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On Weak Convergence of Norm Preserving Operators on $L^2(X)$ and its Application to Measure Preserving Transformations

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1. Introduction

Let $(X, \mathfrak{M}, \mu)$ be an arbitrary measure space and let $L^2(X)$ be the Hilbert space of all square integrable complex-valued functions on X, with $\| \ \|$ denoting the L^2-norm. The inner product of $f, g \in L^2(X)$ will be denoted by (f, g), i.e.,

$$(f, g) = \int f \bar{g} \, d\mu,$$

where $\bar{g}$ denotes the complex conjugate function of g. Let $\mathfrak{X}$ be the set of all bounded operators on $L^2(X)$. The strong topology of $\mathfrak{X}$ is the one for which a net $\{A_\alpha\}$ in $\mathfrak{X}$ converges to $A \in \mathfrak{X}$ if and only if $\|A_\alpha f - Af\|$ converges to 0 for each $f \in L^2(X)$. The weak topology of $\mathfrak{X}$ is the one for which a net $\{A_\alpha\}$ in $\mathfrak{X}$ converges to $A \in \mathfrak{X}$ if and only if $(A_\alpha f, g)$ converges to (Af, g) for each f and g in $L^2(X)$. Let $\mathfrak{F}$ be the set of all measure preserving transformations of X into itself. Every $T \in \mathfrak{F}$ induces a norm preserving operator $U(T)$ on $L^2(X)$ which is defined by

$$U(T) f(x) = f(Tx) \quad (f \in L^2(X), x \in X).$$

Therefore $\mathfrak{F}$ inherits every topology that the subset $\{U(T);\ T \in \mathfrak{F}\}$ of $\mathfrak{X}$ possesses.

In case μ is a tight measure on a completely regular Hausdorff space X, Papangelou [3] studied the relation between weak convergence and convergence in measure of transformations in $\mathfrak{G}$, where $\mathfrak{G}$ is the group of all invertible measurable and measure preserving transformations of X. The author [5, 6] extended then one of his results and gave counterexamples which show that the hypotheses can not be weakened. The following are known:

(a) Let $\mathfrak{A}$ be any uniformity of X compatible with the topology of X. If a net $\{T_\alpha\}$ in $\mathfrak{G}$ converges to $T \in \mathfrak{G}$ weakly then $\{T_\alpha\}$ $\mathfrak{A}$-converges in measure to T. Moreover, the converse holds if μ is locally finite.

(b) Let X be metrizable and let μ be σ-finite. If a sequence $\{T_n\}$ in $\mathfrak{G}$ converges to $T \in \mathfrak{G}$ weakly then every subsequence $\{T_{k(n)}\}$ of $\{T_n\}$ has a subsequence which converges to T almost everywhere. Moreover, if μ is, in addition, locally finite, the converse is true.

(c) Let X be metrizable and let μ be σ-finite and locally finite. Then the weak topology on the group $\mathfrak{G}$ is the finest topology $\mathscr{T}$ such that if a sequence $\{T_n\}$ in $\mathfrak{G}$ converges to $T \in \mathfrak{G}$ almost everywhere then $\mathscr{T}$-lim $T_n = T$.

In the present paper we shall study the weak topology on some subspaces of $\mathfrak{X}$ which are larger than $\mathfrak{G}$. In Section 2, the weak topology on the space $\mathfrak{H}$ of all norm preserving operators on an arbitrary $L^2(X)$ is considered. A necessary and sufficient condition for weak convergence of a net $\{A_\alpha\}$ in $\mathfrak{H}$ to $A \in \mathfrak{H}$ will be given. Concerning weak convergence of a sequence $\{A_n\}$ to A, another condition is given too. In Section 3, the weak topology on the space of all measure preserving transformations of X into itself is investigated exclusively. The metrizability, separability and completeness of the topology come into question here. The above results (a), (b) and (c) are all generalized to the space of measure preserving transformations.

2. Weak Convergence of Norm Preserving Operators

Theorem 1. *Let $\{A_\alpha\}$ be a net in $\mathfrak{H}$ and let $A \in \mathfrak{H}$. Then the following two statements are equivalent:*

(I) *A_α converges to A weakly.*

(II) *$A_\alpha f$ converges to Af in measure for each $f \in L^2(X)$.*

Proof. (I) implies (II): Since the weak topology and the strong topology are coincident on $\mathfrak{H}$, this is clear.

(II) implies (I): Let $f \in L^2(X)$. For $\varepsilon > 0$, there exists a measurable set E of finite measure and a positive number M such that $\|Af - \chi_E \cdot Af\| < \varepsilon$ and $|Af(x)| < M$ for all $x \in E$, where χ_E denotes the indicator function of E. By hypothesis (II), $A_\alpha f$ converges to Af in measure. Since $\|A_\alpha f\| = \|Af\|$ for all α, it follows that $A_\alpha f$ converges to Af in the L^2-norm topology. The proof is complete.

Theorem 2. *Let $\{A_n\}$ be a sequence in $\mathfrak{H}$ and let $A \in \mathfrak{H}$. Then the following two statements are equivalent:*

(I) *A_n converges to A weakly.*

(II) *Let $f \in L^2(X)$. Then every subsequence $\{A_{k(n)} f\}$ of $\{A_n f\}$ has a subsequence $\{A_{k(j(n))} f\}$ which converges to Af almost everywhere.*

Proof. (I) implies (II): Obvious.

(II) implies (I): Let $A_n f$ converge to Af almost everywhere. Define an integrable function g_n by $g_n = 2(|A_n f|^2 + |Af|^2)$. Clearly g_n converges to $4|Af|^2$ almost everywhere, $|A_n f - Af|^2 \leqq g_n$,

$$\lim_{n \to \infty} |A_n f(x) - Af(x)| = 0 \text{ a.e.} \quad \text{and} \quad \lim_{n \to \infty} \int g_n \, d\mu = 4 \|f\|^2 .$$

Therefore we can apply Lebesgue's theorem on dominated convergence to infer that $\lim_{n \to \infty} \|A_n f - Af\| = 0$. This together with hypothesis (II) implies that every subsequence $\{A_{k(n)} f\}$ of $\{A_n f\}$ has a subsequence converging to Af in the L^2-norm topology. Hence the whole sequence $\{A_n f\}$ must converge to Af in the L^2-norm topology. The proof is complete.

3. Weak Convergence of Measure Preserving Transformations

In this section, X will be assumed to be a completely regular Hausdorff space and μ a tight measure on X, unless the contrary is explicitly specified. Here we mean that a measure μ on a σ-field $\mathfrak{M}$ of subsets of X is tight if $\mathfrak{M}$ contains the σ-field $\mathfrak{B}$ generated by all open sets in X, μ on $\mathfrak{M}$ is the completion of μ on $\mathfrak{B}$, and

$$\mu(E)=\sup\{\mu(K);\ K\subset E \text{ and } K \text{ is compact}\}$$

for every measurable set E of finite measure. μ is locally finite if any point of X has an open neighborhood of finite measure.

Let $\mathfrak{A}$ be any uniformity on X compatible with the topology of X (cf. [2, p. 188, Corollary 17]). We denote by $\mathfrak{F}$ the space of all measure preserving transformations of X into itself. If D is an arbitrary subset of X, $\mu^*(D)$ will denote the outer measure of D, i.e.,

$$\mu^*(D)=\inf\{\mu(E);\ D\subset E \text{ and } E \text{ is measurable}\}.$$

Definition 1. The net $\{T_\alpha\}$ in $\mathfrak{F}$ $\mathfrak{A}$-*converges in measure* to $T\in\mathfrak{F}$ if for every measurable set E of finite measure and any member $V\in\mathfrak{A}$

$$\lim_\alpha \mu^*(\{x\in T^{-1}(E);\ (T_\alpha x, Tx)\notin V\})=0. \tag{1}$$

It is sufficient that (1) be true for all compact sets of finite measure.

Theorem 3. *Let $\{T_\alpha\}$ be a net in $\mathfrak{F}$ and let $T\in\mathfrak{F}$. Then* (I) *below implies* (II). *In addition, if μ is locally finite then* (I) *and* (II) *are equivalent:*

(I) *T_α converges to T weakly.*

(II) *T_α $\mathfrak{A}$-converges in measure to T.*

Proof. (I) implies (II): Let E be a measurable set of finite measure. For $\varepsilon>0$, there exists a compact set K such that $K\subset E$ and $\mu(E-K)<\varepsilon$. Put

$$E_0=T^{-1}(K).$$

Let V be any member of $\mathfrak{A}$ and define

$$Q_\alpha(T^{-1}(E))=\{x\in T^{-1}(E);\ (T_\alpha x, Tx)\notin V\}.$$

Then

$$\begin{aligned} Q_\alpha(T^{-1}(E))\subset &(T^{-1}(E)-E_0)\cup\{x\in E_0;\ T_\alpha x\notin K\}\\ &\cup\{x\in E_0;\ T_\alpha x\in K \text{ and } (T_\alpha x, Tx)\notin V\}. \end{aligned} \tag{2}$$

Since $\{x\in E_0;\ T_\alpha x\notin K\}\subset T^{-1}(K)\triangle T_\alpha^{-1}(K)$, where the sign $\triangle$ refers to symmetric difference, it follows that

$$\lim_\alpha \mu(\{x\in E_0;\ T_\alpha x\notin K\})\leqq\lim_\alpha \mu(T^{-1}(K)\triangle T_\alpha^{-1}(K))=0. \tag{3}$$

Clearly the sets of the form

$$\{(u, v)\in K\times K;\ |f_i(u)-f_i(v)|<\varepsilon,\ i=1, 2, \ldots, n\},$$

where f_i is a continuous real-valued function on K for each i, are a basis for a uniformity on K compatible with the topology of the subspace K. By the uniqueness theorem on uniformity of a compact space there exist continuous real-valued functions f_i $(i=1, 2, \ldots, m)$ on K and $\delta>0$ such that

$$\{(u, v)\in K\times K;\ |f_i(u)-f_i(v)|<\delta,\ i=1, 2, \ldots, m\}\subset V.$$

Define $f_i^*(x)=f_i(x)$ if $x\in K$ and $f_i^*=0$ otherwise for each i. Then f_i^* is a function in $L^2(X)$. By Theorem 1, $f_i^*(T_\alpha x)$ converges to $f_i^*(Tx)$ in measure. It follows that there exists α_1 such that if $\alpha\geqq\alpha_1$ then

$$\mu\big(\{x\in X;\ |f_i^*(T_\alpha x)-f_i^*(Tx)|\geqq\delta\}\big)<\varepsilon/m$$

for each i. Therefore $\alpha\geqq\alpha_1$ implies

$$\begin{aligned}&\mu^*\big(\{x\in E_0;\ T_\alpha x\in K \text{ and } (T_\alpha x, Tx)\notin V\}\big)\\ &\qquad\leqq\sum_{i=1}^{m}\mu\big(\{x\in X;\ |f_i^*(T_\alpha x)-f_i^*(Tx)|\geqq\delta\}\big)<\varepsilon.\end{aligned}\tag{4}$$

From (2), (3) and (4) it follows that there exists $\alpha(\varepsilon)$ such that if $\alpha\geqq\alpha(\varepsilon)$ then $\mu^*\big(Q_\alpha(T^{-1}(E))\big)<3\varepsilon$.

If μ is locally finite then (II) implies (I): Let E be a measurable set of finite measure. For a positive number ε with $\varepsilon<1$, choose a compact set K such that $K\subset E$ and $\mu(E-K)<(\varepsilon/2)^2$. By [4, Lemma] there exists an open set G such that $K\subset G$ and $\mu(G-K)<(\varepsilon/2)^2$. Let f be a continuous function on X such that $f(K)=1$, $f(X-G)=0$ and $0\leqq f(x)\leqq 1$ for all $x\in X$. Then

$$\|f-\chi_E\|\leqq\|f-\chi_K\|+\|\chi_K-\chi_E\|<\varepsilon.$$

Since f is uniformly continuous on K, there exists a symmetric member V of $\mathfrak{U}$ such that

$$(u, v)\in(K\times K)\cap V \quad\text{implies}\quad |f(u)-f(v)|<\varepsilon.\tag{5}$$

Here it may and will be assumed by virtue of the compactness of K that

$$\{y\in X;\ (x, y)\in V \text{ for some } x\in K\}\subset G\tag{6}$$

(cf. [2, p. 199, Theorem 33]).

Put $E_0=T^{-1}(K)$. Then by hypothesis (II), it follows that

$$\lim_\alpha\mu^*\big(Q_\alpha(E_0)\big)=\lim_\alpha\mu^*\big(\{x\in E_0;\ (T_\alpha x, Tx)\notin V\}\big)=0.$$

Let E_α be a measurable set such that

$$Q_\alpha(E_0)\subset E_\alpha\subset E_0 \quad\text{and}\quad \lim_\alpha\mu(E_\alpha)=0.\tag{7}$$

On the other hand it follows from (5) and (6) that

$$\{x\in X;\ |f(T_\alpha x)-f(Tx)|\geqq\varepsilon\}$$
$$\subset(T^{-1}(G)-E_0)\cup E_\alpha\cup T_\alpha^{-1}(G-K)\cup(T_\alpha^{-1}(G)-(E_0-E_\alpha)).$$

Thus $\mu(\{x\in X;\ |f(T_\alpha x)-f(Tx)|\geqq\varepsilon\})\leqq\varepsilon+\mu(E_\alpha)+\varepsilon+(\varepsilon+\mu(E_\alpha))$. This together with (7) demonstrates that $f(T_\alpha x)$ converges to $f(Tx)$ in measure, and so the same argument as in the proof of Theorem 1 implies that $f(T_\alpha x)$ converges to $f(Tx)$ in the L^2-norm topology. Since the set of all functions f of the form

$$f=\sum_{i=1}^{n}a_i f_i,$$

where f_i are bounded continuous functions such that $\mu(\{x\in X;\ f_i(x)\neq 0\})<\infty$ and a_i are complex numbers, is dense in $L^2(X)$ with respect to the L^2-norm topology, the standard argument of linearity and continuity implies that $g(T_\alpha x)$ converges to $g(Tx)$ in the L^2-norm topology for each $g\in L^2(X)$. This completes the proof.

Remark. Our definition of $\mathfrak{A}$-convergence in measure is somewhat different from that of Papangelou [3]. But if T is invertible measurable and measure preserving, or if μ is σ-finite, then they are coincident. In the former case this is immediate. Suppose now that μ is σ-finite. Let E be a measurable set of finite measure. For $\varepsilon>0$ there exists a measurable set F of finite measure such that $\mu(E-T^{-1}(F))<\varepsilon$. Let V be any member of $\mathfrak{A}$ and suppose that

$$\lim_\alpha \mu^*(Q_\alpha(T^{-1}(F)))=\lim_\alpha \mu^*(\{x\in T^{-1}(F);\ (T_\alpha x, Tx)\notin V\})$$
$$=0.$$

Then the formula

$$\mu^*(Q_\alpha(E))\leqq\mu(E-T^{-1}(F))+\mu^*(Q_\alpha(T^{-1}(F)))$$

implies that there exists $\alpha(\varepsilon)$ such that if $\alpha\geqq\alpha(\varepsilon)$ then $\mu^*(Q_\alpha(E))<2\varepsilon$. Hence the two definitions are coincident in this case.

Theorem 4. *Let μ be a locally finite tight measure on X. Let T and T_n $(n=1,2,\ldots)$ be measure preserving transformations of X into itself. If every subsequence $\{T_{k(n)}\}$ of $\{T_n\}$ has a subsequence $\{T_{k(j(n))}\}$ which converges to T almost everywhere then T_n converges to T weakly.*

Proof. Let T_n converges to T almost everywhere. If f is a continuous function in $L^2(X)$ then $f(T_n x)$ converges to $f(Tx)$ almost everywhere. Since $\|f(T_n x)\|=\|f(Tx)\|$ for all n, the same argument as in the proof of Theorem 2 implies that $f(T_n x)$ converges to $f(Tx)$ in the L^2-norm topology. Since the set of functions f in $L^2(X)$ which are continuous is dense in $L^2(X)$ with respect to the L^2-norm topology, it follows by the standard argument of linearity and continuity that $g(T_n x)$ converges to $g(Tx)$ in the L^2-norm topology for each $g\in L^2(X)$. There-

fore T_n converges to T weakly. This together with the hypothesis of Theorem 4 shows that every subsequence $\{T_{k(n)}\}$ of $\{T_n\}$ has a subsequence which converges to T weakly. Hence the whole sequence $\{T_n\}$ must converge to T weakly. The proof is completed.

The converse of Theorem 4 does not hold in general. In fact it is known [5] that there exists a compact abelian group H with the normalized Haar measure μ for which there exists a sequence $\{T_n\}$ of invertible measurable and measure preserving transformations of H such that $\{T_n\}$ converges to the identity transformation I of H weakly, but for any subsequence $\{T_{k(n)}\}$ of $\{T_n\}$ the limit of $\{T_{k(n)}\, x\}$ does not exist for all $x \in X$.

Theorem 5. *Let X be metrizable and let μ be a σ-finite tight measure on X. Suppose that T and T_n $(n=1, 2, \ldots)$ are measure preserving transformations of X into itself. If T_n converges to T weakly then every subsequence $\{T_{k(n)}\}$ of $\{T_n\}$ has a subsequence $\{T_{k(j(n))}\}$ which converges to T almost everywhere.*

Proof. Let d be a metric on X compatible with the topology of X. Then the family of subsets of $X \times X$ of the form $\{(u, v) \in X \times X;\ d(u, v) < \varepsilon\}$ is a basis for a uniformity $\mathfrak{A}$ of X compatible with the topology of X. Therefore if T_n converges to T weakly then T_n $\mathfrak{A}$-converges in measure to T by virtue of Theorem 3. Let $\{E_i\}$ be a sequence of mutually disjoint measurable sets of finite measure such that $X = \bigcup \{E_i;\ i=1, 2, \ldots\}$. By remark it follows that there exists a subsequence $\{T_{k_i(n)}\}$ of $\{T_n\}$ for each i such that

$$\mu^*(\{x \in E_i;\ d(T_{k_i(n)}\, x,\ Tx) \geqq 2^{-n}\}) < 2^{-n}.$$

Choose a measurable set F_n such that $\{x \in E_i;\ d(T_{k_i(n)}\, x,\ Tx) \geqq 2^{-n}\} \subset F_n \subset E_i$ and $\mu(F_n) < 2^{-n}$. Clearly

$$\mu(\limsup_{n \to \infty} F_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} T_{k_i(n)}\, x = Tx$$

for all $x \in E_i - \limsup_{n \to \infty} F_n$. Hence $T_{k_i(n)}$ converges to T almost everywhere on E_i. The diagonal method can now be applied in order to obtain a subsequence $\{T_{k(n)}\}$ of $\{T_n\}$ which converges to T almost everywhere. This establishes Theorem 5.

A counter-example in [6] shows that the hypotheses of Theorem 5 can not be weakened.

Let $(X, \mathfrak{M}, \mu)$ be an arbitrary measure space. If E is a measurable set then we will say that the measure space $(E, \mathfrak{M}_E, \mu_E)$, where $\mathfrak{M}_E = \{F \in M;\ F \subset E\}$ and $\mu_E(F) = \mu(F)$ for $F \in \mathfrak{M}_E$, is a subspace of the measure space $(X, \mathfrak{M}, \mu)$.

Definition 2. A measure space $(X, \mathfrak{M}, \mu)$ is a *σ-finite Lebesgue space* if there exists a countable family $\Gamma = \{X_n\}$ of mutually disjoint measurable sets of finite measure such that $(X_n, \mathfrak{M}_{X_n}, \mu_{X_n})$ is a Lebesgue space in the sense of Rohlin [4] for each n, and

$$X = \bigcup_n X_n.$$

Proposition 1. *Let $(X, \mathfrak{M}, \mu)$ be an arbitrary σ-finite Lebesgue space. Then the weak topology on the space $\mathfrak{F}$ of all measure preserving transformations of X into itself is metrizable and separable.*

Proof. Since $L^2(X)$ is a separable Hilbert space, the weak topology on the closed unit sphere in the space $\mathfrak{X}$ of all bounded operators on $L^2(X)$ is compact and metrizable. Hence if it is proved that the mapping $T \to U(T)$ is one-to-one then the proof is complete. Here we mean that two measure preserving transformations T and T^* are equal provided $Tx = T^*x$ for almost all $x \in X$. It is known [6] that $(X, \mathfrak{M}, \mu)$ is a subspace of some Radon measure space $(\tilde{X}, \tilde{\mathfrak{M}}, \tilde{\mu})$, where $\tilde{X}$ is a locally compact, σ-compact and metrizable space, for which $\tilde{\mu}(\tilde{X} - X) = 0$. Hence we may consider without loss of generality that $T \in \mathfrak{F}$ is a measure preserving transformation of $\tilde{X}$ into itself. Theorem 5 implies now that the mapping $T \to U(T)$ is one-to-one. This completes the proof.

Theorem 6. *Let X be metrizable and let μ be a σ-finite locally finite tight measure on X. Then the weak topology on the space $\mathfrak{F}$ of all measure preserving transformations of X into itself is the finest topology $\mathscr{T}$ such that if a sequence $\{T_n\}$ in $\mathfrak{F}$ converges to $T \in \mathfrak{F}$ almost everywhere then $\mathscr{T}$-$\lim T_n = T$.*

Proof. Let $(Y, \mathfrak{M}_Y, \mu_Y)$ be a subspace of $(X, \mathfrak{M}, \mu)$ such that $(Y, \mathfrak{M}_Y, \mu_Y)$ is a σ-finite Lebesgue space and $\mu(X - Y) = 0$ (cf. [6, p. 167]). Then $\mathfrak{F}$ may be considered to be the space of all measure preserving transformations of Y into itself. It follows from Proposition 1 that the weak topology on $\mathfrak{F}$ is metrizable and separable. Let $\{T_n\}$ be a sequence in $\mathfrak{F}$ which converges to $T \in \mathfrak{F}$ almost everywhere. Then, by Theorem 4, T_n converges to T weakly.

Now let $\mathscr{P}$ be any topology on $\mathfrak{F}$ such that if a sequence $\{T_n\}$ in $\mathfrak{F}$ converges to $T \in \mathfrak{F}$ almost everywhere then $\mathscr{P}$-$\lim_{n \to \infty} T_n = T$. Suppose that $\mathfrak{E}$ is a closed set in the topology $\mathscr{P}$. In order to complete the proof, it is sufficient to show that $\mathfrak{E}$ is closed with respect to the weak topology. Let $T \in \mathfrak{F}$ be a cluster point of $\mathfrak{E}$ with respect to the weak topology. There exists a sequence $\{T_n\}$ in $\mathfrak{E}$ which converges to T weakly. Then Theorem 5 implies that the sequence $\{T_n\}$ has a subsequence $\{T_{k(n)}\}$ which converges to T almost everywhere. Thus $\mathscr{P}$-$\lim T_{k(n)} = T$. Hence $T \in \mathfrak{E}$. This completes the proof.

Proposition 2. *Let $(X, \mathfrak{M}, \mu)$ be an arbitrary Lebesgue space with $\mu(X) = 1$. Then the weak topology on the space $\mathfrak{F}$ is metrically topologically complete, and hence $\mathfrak{F}$ is of the second category in itself with respect to the weak topology.*

Proof. Let $\{f_n; n = 1, 2, \ldots\}$ be a dense set in $L^2(X)$ with respect to the L^2-norm topology. Define $\Omega_n = \{f \in L^2(X); \|f\| = \|f_n\|\}$ for each n. Clearly the space Ω_n is a complete metric space under a metric d_n which is defined by

$$d_n(f, g) = \frac{\|f - g\|}{\|f_n\|} \quad \text{for } f, g \in \Omega_n.$$

Since the weak topology and the L^2-norm topology are coincident on Ω_n, it follows that under the mapping φ: $T \to (U(T) f_n)_n$ of $\mathfrak{F}$ into the topological

product space $\prod_{n=1}^{\infty} \Omega_n$ $\mathfrak{F}$ and $\varphi(\mathfrak{F})$ are homeomorphic. Thus if we define a metric d on $\mathfrak{F}$ as follows then it is compatible with the weak topology on $\mathfrak{F}$:

$$d(T, T^*) = \sum_{n=1}^{\infty} 2^{-n}(d_n(U(T) f_n, U(T^*) f_n)) \quad \text{for } T, T^* \in \mathfrak{F}.$$

Let $\{T_i\}$ be a Cauchy sequence in $\mathfrak{F}$ with respect to the metric d. Then the sequence $\{U(T_i) f_n\}$ is a Cauchy sequence in the L^2-norm for each n. Since $\{f_n; n=1, 2, \ldots\}$ is dense in $L^2(X)$ with respect to the L^2-norm topology, we infer that for every $g \in L^2(X)$ the sequence $\{U(T_i) g\}$ is a Cauchy sequence in the L^2-norm. Let $U g$ be the function in $L^2(X)$ to which the sequence $\{U(T_i) g\}$ converges in the L^2-norm. Clearly $\|U g\| = \|g\|$. In order to complete the proof, it is sufficient to show that there exists $T \in \mathfrak{F}$ such that $U(T) = U$.

Let $f, g \in L^2(X)$ and let c be a complex number. It follows at once that $U(f+g) = Uf + Ug$ and $U(cf) = c\,U(f)$, whence U is a norm preserving operator on $L^2(X)$.

Let $f \in L^2(X)$ be bounded and let M be a positive number such that $|f(x)| < M$ for all $x \in X$. There exists a subsequence $\{U(T_{k(i)}) f\}$ of $\{U(T_i) f\}$ which converges to Uf almost everywhere. Hence $|Uf(x)| < M$ almost everywhere. Since $U(T_i)(fg) = (U(T_i) f)(U(T_i) g)$ for bounded functions f and g, it follows that $U(fg) = (Uf)(Ug)$ for bounded functions f and g. Thus if f is an indicator function, corresponding to $E \in \mathfrak{M}$, then Uf is an indicator function too, corresponding to $F \in \mathfrak{M}$, say. If we denote by $[U]\,E$ the set F then $[U]$ is a measure preserving mapping of the measure algebra $X(\mu)$ associated with $(X, \mathfrak{M}, \mu)$ into itself. It is now a routine matter to investigate that $[U]$ is a σ-endomorphism of $X(\mu)$.

Let $\Lambda = \{E_n; n=1, 2, \ldots\}$ be any basis of $(X, \mathfrak{M}, \mu)$. We consider the set of elements in $X(\mu)$ which correspond to the sets in Λ. The images of these elements in $X(\mu)$ under the mapping $[U]$ are classes of measurable sets in X. In each of these classes we select a certain set. In this way we obtain a certain countable set $\{M_n; n=1, 2, \ldots\} \subset \mathfrak{M}$. Then clearly $\Lambda_0 = \{E_n; n=1, 2, \ldots\} \cup \{M_n; n=1, 2, \ldots\}$ is a basis of $(X, \mathfrak{M}, \mu)$. Since $(X, \mathfrak{M}, \mu)$ is a Lebesgue space, we may assume without loss of generality that X is complete with respect to Λ (not Λ_0). Let $x \in X$. There correspond F_n and N_n for each n, where F_n stands for one of the two sets E_n, $X - E_n$, and N_n stands for one of the two sets M_n, $X - M_n$, such that $x \in F_n$ and $x \in N_n$. Then the set

$$\bigcap_{n=1}^{\infty} Q_n,$$

where $Q_n = E_n$ if $N_n = M_n$, and $Q_n = X - E_n$ if $N_n = X - M_n$, consists of exactly one element in X. Let $y \in X$ be the element of $\bigcap_{n=1}^{\infty} Q_n$. If we denote by Tx the element y, then T is a mapping of X into itself.

Let E be a measurable set of the form

$$E=\bigcap_{k=1}^{m} F_k,$$

where F_k stands for one of the two sets E_k and $X-E_k$ for each k. From the definition of the mapping T it follows that

$$T^{-1}(E)=T^{-1}\left(\bigcap_{k=1}^{m} F_k\right)=\bigcap_{k=1}^{m} T^{-1}(F_k)=\bigcap_{k=1}^{m} N_k,$$

where $N_k=M_k$ if $F_k=E_k$, and $N_k=X-M_k$ if $F_k=X-E_k$. Hence we observe that $T^{-1}(E)$ is measurable and $\mu(E)=\mu(T^{-1}(E))$. This together with the unique extension theorem of a finite measure on a field of subsets of a space implies that if B is a member of the σ-field $\mathfrak{B}$ which is generated by the sets E of the form $E=\bigcap_{k=1}^{m} F_k$ then $T^{-1}(B)$ is measurable and $\mu(B)=\mu(T^{-1}(B))$. Since $(X, \mathfrak{M}, \mu)$ is the completion of $(X, \mathfrak{B}, \mu)$, it follows that T is a measure preserving transformation of X into itself.

Let $[T]$ be the σ-endomorphism of $X(\mu)$ induced by T, and let $\mathscr{S}$ be the set of elements in $X(\mu)$ corresponding to the sets which are finite disjoint union of sets E of the form

$$E=\bigcap_{k=1}^{m} F_k,$$

where F_k stands for one of the two sets E_k and $X-E_k$ for each k. Then $\mathscr{S}$ is dense in $X(\mu)$. It is now easy to see that $[U]=[T]$ on $\mathscr{S}$. Therefore $[U]=[T]$ on the whole space $X(\mu)$. Consequently, the operator $U(T)$ on $L^2(X)$ induced by T is identical with U. This completes the proof of Proposition 2.

Theorem 7. *Let X be metrizable and let μ be a finite tight measure on X. Then the weak topology on the space $\mathfrak{F}$ is metrically topologically complete, and hence $\mathfrak{F}$ is of the second category in itself with respect to the weak topology.*

Proof. Since there exists a measurable set E such that $\mu(X-E)=0$ and the subspace $(E, \mathfrak{M}_E, \mu_E)$ is a Lebesgue space (cf. [6]), this follows immediately from Proposition 2.

Theorem 8. *Let X be metrizable and let μ be a finite tight measure on X. Suppose that $\{T_n\}$ is a sequence of measure preserving transformations of X into itself such that $\lim_{n\to\infty} T_n x$ exists for almost all $x\in X$. Then the mapping $T: x\to \lim_{n\to\infty} T_n x$ is a measure preserving transformation of X into itself, and hence T_n converges to T weakly.*

Proof. Without loss of generality, we may assume that $\lim_{n\to\infty} T_n x$ exists for every $x\in X$. Let $f\in L^2(X)$ be a bounded continuous function on X. Clearly $\lim_{n\to\infty} f(T_n x)=f(Tx)$ for all $x\in X$, and so Lebesgue's theorem on dominated

convergence implies that

$$f(Tx)\in L^2(X) \quad \text{and} \quad \|f(Tx)\| = \lim_{n\to\infty} \|f(T_n x)\| = \|f\|. \tag{8}$$

Let F be any closed set in X, and let $\{f_n\}$ be a monotone decreasing sequence of bounded continuous functions on X such that

$$\lim_{n\to\infty} f_n(x) = \chi_F(x)$$

for every $x\in X$. Then $\chi_{T^{-1}(F)}(x) = \chi_F(Tx) = \lim_{n\to\infty} f_n(Tx)$. It follows from (8) that $T^{-1}(F)$ is measurable and

$$\mu(T^{-1}(F)) = \lim_{n\to\infty} \|f_n(Tx)\|^2 = \lim_{n\to\infty} \|f_n\|^2 = \mu(F). \tag{9}$$

Now let E be any measurable set. It is known that for any positive number ε there exists an open set G and a closed set F such that $F\subset E\subset G$ and $\mu(G-F)<\varepsilon$ (for example, see [1, Theorem 1.1]). Thus there exist two monotone increasing sequences $\{F_n\}$ and $\{F_n^*\}$ of closed sets such that $F_n\subset E$, $F_n^*\subset X-E$, $\lim_{n\to\infty}\mu(F_n) = \mu(E)$ and $\lim_{n\to\infty}\mu(F_n^*) = \mu(X-E)$. It follows then from (9) that

$$\mu\left(\bigcup_{n=1}^{\infty} T^{-1}(F_n)\right) = \mu(E) \quad \text{and} \quad \mu\left(\bigcup_{n=1}^{\infty} T^{-1}(F_n^*)\right) = \mu(X-E).$$

Hence $T^{-1}(E)$ is measurable and $\mu(T^{-1}(E)) = \mu(E)$. This completes the proof of Theorem 8.

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