klassische Definition von H. Weyl.

Helmberg [7] gibt folgende Definition:

Definition 2. Sei μ ein normiertes Borelmaß auf X. Die Folge $(x_n)_{n=1}^{\infty}$ von Elementen aus X heißt *$d\mu$-glv.*, wenn

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n \leqq N} \chi_A(x_n) \text{ exist.} = \mu(A) \tag{4}$$

für alle kompakten Borelmengen A in X, deren Rand das Maß Null hat.

Die letzte Definition ist auch noch für beschränktes Maß λ ($\lambda(X) < \infty$) sinnvoll, denn wir können ja in diesem Fall das normierte Maß

$$\mu := \frac{1}{\lambda(X)} \lambda$$

betrachten.

Wir werden zeigen, daß Definition 1 allgemeiner ist als Definition 2 (und zwar auch im Falle normierter Maße).

1) Sei λ ein normiertes Borelmaß auf X (da X eine abzählbare Umgebungsbasis besitzt, ist λ regulär [5]), $(x_n)_{n=1}^{\infty}$ eine $d\lambda$-glv. Folge in X. Dann folgt aus (4) für alle kompakten Borelmengen A, B (mit $\lambda(B) > 0$), deren Rand das Maß Null hat:

$$\lim_{N \to \infty} \frac{\sum_{n \leqq N} \chi_A(x_n)}{\sum_{n \leqq N} \chi_B(x_n)} \text{ exist.} = \frac{\lambda(A)}{\lambda(B)}. \tag{5}$$

Nun definiert ein reguläres Borelmaß λ in eindeutiger Weise ein Radonmaß, das wir ebenfalls mit λ bezeichnen, und beide Maße stimmen auf den Borelmengen überein ([2]). Ein Borelmaß λ auf X ist stets regulär. Dann gilt (5) auch für jede λ-Menge A, die das Maß Null bezüglich des Radonmaßes λ hat, denn jede solche Nullmenge ist in einer Borelmenge enthalten, die das λ-Maß Null hat. Damit gilt aber (5) (oder (3)) für alle λ-Mengen A, B (mit $\lambda(B)>0$), wie man aus der Beziehung:

$$(\lambda\text{-Menge}) = (\text{kompakte Menge}) \setminus (\text{Nullmenge})$$

sieht.

Es ist also (x_n) λ-rel. glv.

2) Sei λ ein normiertes Maß mit nichtkompaktem Träger auf X, (x_n) λ-rel. glv. Wir werden in § 7 (Satz 7.1) zeigen, daß dann

$$\sum_{n \leqq N} \chi_A(x_n) = \lambda(A)\, N + o(N)$$

für alle λ-Mengen $A \subset X$ gilt. Damit ist aber gezeigt: Es ist die Folge (x_n) auch $d\lambda$-glv. in X.

Mit der Bemerkung nach der Definition 2 haben wir damit also gezeigt:

Ist λ ein beschränktes Maß auf X mit nicht kompaktem Träger, so ist jede λ-rel. glv. Folge von Elementen aus X auch $\frac{1}{\lambda(x)}\, d\lambda$-glv. und umgekehrt. Ist X nicht kompakt (lokalkompakt) und hat das Maß λ einen kompakten Träger, dann werden wir in § 7 zeigen:

Es gibt λ-rel. glv. Folgen, die nicht $d\lambda$-glv. sind.

Somit ist also Definition 1 allgemeiner als Definition 2. (Im Falle beschränkter Maße.)

Im Falle eines unbeschränkten Maßes λ $(\lambda(x)=\infty)$ ist alles anders: Hier kann für eine λ-rel. glv. Folge (x_n) keine Beziehung der Gestalt (4) gelten; wir werden in § 7 zeigen, daß für jede λ-rel. glv. Folge (x_n) einerseits

$$\sum_{n \leqq N} \chi_A(x_n) = o(N) \quad \text{für alle } \lambda\text{-Mengen } A \subset X$$

gilt und andererseits noch, daß für geeignete λ-rel. glv. Folgen (x_n) der Ausdruck

$$\sum_{n \leqq N} \chi_A(x_n)$$

beliebig langsam mit N wächst. (Ist das Maß λ nicht atomar, dann muß ja dieser Ausdruck mit N gegen Unendlich gehen und zwar sicher für jedes A positiven λ-Maßes.)

§ 4. Kriterien für relative Gleichverteilung

Wie bei der gewöhnlichen Gleichverteilung gilt auch hier der

Satz 4.1. *Sei λ ein Maß auf X, $(x_n)_{n=1}^{\infty}$ eine Folge in X. Die folgenden beiden Aussagen sind äquivalent:*

1) (x_n) *ist* λ*-rel.glv. in* X,

2) *für alle auf* X *stetigen, reellwertigen Funktionen* f, g *mit kompaktem Träger und* $\int_X g\,d\lambda \neq 0$ *gilt:*

$$\lim_{N\to\infty} \frac{\sum_{n\leqq N} f(x_n)}{\sum_{n\leqq N} g(x_n)} = \frac{\int_X f\,d\lambda}{\int_X g\,d\lambda}.$$

Beweis. Wird analog geführt wie in [7] (Beweis von Satz 13) bzw. [8] (Beweis von Theorem 5.2.1).

Es folgen nun zwei für das Weitere sehr praktische Kriterien. Das erste schränkt die Klasse den zu betrachtenden λ-Mengen ein:

Satz 4.2. *Sei* λ *ein Maß auf* X, $(x_n)_{n=1}^{\infty}$ *eine Folge in* X. *Die folgenden beiden Aussagen sind äquivalent:*

1) (x_n) *ist* λ*-rel.glv. in* X,

2) *für alle* λ*-Mengen* $A, B \subset X$ *mit* $\lambda(A)>0$, $\lambda(B)>0$ *gilt:*

$$\lim_{N\to\infty} \frac{\sum_{n\leqq N} \chi_A(x_n)}{\sum_{n\leqq N} \chi_B(x_n)} = \frac{\lambda(A)}{\lambda(B)}. \tag{6}$$

Beweis. 1)$\Rightarrow$2) trivial.

2)$\Rightarrow$1): Sei $C \subset X$ eine λ-Menge mit $\lambda(C)=0$, seien weiter $A, B \subset X$ λ-Mengen mit $\lambda(A)>0$, $\lambda(B)>0$. Dann ist auch $A \cup C$ eine λ-Menge und $\lambda(A \cup C)>0$. Daher gilt nach Voraussetzung:

$$\lim_{N\to\infty} \frac{\sum_{n\leqq N} \chi_{A\cup C}(x_n)}{\sum_{n\leqq N} \chi_B(x_n)} = \frac{\lambda(A\cup C)}{\lambda(B)} = \frac{\lambda(A)}{\lambda(B)} = \lim_{N\to\infty} \frac{\sum_{n\leqq N} \chi_A(x_n)}{\sum_{n\leqq N} \chi_B(x_n)}$$

und also

$$\lim_{N\to\infty} \frac{\sum_{n\leqq N} (\chi_{A\cup C}(x_n) - \chi_A(x_n))}{\sum_{n\leqq N} \chi_B(x_n)} = 0. \tag{7}$$

Nun ist $X \setminus \bar{C} \neq \emptyset$ offen; es gibt also eine relativ kompakte Menge D mit $\lambda(D)>0$ und ein $\varepsilon>0$, so daß

$$D \subset V_\varepsilon(D) \subset X \setminus \bar{C}.$$

Nach Hilfssatz 2.1 gibt es daher eine λ-Menge E mit

$$D \subset E \subset X \setminus \bar{C},$$

woraus folgt $\lambda(E)>0$ und $E \cap C = \emptyset$. Daher gilt

$$0 \leqq \chi_{E\cup C}(x) - \chi_E(x) = \chi_C(x).$$

Damit erhalten wir aus (7) für $A=E$:

$$\lim_{N\to\infty}\frac{\sum\limits_{n\leqq N}\chi_C(x_n)}{\sum\limits_{n\leqq N}\chi_B(x_n)}=0,$$

womit sich in Verbindung mit (6) die λ-rel.Glv. von (x_n) ergibt. W.z.b.w.

Definition. Sei (x_n) eine Folge in X, $A\subset X$ eine Teilmenge. Wir bezeichnen mit $(x_n)|A$ die Teilfolge der $x_n\in A$.

Dann gilt der

Satz 4.3. *Sei λ ein Maß auf X, $(x)_{n=1}^{\infty}$ eine Folge in X. Die folgenden beiden Aussagen sind äquivalent:*

1) *(x_n) ist λ-rel.glv. in X,*

2) *für alle kompakten λ-Mengen $C\subset X$ mit $\lambda(C)>0$ ist die Folge*

$$(x_n)|C\ \frac{1}{\lambda(C)}\ \lambda\text{-glv. in } C.$$

Beweis. a) Sei (x_n) λ-rel.glv. in X, $C\subset X$ eine kompakte λ-Menge mit $\lambda(C)>0$, $A\subset C$ eine λ-Menge. Dann gilt also:

$$\lim_{N\to\infty}\frac{\sum\limits_{n\leqq N}\chi_A(x_n)}{\sum\limits_{n\leqq N}\chi_C(x_n)}=\frac{\lambda(A)}{\lambda(C)}. \tag{8}$$

Sei $(y_n)=(x_n)|C$. Es ist somit $\sum\limits_{n\leqq N}\chi_C(y_n)=N$. Es sei i die kleinste natürliche Zahl mit $\chi_C(x_i)=1$ (also $x_i=y_1$). Dann betrachten wir die beiden Folgen:

$$\frac{\chi_A(x_i)}{\chi_C(x_i)},\ldots,\frac{\sum\limits_{i\leqq k\leqq N}\chi_A(x_k)}{\sum\limits_{i\leqq k\leqq N}\chi_C(x_k)},\ldots \tag{9}$$

und

$$\frac{\chi_A(y_1)}{\chi_C(y_1)},\ldots,\frac{\sum\limits_{k\leqq M}\chi_A(y_k)}{\sum\limits_{k\leqq M}\chi_C(y_k)},\ldots. \tag{10}$$

Beide Folgen enthalten dieselben Zahlen, nur kommen diese Zahlen in der Folge (9) im allgemeinen mehrmals vor. Folge (10) entsteht nämlich aus Folge (9) durch Streichen aller Glieder in Folge (9), deren Nenner gleich dem Nenner des vorigen Gliedes ist (dann sind die Zähler ebenfalls gleich). Beide Folgen haben daher dasselbe Konvergenzverhalten. Wegen (8) konvergiert

die Folge (9), also auch (10); das heißt:

$$\lim_{N\to\infty}\frac{\sum\limits_{n\leqq N}\chi_A((x_n)|C)}{\sum\limits_{n\leqq N}\chi_C((x_n)|C)}=\lim_{N\to\infty}\frac{1}{N}\sum_{n\leqq N}\chi_A((x_n)|C)\ \text{exist.}=\frac{1}{\lambda(C)}\cdot\lambda(A)$$

für alle λ-Mengen $A\subset C$. Also ist die Folge $(x_n)|C$ $\frac{1}{\lambda(C)}$ λ-glv. in C.

b) Wir setzen 2) voraus. Seien $A, B\subset X$ λ-Mengen mit $\lambda(B)>0$. Dann ist $\overline{A\cup B}=C$ eine kompakte λ-Menge. Nach Voraussetzung ist also $(x_n)|C$ $\frac{1}{\lambda(C)}$ λ-glv. in C; es gilt also:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n\leqq N}\chi_D((x_n)|C)=\frac{\lambda(D)}{\lambda(C)}\quad\text{für } D=A \text{ und } D=B. \tag{11}$$

Daraus folgt aber

$$\lim_{N\to\infty}\frac{\sum\limits_{n\leqq N}\chi_A((x_n)|C)}{\sum\limits_{n\leqq N}\chi_B((x_n)|C)}=\frac{\lambda(A)}{\lambda(B)},$$

und weiter wie in a):

$$\lim_{N\to\infty}\frac{\sum\limits_{n\leqq N}\chi_A(x_n)}{\sum\limits_{n\leqq N}\chi_B(x_n)}\ \text{exist.}=\frac{\lambda(A)}{\lambda(B)}.\qquad \text{W.z.b.w.}$$

§ 5. Existenz relativ gleichverteilter Folgen

Wir konstruieren zunächst relativ gleichverteilte Folgen in der Menge der natürlichen Zahlen.

Satz 5.1. *Sei $a(k)$ $(k=1,2,\ldots)$ eine Folge natürlicher Zahlen mit $a(k)\geqq k$, $m(k)$ definiert durch*

$$m(1)=1,\qquad m(k+1)=m(k)+(k+1)\,a(k)\qquad (k=2,3,\ldots).$$

Setzen wir

$$l_n=\left[\frac{n-m(k)}{k+1}\right]+1\qquad \textit{für } m(k)\leqq n<m(k+1)\qquad (k=1,2,\ldots)$$

($[\alpha]=$ größte ganze Zahl $\leqq\alpha$), so ist die Folge (l_n) gleichverteilt in $N=\{1,2,\ldots\}$, d.h. es gilt

$$\lim_{M\to\infty}\frac{\#\limits_{n\leqq M}(l_n=i)}{\#\limits_{n\leqq M}(l_n=k)}=1\qquad \textit{für alle } i,k\in N$$

und jede natürliche Zahl i kommt in der Folge (l_n) unendlich oft vor.

Beweis. In der endlichen Folge (l_n) $(n=m(k), m(k)+1, \ldots, m(k+1)-1)$ kommt jede der Zahlen $1, 2, \ldots, a(k)$ genau $(k+1)$mal vor.

Sei nun eine hinreichend große natürliche Zahl M gegeben; wir definieren die natürliche Zahl b durch:

$$m(b) \leqq M < m(b+1).$$

Dann gilt für ein $i \in N$:

$$\begin{aligned}\#_{n \leqq M}(l_n=i) &= \Big(\#_{m(1) \leqq n < m(2)}(l_n=i) + \cdots + \#_{m(i) \leqq n < m(i+1)}(l_n=i)\Big) \\ &\quad + \Big(\#_{m(i+1) \leqq n < m(i+2)} + \cdots + \#_{m(b-1) \leqq n < m(b)}\Big) + \Big(\#_{m(b) \leqq n \leqq M}(l_n=i)\Big) \\ &= (1)+(2)+(3).\end{aligned}$$

Nun haben wir:

$$0 \leqq (1) \leqq \frac{i(i+3)}{2},$$

$$(2) = (i+2) + \cdots + b = \frac{(i+b+2)(b-i-1)}{2},$$

$$0 \leqq (3) \leqq b+1.$$

Damit ergibt sich

$$\#_{n \leqq M}(l_n=i) = (1)+(3)+\frac{(i+b+2)(b-i-1)}{2} = \frac{b^2}{2} + O(b).$$

Aus dieser Beziehung folgt sofort die Behauptung. W.z.b.w.

Dieser Satz gestattet es nun, λ-rel.glv. Folgen in X zu konstruieren. Wir betrachten zunächst den Fall

$$X = R_1 = \{x \in R \mid x \geqq 1\}.$$

Satz 5.2. *Es gibt Folgen $(y_n)_{n=1}^{\infty}$ in R_1, die bezüglich des Lebesgueschen Maßes λ relativ gleichverteilt sind.*

Beweis. Sei $X_i = [i, i+1]$ (abgeschlossenes Intervall) $(i=1, 2, \ldots)$; in jedem X_i gibt es λ-glv. Folgen $(x_n^{(i)})$ (man kann sie z.B. durch Translation aus in $[0,1]$ λ-glv. Folgen erhalten). Sei $(l_n)_{n=1}^{\infty}$ eine in N rel.glv. Folge wie in Satz 5.1.

Wir definieren nun eine Folge $(y_k)_{k=1}^{\infty}$ in R_1 durch:

$$y_k := x_r^{(s)}, \quad \text{wenn } l_k = s \quad \text{und} \quad \#_{m \leqq k-1}(l_n=s) = r-1$$

(das heißt, wenn die Zahl s in der Folge (l_m) zum r-ten Mal am Platz k vorkommt).

Wir zeigen: (y_k) ist λ-rel.glv. in R_1.

1) Sei $A \subset X_i$ eine λ-Menge. Wir schreiben

$$f_s(N) = \#_{k \in N}(l_k=s) = \sum_{n \leqq N} \chi_{X_s}(y_n) \qquad (s=1, 2, \ldots).$$

Dann gilt nach Satz 5.1:

$$f_s(N)=f_1(N)+o(f_1(N)) \quad (s=1,2,\ldots). \tag{12}$$

Und nun haben wir weiter

$$\begin{aligned}\sum_{n\leqq N}\chi_A(y_n)&=(\text{nach Definition von } (y_k))=\sum_{m\leqq f_i(N)}\chi_A(x_m^{(i)})\\&=(\text{Glv. von } (x_m^{(i)}))=\lambda(A)f_i(N)+o(f_i(N))\\&\overset{(12)}{=}\lambda(A)f_1(N)+o(f_1(N)).\end{aligned} \tag{13}$$

2) Seien $A, B\subset R_1$ λ-Mengen mit $\lambda(B)>0$. Es sei

$$\begin{aligned}A\cap X_i\neq\emptyset \quad &\text{für } i=i_1,\ldots,i_a,\\ B\cap X_j\neq\emptyset \quad &\text{für } j=j_1,\ldots,j_b.\end{aligned}$$

Dann gilt nach (13):

$$\sum_{n\leqq N}\chi_{A\cap X_i}(y_n)=\lambda(A\cap X_i)f_1(N)+o(f_1(N)),$$

und daher (da die Randpunkte der Intervalle X_i keine Rolle spielen)

$$\sum_{n\leqq N}\chi_A(y_n)=\lambda(A)f_1(N)+o(f_1(N))$$

und analog für B.

Wir erhalten also

$$\frac{\sum\limits_{n\leqq N}\chi_A(y_n)}{\sum\limits_{n\leqq N}\chi_B(y_n)}=\frac{\lambda(A)+o(1)}{\lambda(B)+o(1)};$$

und das bedeutet die λ-rel. Glv. der Folge (y_n) in R_1. W.z.b.w.

Den Beweis des Satzes werden wir als Modell für den Beweis des allgemeinen Existenzsatzes verwenden. Die Zerlegung von R_1 in Intervalle X_i ersetzen wir durch die im Hilfssatz 2.3a gegebene Zerlegung von X. Im allgemeinen Fall haben aber diese X_i nicht alle das gleiche Maß eins; wir zeigen daher zunächst einen weiteren Satz über Folgen natürlicher Zahlen.

Satz 5.3. *Sei*

$$r_1, r_2, \ldots \left(r_i>0, \sum_{i=1}^{\infty} r_i=\infty\right)$$

eine Folge reeller Zahlen. Dann gibt es stets Folgen (n_k) $(n_k\in N)$ *natürlicher Zahlen mit:*

1) *Jede natürliche Zahl kommt unendlich oft vor,*

2) *für alle* $m, t\in N$ *gilt:*

$$\lim_{M\to\infty}\frac{\#\limits_{k\leqq M}(n_k=m)}{\#\limits_{k\leqq M}(n_k=t)}=\frac{r_m}{r_t}. \tag{14}$$

Beweis. Wir bezeichnen wieder mit λ das Lebesguesche Maß auf R_1. Sei $r_0=0$ und

$$I_k=\{x\in R_1 | 1+r_1+\cdots+r_{k-1}\leqq x<1+r_1+\cdots+r_k\} \quad (k=1,2,\ldots).$$

Dann ist also (I_k) eine Partition von R_1, alle I_k sind λ-Mengen und $\lambda(I_k)=r_k>0$.

Sei (x_n) λ-rel. glv. in R_1 (Satz 5.2). Dann definieren wir eine Folge (n_k) natürlicher Zahlen durch:

$$n_k:=s \quad \text{wenn } x_k\in I_s \quad (k=1,2,\ldots).$$

Diese Folge (n_k) natürlicher Zahlen erfüllt:

1) Jede natürliche Zahl kommt unendlich oft vor, denn die Folge (x_n) ist λ-rel. glv. und daher enthält jedes Intervall positiver Länge unendlich viele x_n.

2) Wir haben

$$\#_{k\leqq M}(n_k=m)=\sum_{k\leqq M}\chi_{I_m}(x_k),$$

also ist auch die zweite verlangte Eigenschaft erfüllt. W.z.b.w.

Wählen wir in dem eben gegebenen Beweis für (x_n) eine in einem kompakten Intervall $[1, r+1]$ λ-glv. Folge, dann erhalten wir wie vorhin:

Satz 5.3a. *Sei* $r_1, r_2, \ldots$ $\left(r_i>0, \sum_i r_i=r\right)$ *eine endliche oder unendliche Folge reeller Zahlen. Dann gibt es stets Folgen* (n_k) *natürlicher Zahlen mit:*

1) *Jede natürliche Zahl* ($\leqq$ *Anzahl der gegebenen* r_i) *kommt unendlich oft vor,*

2) *für alle natürlichen Zahlen* m, t, *die in der Folge* (n_k) *vorkommen, gilt* (14).

Und nun können wir den allgemeinen Existenzsatz beweisen.

Satz 5.4. *Sei* λ *ein Maß auf* X. *Dann gibt es stets* λ*-rel. glv. Punktfolgen* $(x_n)_{n=1}^{\infty}$ *in* X.

Bemerkung. Im Falle eines normierten (endlichen) Maßes λ mit nichtkompaktem Träger auf X folgt dieser Satz wegen § 3 aus Satz 14 in [7]. Der Beweis, den wir geben werden, ist konstruktiv und funktioniert für endliches und unbeschränktes Maß; wir werden ihn im zweiten Fall $(\lambda(X)=\infty)$ durchführen und in eckigen Klammern die nötigen Änderungen bei endlichem Maß angeben.

Beweis. Sei $(X_i)_{i=1}^{\infty}$ ein System von Teilmengen von X mit den in Hilfssatz 2.3a angegebenen Eigenschaften. Wir setzen $\lambda(X_i)=r_i$. Dann erfüllt die Folge (r_i) die Voraussetzungen von Satz 5.3 [5.3a]. Es gibt also eine Folge (n_k) natürlicher Zahlen mit den dort angegebenen Eigenschaften 1) und 2). Es sei λ_i $(i=1,2,\ldots)$ das durch

$$\lambda_i(C)=\frac{1}{r_i}\lambda(C) \quad (C\subset X_i)$$

gegebene normierte Maß auf X_i.

Nach Hilfssatz 2.3a sind alle X_i kompakt, λ_i ist ein normiertes Maß auf X_i; es gibt daher nach [10], Satz 11 λ_i-glv. Folgen $(x_n^{(i)})$ in X_i $(i=1,2,\ldots)$.

Wegen Eigenschaft 4) ist für festes i

$$\lambda\Big(\bigcup_l (X_i \cap X_l)\Big) = \lambda_i\Big(\bigcup_l (X_i \cap X_l)\Big) = 0.$$

Streichen wir daher in der Folge $(x_n^{(i)})$ alle Glieder, die in $\bigcup_l (X_i \cap X_l)$ enthalten sind, so ist die neue Folge noch immer λ_i-glv. Diese neue Folge bezeichnen wir wieder mit $(x_n^{(i)})$. Wir haben also damit erreicht, daß kein Element einer Folge $(x_n^{(i)})$ in einer Menge der Form $X_l \cap X_m$ $(l \neq m)$ liegt. Aus der Gleichverteilung von $(x_n^{(i)})$ in X_i folgt:

$$\sum_{n \leqq N} \chi_A(x_n^{(i)}) = \lambda_i(A)\, N + o(N) \quad \text{für alle } \lambda\text{-Mengen } A \subset X_i \ (i = 1, 2, \ldots). \tag{15}$$

Wir definieren nun eine Folge (y_k) in X durch:

$$y_k := x_r^{(s)}, \quad \text{wenn } n_k = s \text{ und } \underset{l \leqq k-1}{\#} (n_l = s) = r - 1.$$

Wir zeigen von dieser Folge (y_k), daß sie λ-rel. glv. in X ist.

1) Sei $A \subset X_i$ eine λ-Menge. Wir schreiben

$$f_s(N) = \underset{k \leqq N}{\#} (n_k = s) = \sum_{n \leqq N} \chi_{X_s}(y_n) \qquad (s = 1, 2, \ldots).$$

Dann gilt nach Satz 5.3 [5.3a]:

$$f_s(N) = \frac{r_s}{r_1} f_1(N) + o\big(f_1(N)\big) \qquad (s = 1, 2, \ldots).$$

Und nun haben wir weiter (wie im Beweis von Satz 5.2):

$$\sum_{n \leqq N} \chi_A(y_n) = \frac{\lambda(A)}{r_1} f_1(N) + o\big(f_1(N)\big). \tag{16}$$

2) Seien $A, B \subset X$ λ-Mengen mit $\lambda(B) > 0$.

Es sei $A \cap X_i \neq \emptyset$ für $i = i_1, \ldots, i_a$. Dann gilt nach (16):

$$\sum_{n \leqq N} \chi_{A \cap X_i}(y_n) = \frac{\lambda(A \cap X_i)}{r_1} f_1(N) + o\big(f_1(N)\big)$$

und daher, da $\lambda(A) = \lambda\Big(\bigcup_i (A \cap X_i)\Big)$ und kein y_n in einer Menge der Form $X_i \cap X_k$ $(i \neq k)$ liegt:

$$\sum_{n \leqq N} \chi_A(y_n) = \frac{\lambda(A)}{r_1} \big(f_1(N) + o\big(f_1(N)\big)\big)$$

und analog für B. Wir erhalten also

$$\frac{\sum\limits_{n \leqq N} \chi_A(y_n)}{\sum\limits_{n \leqq N} \chi_B(y_n)} = \frac{\lambda(A) + o(1)}{\lambda(B) + o(1)}$$

und damit ist die λ-rel. Glv. der Folge (y_n) in X gezeigt. W.z.b.w.

§ 6. Ein weiterer Existenzsatz

Ist Y ein kompakter Raum, μ ein normiertes Maß auf Y, dann sei

$$Y^\infty = \bigtimes_{n=1}^{\infty} Y_n \quad (Y_n = Y)$$

das kartesische Produkt von abzählbar vielen Exemplaren von Y mit der üblichen Produkttopologie,

$$\mu^\infty = \bigotimes \mu$$

das mit μ konstruierte Produktmaß auf Y^∞.

Satz 6.1. *Sei X lokalkompakt, Y kompakt mit abzählbarer Umgebungsbasis, λ ein Maß auf X, μ ein normiertes Maß auf Y und $(x_n)_{n=1}^\infty$ eine λ-rel.glv. Folge in X. Dann ist für μ^∞-fast alle Folgen (y_n) in Y die Folge (x_n, y_n) $\lambda \otimes \mu$-rel.glv. in $X \times Y$.*

Beweis. 1) Wir setzen zunächst X und Y als kompakt voraus; λ sei ein normiertes Maß auf X, (x_n) λ-glv. in X.

Sei $A \subset X$ eine λ-Menge mit $\lambda(A) > 0$; wir definieren

$$a_{nk} = \begin{cases} \dfrac{\chi_A(x_k)}{\lambda(A) \cdot n} & \text{für } k \leqq n \\ 0 & \text{für } k > n \end{cases} \qquad (k, n \geqq 1).$$

Dann gilt:

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = \lim \sum_{k=1}^{n} \frac{\chi_A(x_k)}{n \cdot \lambda(A)} = \frac{1}{\lambda(A)} \lim \frac{1}{n} \sum_{k \leqq n} \chi_A(x_k) = 1,$$

da (x_k) λ-glv. ist; weiter gilt

$$\sum_{k=1}^{\infty} |a_{nk}| = \sum_{k=1}^{n} a_{nk} < \infty.$$

Sei $M(A) = ((a_{nk}))$ (unendliche Matrix).

$M(A)$ erfüllt die Hillsche Bedingung

$$\sum_{n=1}^{\infty} e^{-\frac{\delta^2}{\alpha_n}} < \infty \quad \text{für jedes } \delta > 0, \text{ wobei } \alpha_n = \sum_{k=1}^{\infty} a_{nk}^2. \tag{17}$$

Es gilt nämlich

$$\alpha_n = \sum a_{nk}^2 = \frac{1}{\lambda^2(A)} \cdot \frac{1}{n^2} \sum_{k \leqq n} \chi_A(x_k) = \frac{1}{\lambda^2(A)} \cdot \frac{1}{n} \left(\lambda(A) + o(1)\right) < \frac{C}{n}$$

für genügend große n, und daher gilt (17).

Damit sind alle Voraussetzungen von [10], Satz 11 erfüllt und es gilt daher:

Für μ^∞-fast alle Folgen (y_n) in Y ist

$$\lim_{N\to\infty} \sum_{n \leqq N} a_{Nn} \chi_B(y_n) = \mu(B) \quad \text{für alle } \mu\text{-Mengen } B \subset Y.$$

Also haben wir schließlich nach Definition von (a_{Nn}):

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n\leqq N}\chi_A(x_n)\,\chi_B(y_n)=\lambda(A)\,\mu(B)$$

für μ^∞-fast alle (y_n), für alle λ-Mengen $A\subset X$ mit $\lambda(A)>0$, für alle μ-Mengen $B\subset Y$. Daraus folgt weiter durch Approximation:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n\leqq N}\chi_C(x_n,y_n)=(\lambda\otimes\mu)(C)$$

für μ^∞-fast alle (y_n), für alle $\lambda\otimes\mu$-Mengen $C\subset X\times Y$.

2) Sei jetzt X lokalkompakt, λ ein Maß auf X, $C\subset X$ eine kompakte λ-Menge mit $\lambda(C)>0$. Dann ist $C\times Y$ eine kompakte $\lambda\otimes\mu$-Menge. Die Folge (x_n) ist λ-rel.glv. in X, daher ist nach Satz 4.3 die Folge $(x_{n_k})=(x_n)|C$ $\frac{1}{\lambda(C)}$ λ-glv. in C. Wir haben also nach dem in 1) Bewiesenen: Für μ^∞-fast alle Folgen (y_n) in Y ist (x_{n_k},y_k) $\frac{1}{\lambda(C)}$ $\lambda\otimes\mu$-glv. in $C\times Y$. Da auch die Menge aller Folgen (y_n) in Y^∞, für die (y_{n_k}) in einer Menge vom μ^∞-Maß eins liegt, das μ^∞-Maß eins hat, so können wir ebenso sagen:

Für μ^∞-fast alle Folgen (y_n) in Y ist die Folge (x_{n_k},y_{n_k}) $\frac{1}{\lambda(C)}$ $\lambda\otimes\mu$-glv. in $C\times Y$ (und das gilt für alle C wie oben).

3) Es sei X lokalkompakt, (X_i) ein System von Teilmengen von X mit den in Hilfssatz 2.3 angegebenen Eigenschaften. Weiter sei (x_n) λ-rel.glv. in X.

Sei $C\subset X\times Y$ eine kompakte $\lambda\otimes\mu$-Menge mit $(\lambda\otimes\mu)(C)>0$. Es ist natürlich auch das System $(X_i\times Y)$ lokalendlich und daher nach Hilfssatz 2.4

$$C\cap(X_i\times Y)\neq\emptyset \quad \text{nur für } i=i_1,\ldots,i_k.$$

Dann ist $\bigcup_{t=1}^{k}X_{i_t}=D$ eine kompakte λ-Menge und daher gilt nach 2):

Die Folge

$$(x_n,y_n)|(D\times Y) \quad \text{ist } \frac{1}{\lambda(D)}\,\lambda\otimes\mu\text{-glv. in } D\times Y \text{ für } (y_n)\in T_{i_1\ldots i_k}, \tag{18}$$

wobei $\mu^\infty(T_{i_1\ldots i_k})=1$ ist.

Es sei nun $A\subset C$ eine $\lambda\otimes\mu$-Menge. Dann gilt nach (18) für $(y_n)\in T_{i_1\ldots i_k}$:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n\leqq N}\chi_E\big((x_n,y_n)|D\times Y\big)=\frac{1}{\lambda(D)}(\lambda\otimes\mu)(E) \quad \text{für } E=A,\,C \tag{19}$$

und daher ist

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n\leqq N}\chi_A\big((x_n,y_n)|C\big)=\text{(vgl. Beweis von Satz 4.3)}$$

$$=\lim\frac{\sum_{n\leqq N}\chi_A\big((x_n,y_n)|(D\times Y)\big)}{\sum_{n\leqq N}\chi_C\big((x_n,y_n)|(D\times Y)\big)}\overset{(19)}{=}\frac{(\lambda\otimes\mu)(A)}{(\lambda\otimes\mu)(C)}.$$

Diese letzte Beziehung zeigt: Die Folge $(x_n, y_n)|C$ ist $\frac{1}{(\lambda\otimes\mu)(C)}(\lambda\otimes\mu)$-glv. auf C für alle Folgen $(y_n)\in T_{i_1\dots i_k}$.

Wir schreiben nun

$$T=\bigcap T_{i_1\dots i_k},$$

wobei wir den Durchschnitt über alle endlichen Folgen $i_1, \dots, i_k$ natürlicher Zahlen erstrecken. Dann gilt auch

$$\mu^\infty(T)=1$$

und weiter für jede kompakte $\lambda\otimes\mu$-Menge $C\subset X\times Y$ mit $(\lambda\otimes\mu)(C)>0$: $(x_n, y_n)|C$ ist $\frac{1}{(\lambda\otimes\mu)(C)}(\lambda\otimes\mu)$-glv. auf C für alle Folgen $(y_n)\in T$. Das bedeutet aber nach Satz 4.3 die $\lambda\otimes\mu$-rel. Glv. der Folge (x_n, y_n) für $(y_n)\in T$, also für μ^∞-fast alle Folgen (y_n) in Y. Damit ist alles gezeigt. W.z.b.w.

Dieser Satz ist meines Wissens nach auch für die einfachsten Spezialfälle neu. Wohl wurde gezeigt, daß in jedem kompakten Raum (also auch im Produkt zweier kompakter Räume) mit abzählbarer Umgebungsbasis μ^∞-fast alle Folgen μ-glv. sind (μ ist ein normiertes Maß), aber es wurde nicht diese Frage untersucht: Ist eine gleichverteilte Folge (x_n) in X (z.B. kompakt) gegeben, gibt es dann dazu stets Folgen (y_n) in Y, so daß (x_n, y_n) gleichverteilt in $X\times Y$ ist?

Betrachten wir einen ganz einfachen Spezialfall von Satz 6.1:

Sei $X=Y=T^1$ (eindimensionaler Torus), $\lambda=\mu$ das Lebesguesche Maß auf T^1 und (x_n) λ-glv. in T^1. Dann gilt: (x_n, y_n) ist $\lambda\otimes\lambda$-glv. in T^2 für λ^∞-fast alle Folgen (y_n) in T^1.

§ 7. Asymptotisches Verhalten von $\sum_{n\leqq N}\chi_C(x_n)$

Wir beginnen zunächst mit einer Umkehrung von Satz 5.3a. Und zwar zeigen wir den

Hilfssatz 7.1. *Sei (n_k) eine Folge natürlicher Zahlen mit den Eigenschaften 1) und 2) von Satz 5.3a; sei (r_i) eine Folge positiver, reeller Zahlen mit $\sum_i r_i=r<\infty$ und die Länge der Folge (r_i) sei gleich der Anzahl der verschiedenen natürlichen Zahlen in (n_k). Dann gibt es eine in $[1, r+1]$ λ-glv. Folge (x_n) ($\lambda=$ Lebesguesches Maß auf R), so daß*

$$x_s\in I_k \text{ äquivalent ist mit } n_s=k \quad (s=1, 2, \dots)$$

für

$$I_k=\{x\in R_1 | 1+r_1+\cdots+r_{k-1}\leqq x<1+r_1+\cdots+r_k\}.$$

Beweis. Sei $(y_n^{(k)})$ λ-glv. in I_k $(k=1, 2, \dots)$. Wir definieren eine neue Folge (x_n) in $[1, r+1]$ durch:

$$x_k=y_r^{(s)} \text{ wenn } n_k=s \quad \text{und} \quad \#_{m\leqq k-1}(n_m=s)=r-1.$$

Dann folgt wie im Beweis von Satz 5.2, daß die Folge (x_n) λ-glv. in $[1, r+1]$ ist. Für diese Folge (x_n) gilt nun nach Definition dieser Folge:

$$x_s \in I_k \text{ ist äquivalent mit } n_s = k. \quad \text{W.z.b.w.}$$

Ganz analog kann man die entsprechende Umkehrung von Satz 5.3 zeigen. Wir erhalten damit

Folgerung 1. *Die im Beweis von Satz* 5.3 (5.3a) *gegebene Konstruktion von Folgen* (n_k) *natürlicher Zahlen mit den in diesem Satz genannten Eigenschaften liefert alle solchen Folgen.*

Folgerung 2. *Aus dem Hilfssatz* 7.1 *entnimmt man sofort (infolge der* λ*-Glv. der Folge* (x_n)*):*

$$\#_{s \leqq N}(n_s = k) = \sum_{s \leqq N} \chi_{I_k}(x_s) = \frac{r_k}{r} N + o(N) \qquad (k = 1, 2, \ldots).$$

A) Es sei λ ein normiertes Maß auf X, (x_n) λ-rel.glv. in X. Sei (X_n) ein System von Teilmengen von X mit den in Hilfssatz 2.3 b angegebenen Eigenschaften; wir zerlegen das System (X_n) in zwei fremde Teilsysteme (Y_n) und (Z_n), wobei in (Y_n) alle Mengen X_n positiven λ-Maßes, in (Z_n) alle λ-Nullmengen von (X_n) liegen sollen. Sei $\lambda(Y_i) = r_i$ $(i = 1, 2, \ldots)$. Dann ist also $\sum_i r_i = 1$.

Wir unterscheiden jetzt verschiedene Fälle:

1) (Y_n) und (Z_n) sind endlich.

Dann ist also

$$X = \bigcup_m Y_m \cup \bigcup_k Z_k$$

selbst eine λ-Menge (also kompakt) und es liegt λ-Glv. in X vor (die Folge (x_n) ist λ-glv.).

2) (Y_n) ist endlich und (Z_n) unendlich.

Dann sei

$$Y = \bigcup_n Y_n;$$

Y ist eine λ-Menge mit $\lambda(Y) = 1 = \lambda(X)$. X ist lokalkompakt, nicht kompakt (denn sonst müßte nach Hilfssatz 2.4 auch (Z_n) endlich sein). Der Träger des Maßes λ ist kompakt $(\subset \overline{Y})$. Y ist nach Konstruktion lokalkompakt.

In diesem Fall kann vieles passieren. Wir erwähnen einige Möglichkeiten:

a) Sei (y_n) λ-rel. glv. in Y. Dann ist (y_n) auch λ-rel. glv. in $\overline{Y}$ (Y ist eine λ-Menge) und daher λ-glv. in $\overline{Y}$. Es gilt also für alle λ-Mengen $C \subset \overline{Y}$ (und daher auch für alle λ-Mengen in X):

$$\sum_{n \leqq N} \chi_C(y_n) = \lambda(C) N + o(N).$$

Die Folge (y_n) ist also auch $d\lambda$-glv. (§ 3, Def. 2).

b) Sei (y_n) λ-rel.glv. in Y und $z_i \in Z_i$ $(i=1,2,\ldots)$. Wir definieren eine Folge (t_n) in X durch:

$$t_{2n-1}=y_n, \quad t_{2n}=z_n \quad (n=1,2,\ldots).$$

Man sieht sofort, daß diese Folge (t_n) λ-rel.glv. in X ist. Hier haben wir z.B.:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n\leqq N}\chi_Y(t_n)=\tfrac{1}{2}<\lambda(Y)=1.$$

Diese Folge (t_n) ist also nicht $d\lambda$-glv.

c) Sei (y_n) λ-rel.glv. in Y und $z_i \in Z_i$ $(i=1,2,\ldots)$. Wir definieren eine Folge (t_n) in X durch:

$t_{n^2}=y_n$ $(n=1,2,\ldots)$ und für die übrigen x_i wählen wir der Reihe nach $z_1, z_2, \ldots$.

Dann sieht man wieder leicht, daß diese Folge (t_n) λ-rel.glv. in X ist. Hier haben wir z.B.:

$$\sum_{n\leqq N}\chi_Y(t_n)=o(N)$$

und damit wegen der λ-rel.Glv. von (t_n) auch

$$\sum_{n\leqq N}\chi_C(t_n)=o(N) \quad \text{für alle } \lambda\text{-Mengen } C\subset X.$$

Diese Folge (t_n) ist also nicht $d\lambda$-glv.

d) Ersetzt man in c) die Folge (n^2) (die wir in der Definition von (t_n) benutzt haben) durch eine sehr rasch nach unendlich wachsende Folge natürlicher Zahlen, dann sieht man wie vorher, daß

$$\sum_{n\leqq N}\chi_C(t_n)$$

(für jede beliebige λ-Menge $C\subset X$) beliebig langsam wachsen kann.

3) (Y_n) ist unendlich. Hier ist also X lokalkompakt (nicht kompakt) und der Träger von λ ist ebenfalls nicht kompakt. Wir definieren neue Mengen

$$T_n=Y_n\cup Z_n \text{ (oder } T_n=Y_n, \text{ wenn } Z_n=\emptyset) \quad (n=1,2,\ldots).$$

Dann haben wir für das System $(T_i)_{i=1}^{\infty}$ alle Eigenschaften von Hilfssatz 2.3b und noch dazu: $\lambda(T_i)=r_i>0$ für alle i.

Wir definieren eine Folge (n_k) natürlicher Zahlen durch:

$$n_k=l \text{ genau dann, wenn } x_k\in T_l \quad (k=1,2,\ldots).$$

Die Folge (n_k) erfüllt die Voraussetzungen von Hilfssatz 7.1 (und ebenso die Folge (r_i)); wir haben daher nach Folgerung 2:

$$\sum_{l\leqq N}\chi_{T_k}(x_l)=\#_{l\leqq N}(n_l=k)=r_kN+o(N) \quad (k=1,2,\ldots).$$

Daraus folgt zunächst:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n\leqq N}\chi_{T_k}(x_n)=r_k.$$

Sei $C\subset T_k$ eine λ-Menge; dann wird

$$\lim_{N\to\infty}\frac{1}{N}\sum\chi_C(x_n)=\lim_{N\to\infty}\frac{\sum_{n\leqq N}\chi_C(x_n)}{\sum_{n\leqq N}\chi_{T_1}(x_n)}\cdot\lim_{N\to\infty}\frac{1}{N}\sum_{n\leqq N}\chi_{T_1}(x_n)=\frac{\lambda(C)}{r_1}\cdot r_1=\lambda(C)$$

und dieselbe Relation gilt für jede beliebige λ-Menge $C\subset X$, da ja das System (T_i) lokalendlich ist.

Zusammenfassen erhalten wir also den

Satz 7.1. *Sei λ ein normiertes (beschränktes) Maß auf X mit nichtkompaktem Träger. Dann ist jede λ-rel.glv. Folge in X auch $d\lambda$-glv. und umgekehrt. Hat λ kompakten Träger und ist X nicht kompakt, dann gibt es λ-rel.glv. Folgen (x_n) in X, die nicht $d\lambda$-glv. sind.*

B) Wir betrachten jetzt den Fall eines unbeschränkten Maßes auf X. Dann kann der Träger von λ nicht kompakt sein.

1) Wir verwenden die Eigenschaft E 1 von X: Dann gilt zunächst:

$$\lim_{n\to\infty}\lambda\left(\bigcup_{i=1}^{n}Q_i\right)=\lim_{n\to\infty}\lambda(Q_n)=\infty. \tag{20}$$

Wir haben daher für ein beliebiges (festes) i und für eine λ-rel. glv. Folge (x_n) in X:

$$N\geqq\sum_{n\leqq N}\chi_{Q_i}(x_n)=\frac{\lambda(Q_i)}{\lambda(Q_1)}\sum_{n\leqq N}\chi_{Q_1}(x_n)(1+o(1))$$

und weiter

$$\lim_{N\to\infty}\frac{N}{\sum_{n\leqq N}\chi_{Q_1}(x_n)}\geqq\frac{\lambda(Q_i)}{\lambda(Q_1)}.$$

Daraus folgt wegen (20) aber

$$\lim_{N\to\infty}\frac{N}{\sum_{n\leqq N}\chi_{Q_1}(x_n)}=\infty$$

oder wegen der λ-rel. Glv. von (x_n):

$$\sum_{n\leqq N}\chi_C(x_n)=o(N)\qquad\text{für alle }\lambda\text{-Mengen }C\subset X.$$

Und diese letzte Beziehung gilt für alle λ-rel.glv. Folgen (x_n) in X. Zusammenfassend erhalten wir also:

Satz 7.2. *Sei X nicht kompakt und λ ein Maß auf X mit nicht kompaktem Träger. Dann gilt:*

1) $\lambda(X)<\infty \Leftrightarrow \sum_{n\leqq N}\chi_C(x_n)=\frac{\lambda(C)}{\lambda(X)}N+o(N)$ *für alle λ-rel.glv. Folgen (x_n) in X und für alle λ-Mengen $C\subset X$.*

2) $\lambda(X)=\infty \Leftrightarrow \sum_{n\leqq N}\chi_C(x_n)=o(N)$ *für alle λ-rel.glv. Folgen (x_n) in X und für alle λ-Mengen $C\subset X$.*

2) Wir wollen die Aussage 2) von Satz 7.2 jetzt genauer untersuchen.

Sei $g(x)\geqq 1$ für $x\geqq 1$ eine stetige, monoton (im starken Sinn) wachsende Funktion mit $g(x+1)\leqq x^2$ und $\lim_{x\to\infty} g(x)=\infty$, sei h die Umkehrfunktion von $\sqrt{g}$. Wir setzen in Satz 5.1 $a(k)=[h(k+1)]$. Dann gilt $a(k)\geqq k$ und wir haben nach Satz 5.1 für die dort definierte Folge (l_n):

$$\#_{n\leqq M}(l_n=i)=\frac{b^2}{2}+O(b), \quad \text{wenn } m(b)\leqq M<m(b+1). \tag{21}$$

Nun gilt:

$$m(b)=m(b-1)+ba(b-1)>a(b-1)+1\geqq h(b)$$

und daher folgt aus

$$m(b)\leqq M$$

die Beziehung

$$h(b)<M$$

und weiter

$$b^2<g(M).$$

Wir erhalten also aus (21):

$$\#_{n\leqq M}(l_n=i)<g(M)+o(g(M)) \quad (i=1,2,\ldots). \tag{22}$$

Aus Satz 5.2 (und Beweis) ergibt sich damit in den dort verwendeten Bezeichnungen ((y_n) ist rel.glv. im R_1 bezüglich des Lebesgueschen Maßes μ):

$$\sum_{n\leqq N}\chi_A(y_n)=\mu(A)f_1(N)+o(f_1(N)) \overset{(22)}{<} \mu(A)\,g(N)+o(g(N)).$$

Für die in Satz 5.3 konstruierte Folge (n_k) haben wir daher:

$$\#_{k\leqq N}(n_k=m)<r_m\,g(N)+o(g(N))$$

und schließlich ergibt sich aus Satz 5.4 (und Beweis):

$$\sum_{n\leqq N}\chi_C(x_n)<\frac{\lambda(C)}{r_1}g(N)+o(g(N))$$

für eine λ-rel.glv. Folge (x_n) in X und für alle λ-Mengen $C \subset X$. Daraus folgt also (wenn wir noch A) 2) dazunehmen):

Satz 7.3. *Sei λ ein Maß auf X (X nicht kompakt), so daß der Träger von λ entweder kompakt ist oder λ ein unbeschränktes Maß ist. Dann gibt es zu jeder positiven, beliebig langsam nach unendlich wachsenden Funktion $g(N)$ stets eine λ-rel.glv. Folge (x_n) in X, so daß*

$$\sum_{n \leqq N} \chi_C(x_n) = o(g(N))$$

für alle λ-Mengen $C \subset X$ gilt.

§ 8. Approximation und Umordnung

Wir beginnen mit einem Hilfssatz (d ist eine Metrik, die die Topologie von X liefert).

Hilfssatz 8.1. *Sei λ ein Maß auf X, (x_n) λ-rel.glv. in X, (y_n) eine Folge in X mit $d(x_n, y_n) \to 0$ für $n \to \infty$.*

Sei $A_1 \subset X$ eine kompakte λ-Menge mit $\lambda(A_1) > 0$; sei $A_2 \subset A_1$ eine λ-Menge mit $\lambda(A_1) = \lambda(A_2)$.

Sei $(x_{n_k}) = (x_n)|A_1$.

Dann gilt

$$\#_{k \leqq N}(y_{n_k} \notin A_2) = o(N).$$

Beweis. Sei

$$U_t = \{y | d(y, X \setminus A_2) < t\} \quad \text{und} \quad t_k = \max_{2^k < l \leqq 2^{k+1}} d(x_{n_l}, y_{n_l}).$$

Dann haben wir nach Voraussetzung: $\lim_{k \to \infty} t_k = 0$. Nun gilt:

$$\#_{2^{s-1} < k \leqq 2^s}(y_{n_k} \notin A_2) \leqq \#_{2^{s-1} < k \leqq 2^s}(x_{n_k} \in A_1 \setminus U_{t_{s-1}}).$$

Daher erhalten wir für $2^l < N \leqq 2^{l+1}$:

$$\begin{aligned}
\#_{k \leqq N}(y_{n_k} \notin A_2) &= \#_{0 < k \leqq 2^1}(y_{n_k} \notin A_2) + \#_{2^1 < k \leqq 2^2} + \cdots + \#_{2^{l-1} < k \leqq 2^l} + \#_{2^l < k \leqq N} \\
&\leqq \#_{0 < k \leqq 2^1}(x_{n_k} \in A_1 \setminus U_{t_0}) + \cdots + \#_{2^{l-1} < k \leqq 2^l}(x_{n_k} \in A_1 \setminus U_{t_{l-1}}) \\
&\quad + \#_{2^l < k \leqq N}(x_{n_k} \in A_1 \setminus U_{t_l}) = (*).
\end{aligned}$$

Da die Folge (x_n) λ-rel.glv. in X ist, so ist $(x_n)|A_1 = (x_{n_k})$ $\dfrac{1}{\lambda(A_1)}$ λ-glv. in A_1; es gilt daher

$$\#_{2^{s-1} < k \leqq 2^s}(x_{n_k} \in A_1 \setminus U_{t_{s-1}}) = \frac{\lambda(A_1 \setminus U_{t_{s-1}})}{\lambda(A_1)} 2^{s-1} + \alpha_{s-1}$$

mit

$$\lim_{s\to\infty}\frac{\alpha_{s-1}}{2^s}=0.$$

Damit erhalten wir:

$$\begin{aligned}(*)&=\sum_{i=0}^{l-1}\frac{\lambda(A_1\setminus U_{t_i})}{\lambda(A_1)}2^i+\frac{\lambda(A_1\setminus U_{t_l})}{\lambda(A_1)}(N-2^l)+\sum_{i=0}^{s}\alpha_i\\&\leqq\sum_{i=0}^{l-1}\frac{\lambda(A_1\setminus U_{t_i})}{\lambda(A_1)}2^i+\frac{\lambda(A_1\setminus U_{t_l})}{\lambda(A_1)}N+\sum_{i=0}^{s}\alpha_i\\&=\sum_{i=0}^{l-1}\frac{\lambda(A_1\setminus U_{t_i})}{\lambda(A_1)}2^i+o(N)+o(N)=o(N),\end{aligned}$$

denn es ist ja

$$\lim_{l\to\infty}\lambda(A_1\setminus U_{t_l})=\lambda(A_1)-\lim_{l\to\infty}\lambda(U_{t_l})=\lambda(A_2)-\lambda(A_2)=0.\qquad\text{W.z.b.w.}$$

Wir zeigen jetzt

Satz 8.1. *Sei λ ein nicht atomares Maß ($\lambda(\{x\})=0$ für alle $x\in X$) auf X, (x_n) λ-rel.glv. in X, (y_n) eine Folge in X mit*

$$\lim_{n\to\infty}d(x_n,y_n)=0.$$

Dann ist auch (y_n) λ-rel.glv. in X.

Beweis. 1) Sei zunächst X kompakt (mit abzählbarer Umgebungsbasis), λ ein normiertes Maß auf X ($\lambda(X)=1$), (x_n) λ-glv. in X und (y_n) wie oben.

(x_n) ist λ-glv.; daraus folgt für alle stetigen Funktionen $f\colon X\to R$ (vgl. [11])

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n\leqq N}f(x_n)=\lambda(f);\tag{23}$$

diese Funktionen f sind auch gleichmäßig stetig, das heißt, zu jedem $\varepsilon>0$ gibt es ein $\delta=\delta(\varepsilon)>0$, so daß für

$$d(x,y)<\delta\quad\text{gilt}\quad|f(x)-f(y)|<\varepsilon.\tag{24}$$

Sei nun $f\colon X\to R$ stetig, $\varepsilon>0$ gegeben; wir suchen ein dazugehöriges $\delta=\delta(\varepsilon)$ (aus der gleichmäßigen Stetigkeit von f); es gibt dann ein $n=n(\delta)=n(\varepsilon)\in N$, so daß

$$d(x_n,y_n)<\delta\qquad\text{für } n\geqq n(\varepsilon).$$

Daraus folgt nach (24):

$$|f(x_n)-f(y_n)|<\varepsilon\qquad\text{für } n\geqq n(\varepsilon).$$

Damit ergibt sich:

$$\begin{aligned}\frac{1}{N}\sum_{n\leqq N} f(y_n) &= \frac{1}{N}\sum_{n<n(\varepsilon)} f(y_n)+\frac{1}{N}\sum_{N\geqq n\geqq n(\varepsilon)} f(y_n)\\ &\leqq \frac{1}{N}\sum_{n<n(\varepsilon)} f(y_n)+\frac{1}{N}\sum_{N\geqq n\geqq n(\varepsilon)} \big(f(x_n)+\varepsilon\big)\\ &= \frac{1}{N}\sum_{n<n(\varepsilon)} \big(f(y_n)-f(x_n)\big)+\frac{1}{N}\sum_{n\leqq N} f(x_n)+\frac{N-n(\varepsilon)+1}{N}\varepsilon\\ &\leqq \frac{1}{N}\sum_{n\leqq N} f(x_n)+2\varepsilon \qquad \text{für } N \text{ genügend groß.}\end{aligned}$$

Analog schätzt man $\frac{1}{N}\sum_{n\leqq N} f(x_n)$ nach unten ab und es folgt also damit die λ-Glv. von (y_n) nach (23).

2) Es sei jetzt X lokalkompakt, λ ein nicht atomares Maß auf X, (x_n) und (y_n) wie in Satz 8.1.

Sei $A\subset X$ eine kompakte λ-Menge mit $\lambda(A)>0$. Sei $(y_{m_k})=(y_n)|A$.

Wir betrachten die Folge (x_{m_k}) und bezeichnen mit $x_{m_{k_l}}$ alle Elemente dieser Folge für die

$$x_{m_{k_l}}\notin A.$$

Wir setzen

$$A'=A\cup\bigcup_l\{x_{m_{k_l}}\}.$$

Dann haben wir (λ nicht atomar): $\lambda(A)=\lambda(A')$ und $A\subset A'$. Sei $\varepsilon_l=d(x_{m_{k_l}},A)$; da $y_{m_{k_l}}\in A$, so haben wir nach Voraussetzung

$$\lim \varepsilon_l=0.$$

a) Es gibt nur endlich viele l, für die $x_{m_{k_l}}\notin A$. Dann ist A' kompakt.

b) Es sei $x_{m_{k_l}}\notin A$ für $l=1,2,\ldots$.

Es gibt ein $s>0$, so daß $\overline{V_s(A)}$ kompakt ist und ein $l(s)$, so daß

$$x_{m_{k_l}}\in\overline{V_s(A)} \qquad \text{für } l\geqq l(s).$$

Alle Häufungspunkte der Menge $\{x_{m_{k_l}}\}_{l=1}^{\infty}$ liegen in $\overline{V_s(A)}$ und sogar in A (denn A ist kompakt und $\lim\limits_{l\to\infty}\varepsilon_l=0$). Es liegen daher alle Häufungspunkte von A' in A', also ist A' abgeschlossen und daher kompakt. A' ist eine λ-Menge.

Sei nun

$$(x_{n_k})=(x_n)|A'.$$

Dann folgt aus Hilfssatz 8.1 ($A_1=A'$, $A_2=A$):

$$\#_{k\leqq N}(y_{n_k}\notin A)=o(N).$$

Streichen wir in der Folge (x_{n_k}) alle jene Elemente, für die $y_{n_k} \notin A$ liegt, so streichen wir $o(N)$ unter den ersten N Gliedern. Die so aus (x_{n_k}) entstehende Folge bezeichnen wir mit $(x_{\overline{n_k}})$. Nun ist (Satz 4.3) die Folge (x_{n_k}) $\frac{\lambda}{\lambda(A)}$-glv. in A', daher ist auch die neue Folge $(x_{\overline{n_k}})$ $\frac{\lambda}{\lambda(A)}$-glv. in A'.

Die Folge natürlicher Zahlen $(\overline{n_k})$ hat also folgende Eigenschaften:

α) $(x_{\overline{n_k}})$ ist $\frac{\lambda}{\lambda(A)}$-glv. in A',

β) $x_{\overline{n_k}} \in A'$ und

γ) $y_{\overline{n_k}} \in A$.

Sei $(y_{m_k}) = (y_n)|A$.

Die Folge natürlicher Zahlen (m_k) erfüllt:

$$y_{m_k} \in A \quad \text{und} \quad x_{m_k} \in A'.$$

Beide Folgen (m_k) und $(\overline{n_k})$ sind nach Konstruktion eindeutig festgelegt. Sie sind identisch; denn ist $a \in (\overline{n_k})$, so ist $x_a \in A'$ und $y_a \in A$, also $a \in (m_k)$ und ist

$$b \in (m_k), \quad \text{so ist } y_b \in A \quad \text{und} \quad x_b \in A', \quad \text{also } b \in (\overline{n_k}).$$

Es ist also

$$(\overline{n_k}) \equiv (m_k);$$

nun ist $(x_{\overline{n_k}}) = (x_{m_k})$ nach α) $\frac{\lambda}{\lambda(A)}$-glv. in A', A' ist kompakt; daher folgt aus 1):

(y_{m_k}) ist $\frac{\lambda}{\lambda(A)}$-glv. in A' und daher auch $(\lambda(A) = \lambda(A'))$: $(y_{m_k}) = (y_n)|A$ ist $\frac{\lambda}{\lambda(A)}$-glv. in A. Das gilt für alle kompakten λ-Mengen $A \subset X$ mit $\lambda(A) > 0$, daher ist die Folge (y_n) λ-rel. glv. in X nach Satz 4.3. W. z. b. w.

Aus dem eben gegebenen Beweis sieht man, daß in Satz 8.1 die Voraussetzung λ nicht atomar für kompaktes X nicht notwendig ist.

Ist X eine kompakte Gruppe, so wurde dieser Satz bereits von Hlawka [9] bewiesen (mit einer anderen Methode).

Für $X = T^1$ (eindimensionaler Torus) hat als erster Neumann [12] bewiesen: Jede in X direkte Folge kann zu einer in X glv. Folge umgeordnet werden.

Wir beweisen dieses Ergebnis hier allgemein.

Satz 8.2. *Sei λ ein nicht atomares Maß auf X, (y_n) eine in X dicht liegende Folge. Dann kann (y_n) zu einer in X λ-rel. glv. Folge umgeordnet werden.*

Beweis. Sei (x_n) λ-rel. glv. in X (Satz 5.4); sei (ε_n) eine Folge reeller, positiver Zahlen mit $\lim_{n \to \infty} \varepsilon_n = 0$. Wir können dann zu jedem $n \in N$ ein $k(n) \in N$ finden, so daß

$$d(x_n, y_{k(n)}) < \varepsilon_n$$

wird. Nach Satz 8.1 ist dann die Folge $(y_{k(n)})$ λ-rel. glv. in X. Sei $C \subset X$ eine λ-Menge. Wir schreiben

$$G_C(N) = \sum_{n \leqq N} \chi_C(y_{k(n)});$$

es gilt dann

$$G_C(N) = \frac{\lambda(C)}{\lambda(D)} \left(G_D(N) + o(G_D(N))\right)$$

und

$$G_D(N) \to \infty \text{ für } N \to \infty \quad \text{für alle } \lambda\text{-Mengen } D \subset X \text{ mit } \lambda(D) > 0.$$

Es gibt daher eine Folge $(f(N))$ natürlicher Zahlen, so daß $f(N)$ monoton wachsend ist, $\lim_{N \to \infty} f(N) = \infty$ und

$$\lim_{N \to \infty} \frac{f(N)}{G_C(N)} = 0 \quad \text{für alle } \lambda\text{-Mengen } C \subset X \text{ mit } \lambda(C) > 0.$$

Sei $(y_{l(n)})$ die Folge aller y_n, die nicht in $(y_{k(n)})$ vorkommen. Wir konstruieren eine neue Folge (z_n) durch:

Die Teilfolge der (z_n), bestehend aus $z_1, z_2, \ldots, z_{t(N)}$ mit $t(N) = N + f(N)$ enthält $y_{k(1)}, \ldots, y_{k(N)}$ und $y_{l(1)}, \ldots, y_{l(f(N))}$. Die Folge (z_n) ist eine Umordnung der Folge (y_n). Sei M eine natürliche Zahl, L eine natürliche Zahl mit

$$L + f(L) \leqq M < L + 1 + f(L+1).$$

Dann gilt für eine beliebige λ-Menge $C \subset X$:

$$\sum_{n \leqq M} \chi_C(z_n) \leqq \sum_{n \leqq L+1+f(L+1)} = \sum_{n \leqq L+1} \chi_C(y_{k(n)}) + \sum_{n \leqq f(L+1)} \chi_C(y_{l(n)})$$
$$\leqq G_C(L+1) + f(L+1)$$

und

$$\sum_{n \leqq M} \chi_C(z_n) \geqq \sum_{n \leqq L+f(L)} = \sum_{n \leqq L} \chi_C(y_{k(n)}) + \sum_{n \leqq f(L)} \chi_C(y_{l(n)}) \geqq G_C(L).$$

Damit erhalten wir

$$\frac{G_C(L)}{G_D(L+1) + f(L+1)} \leqq \frac{\sum_{n \leqq N} \chi_C(z_n)}{\sum_{n \leqq N} \chi_D(z_n)} \leqq \frac{G_C(L+1) + f(L+1)}{G_D(L)}$$

für λ-Mengen $C, D \subset X$ mit $\lambda(D) > 0$. Beide Seiten dieser Ungleichung streben für L (und N) nach unendlich gegen $\frac{\lambda(C)}{\lambda(D)}$, woraus also die λ-rel. Glv. der Folge (z_n) folgt. W. z. b. w.

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Restricted Regular Rings

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A ring is called restricted regular if each of its proper homomorphic images is regular in the sense of von Neumann. Such rings will be called r-rings. Simple examples exist that an r-ring may not be regular. The object of this article is to characterize right noetherian r-rings. It has been possible to characterize them for the class of non-prime rings. The prime case remains open. However, it can be shown that if R is a prime right noetherian r-ring then each non-trivial ideal is a unique product of maximal ideals. It is also shown that if R is a right duo r-ring then either R is regular or R has exactly one non-trivial ideal. An application to group-rings is given in the last section. Throughout J will denote the Jacobson radical of a ring, and $l(J)$, $r(J)$ respectively will denote the left, right annihilator of J in R. A ring R is called a right duo ring if each of its right ideals is two-sided. We begin with

Theorem 1. *Let R be a nonprime right noetherian ring. Then R is an r-ring if and only if*

(i) *R is semisimple artinian, or*

(ii) *R has exactly one non-trivial ideal, namely the radical J and is isomorphic to an $n \times n$ matrix ring over a local ring, or*

(iii) *R has exactly three non-trivial ideals namely J, $l(J)$, $r(J)$ and is isomorphic to $\begin{pmatrix} U & N \\ O & V \end{pmatrix}$, where U, V are simple artinian and N is irreducible $(U-V)$ bimodule.*

Proof. Suppose $J=0$. Since R is right noetherian, each ideal is a product of prime ideals (proof similar to commutative noetherian rings ([4], p. 200)). As prime ideals are maximal, we have in particular, $0=M_1 \dots M_t$, where M_i are maximal ideals. Now

$$(M_1 \cap M_2 \cap \cdots \cap M_t)^t \subset M_1 M_2 \dots M_t = 0,$$

implies

$$M_1 \cap M_2 \cap \cdots \cap M_t \subset J = 0.$$

Therefore R is a finite direct sum of simple artinian rings R/M_i. Hence R is artinian, proving (i). So let $J \neq 0$. If $J^2 \neq 0$, then R/J^2 regular implies $J=J^2$, which is not possible in a right noetherian ring. Thus $J^2=0$. Hence $l(J) \neq 0$, $r(J) \neq 0$. Since J is the unique minimal ideal, $l(J)$ and $r(J)$ are prime ideals and hence maximal. Also $J \neq 0$ implies R is bound to J ([1], Theorem 6), that is $l(J) \cap r(J) \subset J$. This implies $J = l(J) \cap r(J)$. In case $l(J)=r(J)$, then J is the only

ideal of R. Hence by ([2], Theorem 1, p. 56), $R=S_n$, where S is local r-ring, proving (ii).

Suppose $l(J)\neq r(J)$. Let M be a maximal ideal of R. Then $l(J)\cdot r(J)\subset l(J)\cap r(J)\subset M$ implies $l(J)=M$ or $r(J)=M$. This shows that $l(J)$ and $r(J)$ are the only maximal ideals. It is clear that in a right noetherian r-ring, each non-zero ideal is a finite intersection of maximal ideals. Hence J, $l(J)$ and $r(J)$ are the only proper ideals in R.

We now proceed to show that R is isomorphic to $\begin{pmatrix} U & N \\ O & V \end{pmatrix}$, where U and V are simple artinian. We have $R/J\cong l(J)/J\oplus r(J)/J$, therefore $\bar{1}=\bar{e}+\bar{f}$ where $\bar{e}$, $\bar{f}$ are central orthogonal idempotents in R/J. We can assume that e and f are orthogonal idempotents in R such that $1=e+f$. Next from the above decomposition of R/J

$$eR+J=l(J)=Re+J,$$

and

$$fR+J=r(J)=Rf+J,$$

which immediately yields

$$eJ=0, \quad Jf=0, \quad eae=ea \quad \text{and} \quad faf=fa \quad \text{for all } a\in R.$$

We define

$$\sigma\colon R\to\begin{pmatrix} fRf & fRe \\ 0 & eRe \end{pmatrix},$$

by

$$a\to\begin{pmatrix} faf & fae \\ 0 & eae \end{pmatrix}.$$

Then it is easy to check that σ is an isomorphism of R onto S where

$$S=\begin{pmatrix} fRf & fRe \\ 0 & eRe \end{pmatrix}.$$

Now $S/J(S)=fRf\oplus eRe$, which implies that fRf and eRe are semisimple artinian. But since R/J which is isomorphic to $S/J(S)$ has only two ideals, eRe and fRf are simple artinian. Finally it is clear that $N=fRe$ is irreducible $(U-V)$ bimodule where $U=fRf$ and $V=eRe$ are simple artinian. Converse is obvious when R is of type (i) or (ii). So suppose now that $R\cong S$, where

$$S=\begin{pmatrix} U & N \\ O & V \end{pmatrix}$$

and U, V are simple artinian rings and N is irreducible $(U-V)$ bimodule. Since $S/J(S)$ is regular, it is enough to show that each nonzero ideal A of S contains $J(S)$. Let $a\in A$, then

$$a=ue_{11}+xe_{12}+ve_{22}, \quad u\in U,\ v\in V,\ x\in N$$

$$u'e_{11}av'e_{22}=u'xv'e_{12}\in A, \quad \text{for all } u'\in U,\ v'\in V.$$

Thus $U x V e_{12} \subset A$. Since N is irreducible $(U-V)$ bimodule, $U x V = N$. Thus $N e_{12} \subset A$. Hence $J(S) \subset A$, completing the proof of the theorem.

Assume now that R is a prime right noetherian r-ring. Let A be any nonzero ideal in R. Then $A^2 \neq 0$. Since A/A^2 is a nilpotent ideal in the regular ring R/A^2, we must have $A = A^2$. In particular R must be semi-simple. Furthermore since R/A is regular, there exist a finite set of maximal ideals M_i such that $A = M_1 \cap M_2 \cap \cdots \cap M_n$. Then $A = A^2 = (M_1 \cap M_2 \cdots \cap M_n)^2 \subset M_1 M_2 \ldots M_n \subset A$ gives that $A = M_1 M_2 \ldots M_n$. That is if A is also equal to $M_1' M_2' \ldots M_k'$, then $k = n$ and $M_i = M'_{\sigma(i)}$ where $1 \leqq i \leqq n$ and $\sigma(i)$ is a permutation of $1, 2, \ldots, n$, is clear. This result may be stated in

Proposition. *If R is a prime right noetherian r-ring then R is semi-simple and each non-trivial ideal is a unique product of maximal ideals.*

The next theorem characterizes right duo restricted regular rings without chain condition.

Theorem 2. *A right duo ring R is an r-ring if and only if R is strongly regular or R has exactly one proper ideal.*

Proof. First assume $J = 0$. Then $(x R)^2 \neq 0$, for any nonzero x in R. This implies $R/(x R)^2$ is regular and thus $(x R)^2 = x R$. Therefore $x \in x^2 R$, proving that R is strongly regular. Next let $J \neq 0$. Then J is the minimal ideal and thus $l(J)$, $r(J)$ are prime ideals. If $J^2 \neq 0$, then $J = J^2$ which is not possible. Thus $J^2 = 0$. Hence both $l(J)$ and $r(J)$ are not zero, and being prime ideals they must be maximal. Further R is bound to J ([1], Theorem 6), that is, $l(J) \cap r(J) \subset J$. This gives in our case $l(J) \cap r(J) = J$. Therefore, if $l(J) \neq r(J)$ then $R/J = l(J)/J \oplus r(J)/J$. In particular, $\bar{1} = \bar{e} + \bar{f}$, where $\bar{e}$ and $\bar{f}$ are orthogonal idempotents in R/J. As $J^2 = 0$, idempotents modulo J can be lifted and thus we can assume that e, f are orthogonal idempotents in R such that $1 = e + f$. But then $R = eR \oplus fR$ which yields that R is regular, a contradiction. Hence $l(J) = r(J) = J$, completing the "only if" part. Converse is trivial.

A special case of the above is

Corollary. *A commutative ring is an r-ring iff it is either strongly regular or it has exactly one proper ideal.*

2. In this section we give a characterisation of the group of a right-noetherian group-ring which is restricted regular. Let A be any ring and G a group. Let $R = AG$ be a group-ring and $w G$ the augmentation ideal.

Theorem 3. *Let R be a right noetherian group-ring which is restricted regular. Then either*

(i) *G is finite and $O(G)$ is unit in A, or*

(ii) *G is infinite group and each normal subgroup is of finite index. Further if $Z(G) \neq (e)$, then G is cyclic, or*

(iii) *G is finite simple group.*

Proof. Since $A = R/w G$, A is semi-simple artinian. First let R be non-prime. If $J = 0$, we get (i). Assume $J \neq 0$. Then the only possible non-trivial ideals

are J, $l(J)$ and $r(J)$. We assert that in group-ring R, J is the only non-trivial ideal. Consider the augmentation ideal wG. Then $wG=J$, $r(J)$, or $l(J)$. Since $J^2=0$, the right or the left annihilator of wG is not zero. Thus G is finite. Hence R is self-injective which implies that R is quasi-frobenious. But then $l(J)=r(J)$, proving that J is the only non-trivial ideal in R. Next, let H be any normal subgroup of G. Then $wH=J=wG$. Since w is faithful, this implies, G is a simple group. We now consider the case when R is prime. In this case we know that A is prime and G is infinite. Let H be a normal subgroup of G. Then $A\dfrac{G}{H}\cong\dfrac{AG}{wH}$ is semisimple artinian, hence G/H is finite, proving that each normal subgroup is of finite index. As a special case let the centre $Z(G)\neq(e)$, then $G/Z(G)$ finite will give that the derived group G' of G is finite ([3], 15.1.13, p. 453). This implies $G'=(e)$, since G is infinite. Hence G is abelian, and thus must be then cyclic.

Remark 1. If R is a right noetherian ring such that either (i) or (iii) holds then R can be shown to be restricted regular.

Remark 2. Examples can be given to show that there exist infinite groups each of whose normal subgroup is of finite index but the groups have trivial centres.

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Zur Existenz total ganz abgeschlossener Ringerweiterungen

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1. Ein Ring[1] A heißt *total ganz abgeschlossen*, wenn sich jeder Ringhomomorphismus $f\colon R \to A$ auf jede ganze Erweiterung R' von R fortsetzen läßt. Der Begriff wurde von Enochs in [1] eingeführt, um die Theorie vom *algebraischen Körperabschluß* auf kommutative Ringe zu verallgemeinern. Ein „totaler ganzer Abschluß" des Ringes R läßt sich definieren und existiert, wenn es überhaupt eine total ganz abgeschlossene Erweiterung von R gibt. Die Existenzfrage beantwortet der

Satz [1; Th. 1]. *Ein Ring R kann dann und nur dann in einen total ganz abgeschlossenen Ring A eingebettet werden, wenn R reduziert ist (d.h. keine nilpotenten Elemente enthält).*

Der Beweis des „nur-dann"-Teils wird in [1] durch einen Fehler hinfällig und soll deshalb hier durch einen neuen ersetzt werden. Der Vollständigkeit halber geben wir auch einen kurzen Beweis der anderen Hälfte des Satzes. Der Fehler in [1] liegt darin, daß die auf S. 703 angegebene Multiplikationstafel immer eine nicht-assoziative Algebra definiert. Mit $\iota \in I$ und $\nu = \{\iota\}$ ist dort nämlich

$$Z_\nu^2\, Y_\iota = Z_\nu\, Y_\iota = X \neq 0,$$

aber

$$Z_\nu (Z_\nu\, Y_\iota) = Z_\nu\, X = 0.$$

2. Wir nennen eine Ringerweiterung B von R wesentlich („tight" in [1]), wenn jeder Homomorphismus $f\colon B \to C$, der R monomorph abbildet, ganz B monomorph abbildet. Gleichbedeutend ist: Für jedes Ideal $\mathfrak{b} \neq 0$ in B gilt $R \cap \mathfrak{b} \neq 0$.

Lemma. *Ist R ein Ring mit Nilradikal* rad $R \neq 0$, *so gibt es zu jeder Kardinalzahl γ eine wesentliche ganze Erweiterung B von R mit* Card $B \geqq \gamma$.

Beweis. rad R enthält ein Element $c \neq 0$ mit $c^2 = 0$. Sei I eine Indexmenge der vorgegebenen Mächtigkeit γ, und seien e sowie x_i $(i \in I)$ freie Erzeugende des unitären R-Moduls

$$A_I := R\,e \oplus \bigoplus_{i \in I} R\,x_i.$$

[1] „Ring" bedeutet in dieser Note: kommutativer assoziativer Ring mit Einselement.

Mit der Multiplikationstafel

$$e^2=e, \qquad e\,x_i=x_i\,e=x_i,$$
$$x_i^2=c\,e, \qquad x_i\,x_j=0 \quad \text{für } i,j\in I,\ i\neq j,$$

wird A_I zu einer kommutativen, i.A. nicht assoziativen R-Algebra. Der R-Modul

$$N_I:=\bigoplus_{i\in I} N\,x_i,$$

wo N den Annullator von c in R bezeichnet, ist ein Ideal in A_I. In der Algebra $B_I:=A_I/N_I$ seien e bzw. y_i $(i\in I)$ die kanonischen Bilder von e bzw. x_i $(i\in I)$. Man prüft leicht nach, daß B_I assoziative R-Algebra ist; z.B. wird mit $i\neq j$:

$$y_i^2\,y_j=c\,e\,y_j=c\,y_j=0=y_i(y_i\,y_j),$$

da $c\in N$ wegen $c^2=0$. Nach Identifizieren von R mit Re (vermöge $r\mapsto re$ für $r\in R$) wird B_I eine (unitäre) *Ringerweiterung* von R. B_I ist *ganze* Erweiterung von R, da für die Erzeugenden y_i $(i\in I)$ die Gleichung $y_i^2=c$ besteht. Schließlich ist B_I auch *wesentliche* Erweiterung von R. Sei nämlich $\mathfrak{b}$ ein Ideal in B_I, und sei $0\neq b\in\mathfrak{b}$, etwa

$$b=r+\sum_{i\in I} r_i\,y_i, \qquad \text{mit } r,r_i\in R,\ i\in I.$$

Wir unterscheiden drei Fälle:

a) $r\notin N$, also $rc\neq 0$. Dann ist $bc=rc\in\mathfrak{b}\cap R$, und folglich $\mathfrak{b}\cap R\neq 0$.

b) $r\in N$, alle $r_i\in N$. Dann ist $0\neq b=r\in\mathfrak{b}\cap R$, da $r_i\,y_i=0$ für alle $i\in I$.

c) $r\in N$, $r_j\notin N$ für ein $j\in I$. Dann wird $b\,y_j=r_j\,y_j^2=r_j\,c\neq 0$ und $r_j\,c\in\mathfrak{b}\cap R$. Der Ring B_I hat mindestens die Mächtigkeit $\gamma=\operatorname{Card} I$ und besitzt damit alle behaupteten Eigenschaften. Q.e.d.

Beweis des Satzes: 1) Sei $f\colon R\to A$ eine Einbettung des nicht reduzierten Ringes R in einen total ganz abgeschlossenen Ring A. Jede wesentliche ganze Erweiterung B von R läßt sich in A einbetten und erfüllt deshalb

$$\operatorname{Card} B\leqq\operatorname{Card} A.$$

Dies widerspricht dem Lemma.

2) Sei R reduziert, also $\operatorname{rad} R=0$. P sei die Menge aller Primideale von R. Für jedes $\mathfrak{p}\in P$ läßt sich $R/\mathfrak{p}$ als Integritätsbereich in einen total ganz abgeschlossenen Ring $A(\mathfrak{p})$ einbetten [1; Prop. 3]. Nach [1; Prop. 1] ist

$$A:=\prod_{\mathfrak{p}\in P} A(\mathfrak{p})$$

total ganz abgeschlossen. Das direkte Produkt der natürlichen Homomorphismen $R\to R/\mathfrak{p}\subseteq A(\mathfrak{p})$ über alle $\mathfrak{p}\in P$ liefert eine Einbettung $f\colon R\to A$, denn

$$\ker f=\bigcap_{\mathfrak{p}\in P}\mathfrak{p}=\operatorname{rad} R=0$$

nach einem wohlbekannten Satz [2; Th. (2.3)]. Q.e.d.

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The Divisor Class Groups of Some Rings of Holomorphic Functions*

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Abstract

Let $(X, {}_X\mathcal{O})$ be a normal complex analytic space and $A \Subset X$ a connected Stein compact set, i.e. a compact subset of X which has a basis of open neighborhoods which are Stein spaces. We restrict attention to those A such that $R = H^0(A, {}_X\mathcal{O})$ is Noetherian. In Section I various exact sequences involving the divisor class group of R, denoted $C(R)$, are developed. (If A is a point, one of these sequences is well-known [24], [39].)

Let B be a connected compact Stein set on a normal variety Y such that $S = H^0(B, {}_Y\mathcal{O})$ and $T = H^0(A \times B, {}_{X\times Y}\mathcal{O})$ are Noetherian. In II we give a Künneth-type formula which relates $C(R)$, $C(S)$ and $C(T)$. In III we show certain analytic local rings are unique factorization domains, study the divisor class groups of local rings on the quotient of an analytic space by a finite group, and prove a simple result on the topology of germs of complex analytic sets. We give a function-theoretic proof that complete intersections of dimensions greater than three which have isolated singularities have local rings which are unique factorization domains.

Notation

Given a topological space A, a sheaf $\mathfrak{F}$ of abelian groups over A and an open cover $\mathfrak{U}$ of A, we denote by $C^p(\mathfrak{U}, \mathfrak{F})$ the group of Cech p-cochains with coefficients in $\mathfrak{F}$ of the cover $\mathfrak{U}$. Moreover, $H^p(\mathfrak{U}, \mathfrak{F})$ denotes the p-th cohomology group of the corresponding Cech cochain complex and $H^p(A, \mathfrak{F})$ is the p-th Cech cohomology group of A with coefficients in $\mathfrak{F}$. Given a continuous map $f: A \to B$ there is for each $p \geqq 0$ a sheaf $R^p f(\mathfrak{F})$ on B called the p^{th} direct image of $\mathfrak{F}$ (under f). The stalk of $R^p f(\mathfrak{F})$ at $b \in B$ is

$$R^p f(\mathfrak{F})_b = \lim_{N \to b} H^p(f^{-1}(N), \mathfrak{F})$$

where N runs through the system of open neighborhoods of b. Whenever we are given a group G^p for each $p \geqq 0$, G^* denotes the graded group $\{G^p | p \geqq 0\}$.

If C is a subset of A and γ is a contravariant functor from the category of open neighborhoods of C with inclusion maps to the category of abelian

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groups or any other category with direct limits, we denote by

$$\lim_{N \to C} \gamma(N)$$

the direct limit over the system of open neighborhoods of C.

I. Exact Sequences for Divisor Class Groups

Let $(X, {}_X\mathcal{O})$ be a normal complex-analytic space and $A \Subset X$ a connected Stein compact set. Siu [35] has proved $R = \Gamma(A, {}_X\mathcal{O})$ is Noetherian if and only if $Y \cap A$ has only a finite number of topological components for every complex-analytic subvariety Y defined in any open neighborhood of A. Frisch [11] has shown R is Noetherian if A is a real semi-analytic set. We shall assume henceforth that R is Noetherian, although for some of our results this is unnecessary.

It is easy to see that R is an integrally-closed, Noetherian integral domain. Let $I \subset R$ be an ideal. Let $f_1, \ldots, f_n$ generate I and let N be an open neighborhood of A in X in which there is a holomorphic function F_i which represents each f_i, $1 \leqq i \leqq n$. (Here, we use the fact that for any sheaf of abelian groups $\mathfrak{F}$ on X

$$\lim_{N \to A} H^q(N, \mathfrak{F}) = H^q(A, \mathfrak{F}).$$

See [13, Ch. II, § 4.11.1]. This fact will be used repeatedly. Of course, $\Gamma(A, {}_X\mathcal{O}) = H^0(A, {}_X\mathcal{O})$ is the group of sections over A of ${}_X\mathcal{O}$.) The germ along A of the set of common zeros of the F_i, $i = 1, \ldots, n$, does not depend on the choice of generators for I. We denote this germ by $V(I)$. If Y is the germ along A of a (complex) analytic set defined in an open neighborhood of A, we define

$$\mathrm{Id}(Y) = \{f \in R \mid f \text{ is zero on } Y\}.$$

We now prove elementary theorems which relate ideals of R and germs along A of analytic sets which are defined in open neighborhoods of A. We call such germs A-germs.

Lemma I.1 (Nullstellensatz). *For any ideal $I \subset R$, $\mathrm{Id}(V(I))$ is the radical of I. For any A-germ Y, $V(\mathrm{Id}(Y)) = Y$.*

Proof. Choose generators $g_1, \ldots, g_k$ for I and an open neighborhood N of A in which they have representatives $G_1, \ldots, G_k$. Let $\mathfrak{I}$ be the subsheaf of ${}_X\mathcal{O}|N$ generated by $G_1, \ldots, G_k$. Let $\mathcal{O}^k \to \mathfrak{I}$ be the surjective sheaf map which $G_1, \ldots, G_k$ determine. The set W of common zeros of the G's is a representative of $V(I)$. Let $M \subset N$ be a Stein open neighborhood of A. By Theorem B of Cartan and Oka [7, 26, 17], $\Gamma(M, \mathcal{O}^k) \to \Gamma(M, \mathfrak{I})$ is surjective, i.e. the restrictions to M of the G's generate $\mathfrak{I}|M$. By taking the direct limit over Stein neighborhoods of A, we see that $g_1, \ldots, g_k$ generate $\Gamma(A, \mathfrak{I})$. Hence, $\Gamma(A, \mathfrak{I}) = I$.

Let $f \in \mathrm{Id}(V(I))$. Without loss of generality, we may assume f has a representative F in N and F vanishes on W. For each $a \in A$ there is an integer n such that the germ at a of F^n is in $\mathfrak{I}$, i.e. $(F^n)_a \in \mathfrak{I}$ by the ordinary Nullstellensatz. Then $(F^n)_b \in \mathfrak{I}$ for all $b \in N$ sufficiently near a. Since A is compact, there

exists an integer p such that $(F^p)_a \in \mathfrak{I}$ for all $a \in A$. Then

$$f^p = F^p | A \in \Gamma(A, \mathfrak{I}) = I.$$

This shows the radical of I contains $\mathrm{Id}(V(I))$. The opposite inclusion is obvious, so the first claim of the lemma is proved.

Let Y be an A-germ. Let W be an analytic subset which represents Y in a Stein open set $N \supset A$. The ideal sheaf $\mathfrak{I}$ of W is coherent [6].

By Theorem A of Oka and Cartan [7, 26, 17], W is the set of common zeros of $\Gamma(N, \mathfrak{I})$. But $\Gamma(A, \mathfrak{I}) = \mathrm{Id}(Y)$, so $V(\mathrm{Id}(Y)) = Y$.

The A-germ Y will be called irreducible if it is not the union of two A-germs both distinct from Y. A radical ideal $I \subset R$ is prime if and only if $V(I)$ is irreducible. For any ideals $I, J \subset R$, $V(I \cap J) = V(I) \cup V(J)$. It follows from the Lasker-Noether primary decomposition theorem [44, Vol. I, p. 208] that each A-germ may be written uniquely (up to order) as a finite union of irreducible A-germs none of which contains another. Such a representation of an A-germ is called irredundant.

Lemma I.2. *Let Y be an irreducible A-germ. Every open neighborhood of A which contains a subvariety representing Y contains a globally irreducible subvariety which represents Y. An A-germ Z is irreducible if each open set in some basis of neighborhoods of A contains a globally irreducible subvariety which represents Z.*

Proof. Let W be a subvariety of an open neighborhood of A and a representative of Y. Let $W_1, W_2, \ldots$ be the decomposition of W into globally irreducible branches [25, p. 67]. Since the branch decomposition is locally finite and A is compact, there is a neighborhood of A which intersects only finitely many branches, say $W_1, \ldots, W_s$. Then $Y = W_1^* \cup \cdots \cup W_s^*$, where $*$ means "germ along A". Since Y is irreducible, $Y = W_j^*$ for some $j \leqq s$. Suppose Z has the property in the lemma. We shall show $\mathrm{Id}(Z)$ is prime. Let $f, g \in R$ and suppose $f, g \in \mathrm{Id}(Z)$. Choose a neighborhood of A in which there exists a globally irreducuble variety W which represents Z and holomorphic functions F and G which represent f and g. Then $F \cdot G$ is zero on W. Hence F or G is zero on W. This implies f or $g \in \mathrm{Id}(Z)$.

A minimal prime of R is a minimal element of the set of non-zero proper prime ideals of R.

Lemma I.3. *An ideal $P \subset R$ is a minimal prime if and only if there exists an irreducible non-empty A-germ Y such that $P = \mathrm{Id}(Y)$ and for each $a \in A$ every branch of the germ at a of Y has dimension one less than the dimension at a of X.*

Proof. Suppose $P = \mathrm{Id}(Y)$ for an A-germ Y which satisfies the condition of the lemma. We know P is prime. It is obviously a proper ideal of R. Let $Q \subsetneqq P$ be a prime ideal. Then $Q = \mathrm{Id}(Z)$ for an irreducible A-germ $Z \supsetneqq Y$. Let N be a connected open neighborhood of A which has globally irreducible subvarieties $Y' \subset Z'$ which are representatives of Y and Z. (This is possible by Lemma I.2.) If Y' contains an open non-empty subset of Z', $Y' = Z'$. Therefore Y' is nowhere

dense in Z'. At some $a \in A$ the germ of Y is non-empty and each branch of it has dimension one less than the dimension at a of X. The set Z' must contain a neighborhood of a. Hence $Z' = N$, which gives $Q = 0$.

On the other hand, let P be any minimal prime of R. Choose $f \in P$, $f \neq 0$. Then $V(P) \subset V(f \cdot R) = W_1 \cup \cdots \cup W_k$, where the latter equation is the irredundant decomposition of $V(f \cdot R)$ into irreducible germs. Then $V(P) \subset W_i$ for some $i \leqq k$. Moreover, W_i satisfies the condition of the lemma ([17], p. 115). Since $P \supset \mathrm{Id}(W_i)$ and P is a minimal prime, $P = \mathrm{Id}(W_i)$.

Let $\mathfrak{M}$ be the sheaf of germs of meromorphic fractions on X.

Lemma I.4. *Let $U \subset X$ be open and let S be a nowhere dense subvariety of U. Then the restriction mapping α: $\Gamma(U, \mathfrak{M}) \to \Gamma(U - S, \mathfrak{M})$ is injective. If $\dim_s S \leqq \dim_s U - 2$ for all $s \in S$, then α is surjective.*

Proof. Let $m \in \Gamma(U, \mathfrak{M})$ and $\alpha(m) = 0$. Each $p \in U$ has a neighborhood N such that $m|N = f/g$ where $f, g \in \Gamma(N, \mathcal{O})$. Since $f|N - S = 0$, $f = 0$. Therefore $m = 0$. The final assertion of the lemma is in [25, p. 133].

Let S be the singular locus of X. Let Q be the quotient field of R.

Lemma I.5. *The canonical mapping from Q to $\Gamma(A, \mathfrak{M})$ is an isomorphism.*

Proof. Let $m \in \Gamma(A, \mathfrak{M})$. Choose an open connected Stein set $N \supset A$ for which there exists $M \in \Gamma(N, \mathfrak{M})$ with $M|A = m$. Let E be the closure in N of the pole set of $M|N - S$. By the Remmert-Stein theorem ([25], VII, Theorem 1), E is an analytic set. Since N is Stein and the ideal sheaf of E is coherent, there exists (by Theorem A) a holomorphic function $H \not\equiv 0$ on N such that $H|E = 0$.

Let $U \Subset N$ be an open neighborhood of A. Let $W_1, \ldots, W_n$ be the global irreducible components of W which intersect U. For $i \leqq n$, the order with which M has a pole along $W_i - S$ is constant on $W_i - S$, because it is connected. (Compare [42], p. 160.) Let k be the maximum of these orders. The function MH^k extends to a holomorphic function G on U [25, VI, Prop. 4]. Clearly, $m = G|A/H^k|A$. This shows the map from Q to $\Gamma(A, \mathfrak{M})$ is surjective. It is obviously injective.

Let $Q^* = Q - \{0\}$ and let $F(R)$ be the free abelian group on the minimal primes of R. Let $\mathfrak{M}^*$ be the sheaf of germs of not identically zero meromorphic functions on X and let $\mathcal{O}^*$ be the sheaf of germs of invertible holomorphic functions. The quotient of $\mathfrak{M}^*$ by $\mathcal{O}^*$ is denoted by $\mathfrak{D}^\#$ and called the sheaf of divisors. There is an exact sequence

$$0 \to \mathcal{O}^* \to \mathfrak{M}^* \to \mathfrak{D}^\# \to 0.$$

Lemma I.6. *There is an isomorphism*

$$\Lambda\colon \lim_{N \to A} \Gamma(N - S, \mathfrak{D}^\#) \to F(R)$$

such that the composition

$$Q^* \cong \lim_{N \to A} \Gamma(N - S, \mathfrak{M}^*) \to \lim_{N \to A} \Gamma(N - S, \mathfrak{D}^\#) \xrightarrow{\Lambda} F(R)$$

is the divisor class map.

Proof. The isomorphism involving Q^* follows from Lemmas I.4 and I.5. For any open $N \supset A$ define $\Lambda_N: \Gamma(N-S, \mathfrak{D}^*) \to F(R)$ as follows: Let $D \in \Gamma(N-S, \mathfrak{D}^*)$. The support of D is a subvariety E of $N-S$ such that $1+\dim_e E = \dim_e X$ for each $e \in E$. (See [42], p. 156.) The codimension of S in N is at least two. By the Remmert-Stein theorem the closure $\bar{E}$ of E in N is a subvariety of N. Let $Y_1 \cup \cdots \cup Y_n$ be the irredundant decomposition into irreducible A-germs of the A-germ of $\bar{E}$. For $i \leqq n$, let W_i be a globally irreducible subvariety which represents Y_i in a connected, open M, $A \subset M \subset N$. Let $U \subset M$ be an open neighborhood of A such that

$$W \cap U = \bigcup_{i=1}^{n} W_i \cap U.$$

For $i \leqq n$, $\mathrm{Id}(Y_i)$ is a minimal prime. (Proof: It suffices to show $\mathrm{Id}(Y_1)$ is a minimal prime. Let s be the dimension of M. Clearly, $\dim_x W_1 < s$ for all $x \in W_1$. Suppose $x \in W_1 \cap U$ and $\dim_x W_1 < s-1$. Then in some neighborhood of x, $W_1 \subset W_2 \cup \cdots \cup W_n$. Since W_1 is irreducible, it follows that $W_1 \subset W_2 \cup \cdots \cup W_n$. This is a contradiction.) Furthermore, $W_i - S$ is a global irreducible component of $E \cap M$. Let Z_i be the regular points of $W_i - S - U_{j \neq i} W_j$. For $x \in Z_i$, let $m_i(x)$ be the order with which D vanishes along Z_i at x. The function $m_i(x)$ is locally constant and hence constant, because Z_i is connected. (Compare [42, p. 160].) Let m_i be its value.

Let Π be the set of minimal primes of R. For $P \in \Pi$ define

$$\begin{aligned} m_P(D) &= m_i, \quad \text{if } P = \mathrm{Id}(Y_i) \quad \text{for some } i \leqq n \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

Set $\Lambda_N(D) = \sum_{P \in \Pi} m_P(D) P$. The Λ_N are compatible so we define $\Lambda = \lim_{N \to A} \Lambda_N$. Clearly, Λ is injective. To see that it is surjective, for $P \in \Pi$ choose a representative W of $V(P)$ in an open set $N \supset A$ such that $1+\dim_w W = \dim_w N$ for all $w \in W$. Then $W-S$ defines a divisor $D \in \Gamma(N-S, \mathfrak{D}^*)$ such that $\Lambda_N(D) = P$.

Recall the construction of the divisor class map σ. Let $g \in R$, $g \neq 0$. Let $P \in \Pi$. The local ring R_P is a discrete valuation ring. Let $n_P(g)$ be the order of g in R_P and set $\sigma(g) = \sum_{P \in \Pi} n_P(g) P$. Extend σ to Q^* by $\sigma(h/g) = \sigma(h) - \sigma(g)$. To prove that Λ induces the divisor class map, we show that if $G \in \Gamma(N, \mathcal{O})$ represents g in an open $N \supset A$ and $(G) \in \Gamma(N-S, \mathfrak{D}^*)$ is the divisor of $G|(N-S)$, then for each $P \in \Pi$, $m_P((G)) = n_P(g)$.

Let $P \in \Pi$ and let $M \subset N$ be an open Stein neighborhood of A in which there is a globally irreducible subvariety W which represents $V(P)$. Let $x \in W-S$ be a regular point of W. There exists a holomorphic function H defined on M such that $H|W = 0$ and H vanishes of order one along W at x (by Theorem A and the coherence of the ideal sheaf of H). In some neighborhood of A, $H = T^m U V^{-1}$, where $T|A$ generates PR_P and $U|A$, $V|A \in R-P$. Such an equation is possible only if $m = 1$. This shows $m_P((H)) = n_P(H|A) = 1$.

In some neighborhood of A, $G=H^q U_1 U_2^{-1}$, where $U_1|A$ and $U_2|A$ belong to $R-P$. Since U_1 and U_2 are not identically zero on the component of W which contains $V(P)$,

$$m_P((G))=q\,m_P((H))=q=n_P(g).$$

A resolution of the singularities of X is a complex manifold $(X^*, {}_X\mathcal{O})$ and a proper, onto, holomorphic map $f\colon X^*\to X$ such that $f^{-1}(S)$ is nowhere dense and the restriction $f\colon (X^*-f^{-1}(S))\to X-S$ is biholomorphic. Hironaka has announced a proof that a resolution always exists. The algebraic case is in [18]. We use this result freely.

Fix a resolution $f\colon X^*\to X$. Let ${}_{X^*}\mathfrak{M}$ (resp. ${}_{X^*}\mathfrak{M}^*$, ${}_{X^*}\mathcal{O}^*$) be the sheaf of germs of meromorphic functions (resp. not identically zero meromorphic functions, invertible holomorphic functions) on X^*. Let $\mathfrak{D}$ be the sheaf of divisors on X^*. (There is a short exact sequence

$$0\to {}_{X^*}\mathcal{O}^*\to {}_{X^*}\mathfrak{M}^*\to \mathfrak{D}\to 0.)$$

Lemma I.7. *There is a homomorphism* $\lambda\colon \Gamma(f^{-1}(A),\mathfrak{D})\to F(R)$ *such that:*

1. *The kernel of* λ *consists of the divisors whose support has a germ on* $f^{-1}(A)$ *which is contained in the germ of the set* $f^{-1}(S)$.
2. *The kernel of* λ *is countable.*
3. λ *is surjective.*
4. *The composition*

$$Q^*\cong\Gamma(f^{-1}(A), {}_{X^*}\mathfrak{M}^*)\to\Gamma(f^{-1}(A),\mathfrak{D})\xrightarrow{\lambda} F(R)$$

is the divisor class map.

Proof. We first assert that $\Gamma(f^{-1}(A), {}_{X^*}\mathfrak{M})$ is the quotient field of $\Gamma(f^{-1}(A), {}_{X^*}\mathcal{O})$. Given $m\in\Gamma(f^{-1}(A), {}_{X^*}\mathfrak{M})$ let N be a connected open Stein neighborhood of A such that $M\in\Gamma(f^{-1}(N), {}_{X^*}\mathfrak{M})$ represents m. Let E be the pole set of M and let $\mathfrak{I}$ be the ideal sheaf of E. By [14], p. 59, $R^0 f(\mathfrak{I})$ is coherent. By Theorem A there exists a non-trivial section of $R^0 f(\mathfrak{I})$ over N. This section may be interpreted as a holomorphic function H on $f^{-1}(N)$ such that $H\not\equiv 0$ and $H|E=0$. By using the fact that f is a resolution, we may see as in Lemma I.5 that if $U\Subset N$ is an open neighborhood of A $M|f^{-1}(U)$ is a quotient of a holomorphic function by a power of H. This proves the assertion. The isomorphism $f^*\colon \Gamma(A, {}_X\mathcal{O})\to\Gamma(f^{-1}(A), {}_{X^*}\mathcal{O})$ extends to an isomorphism of quotient fields which maps Q^* onto $\Gamma(f^{-1}(A), {}_{X^*}\mathfrak{M}^*)$. This explains the isomorphism in 4.

Let $N\supset A$ be open. A homomorphism

$$\Gamma(f^{-1}(N),\mathfrak{D})\to\Gamma(N-S,\mathfrak{D}^{\#})$$

is given by first restricting to $f^{-1}(N-S)$ and then using the fact that f maps $f^{-1}(N-S)$ isomorphically onto $N-S$. Call the direct limit of these homo-

morphisms r. There is a commutative diagram

$$\begin{array}{ccccc} \Gamma(f^{-1}(A), {}_{X^*}\mathfrak{M}^*) & \xleftarrow{\cong} & Q^* & \xrightarrow{\cong} & \lim\limits_{N\to A} \Gamma(N-S, {}_X\mathfrak{M}^*) \\ \downarrow & & & & \downarrow \\ \Gamma(f^{-1}(A), \mathfrak{D}) & \xrightarrow{\cong} & \lim\limits_{N\to A} \Gamma(f^{-1}(N), \mathfrak{D}) & \xrightarrow{r} & \lim\limits_{N\to A} \Gamma(N-S, \mathfrak{D}^\#) \xrightarrow{A} F(R). \end{array}$$

We define λ to be the composition of the maps of the bottom row of this diagram. Assertion 4 follows from the final assertion of Lemma I.6. The first assertion of the lemma is evident.

To prove the third assertion, we show Ar is surjective. Let P be a minimal prime and let W be a representative of $V(P)$ in an open set $N \supset A$. We may assume $1+\dim_w W = \dim_w N$ for all $w \in W$. The closure of $f^{-1}(W-S)$ in $f^{-1}(N)$ is an analytic set W^* such that $1+\dim_x W^* = \dim_x X^*$ for all $x \in W^*$. The divisor of W^* determines an element d of $\lim\limits_{N\to A} \Gamma(f^{-1}(N), \mathfrak{D})$ such that $Ar(d) = P$.

Let $U_1, U_2, \ldots$ be a basis of neighborhoods of $f^{-1}(A)$. For each $j \geqq 1$, the compact set $f^{-1}(A)$ can be covered by finitely many open sets $W_{ij} \subset U_j$, $i \leqq n_j$, such that

$$\{d \in \Gamma(W_{ij}, \mathfrak{D}) | \text{support of } d \subset f^{-1}(S)\}$$

is countable. Let $X_j = U_{i \leqq n_j} W_{ij}$. Then there are only countably many divisors on X_j with support contained in $f^{-1}(S)$. Assertion 2 follows from the fact that

$$\Gamma(f^{-1}(A), \mathfrak{D}) = \lim_{j\to\infty} \Gamma(X_j, \mathfrak{D}).$$

Lemma I.8. *The coboundary homomorphism*

$$\delta: \Gamma(f^{-1}(A), \mathfrak{D}) \to H^1(f^{-1}(A), {}_{X^*}\mathcal{O}^*)$$

is onto.

Proof. Let $\ell \in H^1(f^{-1}(A), {}_{X^*}\mathcal{O}^*)$. Let N be a connected Stein open neighborhood of A such that there exists a holomorphic line bundle $\mathfrak{L}$ on $f^{-1}(N)$ which represents ℓ. There is a section $s \in H^0(f^{-1}(N), \mathfrak{L}) \cong H^0(N, R^0 f(\mathfrak{L}))$ which is not zero at $p \in A$. Then s is zero on no non-empty open subset of the connected set $f^{-1}(N)$.

There is an open cover $\{U_\beta\}_{\beta\in B}$ of $f^{-1}(N)$ such that there are holomorphic functions $g_{\beta\gamma}$ on $U_\beta \cap U_\gamma$ which are never zero and which form the cocycle in $H^1(f^{-1}(N), {}_{X^*}\mathcal{O}^*)$ of $\mathfrak{L}$. After perhaps refining the cover, there are holomorphic functions h_β on U_β which represent the section s. On each $U_\beta \cap U_\gamma$, $h_\gamma = g_{\beta\gamma} h_\beta$. The functions $\{h_\beta\}_{\beta\in B}$ represent a divisor D on $f^{-1}(N)$ such that $\delta(D|f^{-1}(A)) = \ell$.

Lemma I.9. *If the codimension of S in X is at least three at each point of S, the coboundary homomorphism*

$$\delta: \lim_{N\to A} \Gamma(N-S, \mathfrak{D}^\#) \to \lim_{N\to A} H^1(N-S, {}_X\mathcal{O}^*)$$

is onto.

Proof. Let N be a connected Stein neighborhood of A and let $\ell \in H^1(N-S, {}_X\mathcal{O}^*)$. Let $\mathfrak{L}$ be the locally invertible sheaf on $N-S$ associated to ℓ. The sheaf $\mathfrak{L}$ extends to a coherent sheaf $\tilde{\mathfrak{L}}$ on N. ([36], or [12, VII.6].) Let $t \in \Gamma(N, \tilde{\mathfrak{L}})$ be non-zero at some point of $N-S$. Then $t|N-S$ is a non-zero section of $\mathfrak{L}$ on the connected manifold $N-S$. As in the previous lemma, we may construct a divisor D on $N-S$ such that $\delta(D)=\ell$. The lemma follows by passing to the direct limit groups.

The divisor class group of R, denoted $C(R)$, is defined to be the cokernel of the divisor class map.

Proposition I.1. *Let G be the subgroup of $H^1(f^{-1}(A), {}_{X^*}\mathcal{O}^*)$ formed by the line bundles of divisors on $f^{-1}(A)$ which have support whose germ along $f^{-1}(A)$ is contained in the germ of $f^{-1}(S)$. There is a short exact sequence*

$$0 \to G \to H^1(f^{-1}(A), {}_{X^*}\mathcal{O}^*) \to C(R) \to 0.$$

G is a countable group.

Proof. This follows immediately from Lemmas I.7 and I.8.

From the cohomology of the sheaf sequence $0 \to Z \to \mathcal{O} \xrightarrow{e} \mathcal{O}^* \to 0$ on X^*, where e is the exponential map, we obtain an exact sequence

$$H^1(f^{-1}(A), Z) \xrightarrow{a} H^1(f^{-1}(A), {}_{X^*}\mathcal{O}) \xrightarrow{b} H^1(f^{-1}(A), {}_{X^*}\mathcal{O}^*) \xrightarrow{c} H^2(f^{-1}(A), Z).$$

From Proposition I.1 we get an exact sequence

$$0 \to H^1(f^{-1}(A), {}_{X^*}\mathcal{O})/b^{-1}(G) \to C(R) \to \text{Image } c/c(G) \to 0.$$

Since the left hand group is divisible, this sequence splits.

Proposition I.2. *If the codimension of S in X is at least three at each point of S, then $C(R) \cong \lim_{N \to A} H^1(N-S, {}_X\mathcal{O}^*)$.*

Proof. This follows from Lemma I.6, Lemma I.9 and the long exact sequence which is obtained from the cohomology sequence over $N-S$ of the short exact sequence

$$0 \to \mathcal{O}^* \to \mathfrak{M}^* \to \mathfrak{D} \to 0$$

by taking the direct limit as $N \to A$.

Corollary. *If X is a manifold, $C(R) \cong H^2(A, Z)$.*

Proof. The proposition shows $C(R) \cong \lim_{N \to A} H^1(N, {}_X\mathcal{O}^*) = H^1(A, {}_X\mathcal{O}^*)$. Since A is Stein, $H^1(A, {}_X\mathcal{O}) \cong H^2(A, {}_X\mathcal{O}) \cong 0$. The cohomology sequence gives $H^1(A, {}_X\mathcal{O}^*) \cong H^2(A, Z)$. (The latter group is the *Cech* cohomology group.)

II. Divisor Class Groups of Products

Let $(X, {}_X\mathcal{O})$, $(Y, {}_Y\mathcal{O})$ be normal analytic spaces. Let $A \subset X$, $B \subset Y$ be connected compact Stein subsets such that $\Gamma(A, {}_X\mathcal{O})$, $\Gamma(B, {}_Y\mathcal{O})$, $\Gamma(A \times B, {}_{X \times Y}\mathcal{O})$ are Noetherian. This section establishes a formula which relates the divisor class groups of these three rings. The proof requires several technical lemmas.

Lemma II.1. *Let X and Y be analytic spaces and let $f: X \to Y$ be a proper holomorphic map. Let $\mathfrak{F}$ be a coherent analytic sheaf on X and suppose Y is a Stein space. Then the direct image map $f^*: H^q(X, \mathfrak{F}) \to \Gamma(Y, R^q f(\mathfrak{F}))$ is an isomorphism of topological vector spaces for each $q \geqq 0$. In particular, each $H^q(X, \mathfrak{F})$ is Hausdorff.*

Proof. It is well known that f^* is an algebraic isomorphism [14, p. 20]. Let Y^* be a relatively compact Stein open subset of Y and set $X^* = f^{-1}(Y^*)$. Let $\mathfrak{U}$ be a countable open cover of X^* by Stein subsets. Let $Z^q(\mathfrak{U}, \mathfrak{F})$ be the space of q-cocycles with its relative topology as a subset of $C^q(\mathfrak{U}, \mathfrak{F})$. Give $H^q(\mathfrak{U}, \mathfrak{F})$ the quotient topology for the natural projection $\pi: Z^q(\mathfrak{U}, \mathfrak{F}) \to H^q(\mathfrak{U}, \mathfrak{F})$. Leray's theorem [17, Chap. V] gives an isomorphism $\gamma: H^q(\mathfrak{U}, \mathfrak{F}) \cong H^q(X^*, \mathfrak{F})$ which can be used to endow the latter with a topology. If this topology is Hausdorff, it is independent of the choice of $\mathfrak{U}$ [20]. Since $R^q f(\mathfrak{F})$ is coherent [14, p. 59], there are finitely many cocycles $\xi_1, \ldots, \xi_n \in Z^q(\mathfrak{U}, \mathfrak{F})$ whose images generate $\Gamma(Y^*, R^q f(\mathfrak{F}))$ as a $\Gamma(Y^*, {}_Y\mathcal{O})$ module. Hence there is a continuous surjection

$$\alpha: C^{q-1}(\mathfrak{U}, \mathfrak{F}) + \Gamma(Y^*, {}_Y\mathcal{O})^n \to Z^q(\mathfrak{U}, \mathfrak{F})$$

given by

$$\alpha(c, g) = \delta c + \sum_{i=1}^{n} g_i \xi_i,$$

where δ is the Cech coboundary and $g = (g_1, \ldots, g_n) \in \Gamma(Y^*, {}_Y\mathcal{O})^n$. The map α induces a continuous surjective map $\beta: \Gamma(Y^*, {}_Y\mathcal{O})^n \to H^q(\mathfrak{U}, \mathfrak{F})$. By the open mapping theorem (applied to α) $H^q(\mathfrak{U}, \mathfrak{F})$ has the quotient topology for β. Since $(f^*|X^*)\gamma\beta: \Gamma(Y^*, {}_Y\mathcal{O})^n \to \Gamma(Y^*, R^q f(\mathfrak{F}))$ is induced by a sheaf homomorphism from ${}_Y\mathcal{O}^n$ to $R^q f(\mathfrak{F})$, it is continuous. Therefore, $f^*|X^*: H^q(X_1, \mathfrak{F}) \to \Gamma(Y^*, R^q f(\mathfrak{F}))$ is continuous.

Choose a sequence $Y_1, Y_2, \ldots$ of relatively compact Stein open sets in Y such that

$$\bigcup_{n=1}^{\infty} Y_n = Y.$$

Choose a countable Stein open cover $\mathfrak{U}_n$ of $f^{-1}(Y_n)$, $n \geqq 1$. Then

$$\bigcup_{n=1}^{\infty} \mathfrak{U}_n = \mathfrak{U}$$

is a countable Stein open cover of X. We endow $H^q(X, \mathfrak{F})$ with a topology by using $\mathfrak{U}$ and each $H^q(X_n, \mathfrak{F})$ with a topology by using $\mathfrak{U}_n$. The inclusion $\mathfrak{U}_n \subset \mathfrak{U}$ induces a continuous restriction $r_n: H^q(X, \mathfrak{F}) \to H^q(X_n, \mathfrak{F})$. Let

$$\rho_n: \Gamma(Y, R^q f(\mathfrak{F})) \to \Gamma(Y_n, R^q f(\mathfrak{F}))$$

be the restriction map. Then

$$\rho_n \circ (f^*|X) = (f^*|X_n) \circ r_n: H^q(X, \mathfrak{F}) \to \Gamma(Y_n, R^q f(\mathfrak{F}))$$

is continuous. Since $\Gamma(Y, R^q f(\mathfrak{F}))$ has the coarsest topology such that each ρ_n is continuous, $f^* = f^*|X$ is continuous. But $\Gamma(Y, R^q f(\mathfrak{F}))$ is Hausdorff [17, Chap. 8], because $R^q f(\mathfrak{F})$ is coherent. Therefore $H^q(X, \mathfrak{F})$ is Hausdorff. The open mapping theorem shows f^* is an isomorphism.

Let X be an analytic space and let $f: X^* \to X$ be a resolution. A special case of the lemma is the following

Corollary. *Let $U \subset X$ be an open Stein set. Let ${}_{X^*}\mathcal{O}$ be the structure sheaf of X^*. There are topological linear isomorphisms*

$$\Gamma(U, R^q f({}_{X^*}\mathcal{O})) \cong H^q(f^{-1}(U), {}_{X^*}\mathcal{O}), \qquad q \geqq 0.$$

The latter spaces are Hausdorff.

For any analytic space X the symbol ${}_X\mathbb{Z}$ (resp. ${}_X\mathcal{O}$, ${}_X\mathcal{O}^*$) denotes the sheaf on X of germs of functions which are: integer-valued and locally constant (resp. holomorphic on X, holomorphic and nowhere equal to zero). We shall suppress the subscript X when so doing should not create confusion. We denote by ${}_X\mathcal{O}_x$ the stalk at x of ${}_X\mathcal{O}$, etc.

Recall that for any two topological vector spaces E and F the projective and injective completed tensor products (π- and ε-products) are defined. If E or F is nuclear they coincide and are both denoted $E \hat{\otimes} F$. (See [20]. A complete exposition may be found in [40].)

Lemma II.2. *Let X, Y be analytic spaces and let $U \subset X$, $V \subset Y$ be open subsets such that the cohomology groups $H^p(U, {}_X\mathcal{O})$ and $H^p(V, {}_Y\mathcal{O})$ are Hausdorff for all $p \geqq 0$. Let $\mathfrak{U} = \{U_i\}_{i \in I}$ (resp. $\mathfrak{V} = \{V_j\}_{j \in J}$) be a countable cover of U (resp. V) by open Stein sets. Let $\mathfrak{W} = \{U_i \times V_j\}_{(i,j) \in I \times J}$ and $W = X \times Y$. Then there exists a commutative diagram*

$$\begin{array}{ccc} H^*(\mathfrak{U}, {}_X\mathbb{Z}) \otimes_{\mathbb{Z}} H^*(\mathfrak{V}, {}_Y\mathbb{Z}) & \xrightarrow{A(\mathfrak{U}, \mathfrak{V})} & H^*(\mathfrak{U}, {}_X\mathcal{O}) \hat{\otimes} H^*(\mathfrak{V}, {}_Y\mathcal{O}) \\ \downarrow S(\mathfrak{U}, \mathfrak{V}) & & \downarrow T(\mathfrak{U}, \mathfrak{V}) \\ H^*(\mathfrak{W}, {}_W\mathbb{Z}) & \xrightarrow[B(\mathfrak{U}, \mathfrak{V})]{} & H^*(\mathfrak{W}, {}_W\mathcal{O}). \end{array}$$

The diagrams of this type are compatible with refinement of $\mathfrak{U}$ and $\mathfrak{V}$ and with restriction to open sets $U' \subset U$, $V' \subset V$ which satisfy the hypothesis of the lemma.

Proof. Let $A(\mathfrak{U}, \mathfrak{V})$ be the composition (of the obvious maps)

$$H^*(\mathfrak{U}, \mathbb{Z}) \otimes H^*(\mathfrak{V}, \mathbb{Z}) \to H^*(\mathfrak{U}, {}_X\mathcal{O}) \otimes_{\mathbb{C}} H^*(\mathfrak{V}, {}_Y\mathcal{O}) \to H^*(\mathfrak{U}, {}_X\mathcal{O}) \hat{\otimes} H^*(\mathfrak{V}, {}_Y\mathcal{O}).$$

Let $B(\mathfrak{U}, \mathfrak{V})$ be the homomorphism which the sheaf map ${}_W\mathbb{Z} \to {}_W\mathcal{O}$ induces.

Construct $S(\mathfrak{U}, \mathfrak{V})$ as follows: The sets $U_i \times V$, $i \in I$, form an open cover $\mathfrak{U}^*$ of $U \times V$. The sets $U \times V_j$, $j \in J$, form another cover $\mathfrak{V}^*$. There is a double complex $C^*(\mathfrak{U}^*, \mathfrak{V}^*; {}_W\mathbb{Z})$ with

$$C^{p,q}(\mathfrak{U}^*, \mathfrak{V}^*; {}_W\mathbb{Z}) = \prod_N \Gamma(N, {}_W\mathbb{Z}),$$

where N runs through the open sets

$$N=(U_{i_0}\cap\cdots\cap U_{i_p})\times(V_{j_0}\cap\cdots\cap V_{j_q})$$

which are products of p-simplices for $\mathfrak{U}$ and q-simplices for $\mathfrak{V}$. The differentials of this double complex are constructed from the differentials of the Cech cochain complex of the covers $\mathfrak{U}$ and $\mathfrak{V}$. (See [20] or [30]. Note that in [20] a sign $(-1)^p$ has been omitted.)

Define $\psi^{p,q}$: $C^p(\mathfrak{U},{}_X\mathbb{Z})\otimes_{\mathbb{Z}} C^q(\mathfrak{V},{}_Y\mathbb{Z})\to C^{p,q}(\mathfrak{U}^\#,\mathfrak{V}^\#;{}_W\mathbb{Z})$ by

$$(\psi^{p,q}(f\otimes g))_{i_0\dots i_p\, j_0\dots j_q}=f_{i_0\dots i_p}\, g_{j_0\dots j_q},$$

the factors on the right side of the equation being regarded as sections of ${}_W\mathbb{Z}$.

Let $\chi^{p,q}$: $C^{p,q}(\mathfrak{U}^\#,\mathfrak{V}^\#;{}_W\mathbb{Z})\to C^{p,q}(\mathfrak{W},{}_W\mathbb{Z})$ assign to a $(p+q)$-simplex $k=(i_0,j_0)\dots(i_{p+q},j_{p+q})$ of the nerve of the cover $\mathfrak{W}$ a section over the corresponding open set by the formula

$$(\chi^{p,q}(h))_k=\text{rest}\, h_{i_0\dots i_p\, j_0\dots j_q},$$

where $h\in C^{p,q}(\mathfrak{U}^\#,\mathfrak{V}^\#;{}_W\mathbb{Z})$ and rest denotes restriction. The $\chi^{p,q}$ form a homomorphism χ from the associated single complex of $C^*(\mathfrak{U}^\#,\mathfrak{V}^\#;{}_W\mathbb{Z})$ to the Cech cochain complex $C^*(\mathfrak{W},{}_W\mathbb{Z})$.

Let μ: $H^*(\mathfrak{U},{}_X\mathbb{Z})\otimes H^*(\mathfrak{V},{}_Y\mathbb{Z})\to H^*\big(C^*(\mathfrak{U},{}_X\mathbb{Z})\otimes_{\mathbb{Z}} C^*(\mathfrak{V},{}_Y\mathbb{Z})\big)$ be the mapping given by the usual Künneth formula. (See [37, p. 237].) Finally, $S(\mathfrak{U},\mathfrak{V})=\chi^*\psi^*\mu$ is the composition of μ and the cohomology maps induced by χ and ψ.

The construction of $T(\mathfrak{U},\mathfrak{V})$ is similar. First, define

$$\psi_1\colon C^*(\mathfrak{U},{}_X\mathcal{O})\otimes_{\mathbb{C}} C^*(\mathfrak{V},{}_Y\mathcal{O})\to C^*(\mathfrak{U}^\#,\mathfrak{V}^\#;{}_W\mathcal{O}),$$
$$\chi_1\colon C^*(\mathfrak{U}^\#,\mathfrak{V}^\#;{}_W\mathcal{O})\to C^*(\mathfrak{W},{}_W\mathcal{O})$$

by analogy with ψ and χ. Since $\chi_1\psi_1$ is continuous, it extends uniquely to a mapping of cochain complexes

$$\theta\colon C^*(\mathfrak{U},{}_X\mathcal{O})\hat{\otimes} C^*(\mathfrak{V},{}_Y\mathcal{O})\to C^*(\mathfrak{W},{}_W\mathcal{O}).$$

As noted by Kaup [20], Grothendieck's Künneth formula for nuclear Frechet spaces provides a canonical isomorphism

$$j\colon H^*(\mathfrak{U},{}_X\mathcal{O})\hat{\otimes} H^*(\mathfrak{V},{}_Y\mathcal{O})\to H^*\big(C^*(\mathfrak{U},{}_X\mathcal{O})\hat{\otimes} C^*(\mathfrak{V},{}_Y\mathcal{O})\big).$$

Let $T(\mathfrak{U},\mathfrak{V})=\theta^* j$, where θ^* is the cohomology map induced by θ.

Upon reading the proof of Grothendieck's Künneth formula ([34], Exp. 24), one sees that there is a commutative diagram:

$$\begin{array}{ccc} H^*(\mathfrak{U},{}_X\mathcal{O})\otimes_{\mathbb{C}} H^*(\mathfrak{V},{}_Y\mathcal{O}) & \xrightarrow{\iota} & H^*(\mathfrak{U},{}_X\mathcal{O})\hat{\otimes} H^*(\mathfrak{V},{}_Y\mathcal{O}) \\ \downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle j} \\ H^*\big(C^*(\mathfrak{U},{}_X\mathcal{O})\otimes_{\mathbb{C}} C^*(\mathfrak{V},{}_Y\mathcal{O})\big) & \xrightarrow[\lambda]{} & H^*\big(C^*(\mathfrak{U},{}_X\mathcal{O})\hat{\otimes} C^*(\mathfrak{V},{}_Y\mathcal{O})\big). \end{array}$$

(The map μ_1 is the Künneth isomorphism; ι is the canonical inclusion; λ is the cohomology map arising from the cochain inclusion.) Then,

$$T(\mathfrak{U}, \mathfrak{V})\,\iota = \theta^* \lambda\, \mu_1 = \chi_1^*\, \psi_1^*\, \mu_1 .$$

Let α be the homomorphism such that $\iota\,\alpha = A(\mathfrak{U}, \mathfrak{V})$. We have

$$T(\mathfrak{U}, \mathfrak{V}) \circ A(\mathfrak{U}, \mathfrak{V}) = \chi_1^*\, \psi_1^*\, \mu_1\, \alpha = B(\mathfrak{U}, \mathfrak{V})\, \chi^*\, \psi^*\, \mu = B(\mathfrak{U}, \mathfrak{V}) \circ S(\mathfrak{U}, \mathfrak{V}).$$

One can see from the definitions of the mappings of the diagram of the lemma that the mappings are compatible with restrictions.

Let X and Y be normal analytic spaces and $f\colon X^* \to X$, $g\colon Y^* \to Y$ be resolutions of singularities. Let $\mathfrak{F}$ (resp. $\mathfrak{G}$) be a coherent sheaf on X^* (resp. Y^*). For each set $U \subset X$ (resp. $V \subset Y$) set $U^* = f^{-1}(U)$ (resp. $V^* = g^{-1}(V)$). Let $A \subset X$ and $B \subset Y$ be compact Stein sets. Fix bases $\{U_n\}_{n \geqq 0}$, $\{V_n\}_{n \geqq 0}$ of Stein open neighborhoods of A and B. By abuse of notation,

$$H^i(A^*, \mathfrak{F}) \hat{\otimes} H^j(B^*, \mathfrak{G}) = \lim_{n \to \infty} H^i(U_n^*, \mathfrak{F}) \hat{\otimes} H^j(V_n^*, \mathfrak{G}).$$

Proposition II.1. i) *There is a commutative diagram of singly-graded groups:*

$$\begin{array}{ccc} H^*(A^*, \mathbb{Z}) \otimes_{\mathbb{Z}} H^*(B^*, \mathbb{Z}) & \xrightarrow{\alpha} & H^*(A^*, {}_{X^*}\mathcal{O}) \hat{\otimes} H^*(B^*, {}_{Y^*}\mathcal{O}) \\ \downarrow \sigma & & \downarrow \tau \\ H^*(A^* \times B^*, \mathbb{Z}) & \xrightarrow[\beta]{} & H^*(A^* \times B^*, {}_{X^* \times Y^*}\mathcal{O}). \end{array}$$

(The doubly-graded groups on the top line are to be given the structure of singly-graded groups in the standard way. The cohomology with integer coefficients is Cech cohomology.)

ii) *The inclusion* $\mathbb{Z} \to {}_{X^* \times Y^*}\mathcal{O}$ *induces* β.

iii) *The map* α *is the sum over pairs* (i, j) *of the obvious composed maps*

$$H^i(A^*, \mathbb{Z}) \otimes_{\mathbb{Z}} H^j(B^*, \mathbb{Z}) \to H^i(A^*, {}_{X^*}\mathcal{O}) \otimes_{\mathbb{C}} H^j(B^*, {}_{Y^*}\mathcal{O}) \to H^i(A^*, {}_{X^*}\mathcal{O}) \hat{\otimes} H^j(B^*, {}_{Y^*}\mathcal{O}).$$

iv) τ *is an isomorphism.*

v) σ *is an isomorphism in degrees* $\leqq 2$.

Proof. Let U and V be Stein open neighborhoods of A and B. The corollary to Lemma II.1 and Lemma II.2 give for any countable Stein covers $\mathfrak{U}$ and $\mathfrak{V}$ of U^* and V^* a certain commutative diagram. Since the covers $\mathfrak{W}$ of Lemma II.2 are cofinal, by passing to the direct limit we get a commutative diagram

$$\begin{array}{ccc} H^*(U^*, \mathbb{Z}) \otimes_{\mathbb{Z}} H^*(V^*, \mathbb{Z}) & \xrightarrow{A(U^*, V^*)} & H^*(U^*, {}_{X^*}\mathcal{O}) \hat{\otimes} H^*(V^*, {}_{Y^*}\mathcal{O}) \\ \downarrow S(U^*, V^*) & & \downarrow T(U^*, V^*) \\ H^*(U^* \times V^*, \mathbb{Z}) & \xrightarrow[B(U^*, V^*)]{} & H^*(U^* \times V^*, {}_{X^* \times Y^*}\mathcal{O}). \end{array}$$

We next take the direct limit over the system of Stein open neighborhoods U_n and V_n of A and B. This gives the diagram i). Statements ii) and iii) follow from the definitions of $B(\mathfrak{U}, \mathfrak{V})$ and $A(\mathfrak{U}, \mathfrak{V})$ of Lemma II.2.

Theorem 1 of [20] and its proof show that any choice of countable Stein covers of U_n^* and V_n^* gives an isomorphism $T(U_n^*, V_n^*)$ which is independent of the choice of the covers. (Recall that ${}_{X^*}\mathcal{O} \varepsilon {}_{Y^*}\mathcal{O} \cong {}_{X^* \times Y^*}\mathcal{O}$, in the notation of [20].) Hence τ is an isomorphism.

To prove v), note that $S(\mathfrak{U}, \mathfrak{V})$ may be defined as in Lemma II.2 whenever $\mathfrak{U}$ and $\mathfrak{V}$ are open coverings of neighborhoods of A^* and B^*. The neighborhoods U_n^* (resp. V_n^*) are cofinal, because f (resp. g) is proper. The $S(\mathfrak{U}, \mathfrak{V})$ formed by letting $\mathfrak{U}$ and $\mathfrak{V}$ range over all open coverings of fixed neighborhoods U^+ and V^+ of A^* and B^* yield a homomorphism

$$S(U^+, V^+)\colon\ H^*(U^+, \mathbb{Z}) \otimes_{\mathbb{Z}} H^*(V^+, \mathbb{Z}) \to H^*(U^+ \times V^+, \mathbb{Z}).$$

It suffices to exhibit a cofinal system of open neighborhoods U^+ and V^+ for which $S(U^+, V^+)$ is an isomorphism in degrees $\leqq 2$.

Recall that any open cover of a manifold has a refinement which is a simple open cover.

(An open cover $\mathfrak{U} = \{U_i\}_{i \in I}$ is called simple if for every finite subset $i_0, \ldots, i_n$ of I, $U_{i_0} \cap \cdots \cap U_{i_n}$ is connected and has cohomology which vanishes in dimensions greater than zero.) Consequently (using the compactness of A^*), for every open neighborhood N of A^* there are open subsets forming a finite simple open cover of an open neighborhood $U^+ \subset N$ of A^*. Given a neighborhood V_1 of B^*, $\mathfrak{V}$ and V^+ may be constructed by an analogous process. Note that $\mathfrak{W}$ is then a finite simple cover of $U^+ \times V^+ = W^+$. Since $\mathfrak{U}$ and $\mathfrak{V}$ are finite simple covers, ψ is an isomorphism of chain complexes. (This is seen by liberal use of the isomorphism between finite direct sum and finite direct product and the commutativity of direct sums with tensor products.) Therefore ψ^* is an isomorphism.

The map χ^* is always an isomorphism [30, Prop. 9].

By Leray's theorem ([17], Chap. 6), the canonical map $H^*(\mathfrak{U}, \mathbb{Z}) \to H^*(U^+, \mathbb{Z})$ is an isomorphism. Similar statements hold for the pairs $\mathfrak{V}, V^+$ and $\mathfrak{W}, W^+$. In particular, the zeroth and first cohomology groups with respect to the three covers are free abelian. The map μ in Lemma II.2 is an isomorphism in degree $\leqq 2$ by the usual Künneth formula ([37], p. 237). Therefore, $S(\mathfrak{U}, \mathfrak{V}) = \chi^* \psi^* \mu$ is an isomorphism. By Leray's theorem, $S(U^+, V^+)$ is also an isomorphism. This proves v) and finishes the proof of the proposition.

Choose $x \in A$, $y \in B$. Let m_x (resp. m_y, m_w) be the ideal sheaf of functions which vanish at x (resp. y, $w = (x, y)$). Let $\tilde{\mathcal{O}}$ denote the structure sheaf of $X \times Y$.

Let K be the kernel of the obvious map

$$H^1(A^*, \mathbb{Z}) \otimes_{\mathbb{Z}} H^1(B^*, \mathbb{Z}) \to H^1(A^*, {}_{X^*}\mathcal{O}) \otimes_{\mathbb{C}} H^1(B^*, {}_{Y^*}\mathcal{O}).$$

Theorem II.1. *Let $A \subset X$, $B \subset Y$ be connected Stein compact sets in normal complex spaces. Suppose that $R = \Gamma(A, {}_X\mathcal{O})$, $S = \Gamma(B, {}_Y\mathcal{O})$, $T = \Gamma(A \times B, {}_{X \times Y}\mathcal{O})$ are Noetherian.*

There is an isomorphism

$$C(T) \cong C(R) + C(S) + K + M,$$

where

$$M = \Gamma(A \times B, m_x \tilde{\mathcal{O}} \otimes_{Y^*\mathcal{O}} R^1 g({}_{Y^*}\mathcal{O})) + \Gamma(A \times B, R^1 f({}_{X^*}\mathcal{O}) \otimes_{X^*\mathcal{O}} m_y \tilde{\mathcal{O}})$$

such that the projections of $X^* \times Y^*$ on X^* and Y^* induce the obvious inclusion of $C(R) + C(S)$ in $C(T)$. There is an isomorphism $C(T) \cong C(R) + C(S) + K$.

Proof. If $R^1 f({}_{X^*}\mathcal{O})|A$ and $R^1 g({}_{Y^*}\mathcal{O})|B$ are trivial, the first isomorphism implies the second one. If $R^1 f({}_{X^*}\mathcal{O})|A$ is not trivial,

$$H^1(f^{-1}(A), {}_{X^*}\mathcal{O}) = \Gamma(A, R^1 f({}_{X^*}\mathcal{O}))$$

is a non-trivial finitely generated R-module and hence a complex vector space of countable dimension. Consider the split exact sequence

$$0 \to H^1(A^*, {}_{X^*}\mathcal{O})/b^{-1}(G) \to C(R) \to \text{Image } c/c(G) \to 0$$

explained after Proposition I.1. The group $b^{-1}(G)$ is countable. Let $F \subset H^1(A^*, {}_{X^*}\mathcal{O})$ be the vector space over $\mathbb{Q}$ (the rational numbers) which it generates.

An algebraic complement L has dimension c (the cardinality of the continuum) over $\mathbb{Q}$. Then

$$C(R) \cong \text{Image } c/c(G) + F/b^{-1}(G) + L.$$

The two terms of M are finitely generated modules over $\Gamma(A \times B, \tilde{\mathcal{O}})$ and therefore $\mathbb{Q}$-vector spaces of dimension 0 or c. Then $L \cong L + M$. Thus the first isomorphism for $C(T)$ implies the second one.

We now prove the first isomorphism. For each n there is a direct sum splitting $\Gamma(U_n, {}_X\mathcal{O}) = \mathbb{C} + \Gamma(U_n, m_x)$, where $\mathbb{C}$ represents the constant functions. This yields for each $q \geqq 0$

$$\begin{aligned}
\Gamma(A^*, {}_{X^*}\mathcal{O}) \hat{\otimes} H^q(B^*, {}_{Y^*}\mathcal{O}) &= \lim_{n \to \infty} \Gamma(U_n^*, {}_{X^*}\mathcal{O}) \hat{\otimes} H^q(V_n^*, {}_{Y^*}\mathcal{O}) \\
&= H^q(B^*, {}_{Y^*}\mathcal{O}) + \lim_{n \to \infty} \Gamma(U_n, m_x) \hat{\otimes} \Gamma(V_n, R^q g({}_{Y^*}\mathcal{O})) \\
&= H^q(B^*, {}_{Y^*}\mathcal{O}) + \lim_{n \to \infty} \Gamma(U_n \times V_n, m_x \tilde{\mathcal{O}} \otimes_{Y\mathcal{O}} R^q g({}_{Y^*}\mathcal{O})) \\
&= H^q(B^*, {}_{Y^*}\mathcal{O}) + \Gamma(A \times B, m_x \tilde{\mathcal{O}} \otimes_{Y\mathcal{O}} R^q g({}_{Y^*}\mathcal{O})).
\end{aligned}$$

The second last equation is given by one of the main results of [21]. A similar result with the roles of A and B interchanged holds. Set $W^* = X^* \times Y^*$ and note that $f \times g$ is a resolution of the singularities of W^*. Denote by π_x and π_y any mappings induced by the projections of W^* on X^* and Y^*.

Let $1 \in H^0(B^*, {}_{Y^*}\mathcal{O})$ be the constant function 1 and let $\alpha, \beta, \sigma, \tau$ be as in Proposition II.1. Then π_x: $H^q(A^*, {}_{X^*}\mathcal{O}) \to H^q(A^* \times B^*, {}_{W^*}\mathcal{O})$ is given by $\pi_x(s) = \tau(s \hat{\otimes} 1)$. Similarly π_x: $H^q(A^*, \mathbb{Z}) \to H^q(A^* \times B^*, \mathbb{Z})$ is given by $\pi_x(t) = \sigma(t \otimes 1)$ and analogous formulas involving π_y hold. Let $0 \to G_x \to H^1(A^*, {}_{X^*}\mathcal{O}) \to C(R) \to 0$

be the exact sequence of Proposition I.1 and let

$$H^1(A^*, \mathbb{Z}) \xrightarrow{a_x} H^1(A^*, {}_{X^*}\mathcal{O}) \xrightarrow{b_x} H^1(A^*, {}_{X^*}\mathcal{O}^*) \xrightarrow{c_x} H^2(A^*, \mathbb{Z}) \xrightarrow{d_x} H^2(A^*, {}_{X^*}\mathcal{O})$$

be the exact cohomology sequence. Denote the analogous of G_x and these homomorphisms for Y^* and W^* by G_y, a_y, G_w, a_w, etc. Since $H^0(A^*, \mathbb{Z}) \cong \mathbb{Z}$, $H^0(A^*, \mathbb{Z}) \otimes_{\mathbb{Z}} H^q(B^*, \mathbb{Z}) \cong H^q(B^*, \mathbb{Z})$ for each $q \geqq 0$ and a similar isomorphism with A^* and B^* switched holds. From the exact sequence

$$H^1(A^* \times B^*, \mathbb{Z}) \xrightarrow{a_w} H^1(A^* \times B^*, {}_{W^*}\mathcal{O}) \xrightarrow{b_w} H^1(A^* \times B^*, {}_{W^*}\mathcal{O}^*)$$
$$\xrightarrow{c_w} H^2(A^* \times B^*, \mathbb{Z}) \xrightarrow{d_w} H^2(A^* \times B^*, {}_{W^*}\mathcal{O})$$

(by applying Proposition II.1 and the above isomorphisms) we get a split exact sequence

$$0 \to H^1(A^*, {}_{X^*}\mathcal{O})/\text{Image } a_x + H^1(B^*, {}_{Y^*}\mathcal{O})/\text{Image } a_y + M \xrightarrow{\psi} H^1(A^* \times B^*, {}_{W^*}\mathcal{O}^*)$$
$$\xrightarrow{\gamma} \text{Ker } d_x + \text{Ker } d_y + K \to 0$$

where all sums are direct.

Note that G_w is generated by $\pi_x(G_x)$ and $\pi_y(G_y)$. Thus,

$$\gamma(G_w) \cong c_x(G_x) + c_y(G_y) + 0 \subset \text{Ker } d_x + \text{Ker } d_y + K.$$

Suppose $g \in G_x$, $h \in G_y$ and $\gamma(\pi_x(g) + \pi_y(h)) = 0$. Then with σ as in Proposition II.1,

$$0 = \sigma_2^{-1} c_w(\pi_x(g) + \pi_y(h)) = \sigma_2^{-1}(\pi_x c_x(g) + \pi_y c_y(h)).$$

Since σ_2^{-1} is injective, $\sigma_2^{-1} \pi_x c_x(g) \in \text{Ker } d_x$, and $\sigma_2^{-1} \pi_y c_y(h) \in \text{Ker } d_y$, we have $c_x(g) = 0$ and $c_y(h) = 0$. Therefore there are $k \in b_x^{-1}(G_x)$ and $\ell \in b_y^{-1}(G_y)$ such that $b_x(k) = g$ and $b_y(\ell) = h$. Then $\pi_x(g) + \pi_y(h) = \psi(\bar{k} + \bar{\ell})$, where

$$\bar{k} \in H^1(A^*, {}_{X^*}\mathcal{O})/\text{Image } a_x \quad \text{and} \quad \bar{\ell} \in H^1(B^*, {}_{Y^*}\mathcal{O})/\text{Image } a_y.$$

In other words,

$$\psi^{-1}(G_w) \subset b_x^{-1}(G_x)/\text{Image } a_x + b_y^{-1}(G_y)/\text{Image } a_y.$$

The opposite inclusion holds also, so

$$\psi^{-1}(G_w) = b_x^{-1}(G_x)/\text{Image } a_x + b_y^{-1}(G_y)/\text{Image } a_y.$$

From the short exact sequence

$$0 \to G_w \to H^1(A^* \times B^*, {}_{W^*}\mathcal{O}^*) \to C(T) \to 0$$

and the above, we have a split exact sequence

$$0 \to H^1(A^*, {}_{X^*}\mathcal{O})/b_x^{-1}(G_x) + H^1(B^*, {}_{Y^*}\mathcal{O})/b_y^{-1}(G_y) + M \to C(T)$$
$$\to \text{Ker } d_x/c_x(G_x) + \text{Ker } d_y/c_y(G_y) + K \to 0.$$

Rearrangement of terms and the remarks after Proposition I.1 give

$$C(T) \cong C(R) + C(S) + K + M$$

which was to be proved.

Remarks. A similar (but easier) argument shows: Let X and Y be compact manifolds. Then

$$H^1(X \times Y, {}_{X\times Y}\mathcal{O}^*) \cong H^1(X, {}_X\mathcal{O}^*) + H^1(Y, {}_Y\mathcal{O}^*) + K,$$

where K is the kernel of the homomorphism

$$H^1(X, \mathbb{Z}) \otimes_{\mathbb{Z}} H^1(Y, \mathbb{Z}) \to H^1(X, {}_X\mathcal{O}) \otimes_{\mathbb{C}} H^1(Y, {}_Y\mathcal{O}).$$

This is probably a well-known classical result.

The assumption in the theorem that T is Noetherian is not superfluous. If $A = \{x \in \mathbb{C} \mid x \text{ is real}, |x| \leqq 1\}$ and $B = \{x + iy \in \mathbb{C} \mid x, y \text{ are real}, y = x^2 \sin(x^{-1}), |x| \leqq 1\}$, then R and S are Noetherian by Siu's criterion. The intersection of the diagonal of $\mathbb{C}^2$ with $A \times B$ has infinitely many components. By Siu's criterion, $T = \Gamma(A \times B, {}_{\mathbb{C}^2}\mathcal{O})$ is not Noetherian.

Corollary. *Let $x \in X$ and $y \in Y$ be points in normal analytic spaces and let $f\colon X^* \to X$ and $g\colon Y^* \to Y$ be resolutions of singularities. Then*

$$C({}_{X\times Y}\mathcal{O}_{(x,y)}) \cong C({}_X\mathcal{O}_x) + C({}_Y\mathcal{O}_y) + K,$$

where K is the kernel of

$$R^1 f(\mathbb{Z})_x \otimes_{\mathbb{Z}} R^1 g(\mathbb{Z})_y \to R^1 f({}_{X^*}\mathcal{O})_x \otimes_{\mathbb{C}} R^1 g({}_{Y^*}\mathcal{O})_y.$$

Remark. This generalizes [39] and answers several questions raised there.

Example 1. Let A and B be connected compact subsets of $\mathbb{C}$. Assume $T = \Gamma(A \times B, {}_{\mathbb{C}^2}\mathcal{O})$ is Noetherian. This is the case (for example) if A and B are real semi-analytic sets. Then $C(T) \cong H^1(A, \mathbb{Z}) \otimes_{\mathbb{Z}} H^1(B, \mathbb{Z})$, where the cohomology groups are Cech groups.

Example 2. Let X be a connected projective manifold and $f\colon L \to X$ a weakly negative vector bundle [15]. The zero cross-section of L can be blown down by a map $g\colon L \to L'$ onto a normal analytic space whose only singular point, x, is the image of the zero cross-section. Clearly, $f^*\colon H^1(X, \mathbb{Z}) \to R^1 g({}_L\mathbb{Z})_x$ is an isomorphism. Since f is a left inverse to the inclusion $X \hookrightarrow L$ which identifies X with the zero cross-section, $f^*\colon H^1(X, {}_X\mathcal{O}) \to H^1(L, {}_L\mathcal{O})$ is injective. Rossi [29], Corollary 2.2, observes that in this case $H^1(L, {}_L\mathcal{O}) \cong R^1 g({}_L\mathcal{O})_x$. If $f_1\colon L_1 \to X_1$, $g_1\colon L_1 \to L'$, and x' are as above (for $(x, x') \in L' \times L'_1$), K is the kernel of the map

$$H^1(X, \mathbb{Z}) \otimes_{\mathbb{Z}} H^1(X_1, \mathbb{Z}) \to H^1(X, {}_X\mathcal{O}) \otimes_{\mathbb{C}} H^1(X_1, {}_{X_1}\mathcal{O}).$$

Consider the special case where $X = X_1$ is a one-dimensional torus. Then $H^1(X, {}_X\mathcal{O}) \cong \mathbb{C}$ and $H^1(X, \mathbb{Z}) \subset H^1(X, {}_X\mathcal{O})$ is a lattice with generators a and b.

If (b/a) satisfies a quadratic equation with rational coefficients $K \cong \mathbb{Z}+\mathbb{Z}$. Otherwise, $K \cong \mathbb{Z}$.

Example 3. Theorem II.1 makes it easy to construct a normal analytic local ring (that is, the local ring of a point on a normal analytic space) with any finitely generated abelian group as divisor class group. If $H \subset GL(2, \mathbb{C})$ is a finite cyclic group whose generator fixes only $\{0\}$, then the local ring at the singular point of $\mathbb{C}^2/H$ has divisor class group H. (See [5]. Another proof is given by Proposition III.4.) The local ring of the vertex of the Segre cone has divisor class group $\mathbb{Z}$. (See [39].) Theorem II.1 shows every finitely generated abelian group is the divisor class group of the local ring of a point on a (finite) Cartesian product of spaces of the above types. (Note that if $f\colon X^* \to X$ is a resolution of a normal singularity (X, x), then $H^1(f^{-1}(x), \mathbb{Z}) \to H^1(f^{-1}(x), {}_{X^*}\mathcal{O})$ is injective. Consequently, $H^1(f^{-1}(x), \mathbb{Z})=0$ if the divisor class group of ${}_X\mathcal{O}_x$ is finitely generated.)

Example 4. Each group $\mathbb{Q}^c+(\mathbb{Q}/\mathbb{Z})^N$, $0 \leqq N \leqq \infty$, is the divisor class group of a normal analytic local ring. This requires some results from the theory of analytic surfaces which we shall summarize.

Let (X, x) be the germ of a normal surface and let $f\colon X^* \to X$ be a resolution such that the irreducible components $C_1, \ldots, C_k$ of $f^{-1}(x) =: C$ are compact Riemann surfaces and each pair of them intersect in at most one point and transversely, if at all. For topological reasons, C is connected. Mumford [24] established an exact sequence

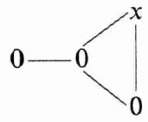
$$0 \to H^1(C, \mathbb{Z}) \to R^1 f({}_{X^*}\mathcal{O})_x \to C_x \to G \to 0$$

where C_x is the divisor class group of ${}_X\mathcal{O}_x$ and G is a finite group. If $H^1(C, \mathbb{Z})=0$, G is the torsion subgroup of $H_1(M, \mathbb{Z})$, M being the boundary of a suitable neighborhood of x. In general, $H^2(C, \mathbb{Z})$ is free abelian of rank k and the group G is the quotient of $H^2(C, \mathbb{Z})$ (regarded as the additive group of k-tuples of integers) by the subgroup generated by the rows of the intersection matrix $(C_i \cdot C_j)$. This matrix is negative definite.

Suppose (for simplicity) that each C_i is a rational curve, i.e. isomorphic to the Riemann sphere. The intersection matrix may be described by a weighted graph. This graph has a vertex p_i for each C_i, $i \leqq k$. The vertices p_i and p_j, $i \neq j$, are joined when $C_i \cdot C_j = 1$. To p_i we assign the weight $C_i \cdot C_i$. For more details, see [5].

Conversely, each weighted graph is the graph associated to a curve C (in a two dimensional complex manifold) whose irreducible components are rational non-singular curves each two of which intersect transversely and in at most one point. (See [27], [41].) The curve C can be blown down to a point if and only if the intersection matrix is negative definite ([15], p. 367). In other words, a curve C of the above type is the total transform of a singular point on a normal analytic surface if and only if its intersection matrix is negative definite.

Consider the graph



where 0 (resp. x) is a rational curve with self-intersection number -2 (resp. -3). The matrix associated to this graph is negative definite and has determinant 1. Consequently, there is a germ (X, x) of a normal analytic surface such that this graph is associated to a resolution f of the singularity. Moreover, the group G in Mumford's sequence is trivial (see [19], § 2) and $H^1(f^{-1}(x), \mathbb{Z}) \cong \mathbb{Z}$ (see [19], § 1). Therefore, the divisor class group of ${}_X\mathcal{O}_x$ is isomorphic to $\mathbb{Q}^c + \mathbb{Q}/\mathbb{Z}$. Let $Y = X \times \cdots \times X$ (N copies) and $y = (x, \ldots, x)$ (N copies). Theorem II.1 and induction show that the divisor class group of ${}_Y\mathcal{O}_y$ is $\mathbb{Q}^c + (\mathbb{Q}/\mathbb{Z})^N$.

Mumford ([24], IV) remarks that the boundary M of a suitable neighborhood of the origin in

$$X = \{(x, y, z) \in \mathbb{C}^3 \mid x^p + y^q + z^r = 0\}$$

has $H_1(M, Z) = 0$ if $p, q, r \geqq 2$ are pairwise relatively prime. (This can be seen by the methods of [23], pp. 65–71.) The "fibration" of M over C (the total transform of 0 in a resolution of the singularity) given an injection $H^1(C, \mathbb{Z}) \to H^1(M, \mathbb{Z})$. (See [19], § 1.) Therefore, $H^1(C, \mathbb{Z}) \cong G \cong 0$ and the divisor class group is isomorphic to $R^1 f({}_{X^*}\mathcal{O})_x$ (f: $X^* \to X$ a good resolution). Distinct unordered triples (p, q, r) correspond to analytic distinct singularities, because Theorem 3.6.10 of [27] enables one to compute the triple from the graph of a minimal resolution of the singularity. (Compare [33].) By [5], the divisor class group is not zero provided $(p, q, r) \neq (2, 3, 5)$. These singularities give examples where the divisor class group is $\mathbb{Q}^c$.

Theorem II.2. *An abelian group is the divisor class group of a normal analytic local ring if and only if it is of one of the types*

1) H,

2) $H + \mathbb{Q}^c + (\mathbb{Q}/\mathbb{Z})^N$, $0 \leqq N < \infty$,

where H is an arbitrary finitely generated abelian group.

Proof. The sufficiency of the condition is seen by applying Theorem II.1 to the Cartesian product of one of the spaces in Example 3 with one of those in Example 4.

Let f: $X^* \to X$ be a resolution of a normal singularity (X, x). By the observation following Proposition I.1, there is a split exact sequence

$$0 \to H^1(f^{-1}(x), {}_{X^*}\mathcal{O})/b^{-1}(G) \to C({}_X\mathcal{O}_x) \to \text{Image } c/c(G) \to 0.$$

Since G and $H^1(f^{-1}(x), \mathbb{Z})$ are finitely generated, $b^{-1}(G)$ is also. The group $H^2(f^{-1}(x), \mathbb{Z})$ is finitely generated so the last group in the short exact sequence is also. The result follows. (See [39].)

III. Divisor Class Groups without Resolutions

Throughout this section, let X be a normal analytic space of dimension n whose singular set S has codimension at least 3 at each point. Let $i: X-S \to X$ be the inclusion.

Let $x \in X$. By Proposition I.2,

$$C(\mathcal{O}_x) \cong R^1 i(\mathcal{O}^*)_x = \lim_{N \to x} H^1(N-S, \mathcal{O}^*),$$

where N ranges through neighborhoods of x. The groups

$$R^q i(\mathbb{Z})_x = \lim_{N \to x} H^q(N-S, \mathbb{Z}), \quad q \geqq 0,$$

may be computed as follows: Triangulate X so that S is a sub-complex and x a vertex (by [22]) and let M be the open star neighborhood of x. One sees easily that $R^q i(\mathbb{Z})_x \cong H^q(M-S, \mathbb{Z})$. Since $M-S=(M \cup \partial M)-(S \cup \partial M)$ is the complement of a sub-complex in a finite complex, these groups are finitely generated. (If A is a sub-complex of a finite complex B, let U be the open star neighborhood of A in the first bary-centric subdivision of B. Then $B-U$ is a finite complex and a deformation retract of $B-A$. Compare [37, p. 124].) In particular, $R^1 i(\mathbb{Z})_x \cong \mathbb{Z}^\ell$ for some $\ell \geqq 0$. We have the exact cohomology sequence

$$0 \to R^1 i(\mathbb{Z})_x \xrightarrow{a} R^1 i(\mathcal{O})_x \to R^1 i(\mathcal{O}^*)_x \to R^2 i(\mathbb{Z})_x \to R^2 i(\mathcal{O})_x.$$

The subgroup $D \subset C(\mathcal{O}_x)$ of divisible elements is isomorphic to the cokernel of a. The group $R^1 i(\mathbb{Z})_x$ may be determined from the divisor class group by noting that for every integer $m \geqq 1$ the cardinality of $\{d \in D \mid md=0\}$ is m^ℓ.

The divisor class group can often be determined by topological considerations.

Proposition III.1. *Let $A \subset X$ be a connected compact Stein set such that $\Gamma(A, \mathcal{O})$ is Noetherian. If $3+\dim_x S \leqq \operatorname{codh}_x \mathcal{O}$ for all $x \in A$, the canonical mapping $C(\Gamma(A, \mathcal{O})) \to \lim_{N \to A} H^2(N-S, \mathbb{Z})$ is injective. If the latter is a torsion group or $4+\dim S \leqq \operatorname{codh}_x \mathcal{O}$ for all $x \in A$, the canonical mapping is an isomorphism.*

Proof. If $3+\dim_x S \leqq \operatorname{codh}_x \mathcal{O}$ for all $x \in A$, this inequality continues to hold throughout a neighborhood U of A. For every Stein open set N with $A \subset N \subset U$, by a theorem of Scheja [32, p. 359] and Theorem B, $H^1(N-S, \mathcal{O}) \cong H^1(N, \mathcal{O})=0$. Hence $\lim_{N \to A} H^1(N-S, \mathcal{O})=0$. Similarly, if $4+\dim_x S \leqq \operatorname{codh}_x \mathcal{O}$ throughout A, $\lim_{N \to A} H^2(N-S, \mathcal{O})=0$. The proposition follows from the isomorphism $C(\Gamma(A, \mathcal{O})) \cong \lim_{N \to A} H^1(N-S, \mathcal{O}^*)$ of Proposition I.2 and the exact sequence

$$(*) \quad \lim_{N \to A} H^1(N-S, \mathcal{O}) \to \lim_{N \to A} H^1(N-S, \mathcal{O}^*) \to \lim_{N \to A} H^2(N-S, \mathbb{Z}) \to \lim_{N \to A} H^2(N-S, \mathcal{O}).$$

Remark. A similar argument is used by Brieskorn in [4, § 3].

In the following lemma and proposition, all homology and cohomology groups have integer coefficients.

It is well-known [3] that for any n-dimensional Stein manifold M, $H_j(M)=0$ for $j>n$ and $H_n(M)$ is free. We have

Lemma III.2. *Let X be a complex n-dimensional manifold which is the union of open subsets $X_0, \ldots, X_k$ which are Stein manifolds. Then $H_j(X)=0$ for $j>n+k$ and $H_{n+k}(X)$ is free.*

Proof. The case $k=0$ is quoted above. Since the intersection of two Stein manifolds is Stein,

$$(X_0 \cup \cdots \cup X_{k-1}) \cap X_k = (X_0 \cap X_k) \cup \cdots \cup (X_{k-1} \cap X_k)$$

is the union of $(k-1)$ Stein manifolds. The lemma follows by induction on k from the Mayer-Vietoris sequence for the homology of $(X_0 \cup \cdots \cup X_{k-1}) \cup X_k$.

In [28] we defined the term "good neighborhood of $x \in X$ (with respect to x)". We showed that for any two good neighborhoods U, V of x, the spaces $U-x$ and $V-x$ have the same homotopy type, denoted $[X, x, x]$. If U is a good neighborhood of x, then $H^k(U-x) \cong \varinjlim_{N \to x} H^k(N-x)$ for all k. If X can be triangulated so that x is a vertex, the open and the closed star neighborhood of x are good neighborhoods of x.

Proposition III.2. *Let $U \subset \mathbb{C}^n$ be an open set and let $f_1, \ldots, f_k$ be holomorphic functions on U. Let $V=\{u \in U \mid f_1(u)=\cdots=f_k(u)=0\}$. Let $p \in V$. Then the reduced cohomology groups $\bar{H}^j([V, p, p])$ are zero for $j \leqq n-k-2$. The group $H^{n-k-1}([V, p, p])$ is free.*

Proof. Triangulate U so that V is a sub-complex and p a vertex [22]. Perform one barycentric subdivision and consider the resulting complex. Let Δ be the closed star neighborhood of p in U and S its boundary. Set $X=\Delta-S-V$. Let $\mathfrak{B}$ be the closed star neighborhood of p in V. Then $\mathfrak{B} \cap S$ is the boundary of $\mathfrak{B}$. It suffices to prove the assertions of the theorem for $\mathfrak{B} \cap S$, because it is a representative of the homotopy type $[V, p, p]$.

Let $\psi: \Delta \to K$ be a simplicial homeomorphism onto an affine complex $K \subset R^N$ such that $\psi(p)=0$. Let $D \subset \Delta$ be an open Stein neighborhood of p. Choose r, $0<r<1$, so small that $\psi^{-1}(r \cdot \psi(\Delta)) \subset D$. Then

$$\psi^{-1}(r \cdot \psi(X)) \subset D-V \subset X.$$

Now $D-V$ is the union of the Stein manifolds $\{d \in D \mid f_i(d) \neq 0\}$, $i=1, \ldots, k$. From the lemma and the fact that $\psi^{-1}(r \cdot \psi(X))$ is a deformation retract of X, $H_{2n-j}(X)$ is zero for $j \leqq n-k$ and free for $j=n-k+1$. The pair $(\Delta, S \cup \mathfrak{B})$ is a compact orientable relative $(2n)$-manifold and $X=\Delta-S-\mathfrak{B}$. By Lefschetz duality [37, p. 297], $H^j(\Delta, S \cup \mathfrak{B})$ is zero for $j \leqq n-k$ and free for $j=n-k+1$. From the reduced cohomology sequence of the pair $(\Delta, S \cup \mathfrak{B})$, $H^j(\Delta, S \cup \mathfrak{B}) \cong \bar{H}^{j-1}(S \cup \mathfrak{B})$ for all $j \geqq 1$. By the Mayer-Vietoris sequence, $\bar{H}^{j-1}(S \cup \mathfrak{B}) \cong \bar{H}^{j-2}(S \cap \mathfrak{B})$ for $j<2n$, because $\mathfrak{B}$ is contractible and S has the homotopy type of a $(2n-1)$ sphere. The proposition follows.

Proposition III.3. *Let $f_1, \ldots, f_k$ be functions holomorphic in a neighborhood $U \subset \mathbb{C}^n$ of 0. Let $V = \{z \in U | f_1(z) = \cdots = f_k(z) = 0\}$. Suppose 0 is an isolated point of the singular set of V. Suppose the germs at 0 of the f's form a ${}_{\mathbb{C}^n}\mathcal{O}_0$-sequence* ([2], *p.* 200). *Suppose $n-k \geqq 4$. Then V is a normal in a neighborhood of 0, the local ring ${}_V\mathcal{O}_0$ is a unique factorization domain, and the groups $\lim_{N \to 0} H^1(N-0, {}_V\mathcal{O})$ and $\lim_{N \to 0} H^2(N-0, \mathbb{Z})$ are zero.*

Proof. By [2, p. 200], $\operatorname{codh}_0 {}_V\mathcal{O}_0 = n-k \geqq 3$. As in Proposition III.1, a theorem of Scheja [32] guarantees the first of the two groups is zero. Proposition III.2 and the paragraph before it show the last group is zero. A theorem of Abhyankar [1] shows V is normal at 0. Proposition III.1 (with $A = \{0\}$) shows $C({}_V\mathcal{O}_0) = 0$. This is equivalent to ${}_V\mathcal{O}_0$ being a unique factorization domain.

Remark. Grothendieck [16, Exp. 11, p. 9] has proved a much stronger result which does not treat only isolated singularities. If V is a complete intersection as in the preceding proposition and the singular locus of V has codimension $\geqq 4$ at 0, [1] shows ${}_V\mathcal{O}_0$ is normal and all localizations of ${}_V\mathcal{O}_0$ at primes of height $\leqq 3$ are unique factorization domains. Grothendieck's theorem then implies that ${}_V\mathcal{O}_0$ is a unique factorization domain. Because V is a complete intersection, $\operatorname{codh}_0 {}_V\mathcal{O}_0 = \dim_0 V$. (This is pointed out in Koestner's thesis, published as Heft 35, 1967, in the Schriftenreihe des mathematischen Instituts der Universität Münster (Westf.).) The hypotheses of Proposition III.1 are satisfied. Thus $\lim_{N \to 0} H^2(N-S, \mathbb{Z}) = 0$. ($S$ is the singular locus of V and N ranges over neighborhoods of 0 in V.) We do not know a direct topological proof of this fact.

Let $(X, {}_X\mathcal{O})$ be an arbitrary normal analytic space. Let G be a finite group of automorphisms of X which leave $x \in X$ fixed. Let $Y = X/G$ and let y be the image of x. By [8], Y is a normal analytic space. Suppose the codimension at x of

$$F = \{z \in X | \exists g \in G, g \neq 1, g(z) = z\}$$

is at least two. Then ${}_X\mathcal{O}_x$ is divisorally unramified over ${}_Y\mathcal{O}_y$.

Proposition III.4 (Samuel). *There is a descent homomorphism j: $C({}_Y\mathcal{O}_y) \to C({}_X\mathcal{O}_x)$. $\operatorname{kernel}(j) \cong G/[G, G]$.*

Proof. Samuel defines j and shows [31, p. 57] that its kernel is $H^1(G, U)$, where U is the group of units of ${}_X\mathcal{O}_x$. An easy modification of the proof of Proposition 1.3 in [31, p. 60], gives

$$H^1(G, U) \cong \operatorname{Hom}(G, \mathbb{C}^*) = G/[G, G].$$

Note that if ${}_X\mathcal{O}_x$ is a unique factorization domain, the proposition gives $C({}_Y\mathcal{O}_y) \cong G/[G, G]$.

Example. Let $G \subset GL(n, \mathbb{C})$ be a finite group and let G act linearly on $\mathbb{C}^n$ in the obvious way. Suppose the fixed points F of G form a sub-variety whose

codimension is everywhere at least two. Let $\tau: \mathbb{C}^n \to \mathbb{C}^n/G = Y$ be the projection. The proposition shows $G/[G, G]$ is the divisor class group of ${}_Y\mathcal{O}_{\tau(0)}$.

In case $\tau(0)$ is the singular locus of Y, $F = \{0\}$. (See [28].) If, moreover, ${}_Y\mathcal{O}_{\tau(0)}$ is a unique factorization domain (i.e. $C({}_Y\mathcal{O}_{\tau(0)}) = 0$), $G = [G, G]$. A theorem of Zassenhaus (see [43], p. 181) claims G is isomorphic to the binary icosahedral group. Unfortunately, it appears that no complete proof of this theorem is available.

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Local Bounds for Solutions of Higher Order Nonlinear Elliptic Partial Differential Equations

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1. Introduction

We shall consider weak solutions of $2m$-th order nonlinear elliptic equations of the form

$$\sum_{|\alpha| \leqq m} (-1)^{|\alpha|} D^\alpha F_\alpha(x, u, Du, \ldots, D^m u) = 0. \tag{1.1}$$

With the nonlinearity structure of (1.1) one usually associates a parameter $p > 1$, and it is then natural to consider solutions which belong to $W^{m,p}_{\text{loc}}$. De Giorgi [1] and Frehse have shown that in general such solutions are not bounded. However, Frehse [2] recently showed that if $p = n/m$ solutions satisfying Dirichlet boundary conditions, i.e. solutions u for which $u \in \mathring{W}^{m,p}(\Omega)$, are bounded, and in [6] the present author showed that for p close enough to n/m solutions are Hölder continuous, this being true also for systems of corresponding type. Here n is the dimension of the space.

In this paper the result of Frehse will be strengthened in that we will show that for $p = n/m$ the following inequality holds

$$\sup_{|x - x_0| < R} |u|^p \leqq K R^{-n} \int_{|x - x_0| < 2R} |u|^p \, dx + K R^n.$$

A global result also holds; if $u \in \mathring{W}^{m,p}(\Omega)$ and Ω is bounded, we have

$$\sup_{\Omega} |u|^p \leqq K \int_{\Omega} |u|^p \, dx + K.$$

Similar inequalities were proved by Moser [3] for linear equations and $m = 1$, and by Serrin [5] for $m = 1$ and arbitrary p.

2. Notations and Statement of Results

We place ourselves in R^n, the points of which are denoted by $x = (x_1, x_2, \ldots, x_n)$. By α we denote integer-valued multi-indices, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, with $\alpha_i \geqq 0$ and $|\alpha| = \sum \alpha_i$. $W^{m,p}(\Omega)$ as usual denotes the Sobolev space of functions in $L^p(\Omega)$ with m-th order distributional derivatives in $L^p(\Omega)$, with $\mathring{W}^{m,p}(\Omega)$ being the closure of $C_0^\infty(\Omega)$ in the $W^{m,p}$-norm and $W^{m,p}_{\text{loc}}(\Omega)$ the space of functions which belong to $W^{m,p}(\Omega')$ for all compact subsets Ω' of Ω. For the theory of $W^{m,p}$-spaces we refer e.g. to [4].

The m-jet $\delta_m \xi$: $W^{m,p}_{\text{loc}}|_x \to R^N$ with components ξ^α, $|\alpha| \leqq m$, is defined by $\xi^\alpha(u) = D^\alpha u(x)$, where

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}.$$

We put $|\xi^j| = \sum_{|\alpha|=j} |\xi^\alpha|$, $|D^j u| = |\xi^j(u)|$, and for short also $\delta_m u = \delta_m \xi(u)$.

The assumptions on the equation will be the following

$$\begin{aligned} &F_\alpha(x, \delta_m \xi) \text{ is measurable in } x \text{ for every fixed } \delta_m \xi \in R^N \\ &\quad \text{and continuous in } \delta_m \xi \text{ for almost all } x \in \Omega, \text{ for all } \alpha, |\alpha| \leqq m, \end{aligned} \tag{2.1}$$

$$\sum_{|\alpha|=m} F_\alpha(x, \delta_m \xi)\, \xi^\alpha \geqq |\xi^m|^p - C \sum_{j=0}^{m-1} |\xi^j|^p - K_1, \tag{2.2}$$

$$|F_\alpha(x, \delta_m \xi)| \leqq C \sum_{j=0}^{m} |\xi^j|^{p-1} + K_2. \tag{2.3}$$

One can actually lessen the restrictions somewhat by letting K_i and C be functions in certain Lebesgue classes, but in order not to overburden the presentation we shall refrain from making this generalization.

By a solution of the equation in Ω we mean a function $u \in W^{m,p}_{\text{loc}}(\Omega)$ which satisfies

$$\sum_{|\alpha| \leqq m} \int_\Omega F_\alpha(x, \delta_m u)\, D^\alpha \varphi \, dx = 0 \tag{2.4}$$

for all $\varphi \in C_0^\infty$. In case $u \in W^{m,p}(\Omega)$ it follows from (2.3) that (2.4) holds for all $\varphi \in \mathring{W}^{m,p}(\Omega)$.

We can now state our main results. To simplify the notation we put $K_3 = K_2^{p/(p-1)} + K_1$.

Theorem 1. *Let u be a solution of Eq. (2.4) in Ω. If*

$$p = n/m \quad \text{and} \quad \{|x - x_0| < 2R\} \subset \Omega$$

then

$$\sup_{|x-x_0|<R} |u|^p \leqq K R^{-n} \int_{|x-x_0|<2R} |u|^p \, dx + K K_3 R^n, \qquad R \leqq 1,$$

where K depends on C, n, and m.

The following global result also holds.

Theorem 2. *Let Ω be a bounded domain and assume that $u \in \mathring{W}^{m,p}(\Omega)$ satisfies (2.4). Then*

$$\sup_\Omega |u|^p \leqq K \int_\Omega |u|^p \, dx + K K_3$$

where K depends on C, n, m, and Ω.

The proof of Theorem 2 is a subset of the proof of Theorem 1, and is in fact considerably less technically involved. Therefore we shall give only the proof of Theorem 1, and that will be done in § 4. For the proof we need several lemmas, and these will be presented in § 3.

Before we go on we make some remarks.

We assume throughout that $m>1$, since the case $m=1$ has been well treated, and since its inclusion would have required some reformulations.

We also note that for $p>n/m$, Theorem 1 follows quite easily, in fact with Sobolev's imbedding theorem and Lemma 2 below.

The method employed is reminiscent of the one used by Moser [3]. However, instead of truncating functions and taking powers we use another method which has been used earlier by us when proving Moser's theorem [7]. A similar, but seemingly less effective trick was also used independently by Frehse [2]. The idea to use for the numbers I_ν integrals involving also derivatives of u instead of powers of u only is borrowed from Frehse. The need to find a good initial estimate in the iteration procedure, (4.1) in the proof below, necessitates the rather awkward formulations of Lemmas 5 and 6; for the iteration itself more streamlined versions would suffice.

Frehse (personal communication) has shown that to every $p<n/m$ there is an equation of the type considered with an unbounded solution. However, by [6] solutions are Hölder continuous if $p>n/m-\kappa(C)$, where C is the constant of (2.3). Whether the inequality of Theorem 1 holds also in this case is doubtful but not known. The present method then certainly does not work, in particular since Lemma 3 does not hold.

3. Lemmata

We shall consider functions defined in balls, say with center at the origin, and with radii R_1 and R_2, where $1\leqq R_1<R_2\leqq 2$, putting $R_2-R_1=\rho$.

We shall need a special type of auxiliary function η which is non-negative, defined and continuous on the positive real axis, identically one for $t\leqq R_1$, identically zero for $t\geqq R_2$. We shall also require η to be C^∞ in $0<t<R_2$, linear in $(R_1+R_2)/2<t<R_2$ so that $c_1(R_2-t)\rho^{-1}\leqq\eta(t)\leqq c_2(R_2-t)\rho^{-1}$ for $R_1-\rho\leqq t\leqq R_2$, with c_i independent of R_1 and R_2. Then for

$$l>m \quad \eta^l\in\mathring{W}^{m,p}(|x|<R_2).$$

It is not difficult to see that η can be constructed such that

$$|D^j\eta^{lp}(|x|)|\leqq K(l,p,m)\,\rho^{-j}\left(\frac{R_2-|x|}{\rho}\right)^{lp-j}, \qquad 1\leqq j\leqq m.$$

The following lemma holds.

Lemma 1. *Let η be defined as above. Then to every $\delta>0$ there is a constant $K_\delta=K(\delta,m,n)$ such that*

$$\int_{|x|<R_2}|D^j u|^p\,\eta^{lp-lp^2}\,|D^{m-j}\eta^{lp}|^p\,dx \leqq\delta\int_{|x|<R_2}|D^m u|^p\,\eta^{lp}\,dx+K_\delta\rho^{-m}\int_{|x|<R_2}|u|^p\,dx, \qquad 0<j<m \tag{3.1}$$

and

$$\int_{|x|<R_2} |D^j u|^p \eta^{lp} \leqq \delta \int_{|x|<R_2} |D^m u|^p \eta^{lp}\, dx + K_\delta \rho^{-n} \int_{|x|<R_2} |u|^p\, dx, \quad 0<j<m. \tag{3.2}$$

Remark. The second inequality is of course valid for more general η.

Proof. Let us begin by noting the following standard interpolationary inequality

$$r^{jp-n} \int_{B_r} |D^j u|^p\, dx \leqq \delta \int_{B_r} |D^m u|^p\, dx + K_\delta r^{-n} \int_{B_r} |u|^p\, dx \tag{3.3}$$

valid for any ball B_r of radius r and any $u \in W^{m,p}(B_r)$.

Then let x_ν, $\nu = 1, 2, \ldots$ be a sequence of points in $\{|x|<R_2\}$, chosen so that

$$\bigcup_{\nu=1}^{\infty} B_\nu = \bigcup_{\nu=1}^{\infty} \{|x-x_\nu|<\alpha_\nu = (R_2 - |x_\nu|)/2\} = \{|x|<R_2\},$$

but also so that no point in $\{|x|<R_2\}$ belongs to more than $K(n)$ of the balls B_ν.

We claim that the following inequality holds

$$\int_{B_\nu} |D^j u|^p\, \eta^{lp-lp^2}\, |D^{m-j}\eta^{lp}|^p\, dx \leqq \delta \int_{B_\nu} |D^m u|^p\, \eta^{lp}\, dx + K_\delta \rho^{-n} \int_{B_\nu} |u|^p\, dx. \tag{3.4}$$

For ν such that $a_\nu > \rho$ it holds since the left hand side is zero. For other values of ν the left hand side is less than or equal to a constant times

$$\left(\frac{a_\nu}{\rho}\right)^{lp} a_\nu^{jp-n} \int_{B_\nu} |D^j u|^p\, dx.$$

Hence by (3.3) and the properties of η, notably the fact that $\eta \geqq K a_\nu \rho^{-1}$ in B_ν and $(a_\nu \rho^{-1})^{lp} a_\nu^{-n} \leqq \rho^{-n}$, the inequality (3.4) follows.

Summing over ν and renaming δ we get the desired inequality (3.1).

To prove (3.2) we note that for $x, y \in B_\nu$ we have $\eta(x) \leqq K \eta(y)$, with K independent of ρ, and proceed as above.

Lemma 2. *Let u be a solution of the equation* (2.4) *in $B = \{|x|<2\}$, and let η be the function of Lemma* 1. *Then for $l > m$*

$$\int_B |D^m u|^p\, \eta^{lp}\, dx \leqq K \rho^{-n} \int_B |u|^p\, dx + K K_3$$

where K depends on $C, n, l,$ and m.

Proof. Put $\varphi = u\eta^{lp}$. It is evident that $\varphi \in \mathring{W}^{m,p}(B)$. By Leibnitz' formula

$$D^\alpha \varphi = D^\alpha u\, \eta^{lp} + \sum_{\beta<\alpha} \binom{\alpha}{\beta} D^\beta u\, D^{\alpha-\beta} \eta^{lp} = D^\alpha u\, \eta^{lp} + A$$

where

$$|A| \leqq K \sum_{j=0}^{|\alpha|-1} |D^j u|\, |D^{|\alpha|-j} \eta^{lp}|.$$

Hence by using φ as a test function in (2.4)

$$\sum_{|\alpha|=m} \int F_\alpha(x, \delta_m u)\, D^\alpha u\, \eta^{lp}\, dx \leqq K \sum_{|\alpha|=m} \int |F_\alpha| \sum_{j=0}^{m-1} |D^j u|\, |D^{m-j}\eta^{lp}|\, dx + K \sum_{|\alpha|<m} \int |F_\alpha| \sum_{j=0}^{|\alpha|} |D^j u|\, |D^{|\alpha|-j}\eta^{lp}|\, dx.$$

The inequalities (2.1)–(2.3) imply

$$\int |D^m u|^p\, \eta^{lp}\, dx \leqq \int \left(C \sum_{j=1}^{m-1} |D^j u|^p + K_1\right) \eta^{lp}\, dx + \int \left(C \sum_{i=1}^{m} |D^i u|^{p-1} + K_2\right) \left(\sum_{j=0}^{m-1} |D^j u|\, |D^{m-j}\eta^{lp}|\right) dx$$

and in view of Young's inequality

$$\tfrac{1}{2} \int |D^m u|^p\, \eta^{lp}\, dx \leqq K \sum_{j=0}^{m-1} \int |D^j u|^p\, |D^{m-j}\eta^{lp}|^p\, \eta^{lp-lp^2}\, dx + K \sum_{j=1}^{m-1} \int |D^j u|^p\, \eta^{lp}\, dx + K K_3. \tag{3.5}$$

Lemma 1 shows that we can bound the integrals with $j>0$ on the right hand side with

$$\frac{1}{8m} \int |D^m u|^p\, \eta^{lp}\, dx + K\, \rho^{-n} \int |u|^p\, dx. \tag{3.6}$$

We also have, due to the choice of η

$$|D^m \eta^{lp}|^p\, \eta^{lp-lp^2} \leqq K\, \rho^{-n}. \tag{3.7}$$

The lemma now follows by inserting (3.6) and (3.7) in (3.5).

Lemma 3. *Let* $g(t)=t\,|t|^a/(\varepsilon\,|t|+1)^b$, $t\in R$, *where* $m-1<a\leqq b<a+1$ *and* $\varepsilon>0$. *Then if* $u\in W^{m,p}$ *we have that* $g(u)\in W^{m,p}$ *and*

$$\left|D^\alpha g(u) - \frac{D^\alpha u\, |u|^a}{(\varepsilon\,|u|+1)^b}\, h(u)\right| \leqq K\, a^{|\alpha|} \sum_{j=1}^{|\alpha|-1} \frac{|D^j u|^{|\alpha|/j}\, |u|^{a+1-|\alpha|/j}}{(\varepsilon\,|u|+1)^b} \quad \text{a.e.}$$

for $|\alpha|\leqq m$, *where* $a+1-b\leqq h(u)\leqq a+1+b$, *and where* K *depends on* n *and* m.

Proof. Since $a>m$ g has bounded derivatives of all orders up to and including m. Formally we have that $D^\alpha g(u)$ is a linear combination of terms of the form

$$g^{(k)}(u) \prod_{i=1}^{k} D^{\beta_i} u, \qquad \sum \beta_i = \alpha,\ k \leqq |\alpha|. \tag{3.8}$$

Using the theory of $W^{1,p}$-spaces, Young's inequality, and the Sobolev imbedding theorem, it is easy to show inductively that $g(u)\in W^{j,p}$ for $j\leqq m$, and that the derivatives are given by a linear combination of (3.8) almost everywhere (see also Frehse [2]).

In this linear combination there is only one term containing $g^{(1)}$, and the corresponding coefficient is 1. Moreover, elementary calculations show that

$$g^{(1)}(t)=\frac{|t|^a}{(\varepsilon|t|+1)^b}\left(a+1-b\frac{\varepsilon t}{\varepsilon|t|+1}\right)$$

and

$$|g^{(k)}(t)|\leqq K a^k\frac{|t|^{a+1-k}}{(\varepsilon|t|+1)^b},\qquad k=2,3,\ldots,m.$$

Hence

$$\left|D^\alpha g(u)-\frac{D^\alpha u\,|u|^a}{(\varepsilon|u|+1)^b}h(u)\right|\leqq\frac{K a^{|\alpha|}}{(\varepsilon|u|+1)^b}\sum\prod_{i=1}^{k}|D^{\beta_i}u|\,|u|^{(a+1-|\alpha|/|\beta_i|)|\beta_i|/|\alpha|}$$

where the summation is extended over all combinations of β_i such that $\sum_1^k\beta_i=\alpha$, and where $h(u)=a+1-b\varepsilon u/(\varepsilon|u|+1)$. Young's inequality completes the proof.

Corollary 3. *Let η be the function of Lemma 1, and let $u\in W^{m,p}(|x|<2)$. Then if $\kappa>m$ and $m-1<a\leqq b<a+1$*

$$\varphi=\frac{u\,|u|^a}{(\varepsilon|u|+1)^b}\eta^\kappa\in\mathring{W}^{m,p}(|x|<2)$$

and

$$\left|D^\alpha\varphi-\frac{D^\alpha u\,|u|^a}{(\varepsilon|u|+1)^b}\eta^\kappa h(u)\right|\leqq K a^{|\alpha|}\sum_{j=1}^{|\alpha|-1}\frac{|D^j u|^{m/j}\,|u|^{a+1-m/j}}{(\varepsilon|u|+1)^b}\eta^\kappa$$
$$+K\rho^{-|\alpha|}\frac{|u|^{a+1}}{(\varepsilon|u|+1)^b}\eta^{\kappa-|\alpha|}\qquad\text{a.e.}$$

for $|\alpha|\leqq m$, where $a+1-b\leqq h(u)\leqq a+1+b$.

Proof. Since $\eta^\kappa\in C_0^m(|x|<2)$ and $g(u)=u\,|u|^a/(\varepsilon|u|+1)^b\in W^{m,p}(|x|<2)$ it is obvious that $\varphi\in\mathring{W}^{m,p}(|x|<2)$.

By Leibnitz' formula and Lemma 3

$$\left|D^\alpha\varphi-\frac{D^\alpha u\,|u|^a}{(\varepsilon|u|+1)^b}\eta^\kappa h(u)\right|\leqq K a^{|\alpha|}\sum_{j=1}^{|\alpha|-1}\frac{|D^j u|^{|\alpha|/j}\,|u|^{a+1-|\alpha|/j}}{(\varepsilon|u|+1)^b}\eta^\kappa$$
$$+K\sum_{\beta<\alpha}\sum_{j=1}^{|\beta|}|a|^{|\beta|}\frac{|D^j u|^{|\beta|/j}\,|u|^{a+1-|\beta|/j}\,|D^{\alpha-\beta}\eta^\kappa|}{(\varepsilon|u|+1)^b}.\tag{3.9}$$

By Young's inequality

$$|D^j u|^{|\beta|/j}\,|u|^{a+1-|\beta|/j}\,|D^{\alpha-\beta}\,\eta^\kappa| \leqq |D^j u|^{m/j}\,|u|^{a+1-m/j}\,\eta^\kappa + |u|^{a+1}\,|D^{\alpha-\beta}\,\eta^\kappa|^{m/(m-|\beta|)}\,\eta^{-\kappa|\beta|/(m-|\beta|)}.$$

An easy calculation shows that

$$\eta^{-\kappa|\beta|/(m-|\beta|)}\,|D^{\alpha-\beta}\,\eta^\kappa|^{m/(m-|\beta|)} \leqq K\,\rho^{-|\alpha|}\,\eta^{\kappa-|\alpha|}$$

which implies the desired estimate for the second term of (3.9). A trivial application of Young's inequality takes care of the first term.

The corollary is proved.

Lemma 4. *Let $u \in W^{m,p}(|x|<R)$, $1 \leqq R \leqq 2$. Then if $s > m^2$*

$$\int |D^i u|^{n/i}\,|u|^{sp-n/i}\,dx \leqq K\,s^\kappa \sum_{k=1}^{i}\sum_{j=1}^{m}\Big[\int |D^j u|^{n/j}\,|u|^{spk/m-n/j}\,dx\Big]^{m/k} + K\,s^\kappa\Big[\int |u|^{sp\gamma}\,dx\Big]^{1/\gamma}, \qquad 1 \leqq i \leqq m-1,$$

and

$$\int |u|^{sp}\,dx \leqq K\,s^{n/\gamma}\Big[\int |Du|^n\,|u|^{sp\gamma-n}\,dx\Big]^{1/\gamma} + K\Big[\int |u|^{sp\gamma}\,dx\Big]^{1/\gamma}$$

where all integrals are over $\{|x|<R\}$, where $\gamma=(m-1)/m$ and where K and κ depend on n and m only.

Proof. We need only consider those u for which the right hand side is finite.

The proof proceeds by induction. We consider Sobolev's inequality in the following form

$$\int |D^i v|^{n/i}\,dx \leqq K\Big[\int |D^m v|^p\,dx\Big]^{m/i} + K\Big[\int |v|^{n\gamma/i}\,dx\Big]^{1/\gamma} \tag{3.10}$$

valid for $v \in W^{m,p}$.

By Lemma 3 the function v defined by

$$v = \frac{u\,|u|^{si/m-1}}{(\varepsilon\,|u|+1)^{(s-1)i/m}}$$

belongs to $W^{m,p}$, and we may apply (3.10) to v. By Lemma 3 again

$$|D^i v| \geqq \frac{i}{m}\,\frac{|D^i u|\,|u|^{si/m-1}}{(\varepsilon\,|u|+1)^{(s-1)i/m}} - K\,s^i \sum_{j=1}^{i-1}\frac{|D^i u|^{i/j}\,|u|^{si/m-i/j}}{(\varepsilon\,|u|+1)^{(s-1)i/m}}$$

and

$$|D^m v| \leqq K\,s^m \sum_{j=1}^{m}\frac{|D^j u|^{m/j}\,|u|^{si/m-m/j}}{(\varepsilon\,|u|+1)^{(s-1)i/m}}.$$

Thus

$$\int \frac{|D^i u|^{n/i}\,|u|^{sp-n/i}}{(\varepsilon\,|u|+1)^{sp-p}}\,dx \leqq K s^n \sum_{j=1}^{i=1} \int \frac{|D^j u|^{n/j}\,|u|^{sp-n/j}}{(\varepsilon\,|u|+1)^{sp-p}}\,dx$$

$$+ K s^{nm/i}\left[\sum_{j=1}^{m} \frac{|D^j u|^{n/j}\,|u|^{spi/m-n/j}}{(\varepsilon\,|u|+1)^{(sp-p)i/m}}\,dx\right]^{m/i} \tag{3.11}$$

$$+ K\left[\int \frac{|u|^{sp\gamma}}{(\varepsilon\,|u|+1)^{sp\gamma-p\gamma}}\,dx\right]^{1/\gamma},$$

where the first term on the right hand side is absent if $i=1$. Hence in view of Fatou's lemma the first step in the induction argument follows by letting ε tend to zero.

To prove the lemma for a general $i \leqq m-1$, assuming it true for $i-1$, we need only estimate the first term on the right hand side of (3.11), and that is done using the induction hypothesis.

The second part of the lemma follows by applying Sobolev's inequality in the form

$$\left[\int |v|^{n/\gamma}\,dx\right]^{\gamma} \leqq K \int |Dv|^n\,dx + K \int |v|^n\,dx,$$

valid for $v \in W^{1,n} \supset W^{m,p}$, to the function

$$v = \frac{u\,|u|^{s\gamma/m-1}}{(\varepsilon\,|u|+1)^{(s-1)\gamma/m}},$$

using Lemma 3, and letting ε tend to zero.

Lemma 5. *Let η be the function of Lemma 1. If $s>m^2$ and $l>m^2$ then to every $\delta>0$ there exists a constant $K_{\delta s}$, depending on δ, s, l, m, and n, such that for all $u \in W^{m,p}(|x|<2)$*

$$\int_{|x|<2} \frac{|D^i u|^{n/i}\,|u|^{sp-n/i}}{(\varepsilon\,|u|+1)^{sp-p}}\,\eta^{lp}\,dx \leqq \delta \int_{|x|<2} \frac{|D^m u|^p\,|u|^{sp-p}}{(\varepsilon\,|u|+1)^{sp-p}}\,\eta^{lp}$$

$$+ K_{\delta s}\,\rho^{-n} \int_{|x|<R_2} |u|^{sp}\,dx + K_{\delta s} \sum_{j=1}^{m} \left[\int_{|x|<R_2} |D^j u|^{n/j}\,dx\right]^{sj/m}, \qquad 1 \leqq i \leqq m.$$

Proof. We shall prove the following inequality

$$\int_{|x|<2} \frac{|D^i u|^{n/i}\,|u|^{sp-n/i}}{(\varepsilon\,|u|+1)^{sp-p}}\,\eta^{lp}\,dx \leqq \delta \sum_{j=i+1}^{m} \int_{|x|<2} \frac{|D^j u|^{n/j}\,|u|^{sp-n/j}}{(\varepsilon\,|u|+1)^{sp-p}}\,\eta^{lp}\,dx \tag{3.12}$$

$$+ K_{\delta s}\,\rho^{-n} \int_{|x|<R_2} |u|^{sp}\,dx + K_{\delta s} \sum_{j=1}^{m} \left[\int_{|x|<R_2} |D^j u|^{n/j}\,dx\right]^{sj/m}, \qquad 1 \leqq i \leqq m-1.$$

The lemma follows from (3.12) directly for $i=m-1$, and for lower values of i by induction.

To prove (3.12) we note the following variant of the Sobolev inequality

$$\int |D^i v|^{n/i}\,dx \leqq K\Big[\int |D^m v|^p\,dx\Big]^{m/i} + K\int |v|^{n/i}\,dx, \tag{3.13}$$

valid for $v\in W^{m,p}$. By Corollary 3 the function

$$v=\frac{u\,|u|^{si/m-1}}{(\varepsilon|u|+1)^{(s-1)\,i/m}}\,\eta^{li/m}$$

belongs to $W^{m,p}$, and we apply (3.13) to v.

By Leibnitz' rule, Lemma 3, and Young's inequality we get

$$\frac{|D^i u|^{n/i}\,|u|^{sp-n/i}}{(\varepsilon|u|+1)^{sp-p}}\,\eta^{lp} \leqq K\,|D^i v|^{n/i} + K\,s^n \sum_{k=1}^{i-1} \frac{|D^k u|^{n/k}\,|u|^{sp-n/k}}{(\varepsilon|u|+1)^{sp-p}}\,\eta^{lp} + K\,\rho^{-n}\,\frac{|u|^{sp}}{(\varepsilon|u|+1)^{sp-p}},$$

where the second term on the right hand side is absent if $i=1$.

By Corollary 3

$$|D^m v| \leqq K\,s^m \sum_{j=1}^{m} \frac{|D^j u|^{m/j}\,|u|^{si/m-m/j}}{(\varepsilon|u|+1)^{(s-1)\,i/m}}\,\eta^{li/m} + K\,\rho^{-m}\,\frac{|u|^{si/m}}{(\varepsilon|u|+1)^{(s-1)\,i/m}}.$$

Using these inequalities in (3.13) we find

$$\int_{|x|<2} \frac{|D^i u|^{n/i}\,|u|^{sp-n/i}}{(\varepsilon|u|+1)^{sp-p}}\,\eta^{lp}\,dx \leqq K\,s^{nm/i}\Big[\sum_{j=1}^{m}\int \frac{|D^j u|^{n/j}\,|u|^{sin/m^2-n/j}}{(\varepsilon|u|+1)^{(sp-p)\,i/m}}\,\eta^{lin/m^2}\,dx\Big]^{m/i}$$

$$+K\,s^n\sum_{k=1}^{i-1}\int_{|x|<2} \frac{|D^k u|^{n/k}\,|u|^{sp-n/k}}{(\varepsilon|u|+1)^{sp-p}}\,\eta^{lp}\,dx + K\int_{|x|<R_2}|u|^{sp}\,dx + K\,\rho^{-n}\int_{|x|<R_2}|u|^{sp}\,dx.$$

By Hölder's inequality with exponents $q=(sp-n/j)/(sip/m-n/j)$ and $q/(q-1)$, and by Young's inequality with exponents $q_1=i(sp-n/j)/(sip-nm/j)$ and $q_1/(q_1-1)$ we get

$$\Big[\int_{|x|<2} \frac{|D^j u|^{n/j}\,|u|^{sip/m-n/j}}{(\varepsilon|u|+1)^{(sp-p)\,i/m}}\,\eta^{lin/m^2}\,dx\Big]^{m/i} \leqq \delta_1 \int_{|x|<2} \frac{|D^j u|^{n/j}\,|u|^{sp-n/j}}{(\varepsilon|u|+1)^{sp-p}}\,\eta^{lp}\,dx + K_{\delta_1}\Big[\int_{|x|<R_2}|D^j u|^{n/j}\,dx\Big]^{sj/m}.$$

The resulting inequality is

$$\int_{|x|<2} \frac{|D^i u|^{n/i}|u|^{sp-n/i}}{(\varepsilon|u|+1)^{sp-p}}\eta^{lp}\,dx \leqq K\delta_1 s^{nm/i}\sum_{j=1}^{m}\int_{|x|<2}\frac{|D^j u|^{n/j}|u|^{sp-n/j}}{(\varepsilon|u|+1)^{sp-p}}\eta^{lp}\,dx$$
$$+K s^n\sum_{k=1}^{i-1}\int\frac{|D^k u|^{n/k}|u|^{sp-n/k}}{(\varepsilon|u|+1)^{sp-p}}\eta^{lp}\,dx \tag{3.14}$$
$$+K_{\delta_1}s^{nm/i}\sum_{j=1}^{m}\Big[\int_{|x|<R_2}|D^j u|^{n/j}\,dx\Big]^{sj/m}+K_{\delta_1}\rho^{-n}\int_{|x|<R_2}|u|^{sp}\,dx,$$

where the second term on the right hand side is absent if $i=1$. Hence the inequality (3.12) follows for $i=1$ if we choose δ_1 such that $K\delta_1 s^{nm/i}=\min(\frac{1}{2},\delta)$. The validity of (3.12) in the general case follows from (3.14) by a standard induction argument: by the induction hypothesis the right hand side of (3.14) is less than or equal to

$$[K\delta_1 s^{nm/i}+(K\delta_1 s^{nm/i}+K s^n)\delta_2]\sum_{j=i+1}^{m}\int_{|x|<2}\frac{|D^j u|^{n/j}|u|^{sp-n/j}}{(\varepsilon|u|+1)^{sp-p}}\eta^{lp}\,dx$$

plus allowable terms. Then δ_1 and δ_2 are chosen so that the coefficient in this expression is less than $\frac{1}{2}$ and δ. The lemma is proved.

Lemma 6. *Let η be the function of Lemma 1 and let $u\in W^{m,p}(|x|<2)$ be a solution of (2.4) in $\{|x|<2\}$. Then for $s>m$ and $l>m$*

$$\int_{|x|<2}\frac{|D^m u|^p|u|^{sp-p}}{(\varepsilon|u|+1)^{sp-p}}\eta^{lp}\,dx\leqq K s^n\sum_{j=1}^{m-1}\int_{|x|<2}\frac{|D^j u|^{n/j}|u|^{sp-n/j}}{(\varepsilon|u|+1)^{sp-p}}\eta^{lp}\,dx$$
$$+K\rho^{-n}\int_{|x|<R_2}|u|^{sp}\,dx+KK_3^s, \tag{3.15}$$

where K depends on n, m, l, and C.

By choosing $l=m+1$, noting that $\eta\equiv 1$ in $|x|<R_1$, and letting ε tend to zero we get the following

Corollary 6. *Let u be a solution of (2.4) in $\{|x|<2\}$. Then for $s>m$*

$$\int_{|x|<R_1}|D^m u|^p|u|^{sp-p}\,dx\leqq K s^n\sum_{j=1}^{m-1}\int_{|x|<R_2}|D^j u|^{n/j}|u|^{sp-n/j}\,dx$$
$$+K\rho^{-n}\int_{|x|<R_2}|u|^{sp}\,dx+KK_3^s$$

for $1\leqq R_1<R_2\leqq 2$, where $\rho=R_2-R_1$ and where K depends on n, m, and C only.

Proof of Lemma 6. It suffices to assume that the right hand side of (3.15) is finite. By Corollary 3 with $a=b=sp-p$ the function φ defined by

$$\varphi=\frac{u|u|^{sp-p}}{(\varepsilon|u|+1)^{sp-p}}\eta^{lp}$$

belongs to $\mathring{W}^{m,p}(|x|<2)$ and if we use φ as a test function in (2.4) we find

$$\sum_{|\alpha|=m} \int F_\alpha(x, \delta_m u) \frac{D^\alpha u\,|u|^{sp}}{(\varepsilon|u|+1)^{sp-p}} h(u)\,\eta^{lp}\,dx$$
$$\leqq K \sum_{|\alpha|\leqq m} \int |F_\alpha(x, \delta_m u)| \left(s^m \sum_{j=1}^{m-1} \frac{|D^j u|^{m/j}\,|u|^{sp-p+1-m/j}}{(\varepsilon|u|+1)^{sp-p}} \eta^{lp} \right.$$
$$\left. + \rho^{-m} \frac{|u|^{sp-p+1}}{(\varepsilon|u|+1)^{sp-p}} \eta^{lp-m} \right) dx,$$

all integrals being taken over $\{|x|<2\}$. By the hypotheses on the equation

$$\int \frac{|D^m u|^p\,|u|^{sp-p}}{(\varepsilon|u|+1)^{sp-p}} \eta^{lp}\,dx \leqq K s \int \left(\sum_{j=0}^{m-1} |D^j u|^p + K_1 \right) \frac{|u|^{sp-p}}{(\varepsilon|u|+1)^{sp-p}} \eta^{lp}\,dx$$
$$+ K \int \left(\sum_{i=0}^{m} |D^i u|^{p-1} + K_2 \right) \left(s^m \sum_{j=1}^{m-1} \frac{|D^j u|^{m/j}\,|u|^{sp-p+1-m/j}}{(\varepsilon|u|+1)^{sp-p}} \right) \eta^{lp} \qquad (3.16)$$
$$+ K \rho^{-m} \frac{|u|^{sp-p+1}}{(\varepsilon|u|+1)^{sp-p}} \eta^{lp-m}\,dx.$$

The term involving $D^m u$ on the right hand side is estimated by Young's inequality (exponents p and $p/(p-1)$) and is majorized by

$$\frac{1}{2} \int \frac{|D^m u|^p\,|u|^{sp-p}}{(\varepsilon|u|+1)^{sp-p}} \eta^{lp}\,dx + K s^n \sum_{j=1}^{n} \int \frac{|D^j u|^{n/j}\,|u|^{sp-n/j}}{(\varepsilon|u|+1)^{sp-p}} \eta^{lp}\,dx$$
$$+ K \rho^{-n} \int \frac{|u|^{sp}}{(\varepsilon|u|+1)^{sp-p}} \eta^{lp-m}\,dx.$$

Here the first term can be absorbed into the left hand side of (3.16), and the remaining terms are trivially majorized by the left hand side of (3.15). The remaining terms in the right hand side of (3.16) allow by Young's inequality estimates of the same type modulo a term of the form

$$\int (K_2^{p/(p-1)} + K_1) \frac{|u|^{sp-p}}{(\varepsilon|u|+1)^{sp-p}} \eta^{lp}\,dx.$$

This integral is, by Young's inequality with exponents s and $s/(s-1)$, less than the right hand side of (3.15), and that concludes the proof of the lemma.

4. Proof of the Theorem

We first note that, by Sobolev's imbedding theorem, $|u|^{sp}$ is locally integrable for all $s>0$.

Now select an arbitrary point of Ω. It is of course no restriction to assume this point to be the origin. We shall also assume the distance from the point to

the boundary of Ω to be greater than two. As we shall see below, the general case can be transformed to this situation.

With η being the function of Lemma 1, we see that a combination of Lemma 5 and Lemma 6 with $l=m^2+1$, say, and $s>m^2$ gives

$$\sum_{i=1}^{m-1}\int_{|x|<2}\frac{|D^i u|^{n/i}|u|^{sp-n/i}}{(\varepsilon|u|+1)^{sp-p}}\eta^{lp}\,dx\leqq K s^n\delta\sum_{i=1}^{m-1}\int_{|x|<2}\frac{|D^i u|^{n/i}|u|^{sp-p}}{(\varepsilon|u|+1)^{sp-p}}\eta^{lp}\,dx$$
$$+K_{\delta s}\sum_{j=1}^{m}\Big[\int_{|x|<R_2}|D^j u|^{n/j}\,dx\Big]^{sj/m}$$
$$+(K_{\delta s}+K\delta)\rho^{-n}\int_{|x|<R_2}|u|^{sp}\,dx+K\delta(K_2^{p/(p-1)}+K_1)^s.$$

Here and in what follows K denotes constants which depend on n, m, and C only.

Choosing δ so that $K s^n\delta=\frac{1}{2}$ we find, after letting ε tend to zero and noting that $\eta\equiv 1$ in $\{|x|<R_1\}$

$$\sum_{i=1}^{n-1}\int_{|x|<R_1}|D^i u|^{n/i}|u|^{sp-n/i}\,dx\leqq K_s\rho^{-n}\int_{|x|<R_2}|u|^{sp}\,dx$$
$$+K_s\sum_{j=1}^{m}\Big[\int_{|x|<R_2}|D^j u|^{n/j}\,dx\Big]^{sj/m}+KK_3^s.$$

By well-known inequalities we can bound the integrals on the right hand side with

$$K_s\rho^{-n}\Big[\int_{|x|<R_2}|D^m u|^p\,dx\Big]^s+K_s\rho^{-n}\Big[\int_{|x|<R_2}|u|^p\,dx\Big]^s.$$

If we choose $R_1<R_2<2$ such that $2-R_2=R_2-R_1=\rho$ we find by Lemma 2

$$\sum_{i=1}^{m-1}\int_{|x|<R_1}|D^i u|^{n/i}|u|^{sp-n/i}\,dx+\int_{|x|<R_1}|u|^{sp}\,dx$$
$$\leqq K_s\rho^{-(s+1)n}\Big[\int_{|x|<R}|u|^p\,dx\Big]^s+KK_3^s \tag{4.1}$$

valid for $s>m^2$.

Now define the balls $B_\nu=\{|x|<\rho_\nu\}$, $\nu=0,1,\ldots$, where $\rho_\nu=1+2^{-\nu}$, and the numbers I_ν by

$$I_\nu=\sup\Big\{\int_{B_\nu}|u|^{\gamma^\nu p}\,dx,\ \int_{B_\nu}|D^i u|^{n/i}|u|^{\gamma^\nu p-n/i}\,dx\Big\},$$

where the supremum is taken over those i for which $1\leqq i\leqq m-1$, and where $\gamma=m/(m-1)$.

Let ν_0 be the smallest integer such that $\gamma^{\nu_0}>m^2$. By (4.1) we have

$$I_{\nu_0}^{\gamma^{-\nu_0}}\leqq K\int_{B_0}|u|^p\,dx+KK_3. \tag{4.2}$$

For $v > v_0$ divide the interval (ρ_v, ρ_{v-1}) into two equal parts, $\rho_v < \rho_v' < \rho_{v-1}$. Let v_1 be the largest integer such that

$$I_{v-1}^{\gamma^{-v+1}} \leqq \int_{B_v} |u|^p \, dx. \tag{4.3}$$

If there is no such integer, we have that (4.3) holds for an infinite number of integers, and letting v tend to infinity through such integers, we find that the inequality of theorem is true, in fact

$$|u(y)|^p \leqq \int_{|x|<1} |u|^p \, dx, \qquad |y|<1.$$

Let v_2 be the largest integer such that

$$I_{v-1}^{\gamma^{-v+1}} \leqq (K_2^{p/(p-1)} + K_1). \tag{4.4}$$

If v_2 does not exist, by the same argument we find

$$|u(y)|^p \leqq (K_2^{p/(p-1)} + K_1)$$

and the desired inequality holds again.

For $v > v' = \max(v_0, v_1, v_2)$ we shall prove

$$I_v \leqq K^v I_{v-1}^{\gamma}. \tag{4.5}$$

To do that we apply Lemma 4 with $R = \rho_v$ and $s = \gamma^v$. Two cases must be distinguished. In case I_v is equal to the integral of $|u|^{\gamma^v p}$ it follows immediately from the second part of that lemma that (4.5) holds. In the second case we have

$$I_v \leqq K^v \sum_{k=1}^{m-1} \sum_{j=1}^{m} \Big[\int_{|x|<\rho_v} |D^j u|^{n/j} \, |u|^{\gamma^v pk/m - n/j} \, dx \Big]^{m/k} + K^v \Big[\int_{|x|<\rho_v} |u|^{\gamma^{v-1} p} \, dx \Big]^{\gamma}.$$

The second term is trivially less than $K^v I_{v-1}^{\gamma}$. To estimate the terms with $j = m$ we use Lemma 6 with $R_1 = \rho_v$, $R_2 = \rho_v'$, and $s = \gamma^v k m^{-1}$ to get

$$\begin{aligned} \int_{|x|<\rho_v} |D^m u|^p \, |u|^{\gamma^v pk/m - p} \, dx \leqq K^v \sum_{i=1}^{m-1} \int_{|x|<\rho_v'} |D^i u|^{n/i} \, |u|^{\gamma^v pk/m - n/i} \, dx \\ + K^v \int_{|x|<\rho_v'} |u|^{\gamma^v pk/m} \, dx + K K_3^{\gamma^v k/m}. \end{aligned}$$

By applying Hölder's inequality to the second integral on the right we find that it is less than $K^v I_{v-1}^{\gamma k/m}$, and the same is true for the last term, since $v > v_2$. Hence we have

$$I_v \leqq K^v \sum_{k=1}^{m-1} \sum_{j=1}^{m-1} \Big[\int_{|x|<\rho_v'} |D^j u|^{n/j} \, |u|^{\gamma^v pk/m - n/j} \, dx \Big]^{m/k} + K^v I_{v-1}^{\gamma}.$$

For $k<m-1$ we apply Hölder's inequality with exponents $q=(\gamma^{\nu-1}p-n/j)/(\gamma^{\nu}pkm^{-1}-n/j)$ and $q/(q-1)$ and find

$$I_\nu \leqq K^\nu \sum_{k=1}^{m-1}\sum_{j=1}^{m-1}\Big[\int_{|x|<\rho'} |D^j u|^{n/j}\,dx\Big]^{m(q-1)/kq} I_{\nu-1}^{m/kq} + K^\nu I_{\nu-1}^{\gamma}$$

where q of course depends on k, j, and ν.

By Lemmas 1 and 2

$$\Big[\int_{|x|<\rho'_\nu} |D^j u|^{n/j}\,dx\Big]^{j/m} \leqq K^\nu \int_{|x|<\rho_{\nu-1}} |u|^p\,dx$$

and since $\nu>\nu_1$ the right hand side here is less than or equal to $K^\nu I_{\nu-1}^{\gamma^{1-\nu}}$. An elementary calculation now gives (4.5).

If $\nu'>\nu_0$ we find in the same way

$$I_{\nu'}^{\gamma^{-\nu'}} \leqq K^{\nu'/\gamma^{\nu'}} \int_{|x|<2} |u|^p\,dx + K^{\nu'/\gamma^{\nu'}}(K_2^{p/(p-1)}+K_1) \leqq K\int_{|x|<2}|u|^p\,dx + KK_3 \quad (4.6)$$

while if $\nu'=\nu_0$ (4.6) follows from (4.2).

A simple iteration procedure gives from (4.5)

$$I_\nu^{\gamma^{-\nu}} \leqq K^{\sum_{\mu=\nu'}^{\nu}\mu/\gamma^\mu} I_{\nu'}^{\gamma^{-\nu'}} \leqq K\int_{|x|<2}|u|^p\,dx + KK_3$$

where the last inequality follows from (4.6).

Letting ν tend to infinity while observing that

$$\int_{|x|<1}|u|^{\gamma^\nu p}\,dx \leqq I_\nu$$

we find that we have proved

$$\sup_{|x-x_0|<1}|u|^p \leqq K\int_{|x-x_0|<2}|u|^p\,dx + KK_3.$$

The full theorem can be deduced from this inequality. Thus if u is a solution of the equation in $\{|x|<2R\}$ with $R\leqq 1$ the function $v(x)=u(Rx)$ satisfies the equation

$$\sum_{|\alpha|\leqq m}\int_{|x|<2}\tilde{F}_\alpha(x,\delta_m u)\,D^\alpha\varphi\,dx=0$$

where $\tilde{F}_\alpha$ satisfies the conditions of the theorem with $\tilde{C}\leqq C$, $\tilde{K}_1=K_1R^n$, and $\tilde{K}_2\leqq R^{n-m}K_2$. Hence

$$\sup_{|x|<1}|v|^p \leqq K\int_{|x|<2}|v|^p\,dx + KR^nK_3.$$

A change of variables in the integral completes the proof of the theorem.

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A Universal Coefficient Theorem for Categories with Kernels

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Let $\mathscr{F}$ be the category whose objects are the free abelian groups with prescribed bases, thus the object F with basis B may be identified with the coproduct $Z\otimes B=\sum_{b\in B} Z_b$ where $Z_b=Z$ for every $b\in B$. $f\colon Z\otimes B\to Z\otimes B'$ is a morphism in $\mathscr{F}$ if on each summand $Z\xrightarrow{1_Z\otimes b} Z\otimes B$ of the coproduct $Z\otimes B$

$$f(1_Z\otimes b)=\sum_{b'\in B'}(z_{b'}\cdot 1_Z)\otimes b'\colon\ Z\to Z\otimes B'$$

where all but a finite number of the integers $z_{b'}$ are zero. The category $\mathscr{F}$ is clearly preadditive. Let $\mathscr{G}$ be the category of abelian groups. Consider the functor $U\colon \mathscr{F}\to\mathscr{G}$ defined by:

(i) UF of the object F with prescribed basis B is the abelian group F (forget the basis).

(ii) Uf is the morphism f viewed as a homomorphism in $\mathscr{G}$.

We observe that the functor U is full and faithful.

Proposition 1. *$\mathscr{F}$ is a category with kernels.*

Proof. Let $f\colon Z\otimes B\to Z\otimes B'$ be a morphism in $\mathscr{F}$, then the kernel of the homomorphism Uf is identified with a subgroup $\bar{K}$ of $U(Z\otimes B)$. The kernel $\bar{K}$ is a free abelian group (p. 143 § 19 [4]). If K is any free basis of $\bar{K}$, then $U(Z\otimes K)=\bar{K}$. The morphism $g\colon Z\otimes K\to Z\otimes B$ defined by $g(1_Z\otimes k)=\sum n_i 1_Z\otimes b_i$ where $\sum n_i\otimes b_i=(\ker Uf)(k)$ is such that $Ug=(\ker Uf)$. It is clear that the class of all pairs $(Z\otimes K, g)$ is the kernel of the morphism $f\colon Z\otimes B\to Z\otimes B'$ in the category $\mathscr{F}$.

Let $\mathscr{A}$ be a preadditive category with kernels and coproducts. As in [3] a sequence $A'\xrightarrow{i}A\xrightarrow{j}A''$ in $\mathscr{A}$ is said to be exact if i admits a factorization $A'\xrightarrow{l}\bar{A}\xrightarrow{k}A$ such that l is an epimorphism and k is a kernel of j. If $T\colon \mathscr{F}\to\mathscr{A}$ is a functor, we have the universal coefficient theorem:

Proposition 2. *If*

$$Z\otimes B_*\colon\ \ldots Z\otimes B_n\xrightarrow{d_n} Z\otimes B_{n-1}\longrightarrow\cdots\xrightarrow{d_1} Z\otimes B_0\to 0$$

is a complex in $\mathscr{F}$ such that $U(Z\otimes B_*)$ is acyclic, then

(i) *$Z\otimes B_*$ is split exact.*

(ii) *$T(Z\otimes B_*)$ is split exact.*

Proof. Since $U(Z\otimes B_*)$ is an acyclic complex of free groups in $\mathscr{G}$, there exist splitting homomorphisms $\bar{s}_n$: $U(Z\otimes B_n)\to U(Z\otimes B_{n+1})$ such that:

$$U d_1 \circ \bar{s}_0 = 1_{U(Z\otimes B_0)},$$
$$U d_{n+1}\circ \bar{s}_n + \bar{s}_{n-1}\circ U d_n = 1_{U(Z\otimes B_n)}.$$

(The construction of the homomorphisms $\bar{s}_n$ is given in Chapter 5, § 1 [2].) Since U is full and faithful, there exist s_n: $Z\otimes B_n\to Z\otimes B_{n+1}$ such that $U s_n=\bar{s}_n$ for $n=0,1,\ldots$. Thus the complex $Z\otimes B_*$ splits. We now use the standard argument to show that if a positive complex K_* splits in a preadditive category with kernels, then K_* is acyclic. To show d_1 is an epimorphism, suppose h,h': $Z\otimes B_0\rightrightarrows Z\otimes H$ are morphisms in $\mathscr{F}$ such that $hd_1=h'd_1$. Then $h=hd_1s_0=h'd_1s_0=h'$. For the exactness at $Z\otimes B_1$ consider the diagram

$$\begin{array}{ccccc} & & Z\otimes K_1 & & \\ & {\scriptstyle l_2}\nearrow & & \searrow{\scriptstyle \ker d_1} & \\ Z\otimes B_2 & \underset{s_1}{\overset{d_2}{\rightleftarrows}} & Z\otimes B_1 & \underset{s_0}{\overset{d_1}{\rightleftarrows}} & Z\otimes B_0. \end{array}$$

To show l_2 is an epimorphism we first observe that

$$\begin{aligned} \ker d_1\circ l_2\circ s_1\circ \ker d_1 &= d_2\circ s_1\circ \ker d_1 \\ &= (1_{Z\otimes B_1}-s_0 d_1)\circ \ker d_1 \\ &= \ker d_1\circ 1_{Z\otimes K_1}. \end{aligned}$$

Since $\ker d_1$ is a monomorphism,

$$l_2\circ s_1\circ \ker d_1 = 1_{Z\otimes K_1}. \tag{1}$$

Suppose $h_1 l_2 = h_2 l_2$, then

$$\begin{aligned} h_1 &= h_1\circ 1_{Z\otimes K_1} \\ &= h_1\circ l_2\circ s_1\circ \ker d_1 \quad \text{by (1)} \\ &= h_2\circ l_2\circ s_1\circ \ker d_1 \\ &= h_2 \quad \text{by (1)}. \end{aligned}$$

The exactness of $Z\otimes B_*$ follows now by induction. Since the positive complex $T(Z\otimes B_*)$ splits, i.e.

$$T d_1\circ T s_0 = 1_{T(Z\otimes B_0)},$$
$$T d_{n+1}\circ T s_n + T s_{n-1}\circ T d_n = 1_{T(Z\otimes B_n)},$$

the proof of (ii) is similar to the proof of (i).

We remark that in the case where the category $\mathscr{A}$ is abelian with coproducts of arbitrary size, we have provided a simple proof of the universal coefficient theorem of André (Lemma 4.2 [1]).

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