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The Ordered Set of Nörlund Methods

CASPER GOFFMAN and HAROLD V. HUNEKE

1. We consider the ordered set of Nörlund summability methods, ordered by inclusion. There is a Nörlund method strictly stronger than convergence but weaker than all the Cesàro methods (C, α), $\alpha > 0$ (e.g., Hardy [1]), and there is a Nörlund method stronger than all the Cesàro methods (e.g., Hardy [1] and Hill [2]).

Using the fact that the Nörlund methods form an ordered semigroup, relative to the appropriate operation, we show that for each $\alpha > 0$ there is a "continuum" of Nörlund methods stronger than (C, α) but weaker than (C, β) for all $\beta > \alpha$. We also show that for any sequence of Nörlund methods there is a Nörlund method which is not weaker than any method in the sequence.

2. Given a sequence $\{p_n\}$ of real numbers with $p_0 > 0$, $p_n \geqq 0$, $n = 1, 2, \ldots$, the corresponding Nörlund method is the matrix method

$$a_{mn} = \begin{cases} p_{m-n}/P_m & n \leqq m \\ 0 & n > m, \end{cases}$$

where $P_n = p_0 + p_1 + \cdots + p_n$, $n = 0, 1, \ldots$.

The facts we need about Nörlund methods are in [1], p. 64 – 70. In particular,

a) (N, p_n) is regular if and only if $\lim_{n \to \infty} p_n/P_n = 0$.

b) Regular Nörlund methods are consistent.

c) If (N, p_n) is regular, the series $p(x) = \sum_{n=0}^{\infty} p_n x_n$ and $P(x) = \sum_{n=0}^{\infty} P_n x_n$ converge for $|x| < 1$.

d) For regular (N, p_n) and (N, q_n), the series $k(x) = \sum_{n=0}^{\infty} k_n x^n = q(x)/p(x)$ and $l(x) = \sum_{n=0}^{\infty} l_n(x) = p(x)/q(x)$ have positive radii of convergence, and

$$q_n = k_0 p_n + \cdots + k_n p_0, \qquad Q_n = k_0 P_n + \cdots + k_n P_0,$$
$$p_n = l_0 q_n + \cdots + l_n q_0, \qquad P_n = l_0 Q_n + \cdots + l_n Q_0.$$

e) For regular (N, p_n) and (N, q_n), (N, q_n) is stronger than (N, p_n) if and only if $\lim_n k_n/Q_n = 0$ and there is an H with $\frac{1}{Q_n}\{|k_0| P_n + \cdots + |k_n| P_0\} < H$ for all n.

f) Regular (N, p_n) and (N, q_n) are equivalent if and only if $\sum_n |k_n|$ and $\sum_n |l_n|$ both converge.

g) If (N, p_n) and (N, q_n) are regular and

(i) $p_{n+1}/p_n \geqq p_n/p_{n-1}$, $n=1, 2, \ldots$; $p_0=1$, $p_n>0$, for all n,

(ii) $q_n>0$ for all n,

(iii) there is an r such that $n>r$ implies $q_{n+1}/q_n \geqq p_{n+1}/p_n$ then (N, q_n) is stronger than (N, p_n).

3. The symmetric product $(N, g_n)=(N, p_n)\circ(N, q_n)$ of Nörlund methods is defined by

$$g_n=p_0\, q_n+\cdots+p_n\, q_0.$$

If (N, p_n) and (N, q_n) are regular, (N, g_n) is easily seen to be stronger than both of them. We prove the following more general fact.

Proposition 1. *If (N, q_n) is stronger than (N, p_n) and (N, r_n) is regular, then $(N, v_n)=(N, q_n)\circ(N, r_n)$ is stronger than $(N, u_n)=(N, p_n)\circ(N, r_n)$.*

Proof. There is an H such that

$$\frac{1}{Q_n}\{|k_0|\, P_n+\cdots+|k_n|\, P_0\}<H, \qquad \text{for all } n,$$

and $\lim_n k_n/Q_n=0$. There are $\{K_n\}$ and $\{L_n\}$ for $K(x)=v(x)/u(x)$ and $L(x)=u(x)/v(x)$. But $v(x)=q(x)\, r(x)$ and $u(x)=p(x)\, r(x)$, so that $K(x)=k(x)$. Similarly, $L(x)=l(x)$. Now

$$\begin{aligned}\frac{1}{V_n}\{|K_0|\, U_n+\cdots+|K_n|\, U_0\}&=\frac{1}{V_n}\{|k_0|\, U_n+\cdots+|k_n|\, U_0\}\\&=\frac{1}{V_n}\{|k_0|(r_0\, P_n+\cdots+r_n\, P_0)+\cdots+|k_n|\, r_0\, P_0\}\\&=\frac{1}{V_n}\{r_0(|k_0|\, P_n+\cdots+|k_n|\, P_0)+\cdots+r_n\, k_0\, P_0\}\\&\leqq\frac{H}{V_n}(r_0\, Q_n+\cdots+r_n\, Q_0)=H,\end{aligned}$$

for all n.

We are more interested in the following version.

Proposition 2. *If (N, q_n) is strictly stronger than (N, p_n) then (N, v_n) is strictly stronger than (N, u_n).*

Proof. Either $\sum_n |k_n|=\infty$ or $\sum_n |l_n|=\infty$ so that either $\sum_n |K_n|=\infty$ or $\sum_n |L_n|=\infty$ and the result follows.

A semigroup with order relation $<$ is an ordered semigroup if (α) $a<b$ and $b<c$ implies $a<c$, and (β) $a<b$ implies $a\, c<b\, c$ for each c. We may make the following statement.

Theorem 1. *With strictly weaker than as order relation and symmetric product as operation, the regular Nörlund methods form an abelian ordered semigroup.*

4. The most important Nörlund methods are the Cesàro methods (C, α), $\alpha > 0$. The (C, α) method is the Nörlund method for which $p(x) = (1-x)^{-\alpha}$. Then $(C, \alpha) \circ (C, \beta) = (C, \alpha+\beta)$ so that $0 < \alpha < \beta$ implies (C, β) is strictly stronger than (C, α). We show that for each $\alpha \geqq 0$ there is a continuum of Nörlund methods stronger than (C, α) but weaker than (C, β) for all $\beta > \alpha$. We need the following fact.

Proposition 3. *If (N, p_n) and (N, q_n) are regular and (N, q_n) is stronger than (N, p_n), and if $\lim_n Q_n/P_n = \infty$, then (N, q_n) is strictly stronger than (N, p_n).*

Proof. Now, $Q_n = k_0 P_n + \cdots + k_n P_0$, so that $Q_n \leqq |k_0| P_n + \cdots + |k_n| P_0$ and so $Q_n/P_n \leqq |k_0| + \cdots + |k_n|$. It follows that $\sum_n |k_n| = \infty$ so that (N, p_n) and (N, q_n) are not equivalent.

Consider the Nörlund methods

$$p_n^\delta = \frac{2}{n+2}\left(\frac{\log 2}{\log(n+2)}\right)^\delta, \qquad \text{with } 0 \leqq \delta \leqq 1.$$

Since,

$$\lim_n p_n^\delta = \lim_n \frac{2}{2n+2}\left(\frac{\log 2}{\log(n+2)}\right) = 0,$$

the methods are regular.

Let $\gamma < \delta$. Then, by an immediate check, $p_0^\delta = 1$ and $p_n^\delta / p_{n-1}^\delta \leqq p_{n+1}^\delta / p_n^\delta$, $p_n^\delta > 0$, $n = 1, 2, \ldots$ and $p_{n+1}^\gamma / p_n^\gamma \geqq p_{n+1}^\delta / p_n^\delta$, $n = 1, 2, \ldots$. So, by g), (N, p_n^γ) is stronger than (N, p_n^δ).

We show it is strictly stronger. Since p_n^δ is monotonically decreasing in n,

$$\int_1^n \frac{2}{t+2}\left(\frac{\log 2}{\log(t+2)}\right)^\delta dt < P_n^\delta < \int_0^n \frac{2}{t+2}\left(\frac{\log 2}{\log(t+2)}\right) dt$$

and by computing the integrals, we have

$$\lim_n \frac{(1-\delta) P_n^\delta}{2(\log 2)^\delta [\log(n+2)]^{1-\delta}} = 1.$$

Similarly,

$$\lim_n \frac{(1-\gamma) P_n^\gamma}{2(\log 2)^\gamma [\log(n+2)]^{1-\gamma}} = 1.$$

Then

$$\lim_n \frac{P_n^\gamma}{P_n^\delta} = \lim_{n\to\infty} \frac{1-\delta}{1-\gamma} (\log 2)^{\gamma-\delta} [\log(n+2)]^{\delta-\gamma} = \infty.$$

The following now follows from Proposition 3.

Proposition 4. *The methods (N, p_n^δ) are regular and if $\gamma < \delta$, then (N, p_n^γ) is strictly stronger than (N, p_n^δ).*

Proposition 5. *For $\alpha>0$ and $0\leqq\delta\leqq 1$, (C,α) is strictly stronger than (N,p_n^{δ}).*

Proof. It suffices to show that (C,α) is stronger than (N,p_n^0) and this is done in [1], p. 110.

The next statement follows from the ordered semigroup property.

Theorem 2. *For each $k>0$, the set of regular Nörlund methods*

$$(N,p_n^{\delta})\circ(C,k),\qquad 0\leqq\delta\leqq 1,$$

is stronger than (C,k) but weaker than $(C,k+\alpha)$ for each $\alpha>0$.

5. It was shown by Hill, [2], and by Hardy, [1], that there is a Nörlund method stronger than all the Cesàro methods. This is the method (N,p_n), $p_n=e^{\sqrt{n}}$. This suggests the question as to whether or not there is a Nörlund method stronger than all the methods in any given sequence of Nörlund methods. A more restricted question is whether for any Nörlund method (N,p_n) there is a single Nörlund method stronger than all the methods in the sequence

$$(N,p_n),\ (N,p_n)\circ(N,p_n),\ (N,p_n)\circ(N,p_n)\circ(N,p_n),\ \ldots .$$

We are not able to answer these questions. However, we show that for any sequence of regular Nörlund methods there is a regular Nörlund method (N,p_n) such that no method in the sequence is stronger than (N,p_n). Accordingly, no countable set of regular Nörlund methods encompasses all regular Nörlund methods.

We simplify the problem. Let $\{(N,p_{mn})\}$, $m=0,1,\ldots$, be a sequence of regular Nörlund methods. The methods

$$(N,q_{mn})=(N,p_{1n})\circ(N,p_{2n})\circ\cdots\circ(N,p_{mn})$$

are regular, increase in strength, and (N,q_{mn}) is stronger than (N,p_{mn}), for each m. Moreover, $q_{m+1,n}\geqq q_{mn}$. Let

$$(N,r_{mn})=(N,q_{mn})\circ(C,1)\circ(C,1),\qquad m=0,1,2,\ldots .$$

The sequence $\{(N,r_{mn})\}$ has the properties

(a) $r_{in}\geqq r_{jn}$ for $i>j$ and any n,

(b) $\lim_n r_{m,n+1}/r_{mn}=1$ for any m,

(c) $r_{m,n+1}>r_{mn}$ for all m and n,

(d) $\lim_n r_{mn}=\infty$ for each m.

The Cesàro methods for integral values of α have properties (a)–(d).

The following lemmas are needed.

Lemma 1. *If (N,p_n) and (N,q_n) are regular and $\lim_n Q_n/P_n=\infty$, then (N,q_n) is either stronger than or not comparable with (N,p_n).*

Proof. If (N, p_n) is stronger than (N, q_n), by (e) there is an H such that

$$\frac{1}{P_n}\{|l_0|\,Q_n+\cdots+|l_n|\,Q_0\}<H \qquad \text{for all } n.$$

Then, $Q_n/P_n<H$.

Lemma 2. *If* $\liminf_n Q_n/P_n>0$ *then* (N, Q_n) *is either stronger than or not comparable with* (N, p_n).

Proof. Since (N, Q_n) is regular

$$\lim_n \frac{Q_n}{\sum_{m=0}^{n} Q_m}=0.$$

Then

$$\lim_n \frac{\sum_{m=0}^{n} Q_m}{P_n}=\infty$$

and the result follows by Lemma 1.

Lemma 3. *If* $\{p_n\}$ *is strictly increasing and* $\lim_n p_{n+1}/p_n=1$ *then, for each* m *and constant* $A>1$, *there is an* N *such that* $n>N$ *implies*

$$\sum_{k=n+m+1}^{n+2m} p_k<A\sum_{k=n+1}^{n+m} p_k.$$

Proof. Let $r=\sqrt[m]{A}$ and N such that $n>N$ implies $p_{n+1}/p_n<r$. Then

$$\sum_{k=n+m+1}^{n+2m} p_k<r\,p_{m+n}\left[\frac{r^m-1}{r-1}\right]$$

and

$$\sum_{k=n+1}^{n+m} p_k>\frac{p_{m+n}}{r^{m-1}}\left[\frac{r^m-1}{r-1}\right].$$

Then,

$$\sum_{k=m+n+1}^{m+2n} p_k<r^m\sum_{k=n+1}^{m+n} p_k=A\sum_{k=n+1}^{m+n} p_k.$$

Theorem 3. *If* $\{(N, p_{mn})\}$ *is a sequence of regular Nörlund methods such that*

(a) $p_{rn}\geqq p_{sn}$, *for* $r>s$ *and any* n,

(b) $\lim_n p_{m,n+1}/p_{mn}=1$ *for each* m,

(c) $p_{m,n+1}>p_{mn}$ *for all* m *and* n,

(d) $\lim_n p_{mn}=\infty$ *for each* m,

there is a regular Nörlund method (N, Q_n) *which is stronger than or not comparable with each method of the sequence.*

Proof. Consider first the methods (N, p_{1n}) and (N, p_{2n}). Since $\lim_n p_{m,n+1}/p_{mn} = 1$, for each m, there is an n_1 such that $n > n_1$ implies $p_{1,n+1}/p_{1n} < 1 + \frac{1}{2} = r_1$ and $p_{2,n+1}/p_{2n} < 1 + \frac{1}{2} = r_1$. Since $p_{1n_1} < p_{2n_2}$, $p_{1n} < p_{1,n+1}$, for all n, and $\lim_n p_{1n} = \infty$ there is an $m_1 > n_1$ such that $p_{1,m_1-1} \leqq p_{2n_1} \leqq p_{1m_1}$. Then

$$\frac{p_{2n_1}}{p_{1,m_1-1}} \leqq \frac{p_{1m_1}}{p_{1,m_1-1}} < r_1.$$

Let $q_n = p_{1n}$ for $n = 0, 1, \ldots, m_1 - 1$ and $q_{m_1} = p_{2n_1}$.

Let $A > 1$, $a_1 = m_1 - n_1$ and $r_2 = \min\left(1 + \frac{1}{2^2}, \sqrt[a_1]{A}\right)$. Consider (N, p_{2n}) and (N, p_{3n}). There is an n_2 such that

(a) for $n > n_2$, $p_{2,n+1}/p_{2n} < r_2$,

(b) for $n > n_2$, $p_{3,n+1}/p_{3n} < r_2$,

(c) $\sum_{i=n_1}^{n_2} p_{2i} > \frac{1}{A} \sum_{i=n_2+1}^{n_2+a_1} p_{2i} > \frac{1}{A} \sum_{i=n_2+1}^{n_2+a_2} p_{1i}$.

Condition (c) follows by Lemma 3 with $n_2 - a_1$ greater than the N of the lemma.

Since $p_{3n_2} > p_{2n_2}$, $p_{2n} < p_{2,n+1}$, for all n, and $\lim_n p_{3n} = \infty$, there is $m_2 > n_2$ such that $p_{2,n_1+m_2-1} \leqq p_{3,n_2} \leqq p_{2,n_1+m_2}$. Then,

$$\frac{p_{3,n_2}}{p_{2,n_1+m_2-1}} \leqq \frac{p_{2,n_1+m_2}}{p_{2,n_1+m_2-1}} < r_2.$$

Let $q_{m_1+n} = p_{2,n_1+n}$ for $n = 0, 1, 2, \ldots, m_2 - 1$.

Suppose now that for any k we have for each $i \leqq k$,

$$a_i = m_1 - n_1 + \sum_{j=2}^{i-1} [m_j - (n_j - n_{j-1})], \qquad r_i = \min\left(1 - \frac{1}{2^i}, A^{\frac{1}{a_{i-1}}}\right),$$

$$\sum_{j=n_{i-1}}^{n_i} p_{ij} > \frac{1}{A} \sum_{j=n_i+1}^{n_i+a_{i-1}} p_{ij} \geqq \frac{1}{A} \sum_{j=n_i+1}^{n_i+a_{i-1}} p_{sj} \qquad \text{for all } s < i,$$

and m_i and q_i such that

$$q_{m_1+\cdots+m_{i-1}+n} = p_{i,n_{i-1}+n} \qquad \text{for } n = 0, 1, \ldots, m_i - 1$$

and

$$\frac{q_{n+1}}{q_n} < r_i \qquad \text{for } m_1 + \cdots + m_{i-1} < n < m_1 + \cdots + m_i.$$

Set $a_{k+1} = m_1 - n_1 + \sum_{i=2}^{k} [m_i - (n_i - n_{i-1})]$ and

$$r_{k+1} = \min\left(1 + \frac{1}{2^{k+1}}, \sqrt[a_k]{A}\right).$$

Relative to the methods $(N, p_{k+1,n})$ and $(N, p_{k+2,n})$ there is an n_{k+1} such that

(a) for $n>n_{k+1}$, $\dfrac{p_{k+1,n+1}}{p_{k+1,n}}<r_{k+1}$,

(b) for $n>n_{k+1}$, $\dfrac{p_{k+2,n+1}}{p_{k+1,n}}<r_{k+1}$,

(c) $\displaystyle\sum_{i=n_k}^{n_{k+1}} p_{k+1,i}>\frac{1}{A}\sum_{n_{k+1}+1}^{n_{k+1}+a_k} p_{k+1,i}\geqq\sum_{i=n_{k+1}+1}^{n_{k+1}+a_k} p_{si}$ for each $s<k+1$.

Since $p_{k+2,n_{k+1}}\geqq p_{k+1,n_{k+1}}$, $\lim_{n\to\infty} p_{k+1,n}=\infty$, and $\{p_{k+1,n}\}$ is increasing, there is an m_{k+1} such that

$$p_{k+1,n_k+m_{k+1}-1}\leqq p_{k+2,n_{k+1}}\leqq p_{k+1,n_k+m_{k+1}}.$$

Then

$$\frac{p_{k+2,n_{k+1}}}{p_{k+1,n_k+m_{k+1}-1}}\leqq\frac{p_{k+1,n_k+m_{k+1}}}{p_{k+1,n_k+m_{k+1}-1}}<r_{k+1}.$$

Let

$$q_{m_1+\cdots+m_k+n}=p_{k+1,n_k+n},\qquad n=0,1,\ldots,m_{k+1}-1.$$

We then obtain:

(α) a sequence of integers $\{a_k\}$ with

$$a_k=(m_1-n_1)+\sum_{i=2}^{k}[m_i-(n_i-n_{i-1})],$$

(β) a sequence $\{r_k\}$ with $\lim_{k\to\infty} r_k=1$,

(γ) a sequence $\{n_k\}$ such that

$$\sum_{i=n_{k-1}}^{n_k} p_{ki}>\frac{1}{A}\sum_{i=n_k+1}^{n_k+a_k-1} p_{ki}\geqq\frac{1}{A}\sum_{i=n_k+1}^{n_k+a_k-1} p_{si},\qquad\text{for all } s<k,$$

(δ) sequences $\{m_k\}$ and $\{q_k\}$ such that

$$\frac{q_{n+1}}{q_n}<r_k\qquad\text{for } m_1+\cdots+m_{k-1}<n<m_1+\cdots+m_k.$$

The method (N, q_n) is regular since $\lim_{n\to\infty}\dfrac{q_{n+1}}{q_n}=1$. We consider the relation between (N, q_n) and each (N, p_{mn}). Now,

$$\frac{P_{rs}}{Q_s}=\frac{\displaystyle\sum_{i=0}^{n_r} p_{ri}+\sum_{i=n_r+1}^{s-a_{r-1}-1} p_{ri}+\sum_{i=s-a_{r-1}}^{s} p_{ri}}{Q_s}.$$

Since $\lim_n Q_n=\infty$, we have

$$\lim_s\frac{\displaystyle\sum_{i=0}^{n_r} p_{ri}}{Q_s}=0.$$

For each j such that $n_r<j<s-a_{r-1}$, there is, for each p_{rj}, a term from $\{q_n\}$ which is greater than p_{rj}. These terms are different for different j. Hence, $\sum_{i=n_r+1}^{s-a_{r-1}} p_{ri}<Q_s$. Finally, $\{q_n\}$ is so formed that $Q_s>\frac{1}{A}\sum_{i=s-a_{r-1}}^{s} p_{ri}$.

It follows that $\limsup_n P_{mn}/Q_n<1+A$ and, by Lemma 2, (N, Q_n) is stronger than or not comparable with (N, p_{mn}), for each m.

6. We now state some unsolved questions which may be of interest.

(i) To find general properties of the ordered set of Nörlund methods, e.g., is there always a method between any two given methods?

(ii) There are maximal chains in the ordered set of Nörlund methods. What can be said about them?

(iii) For any Cesàro method (C, α) there is a continuum of Nörlund methods (probably a much larger ordered set) between (C, α) and all (C, β), $\beta>\alpha$. It is not known if there are any Nörlund methods between (C, α) and all (C, β), $\beta>\alpha$.

(iv) Given a sequence of Nörlund methods, is there always a stronger Nörlund method? Perhaps the following easily proved criterion could be used. If (N, p_n) and (N, q_n) are regular, then (N, q_n) is stronger than (N, p_n) if $\lim_n P_n=\infty$, $\lim_n P_n/Q_n=0$ and all the k_n, $n>n_0$, are non-negative.

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Ends and Free Products of Groups

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Stallings, in his recent work on groups of cohomological dimension one [2], uses Freudenthal's theory of ends of a group [1]. One purpose of this paper is to give a new account of the theory of ends, which is almost entirely algebraic, in contrast to the previous combinatorial and topological approaches. The proofs appear to be slightly simpler that the previous ones.

In § 2, we show how one of Stallings' theorems can be improved by removing a condition. We prove a lemma which, combined with his methods, gives the following theorem. *A finitely generated torsion-free group with infinitely many ends is a non-trivial free product.* (This result has also been proved by Bergman [3] and Stallings and Dunwoody [4]).

I am grateful to Professor Stallings for sending me a preprint of his paper.

1. Ends

Throughout the paper G will denote an infinite finitely generated group.

We say that a property holds *for almost all elements* of a subset B of G if it holds for all but a finite number of elements of B. In particular B is almost contained in C, written $B \overset{a}{\subseteq} C$, if almost all elements of B are in C, and B almost equals C, written, $B \overset{a}{=} C$, if $B \overset{a}{\subseteq} C$ and $C \overset{a}{\subseteq} B$. We say E is almost (right) invariant if $Eg \overset{a}{=} E$ for all g in G. The letter E will always denote an almost invariant set.

Let $\overline{Z_2 G}$ denote $\mathrm{Hom}_Z(ZG, Z_2)$. Then $\overline{Z_2 G}$ may be identified with the set of all subsets of G (where symmetric difference is the operation of addition). $\overline{Z_2 G}$ contains $Z_2 G$ as a submodule, and $Z_2 G$ is identified with the set of all finite subsets of G. Let $\mathscr{E} G = \overline{Z_2 G}/Z_2 G$. Then a subset B of G is almost invariant if and only if its image in $\mathscr{E} G$ is invariant.

Since $\overline{Z_2 G}$ is co-induced, $H^1(G; \overline{Z_2 G}) = 0$. Also $H^0(G; \overline{Z_2 G}) = Z_2$ and $H^0(G; Z_2 G) = 0$, as we see by observing which subsets of G are invariant. The exact sequence $0 \to Z_2 G \to \overline{Z_2 G} \to \mathscr{E} G \to 0$ gives rise to the exact sequence

$$0 \to H^0(G; Z_2 G) \to H^0(G; \overline{Z_2 G}) \to H^0(G; \mathscr{E} G) \to H^1(G; Z_2 G) \to H^1(G; \overline{Z_2 G}).$$

All terms are vector spaces over Z_2, and we see that $1 + \dim H^1(G; Z_2 G) = \dim H^0(G; \mathscr{E} G)$. We call $\dim H^0(G; \mathscr{E} G)$ *the number of ends of* G. In particular G has at least two ends if and only if there exists an almost invariant set E such that E and its complement are both infinite, and G has exactly two ends if, in addition, any almost invariant set E_1 with E_1 and its complement both infinite is almost equal to E or its complement.

Let $x_1, \ldots, x_n$ be a set of generators of G. The *graph* of G, with respect to these generators, is the complex with vertices the element of G and with an edge (labelled x_i) from g to gx_i for any $g \in G$ and any i. G acts (freely) on this complex by left multiplication. Let $B \subseteq G$, and let B^* be its complement. The *coboundary* of B, written δB, is the set of edges with one vertex in B and the other in B^*. There will be a finite number of edges labelled x_i in δB if and only if $Bx_i \stackrel{a}{=} B$, and so δB is finite if and only if $Bx_i \stackrel{a}{=} B$ for all i, which is plainly equivalent to $Bg \stackrel{a}{=} B$ for all g.

(1.1) *The infinite cyclic group has two ends.*

Proof. Let c generate an infinite cyclic group G. Then G has at least two ends, since $\{c^n; n>0\}$ is almost invariant. For any E and almost all n, if $c^n \in E$ then $c^{n+1} \in E$ and $c^{n-1} \in E$. Hence if $c^n \in E$ for infinitely many positive n then $c^n \in E$ for almost all positive n, and similarly for negative n. It follows that either E or E^*, its complement, must be finite or almost equal to $\{c^n; n>0\}$, as required.

(1.2) **Proposition.** *Let H be a subgroup of finite index in G. Then G and H have the same number of ends.*

(1.3) **Corollary.** *If G has an infinite cyclic subgroup of finite index, then G has two ends.*

Proof. Let $b_1 = e, b_2, \ldots, b_r$ be representatives of the right consets of H. We must prove that the numbers of G-invariant elements of $\mathscr{E}G$ equals the number of H-invariant elements of $\mathscr{E}H$. Let E be an almost G-invariant subset of G. Then $E \cap H$ is an almost H-invariant subset of H. Since

$$E = \bigcup_{i=1}^{r} (E \cap Hb_i) \quad \text{and} \quad Eb_i \stackrel{a}{=} E,$$

we see that

$$E \stackrel{a}{=} \bigcup_{i=1}^{r} (Eb_i \cap Hb_i) = \bigcup_{i=1}^{r} (E \cap H)b_i.$$

Hence if $E_1 \cap H \stackrel{a}{=} E_2 \cap H$ then $E_1 \stackrel{a}{=} E_2$. Finally we must show that if C is an almost H-invariant subset of H there is an almost G-invariant subset E of G with $E \cap H \stackrel{a}{=} C$. For then we will have shown that the mapping $E \to E \cap H$ induces a one-one mapping of $H^0(G; \mathscr{E}G)$ onto $H^0(H; \mathscr{E}H)$. We simply define E to be $\bigcup_{i=1}^{r} Cb_i$. For any g the cosets Hb_ig are just the cosets Hb_i in some order, so we can find $h_i \in H$ such that $b_1 g, \ldots, b_r g$ is a permutation of $h_1 b_1, \ldots, h_r b_r$. Hence

$$Eg = \bigcup_{i=1}^{r} Cb_i g = \bigcup_{i=1}^{r} Ch_i b_i \stackrel{a}{=} \bigcup_{i=1}^{r} Cb_i,$$

since $Ch_i \stackrel{a}{=} C$. (We should notice also that by Schreier's theorem H is finitely generated.)

(1.4) **Lemma.** *Given E_0 and E_1, for almost all $g \in E_0$ either $g E_1 \subseteq E_0$ or $g E_1 \supseteq E_0^*$.*

Proof. Let $B = \{h \in E_1$; either $h x_i$ or $h x_i^{-1}$ in E_1^* for some $i\}$ and $C = \{h \in E_0^*$; either $h x_i$ or $h x_i^{-1}$ in E_0 for some $i\}$, where $x_1, \ldots, x_n$ generate G. Then B and C are finite, being subsets of the vertices of δE_1 and δE_0. (It is at this point that we require G to be finitely generated. The definitions and Proposition (1.2) make sense for infinitely generated groups.) Suppose $g \in E_0$ and $g E_1 \cap E_0^* \neq \emptyset \neq g E_1^* \cap E_0^*$. Then we can find a sequence $g_1, \ldots, g_r$ with $g_1 \in g E_1 \cap E_0^*$, $g_r \in g E_1^* \cap E_0^*$ and such that $g_s^{-1} g_{s+1}$ is of form x_i or x_i^{-1} (for some i) for $s = 1, \ldots, r-1$ (we need only take arbitrary elements in the two sets, write them as products of the x_i and x_i^{-1}, and take all their initial segments). Then either (i) for some m, $g_m \in g E_1 \cap E_0^*$ and $g_{m+1} \in g E_1^* \cap E_0^*$ or (ii) for some j, k, $g_j \in g E_1 \cap E_0^*$, $g_k \in g E_1^* \cap E_0^*$ and $g_{j+1} \in E_0$, $g_{k-1} \in E_0$ (for if case (i) does not hold take j as small as possible subject to $g_{j+1} \notin g E_1 \cap E_0^*$ and similarly for k).

In case (i) we have $g_m \in g B \cap E_0^*$. Hence for some $b \in B$ we have $g \in E_0 \cap E_0^* b^{-1} \stackrel{a}{=} E_0 \cap E_0^* = \emptyset$. As B is finite, this means that g is in one of a finite number of finite sets.

In case (ii) we have $g_j \in g E_1 \cap C$ and $g_k \in g E_1^* \cap C$. Hence for some $c, d \in C$ we have $g^{-1} \in E_1 c^{-1} \cap E_1^* d^{-1} \stackrel{a}{=} E_1 \cap E_1^* = \emptyset$. As C is finite we again find that g is in one of a finite number of finite sets, i.e. there are only finitely many $g \in E_0$ with $g E_1 \cap E_0^* \neq \emptyset \neq g E_1^* \cap E_0^*$ as required.

(1.5) **Lemma.** *Let G have at least two ends, and let E and E^* be infinite. If $\{g; g E \stackrel{a}{=} E\}$ is infinite, then G has an infinite cyclic subgroup of finite index, and consequently has exactly two ends.*

Proof. We may assume, replacing E by E^* if necessary, that $\{g \in E; g E \stackrel{a}{=} E\}$ is infinite. Take any $b \in E$. Applying Lemma (1.4) with $E_1 = E$, $E_0 = E - \{b\}$, we can find c with $cE \stackrel{a}{=} E$ and either $cE \subseteq E - \{b\}$ or $cE \supseteq E^* \cup \{b\}$. The latter contradicts $cE \stackrel{a}{=} E$, as E^* is infinite. In particular we have $cE \subset E$ which gives $c^n E \subset E$ for all positive n and hence $c^n \neq e$ for $n > 0$. As we have $c^n b \in E$ for $n > 0$, we have $c^n \in E$ for almost all positive n. Also $cE \subset E$ gives $c^{-1} E^* \subset E^*$, from which we deduce similarly that $c^{-n} \in E^*$ for almost all positive n.

We must have $\bigcap_{n \geqq 0} c^n E = \emptyset$. For if $d \in \bigcap_{n \geqq 0} c^n E$, then $c^{-n} \in E d^{-1}$ $(\stackrel{a}{=} E)$ for all $n \geqq 0$, which is impossible. Then for any $g \in E$, we can find $n \geqq 0$ with $g \in c^n E$, $g \notin c^{n+1} E$, so $E = \bigcup_{n \geqq 0} c^n (E - cE)$ and similarly $E^* = \bigcup_{n \geqq 0} c^{-n} (E^* - c^{-1} E^*)$. As $cE \stackrel{a}{=} E$, the sets $E - cE$ and $E^* - c^{-1} E^*$ are finite, so that the infinite cyclic group generated by c has finite index.

(1.6) **Theorem.** *A group must have one, two, or infinitely many ends. It has two ends if and only if it has an infinite cyclic subgroup of finite index.*

Proof. For suppose G has a finite number of ends. If it does not have one end, we can find $E_1, \ldots, E_r$ infinite with infinite complements such that any infinite E with infinite complement almost equals E_i for some i. In particular,

given $g \in G$, $gE \overset{a}{=} E_i$ for some i. Now $\{g; gE_1 \overset{a}{=} E_1\}$ is plainly a subgroup of G and $\{g; gE_1 \overset{a}{=} E_i\}$ is either empty or a coset of this subgroup. Hence this subgroup has finite index, and so must be infinite which gives the theorem by Lemma (1.5).

(1.7) **Proposition.** *A torsion group has one end.*

Proof. Suppose G has more than one end, and let E, E^* be infinite. Let $E^{-1} = \{g^{-1}; g \in E\}$. If $E^{-1} \overset{a}{\subseteq} E$, taking inverses we see $E \overset{a}{\subseteq} E^{-1}$ and so $E \overset{a}{=} E^{-1}$. As E is almost right invariant, E^{-1} will be almost left-invariant, whence $gE \overset{a}{=} gE^{-1} \overset{a}{=} E^{-1} \overset{a}{=} E$ for all g. Then by Lemma (1.5), G has two ends, and is not torsion.

Hence we may assume $\{g \in E; g^{-1} \in E^*\}$ is infinite. We may also assume e, the identity, is in E. By Lemma (1.4) we can find $c \in E$ with $c^{-1} \in E^*$ and either $cE \subseteq E - \{e\}$ or $cE \supseteq E^* \cup \{e\}$. The latter contradicts $c^{-1} \in E^*$ while the former gives c of infinite order as in (1.6).

(1.8) **Proposition.** *The free product of two finitely generated groups has infinitely many ends unless both groups are of order two, when it has two ends.*

Proof. We know that any free product is infinite. The free product of two groups of order two has an infinite cyclic subgroup of index two, so has two ends.

Let G be the free product of G_1 and G_2 and let $b, c \in G_1 - \{e\}$ be distinct. Let E_b denote the set of elements in G whose normal form begins with b, and similarly for E_c. Take any $g \in G$. Then if $h = bu \in E_b$, we have $hg \in E_b$ unless u^{-1} is an initial segment of g. Hence $E_b g \overset{a}{\subseteq} E_b$, and so E_b and E_c are almost invariant. As they are both infinite and $E_c \subseteq E_b^*$ but $E_c \overset{a}{\neq} E_b^*$, G must have infinitely many ends.

(1.9) **Proposition.** *Let G possess a finitely generated normal subgroup H such that H and G/H are infinite. Then G has one end.*

(1.10) **Corollary.** *The direct product of two finitely generated infinite groups has one end.*

Proof. It is most convenient to prove this result combinatorially. We shall consider the graph of G with respect to a finite set of generators containing a set of generators of H.

If G has at least two ends there is a finite set F such that any connected subset of F^* has infinite complement. We need only take E with E, E^* infinite and let F be the set of vertices of δE. Then any connected subset of F^* lies entirely in E or E^*. We shall show that under the hypotheses of (1.9) any finite set F is contained in a finite set F_1 such that F_1^* is connected.

Let S be a Schreier system of coset representatives for H. Then S is connected. For if $g = x_{i_1}^{\varphi_1} \ldots x_{i_r}^{\varphi_r} \in S$, where $\varphi_i = \pm 1$, then there is a path from e to g whose vertices are the elements $x_{i_1}^{\varphi_1} \ldots x_{i_k}^{\varphi_k} \in S$ for $k = 1, \ldots, r$. Similarly H is connected, since H is generated by a subset of the generators of G. Then, also, for any $h \in H$, hS is connected and, for any $s \in S$, $sH = Hs$ is connected.

Let F be finite. Then we can find $s_1, \ldots, s_n \in S$ and $h_1, \ldots, h_m \in H$ such that any $g \in F$ is $h_i s_j$ for some i, j. Let $F_1 = \{h_i s_j;$ all $i, j\}$. Then F_1^* is the union of the sets Hs for $s \neq s_1, \ldots, s_n$ and the sets hS for $h \neq h_1, \ldots, h_m$. As hS and Hs are connected and $hs \in hS \cap Hs$ we see that $hS \cup Hs$ is connected. As any two sets $hS \cup Hs$ and $h'S \cup Hs'$ intersect, F_1^*, which is the union of such sets, will be connected.

(1.11) **Proposition.** *If H is a finite normal subgroup of G, then G and G/H have the same number of ends.*

Let $\pi: G \to G/H$ be the natural map. Plainly if E and E_1 are almost invariant subsets of G, then πE and πE_1 are almost invariant subsets of G/H, and if $E \overset{a}{=} E_1$ then $\pi E \overset{a}{=} \pi E_1$.

If B and C are any subsets of G/H then $\pi^{-1} B - \pi^{-1} C = \pi^{-1}(B - C)$ has $|H|$ times the number of elements of $B - C$. In particular, if $B \overset{a}{=} C$, then $\pi^{-1} B \overset{a}{=} \pi^{-1} C$ and if B is an almost invariant subset of G/H then $\pi^{-1} B$ is an almost invariant subset of G.

Let E and E_1 be almost invariant subsets of G. Then $\pi^{-1} \pi E = \bigcup_{h \in H} Eh \overset{a}{=} E$, since H is finite and $Eh \overset{a}{=} E$ for all h. So if

$$\pi E \overset{a}{=} \pi E_1 \quad \text{then} \quad E \overset{a}{=} \pi^{-1} \pi E \overset{a}{=} \pi^{-1} \pi E_1 \overset{a}{=} E_1.$$

Hence π induces a one-one map from the G-invariant elements of $\mathscr{E} G$ onto the G/H-invariant elements of $\mathscr{E}(G/H)$, as required.

(1.12) **Proposition.** *Let G be torsion-free, and let H be a subgroup of G. G has a non-trivial almost invariant H-invariant subset if and only if H is contained in a proper free factor of G.*

Proof. Suppose $G = G_1^* G_2$. The set of elements whose normal form begins with an element of G_2 (other than e) is plainly almost invariant and G_1-invariant.

Consider the group $C = A_H^* B$, the free product of A and B with H amalgamated, where $A = G$, $B = G \times Z$. It is finitely generated and torsion-free, and $C \neq Z$. Let E be an almost invariant H-invariant subset of A not containing e. Then the set of elements of C whose normal form begins with an element of E is almost invariant. (The normal form of an element of C is not unique, but if one normal form begins with an element of E they all do, as E is H-invariant.) Stallings' main theorem, extended to finitely generated groups, now shows that C is a non-trivial free product, since (see [2], Proposition (5.2)) the only torsion-free group with two ends is Z.

Let C be the free product of X and Y. By Kuros' theorem, unless B is contained in a conjugate of X or Y, it is either infinite cyclic or a non-trivial free product. But $B \neq Z$, and as B is a non-trivial direct product it cannot be a non-trivial free product. Thus, replacing X and Y by suitable conjugates, we have $B \subseteq X$. Also by Kuros' theorem, $A \cap X$ is a free factor of A. We cannot have $A = X$, as this would give $C = X$, and as $H \subseteq A \cap X$, we have the result.

Finally, we show that the number of ends is the upper bound of the number of infinite components of sets with finite complement, and is also the number

of ultrafilters of almost invariant sets which contain the filter of all sets with finite complement. Such an ultrafilter is called an *end*.

Let C be any component of the almost invariant set E (i.e. C is a maximal connected subset). Then there is no edge with one vertex in C and the other in $E \cap C^*$ (else we could add this vertex to C and get a larger connected subset of E). Thus $\delta C \subseteq \delta E$, so C is almost invariant. If C_1 is another component of E, $\delta C \cap \delta C_1 = \emptyset$, so E can have only finitely many components, as δE is finite. If E is infinite one of these components must be infinite.

Let $E \supseteq E_1$ be infinite almost invariant sets. Then any component of E_1 is contained in a component of E, and any component C of E meets E_1 in the union of components of E_1, one of which will be infinite if $C \cap E_1$ is infinite.

Let $S_1 \subseteq S_2 \subseteq \cdots$ be finite subsets of G whose union is G. Then S_n^* has at least one infinite component, and any such contains an infinite component of S_{n+1}^*. Let $C_1 \supseteq C_2 \supseteq \cdots$ where each C_n is an infinite component of S_n^*. The collection of all almost invariant sets containing C_n for at least one n is plainly a filter containing F^* for any finite F (since $F \subseteq S_n$ for some n).

This filter is an ultrafilter. For let E be almost invariant. Take n so that S_n contains all the vertices of δE. Then any connected subset of S_n^* lies completely in E or completely in E^*. In particular, either $C_n \subseteq E$ or $C_n \subseteq E^*$, i.e. either E or E^* lies in the filter.

Hence any such sequence C_n determines a unique end (and it is only necessary for us to give the sets C_n for infinitely many n).

Conversely, let θ be any end. Let $S_n^* = A_1 \cup \cdots \cup A_r \cup F$, where F is finite and each A_i is a component of S_n^* and so is almost invariant. Then $A_i \in \theta$ for some i. For otherwise $A_i^* \in \theta$, all i, and as S_n^*, $F^* \in \theta$, by definition, we would get $\emptyset = A_1^* \cap \cdots \cap A_r^* \cap F^* \cap S_n^* \in \theta$, which is absurd. As $A_i \cap A_j = \emptyset$ for $i \neq j$, only one A_i is in θ. Consequently θ defines a sequence $\{C_n\}$ where C_n is a component of S_n^*, $C_n \in \theta$ for all n, and $C_n \supseteq C_{n+1}$ for all n, since there is a component of S_n^* containing C_{n+1}, and this will be in θ, so must be C_n. Plainly θ is the end associated with the sequence $\{C_n\}$ by the method of the previous paragraph. Since for each n there are only finitely many choices for C_n, there are at most C ends.

Suppose there is an upper bound to the number of infinite components of a set whose complement is finite. Let k be this bound, attained for F^*, say. If $F \subseteq S_n$, every infinite component of F^* will contain exactly one infinite component of S_n^*, which will in turn contain exactly one component of S_{n+1}^*. Consequently, if we take m so that $F \subseteq S_m$ there are k infinite components of S_m, and if C_m is one of them there is exactly one sequence $C_m \supseteq C_{m+1} \supseteq \cdots$, where each C_n is an infinite component of S_n^*. Consequently, k is the cardinality of the set of ends.

Suppose this upper bound is 1. Let E be any almost invariant set, and let F be the (finite) set of vertices of δE. Then any component of F^* lies either in E or in E^*, so that $E \cap F^*$ and $E^* \cap F^*$ are unions of components of F^*. As F^* has only one infinite component, and has only finitely many components, either E or E^* is finite, so $\dim H^0(G; \mathscr{E}G) = 1$.

Suppose this upper bound is 2. If the finite set F is chosen so that F^* has two infinite components A and B, then A is infinite, almost invariant, and A^* is infinite, as $A^* \supseteq B$. Let E be an almost invariant set. Let C be the set of vertices of δE. Then $(C \cup F)^*$ has finitely many components and exactly two infinite components, one in A and one in B. All these components lie either in E or in E^* and each of $E \cap (C \cup F)^*$ and $E^* \cap (C \cup F)^*$ is the union of components of $(C \cup F)^*$. Consequently (depending on which infinite components lie in E or which in E^*) either $E \overset{a}{=} A$ or $E^* \overset{a}{=} A$ or $E \overset{a}{=} G$ or $E^* \overset{a}{=} G$. Hence $\dim H^0(G; \mathscr{E}G) = 2$.

Suppose that, for some finite F, F^* has at least three infinite components, A, B, C, say. Then A, B, C are almost invariant.

Let E be any infinite almost invariant set. As $(A \cup B) \cap (B \cup C) \cap (C \cup A) = \emptyset$, we can assume, renaming, that $\{g \in E; g^{-1} \notin A \cup B\}$ is infinite. By (1.4), for almost all $g \in E$ either $g(A \cup B) \subseteq E - \{e\}$ or $g(A \cup B) \supseteq E^* \cup \{e\}$. Since the latter implies $g^{-1} \in A \cup B$, we can choose g so that the former holds. If n is such that S_n contains all the vertices of $\delta(gA \cup gB)$, then $E \cap S_n^*$ contains at least two infinite components, one in gA and one in gB, since a connected set meeting both gA and gA^* must contain a vertex of δgA, which is contained in S_n.

From this, we deduce the following, inductively. *There is a sequence $n_1 \leqq n_2 \ldots$ of positive integers and a family of sets $C_{\varphi_1 \ldots \varphi_r}$, $\varphi_i = 0$ or 1, r arbitrary, such that $C_{\varphi_1 \ldots \varphi_r}$ is an infinite component of $S_{n_r}^*$ and such that*

$$C_{\varphi_1 \ldots \varphi_r} \supseteq C_{\varphi_1 \ldots \varphi_r \varphi_{r+1}} \qquad \text{for all } \varphi_1, \ldots, \varphi_{r+1}.$$

Each infinite sequence $\varphi_1, \varphi_2, \ldots$ defines an end. Hence there are exactly c ends.

Also $S_{n_r}^*$ has at least 2^r infinite components (each of which is an almost invariant set). Thus the upper bound of the number of infinite components of a set with finite complement is ∞. Also $\dim H^0(G; \mathscr{E}G) = \infty$, since these 2^r disjoint sets give at least $2^r - 1$ independent elements of $H^0(G; \mathscr{E}G)$ (but not 2^r since their union may have finite complement). Thus we have another proof that G has 1, 2 or ∞ ends.

2. Free Products

The analysis given in §4 of [2] shows that in order to prove that any finitely generated torsion-free group with infinitely many ends is a non-trivial free product it is enough to replace Lemma (4.1) of [2] by the following lemma.

Lemma. *Let G have at least two ends. Then in the graph of G (with respect to some finite set of generators) there is a set E of vertices with E and E^* infinite and $P = \delta E$ finite such that for any g at least one of the sets $E \cap gE$, $E \cap gE^*$, $E^* \cap gE$, $E^* \cap gE^*$ is finite and such that one of these sets is empty if $P \cap gP = \emptyset$. If G is torsion free with infinitely many ends then exactly one of these four sets is finite if $g \neq e$.*

First take any E with E, E^* infinite and δE finite. If $E \cap gE$ and $E^* \cap gE^*$ are finite we see that $gE \overset{a}{=} E^*$. Hence $gE^* \overset{a}{=} E$, and so $g^2 E \overset{a}{=} gE^* \overset{a}{=} E$. As E is

almost invariant, Lemma (1.5) shows that if g has infinite order then G has two ends. The same holds if $E \cap gE^*$ and $E^* \cap gE$ are finite, as these give $gE \overset{a}{=} E$. The other four ways in which two of $E \cap gE$, $E \cap gE^*$, $E^* \cap gE$, $E^* \cap gE^*$ may be finite would give, taking the union of the two sets, that one of E, E^*, gE, gE^* is finite. Hence we have proved the second part of the lemma.

For the first part of the lemma we begin by considering sequences of positive integers $(n, m, x_1, \ldots, x_M)$, where $M = \frac{1}{2}m(m-1)$ for $m > 1$, together with pairs $(n, 1)$ and $(0, 0)$, which correspond to $M = 0$. We define

$$(n_1, m_1, x_1, \ldots, x_{M_1}) < (n_2, m_2, y_1, \ldots, y_{M_2})$$

if $n_1 < n_2$ or $n_1 = n_2$ and $m_1 < m_2$ or $n_1 = n_2$, $m_1 = m_2$ and, for some h, $x_i = y_i$ for $i < h$ but $x_h < y_h$. This is a well-ordering.

We define the distance between two sets A and B of edges as the smallest number of edges in a path starting with a vertex of some edge in A and finishing with a vertex of some edge in B. We define the *weight* of a set A of edges, wt A, to be $(n, m, x_1, \ldots, x_M)$ where A has n edges, m components, and $x_1 \leqq \cdots \leqq x_M$ are the distances between all the pairs of components of A, in increasing order.

We choose E with E and E^* infinite and δE finite in such a way that $P = \delta E$ has as small a weight as possible. Intuitively, P is chosen to have as few edges as possible, as few components as possible, and its components as close together as possible. Let wt $P = (n, m, x_1, \ldots, x_M)$. As $\emptyset \neq E \neq G$, wt $P \neq (0, 0)$.

Then if wt $\delta(E \cap gE) <$ wt δE, either $E \cap gE$ or $(E \cap gE)^*$ must be finite, and the latter is infinite. Hence to show that one of $E \cap gE, E \cap gE^*, E^* \cap gE, E^* \cap gE^*$ is finite we need only show that the coboundary of one of them has weight less than wt P.

The four sets $\delta(E \cap gE)$, etc., are all contained in $P \cup gP$, and each edge of $P \cup gP$ lies in exactly two of these sets. Hence the sum of the numbers of edges in these four sets is twice the number of edges in $P \cup gP$, and so is less than $4n$ if $P \cap gP \neq \emptyset$. So one of the sets has less than n edges, so has weight less than wt P, as required.

Suppose $P \cap gP = \emptyset$. We shall show that either one of the sets $E \cap gE$, etc., is empty, or one of $E \cap gE^*$, $E^* \cap gE$ is finite. A similar argument will show that either one of the sets is empty or one of $E \cap gE$, $E^* \cap gE^*$ is finite. Hence one set must be empty, as otherwise, taking the union of the two finite sets, we find that E, E^*, gE, or gE^* would be finite.

The situation can be represented by the diagram. Here A denotes the set of edges of P with both vertices in $gE \cap g^{-1}E^*$, etc. (as $P \cap gP = \emptyset$, any edge of P has both its vertices in gE or both in gE^*, and similarly both in $g^{-1}E$ or both in $g^{-1}E^*$), and r is the length of the shortest path with all its vertices in $E^* \cap gE$ starting with a vertex of $A \cup B$ and ending with a vertex of $gA \cup gC$, etc. (with $r = \infty$ if no such path exists).

If A has fewer edges than D or has the same number of edges as D but fewer components, then plainly $\delta(E^* \cap gE) = A \cup B \cup gC \cup gA$ has smaller weight than $P = A \cup B \cup C \cup D$. Similarly if D has fewer edges than A or the same number of edges but fewer components, then wt $\delta(E \cap gE^*) <$ wt P. So we may

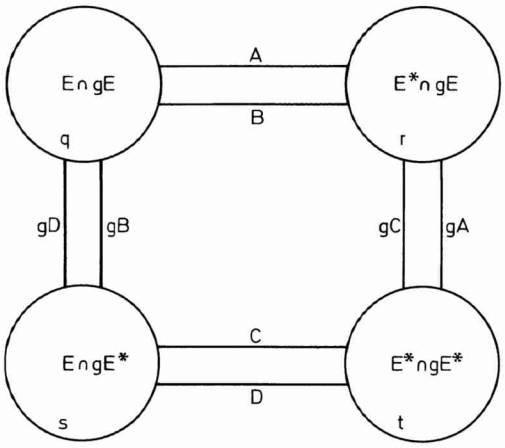

Fig. 1

assume that A and D have the same number of edges and components. We may also assume r and s are positive, since if $r=0$ a component of $A\cup B$ and one of $gA\cup gC$ will unite to form a single component of $\delta(E^*\cap gE)$.

If $E\cap gE\neq\emptyset\neq E^*\cap gE^*$ there will be a path starting in $E\cap gE$ and ending in $E^*\cap gE^*$. The shortest such path can only contain one edge of $\delta(E\cap gE)$ and one of $\delta(E^*\cap gE^*)$. Hence it must consist either of an edge of $A\cup B$, followed by a path in $E^*\cap gE$, followed by an edge of $gA\cup gC$ or else of an edge of $gB\cup gD$, followed by a path in $E\cap gE^*$, followed by an edge of $C\cup D$. It follows that if $r=s=\infty$, when no such paths in $E^*\cap gE$ or $E\cap gE^*$ can be found, then either $E\cap gE$ or $E^*\cap gE^*$ must be empty.

We may assume $r\leqq s$, interchanging E and E^* if necessary, and hence, by the last paragraph, we can take r finite. The shortest path starting with a vertex of $A\cup B$ and ending with a vertex of $C\cup D$ can contain no edges of $A\cup B\cup C\cup D=P$, and so lies entirely in E or entirely in E^*. If it lies entirely in E^* it must begin with a path in $E^*\cap gE$ from a vertex of $A\cup B$ to one of $gA\cup gC$ and continue with an edge of $gA\cup gC$, and so must have length at least $r+1$. Similarly such a path in E has length at least $s+1\geqq r+1$. The shortest path from a vertex of A to one of B (or from one of C to one of D, similarly) has the same length as the shortest path from a vertex of gA to one of gB, which is at least $r+1$ by a similar argument. As $r<\infty$, $A\cup B\neq\emptyset\neq gA\cup gC$, so either $B\neq\emptyset\neq C$ or $A\neq\emptyset$, whence also $D\neq\emptyset$.

Intuitively, we now say that if the components of A are significantly closer together than those of D, then $\delta(E^*\cap gE)=A\cup B\cup gC\cup gA$ has smaller weight than $P=A\cup B\cup C\cup D$, while $\mathrm{wt}\,\delta(E\cap gE^*)<\mathrm{wt}\,P$ if the components of D are significantly closer together than those of A. If there is not much difference then $\mathrm{wt}\,\delta(E^*\cap gE)<\mathrm{wt}\,P$, because any two components from different sets A, B, C, D have distance at least $r+1$ while there are components from different sets A, B, gC, gA whose distance is r.

We observe that if $y_1 \leqq \cdots \leqq y_M$ and $y'_1, \ldots, y'_M$ is a permutation of $y_1, \ldots, y_M$, then $(n, m, y_1, \ldots, y_M) \leqq (n, m, y'_1, \ldots, y'_M)$. For if $y_i = y'_i$ for $i < h$ but $y_h \neq y'_h$ then $y_h < y'_h$. It follows that $\mathrm{wt}\,\delta(E^* \cap gE) < \mathrm{wt}\ P = (n, m, x_1, \ldots, x_M)$ if, for some h, $\delta(E^* \cap gE)$ has pairs of components whose distance is x_i for $i < h$, and a pair closer together than x_h.

Let x_k be the first entry in wt P which is the distance between a component of one of A, B, C, D and a component of a different one of A, B, C, D. By the choice of k, $x_k > x_{k-1}$, and by previous analysis $x_k > r$. Let the entries in $x_1, \ldots, x_{k-1}$ which are the distances between two components of A be $x_{i_1}, \ldots, x_{i_\alpha}$ and those which are the distances between two components of D be $x_{j_1}, \ldots, x_{j_\delta}$. Let $x_{i_\lambda} = x_{j_\lambda}$ for $\lambda \leqq \mu$ but $x_{i_{\mu+1}} \neq x_{j_{\mu+1}}$ if $\mu < \min(\alpha, \delta)$.

If $\mu < \min(\alpha, \delta)$ and $x_{i_{\mu+1}} < x_{j_{\mu+1}}$, then $\delta(E^* \cap gE) = A \cup B \cup gC \cup gA$ has components at distance apart x_h for $h < j_{\mu+1}$ and a pair distance apart $x_{i_{\mu+1}}$, since we are replacing D by gA and C by gC and since x_h for $h < k$ is the distance between two components of the same one of A, B, C, D. Also if $\mu = \delta < \alpha$, then $\delta(E^* \cap gE)$ has components at distance apart x_h for $h < k$ and a pair distance $x_{i_{\delta+1}} < x_k$. In either case we get $\mathrm{wt}\,\delta(E^* \cap gE) < \mathrm{wt}\ P$. Similarly $\mathrm{wt}\,\delta(E \cap gE^*) < \mathrm{wt}\ P$ if $\mu < \min(\alpha, \delta)$ and $x_{j_{\mu+1}} < x_{i_{\mu+1}}$ or if $\mu = \alpha < \delta$.

To complete the proof of the lemma, we need only show that $\mathrm{wt}\,\delta(E^* \cap gE) < \mathrm{wt}\ P$ if $\mu = \alpha = \delta$ (this includes the case $A = D = \emptyset$). In this case $\delta(E^* \cap gE)$ has components at distance apart $x_1, \ldots, x_{k-1}$, and, by the definition of r, has a pair (one in $A \cup B$, one in $gA \cup gC$) whose distance is $r < x_k$, as required.

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Some Properties of Groups Inherited by Normal Subgroups of Finite Index

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1. Introduction

Let $\mathfrak{M}^*$ and $\mathfrak{M}$ denote respectively the classes of groups satisfying the maximal and the minimal conditions for normal subgroups. Thus $G \in \mathfrak{M}^*$ (resp. $\mathfrak{M}$) if and only if G satisfies the maximal (minimal) condition for G-invariant subgroups. More generally, if K is a normal subgroup of a group G and if H is any subgroup of G, we shall write K satisfies Max-H (resp. K satisfies Min-H) if K satisfies the maximal (minimal) condition upon subgroups normalised by H.

It is well known that neither the maximal condition nor the minimal condition for normal subgroups is inherited by arbitrary normal subgroups. For example, if A and B are perfect simple groups, and B infinite, then the wreath product $G = A \wr B$ has just three normal subgroups: 1, G, and a subgroup D which is the direct product of $|B|$ groups isomorphic to A. Obviously D satisfies neither the maximal nor the minimal condition for normal subgroups, although G satisfies both.

However our first theorem will show that the maximal and minimal conditions for normal subgroups are inherited by normal subgroups of finite index (and therefore by arbitrary subgroups of finite index). More precisely, we shall prove

Theorem A. *Let G_0 be a normal subgroup of finite index in the group G. If K is a normal subgroup of G satisfying* Max-G *(resp.* Min-G*), then K satisfies* Max-G_0 *(resp.* Min-G_0*). Therefore the maximal and minimal conditions for normal subgroups are inherited by normal subgroups of finite index.*

It follows from the final statement in Theorem A that any $\mathfrak{M}$-group G is a finite extension of an $\mathfrak{M}$-subgroup G_0 which has no proper subgroups of finite index. G_0, here, is of course the unique minimal (normal) subgroup of finite index in G. This fact is of some importance for the study of $\mathfrak{M}$-groups, and it yields considerably simplified proofs of many of the known theorems about $\mathfrak{M}$-groups.

Our second theorem uses an argument similar to that of Theorem A to exhibit some further properties of groups which are inherited by normal subgroups of finite index. We shall use the notation of classes of groups and closure operations developed by Hall in [1]. We recall the definitions of the closure operations P and Q. If $\mathfrak{X}$ is any class of groups, then $P\mathfrak{X}$ is the class of all groups which have a finite subnormal series with each factor in $\mathfrak{X}$, and $Q\mathfrak{X}$

is the class of all homomorphic images of $\mathfrak{X}$-groups. We shall call a class $\mathfrak{X}$ of groups B-closed if and only if H and K are both $\mathfrak{X}$-groups whenever they are both normal subgroups of their join, with $HK \in \mathfrak{X}$ and $H \cap K \in \mathfrak{X}$. This defines B as a closure operation in the usual manner, and it is clear that any class of groups which is both P-closed and Q-closed is also B-closed. If $\mathfrak{X}$ is a class of groups, $\mathfrak{X}'$ will denote the class of groups all of whose normal subgroups are $\mathfrak{X}$-groups. We may now state Theorem B as follows:

Theorem B. *If $\mathfrak{X}$ is a B-closed class of groups, then a normal subgroup of finite index in an $\mathfrak{X}'$-group is again an $\mathfrak{X}'$-group.*

For $\mathfrak{X}$ here we may take the class $\mathfrak{G}$ of finitely generated groups, $\mathfrak{G}'$, $\mathfrak{G}''$ and so on, $\mathfrak{M}^*$, and various other maximal and minimal conditions. Another obvious candidate for $\mathfrak{X}$ is the class $\mathfrak{P}$ of perfect groups.

Let M be McLain's characteristically simple group (see McLain [2]) defined with respect to, say, the rational numbers in their natural order and the Galois field of p elements, where p is a prime, and let H be the group of automorphisms of M induced by the order-preserving permutations of bounded support of the rationals. H is simple from a result of Higman [3]. It is clear that M has normal subgroups which are not perfect; however the split extension $G = MH$ of M by H has only three normal subgroups, namely 1, M and G, and these are all perfect. Thus $\mathfrak{P}' > \mathfrak{P}''$, and it follows that *$B$-closure of the class $\mathfrak{X}$ is not sufficient to ensure that an arbitrary normal subgroup of an $\mathfrak{X}'$-group will be an $\mathfrak{X}'$-group.*

I should like to thank my research supervisor, Dr. J. E. Roseblade, for his generous help in the preparation of this paper.

2. Proofs of Theorems A and B

If H is a subgroup and T is a subset of a group G, we shall write H^T for the subgroup of G generated by the conjugates of H under T. If H is a normal subgroup of G we write $H \lhd G$, and if $H \lhd H^G$ we write $H \lhd^2 G$.

We shall require the following lemmas; their proofs are easy and will be omitted.

Lemma 1. *If $H \lhd K \leqq G$, and if T is a set of elements of G satisfying $G = KT$, then the subgroups H^t $(t \in T)$ exhaust the conjugates of H under G. Consequently both H^T and $\bigcap \{H^t \mid t \in T\}$ are normal subgroups of G.*

Lemma 2. *Suppose that H_1, H_2, K_1 and K_2 are normal subgroups of the group G with $H_1 \leqq H_2$ and $K_1 \leqq K_2$. If also $H_1 K_1 = H_2 K_2$ and $H_1 \cap K_1 = H_2 \cap K_2$, then $H_1 = H_2$ and $K_1 = K_2$.*

Lemma 3. *Let $H \lhd^2 G$ and let T' be any set of elements of G. Write $T = T' \cup \{1\}$ and $H_1 = H \cap (H^{T'})$.*

Then $(H_1^{T'} H)(H^{T'}) = H^T$, and $(H_1^{T'} H) \cap (H^{T'}) = H_1^T$.

Proof of Theorem A. Since $K/K \cap G_0$ is finite, we need only consider the case $K \leqq G_0$.

Let (K_i) be any ascending (resp. descending) chain of G_0-invariant subgroups of K. If S is a transversal to G_0 in G, then $K_i^S = K_i^G \lhd G$ by Lemma 1, so that the chain (K_i^S) breaks off. Moreover S is finite.

Suppose that the result is false, and choose a finite set T of elements of G with least cardinal such that if (K_i) is any ascending (resp. descending) chain of G_0-invariant subgroups of K, the chain (K_i^T) breaks off. We may suppose $1 \in T$, and since K does not satisfy Max-G_0 (resp. Min-G_0), the set $T' = T \setminus \{1\}$ is not empty.

But if (K_i) is any ascending (resp. descending) chain of G_0-invariant subgroups of K, then so also is (K_i^*), where $K_i^* = K_i \cap (K_i^{T'})$ for each i. Therefore the chains (K_i^T) and (K_i^{*T}) break off. Thus it follows from Lemmas 2 and 3 that the chain $(K_i^{T'})$ breaks off.

However this is a contradiction to the definition of T; the theorem must therefore be true.

We note that if $K \leqq H \leqq G$ and $K \lhd G$, and if K satisfies Max-H (resp. Min-H), then, *a fortiori*, K satisfies Max-G (resp. Min-G). Since every subgroup of finite index in a group contains a normal subgroup of finite index, it follows that the hypothesis $G_0 \lhd G$ may be deleted in Theorem A, and that any subgroup of finite index in an $\mathfrak{M}^*$-group (resp. $\mathfrak{M}$-group) is an $\mathfrak{M}^*$-group (resp. $\mathfrak{M}$-group).

Proof of Theorem B. Suppose that $G \in \mathfrak{X}'$, and that G_0 is a normal subgroup of finite index in G.

Let K be any normal subgroup of G_0. If S is a transversal to G_0 in G, then $K^S \lhd G$ and $K^S \in \mathfrak{X}$.

Suppose that the result is false and that $G_0 \notin \mathfrak{X}'$. Since S is finite, we can choose a finite set T of elements of G with least cardinal, such that for every $K \lhd G_0$ we have $K^T \in \mathfrak{X}$. We may suppose that $1 \in T$, and since $G_0 \notin \mathfrak{X}'$, the set $T' = T \setminus \{1\}$ is not empty.

Choose $K \lhd G_0$, and define $K^* = K \cap (K^{T'})$. We have $K^T \in \mathfrak{X}$ and $K^{*T} \in \mathfrak{X}$. Thus from Lemma 3 and the definition of B-closure it follows that $K^{T'} \in \mathfrak{X}$.

However this contradicts the definition of T; the theorem must therefore be true.

We remark that B-closure is a somewhat stronger condition upon the class $\mathfrak{X}$ than is actually required in order for the proof of Theorem B to show that a normal subgroup of finite index in an $\mathfrak{X}'$-group is an $\mathfrak{X}'$-group. We also remark that the argument of Theorems A and B can be used to prove a number of results of a similar nature.

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Galois Theory for the Family of Partial Differential Equations $\Delta_\alpha \psi = 0$

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Introduction

With attempts to develop for ordinary differential equations a theory analogous to the Galois theory for algebraic equations, the subject of differential algebra has grown over the last few decades, mainly due to the pioneering efforts of Ritt and Kolchin. However, attempts toward the study and development of similar Galois theory for partial differential equations have been restricted to questions about ideals and manifolds of generic zeros of partial differential polynomials by Ritt [1], while in his paper on the Picard-Vessiot theory of partial differential fields, Kolchin [2] had discussed only those aspects of partial differential fields which are closely related to the Picard-Vessiot theory of homogeneous linear ordinary differential equations. In this paper, which is to mark the beginning of a series of communications on the Galois theory for partial differential equations, an examination is made of certain aspects of the Galois theory for the one parameter family of partial differential equations

$$\Delta_\alpha \psi \equiv \alpha \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = 0, \tag{1}$$

which appears in several pure and applied problems [3].

Summary

The partial differential fields most suited for the purpose of construction of Galois theory for the family (1) are endowed with the symmetric bilinear form (2iv) and are called α-differential fields. In Section 1 are defined certain algebraic notions related to the symmetric bilinear form (2iv) and which are necessary for the construction of any Galois theory. Necessary and sufficient condition for the extension of the domain of the operator Δ_α (this operator is not a derivation although it commutes with the partial derivations of the α-differential field) from an α-differential field K to a finitely generated α-differential extension field is given in Theorem 1.

Section 2 defines the notion of α-differential mapping as linear mappings which preserve the symmetric bilinear form and commute with the partial derivations. The group properties of the set of α-differential mappings are discussed and the Galois correspondence theorems set up for α-differential fields.

Section 3 sets up the notion of α-Liouvillian extensions of α-differential fields and briefly discusses the Galois groups associated with these α-Liouvillian extension fields.

Section 4 points to the procedure for the algebraic characterization of ω-simple-α-differential field extensions by elementary solutions of the partial differential equation $\Delta_\alpha \psi_m = 0$.

§ 1. α-Differential Algebraic Systems

An α-differential ring R is a commutative ring with mappings $\delta_1, \delta_2, \delta_3$ (called partial derivatives) of R into R satisfying

(i) $\delta_i(r_1 + r_2) = \delta_i(r_1) + \delta_i(r_2)$. (2i)

(ii) $\delta_i(r_1 r_2) = r_1 \delta_i(r_2) + r_2 \delta_i(r_1)$. (2ii)

(iii) $\delta_i \delta_j(r_1) = \delta_j \delta_i(r_1)$. (2iii)

(iv) a binary operation $\otimes$ in R satisfying

$$r_1 \otimes r_2 = \alpha\, \delta_1(r_1)\, \delta_1(r_2) + \delta_2(r_1)\, \delta_2(r_2) + \delta_3(r_1)\, \delta_3(r_2) \tag{2iv}$$

for all ($i = 1, 2, 3$) $r_1 \in R$, $r_2 \in R$ and $\alpha \in R$ such that $\delta_i(\alpha) = 0$ for $i = 1, 2, 3$. Thus an α-differential ring is a partial differential ring with the binary structure (iv).

R_1 and R_2 are two α-differential rings. Then R_2 is an α-differential extension ring of R_1 (or R_1 is an α-differential subring of R_2) iff R_1 is a subring of R_2 and $\delta_i(r)$ in R_2 coincides with $\delta_i(r)$ in R_1 whenever $r \in R_1 \subset R_2$ (evidently, in this case, $\overset{2}{\otimes}$ in R_2 would coincide with $\overset{1}{\otimes}$ in R_1).

Considering fields K, K_1, K_2 instead of rings R, R_1, R_2 above, we get the notion of α-differential field and α-differential extension field (α-differential subfield).

In an α-differential ring R (field F), the set of all elements $c \in R$ ($c \in F$) such that $\delta_i(c) = 0 (\forall\, i)$ form an α-differential subring (subfield) $\mathscr{C}$, called the ring (field) of constants of R (of F).

R_1 an α-differential subring of α-differential ring R_2, $\phi \neq S \subset R_2$. The set of all polynomials with coefficients in R_1 in a finite number of elements of the form $\delta_1^{i_1} \delta_2^{i_2} \delta_3^{i_3}(s)$ for $s \in S$ and i_1, i_2, i_3 non-negative integers, form an α-differential extension ring $R_1[S]$ of R_1.

K_1 an α-differential subfield of the α-differential field K_2, $\phi \neq S \subset F_2$. The set of all rational functions with coefficients in K_1 in a finite number of elements of the form $\delta_1^{i_1} \delta_2^{i_2} \delta_3^{i_3}(s)$ for $s \in S$ and i_1, i_2, i_3 non-negative integers, form an α-differential extension field $K_1(S)$ of K_1.

R an α-differential ring. An ideal I of R is an α-differential ideal of R iff I is closed under δ_i's (i.e. $\delta_i(r) \in I$, $\forall\, r \in I$, $i = 1, 2, 3$). Evidently I is closed under $\otimes$ if I is closed under δ_i's.

$\phi \neq S \subset R$ an α-differential ring. The intersection of all α-differential ideals of R containing S is an α-differential ideal called the α-differential ideal generated by S.

K an α-differential field of characteristic zero. X any symbol called an Indeterminate. An α-differential polynomial in X with coefficient in K is a polynomial with coefficients in K in a finite number of new symbols $\delta_1^{i_1}\delta_2^{i_2}\delta_3^{i_3}(X)$ (i_1, i_2, i_3 being non-negative integers). Defining derivation of α-differential polynomials in the usual way, we see that the set of all α-differential polynomials in X with coefficients in K is an α-differential ring $K[X]$.

Theorem 1.

K an α-differential field.

$\nabla_\alpha \equiv (\sqrt{\alpha}\,\delta_1 + \delta_2 + \delta_3)$ linear mapping of K into some α-differential extension field $\mathscr{L}$.

$\Delta_\alpha \equiv (\alpha\,\delta_1\,\delta_1 + \delta_2\,\delta_2 + \delta_3\,\delta_3)$ linear mapping of K into some α-differential extension field $\mathscr{L}$.

$M = K(\psi_1, \psi_2, \ldots, \psi_n)$ any finitely generated extension field

$$\{\alpha_{11}, \alpha_{12}, \alpha_{13}, \ldots, \alpha_{1j}, \ldots, \alpha_{1n}\}, \{\alpha_{21}, \alpha_{22}, \ldots, \alpha_{2j}, \ldots, \alpha_{2n}\},$$

$\{\alpha_{31}, \ldots, \alpha_{3j}, \ldots, \alpha_{3n}\}$ *and* $\{\rho_1, \rho_2, \ldots, \rho_j, \ldots, \rho_n\}$ *any four sets of n-elements of $\mathscr{L}$, then there exist unique derivations D_1, D_2, D_3 of M extending $\delta_1, \delta_2, \delta_3$ respectively (and there exists unique α-differential structure $\circledast$ on M extending the α-differential structure $\otimes$ on K, and there exists unique linear mapping $B_\alpha \equiv (\sqrt{\alpha}\,D_1 + D_2 + D_3)$ of M extending ∇_α of K) and there exists unique linear mapping $\square_\alpha \equiv (\alpha\,D_1\,D_1 + D_2\,D_2 + D_3\,D_3)$ of M extending Δ_α and such that*

$$D_i\psi_j = \alpha_{ij} \qquad (i=1,2,3),\ (j=1,2,\ldots,n), \tag{3i}$$

$$\square_\alpha \psi_j = \rho_j. \tag{3ii}$$

$$(\textit{Consequently, } B_\alpha\psi_j \equiv (\sqrt{\alpha}\,D_1 + D_2 + D_3)\psi_j = (\sqrt{\alpha}\,\alpha_j + \beta_j + \gamma_j) \equiv \sigma_j (\textit{say}).) \tag{3iii}$$

if and only if for every α-differential polynomial

$$f(\psi_j) \in K(\psi_1, \ldots, \psi_{j-1}, \psi_{j+1}, \ldots, \psi_n)[\psi_j]$$

such that $f(\psi_j) \equiv q_0 + q_1\psi_j + \cdots + q_k\psi_j^k + \cdots + q_m\psi_j^m = 0$ *we have*

$$\sum_{k=0}^{m} (\delta_i q_k)\psi_j^k + \left(\sum_{k=0}^{m} k\,q_k\psi_j^{k-1}\right)\alpha_{ij} = 0 \qquad (i=1,2,3),\ (j=1,2,\ldots,n) \tag{4i}$$

$$\begin{aligned}\sum_{k=0}^{m} (\Delta_\alpha q_k)\psi_j^k + 2\sum_{k=0}^{m} k\,\psi_j^{k-1}(\nabla_\alpha q_k)\cdot\sigma_j + \left(\sum_{k=0}^{m} k\,q_k\psi_j^{k-1}\right)\cdot\rho_j \\ + \left[\sum_{k=0}^{m} k(k-1)\,q_k\psi_j^{k-2}\right](\sigma_j\cdot\sigma_j) = 0.\end{aligned} \tag{4ii}$$

$$\left(\textit{Consequently, } \sum_{k=0}^{m} (\nabla_\alpha q_k)\psi_j^k + \left(\sum_{k=0}^{m} k\,q_k\psi_j^{k-1}\right)\sigma_j = 0.\right) \tag{4iii}$$

Proof. If D_1, D_2, D_3 are derivations and $\square_\alpha$ linear mapping of M extending the derivations $\delta_1, \delta_2, \delta_3$ and the linear mapping Δ_α of M, respectively, then for any α-differential polynomial

$$f(\psi_j) \equiv q_0 + q_1 \psi_j + q_2 \psi_j^2 + \cdots + q_k \psi_j^k + \cdots + q_m \psi_j^m \in K(\psi_1, \psi_2, \ldots, \psi_{j-1}, \psi_{j+1}, \ldots, \psi_n)[\psi_j]$$

we get by straightforward computation

$$D_i f(\psi_j) = \sum_{k=0}^{m} (D_i q_k) \psi_j^k + \left(\sum_{k=0}^{m} k q_k \psi_j^{k-1} \right) (D_i \psi_j) \quad \text{for } i = 1, 2, 3,\ j = 1, 2, 3, \ldots, n, \tag{5i}$$

$$\begin{aligned} \square_\alpha f(\psi_j) = {} & \sum_{k=0}^{m} (\square_\alpha q_k) \psi_j^k + 2 \sum_{k=0}^{m} k \psi_j^{k-1} (B_\alpha q_k) \cdot (B_\alpha \psi_j) + \left(\sum_{k=0}^{m} k q_k \psi_j^{k-1} \right) \cdot \square_\alpha \psi_j \\ & + \left[\sum_{k=0}^{m} k(k-1) q_k \psi_j^{k-2} \right] (B_\alpha \psi_j) \cdot (B_\alpha \psi_j) \end{aligned} \tag{5ii}$$

whence we see the necessity of conditions (4i)–(4ii). Also (5i) and (5ii) show the uniqueness of D_i's and $\square_\alpha$.

Now if (4i)–(4ii) were true for every α-differential polynomial

$$f(\psi_j) \in K(\psi_1, \ldots, \psi_{j-1}, \psi_{j+1}, \ldots, \psi_n)[\psi_j]$$

such that $f(\psi_j) = 0$, then for any

$$g(\psi_j) \equiv \sum_{l=0}^{m'} p_l \psi_j^l \in K(\psi_1, \psi_2, \ldots, \psi_{j-1}, \psi_{j+1}, \ldots, \psi_n)[\psi_j]$$

we define

$$D_i g(\psi_j) = \sum_{l=0}^{m'} (\delta_i p_l) \psi_j^l + \left(\sum_{l=0}^{m'} l p_l \psi_j^{l-1} \right) \alpha_{ij} \qquad (i = 1, 2, 3),\ (j = 1, 2, \ldots, n), \tag{6i}$$

$$\begin{aligned} \square_\alpha g(\psi_j) = {} & \sum_{l=0}^{m'} (\Delta_\alpha p_l) \psi_j^l + 2 \sum_{l=0}^{m'} l \psi_j^{l-1} (\nabla_\alpha p_l) \cdot \sigma_j + \left(\sum_{l=0}^{m'} l p_l \psi_j^{l-1} \right) \cdot \rho_j \\ & + \left[\sum_{l=0}^{m'} l(l-1) p_l \psi_j^{l-2} \right] (\sigma_j \cdot \sigma_j). \end{aligned} \tag{6ii}$$

One can check that D_i and $\square_\alpha$ so defined by (6i)–(6ii) are well defined. Also, one can check that D_i so defined by (6i) satisfies (2i)–(2iii), (3i) and that $\square_\alpha$ defined by (6ii) is linear and satisfies (3ii).

Corollary. *$\{f_r(\psi_j)\}_{r \in I}$ a basis for the ideal of algebraic relations satisfied by $\psi_1, \psi_2, \ldots, \psi_j, \ldots, \psi_n$ over K, i.e. every*

$$f(\psi_j) \in K(\psi_1, \psi_2, \ldots, \psi_{j-1}, \psi_{j+1}, \ldots, \psi_n)[\psi_j]$$

for which $f(\psi_j)=0$ *can be written as* $f=\sum_r g_r f_r$ *where*

$$g_r \in K(\psi_1, \psi_2, \ldots, \psi_{j-1}, \psi_{j+1}, \ldots, \psi_n)[\psi_j],$$

then for the existence of D_i's *and* $\square_\alpha$ *it is sufficient that the family of polynomials* $\{f_r(\psi_j)\}_{r\in I}$ *satisfy the conditions* (4i) *and* (4ii).

Proof. Straightforward computation.

§ 2. The Galois Theory of α-Differential Fields

K_1 and K_2 are α-differential fields.

A mapping $\pi: K_1 \to K_2$ will be called an α-differential mapping iff

(i) π is a linear mapping, i.e. $\pi(p+q)=\pi(p)+\pi(q)$, (7i)

(ii) π commutes with δ_i (i = 1, 2, 3), i.e. $\pi\delta_i(q)=\delta_i(\pi(q))$, (7ii)

(iii) $\pi(p \overset{1}{\otimes} q)=\pi(p)\overset{2}{\otimes}\pi(q)$ (7iii)

for all $p, q \in K_1$.

If instead of condition (7iii) we assume π to satisfy

(iii′) $\pi(p \times q)=\pi(p)\times\pi(q)$ (7iii′)

then π would be a differential homomorphism. It can be seen that conditions (7i), (7ii), (7iii′) imply condition (7iii), i.e., every differential homomorphism is an α-differential mapping while the converse is not necessarily true.

In the case of ordinary fields (not necessarily differential fields or α-differential fields) K_1 and K_2, the set $\mathscr{L}(K_1, K_2)$ of all linear mappings of K_1 into K_2 is a vector space over K_2 (also, $\mathscr{L}(K_1, K_2)$ is a vector space over K_1). However, for differential fields (and for α-differential fields) K_1 and K_2, the set $\mathscr{L}(K_1, K_2)$ of all linear mappings of K_1 into K_2 satisfying (7ii) is not a vector space over K_2 but it is a vector space over the subfield of constants C_2 of the differential field (and for α-differential fields) K_2. However, the set $\mathscr{L}_\alpha(K_1, K_2)$ of α-differential mappings of the α-differential field K_1 into the α-differential field K_2 is such that $\mathscr{L}_\alpha(K_1, K_2)$ is not closed under the additive operation $(\pi_1 \oplus \pi_2)\, q=\pi_1(q)+\pi_2(q)$, since the linear mapping $(\pi_1 \oplus \pi_2)$ satisfies (7i) and (7ii) but not (7iii) and hence $\mathscr{L}_\alpha(K_1, K_2)$ cannot be made into a vector space over any subset of K_2. In fact, $\langle \mathscr{L}_\alpha(K_1, K_2), \oplus\rangle$ cannot be made into an abelian group. Also, $\langle \mathscr{L}_\alpha(K, K), \oplus\rangle$ cannot be made into an abelian group. However, under the composition $\pi_1 \circ \pi_2$ defined by $\pi_1((\pi_2)(q))=(\pi_1 \circ \pi_2)(q)$, $\langle \mathscr{L}_\alpha(K, K), \circ\rangle$ is a group. It is this group and certain of its subgroups which play the central role in the Galois theory of α-differential fields.

Any differential isomorphism i ($1-1$ differential homomorphism) of an α-differential ring R_1 (field K_1) into another α-differential ring R_2 (field K_2) is an α-differential mapping. Thus the set $\mathscr{S}_\alpha(R_1, R_2)$ of all one-one α-differential mappings of an α-differential ring R_1 (field K_1) into an α-differential ring R_2

(field K_2) is a more extensive class than the set $\mathscr{S}(R_1, R_2)$ of one-one differential homomorphisms of the α-differential ring R_1 (field K_1) into the α-differential ring R_2 (field K_2). Let $\mathscr{S}_\alpha(K, K)=\mathscr{S}_\alpha(K)$ and $\mathscr{S}(K, K)=\mathscr{S}(K)$ whence we have the inclusions

$$\mathscr{S}_\alpha(R_1, R_2)\supset\mathscr{S}(R_1, R_2) \quad \text{and} \quad \mathscr{S}_\alpha(K)\supset\mathscr{S}(K). \tag{8}$$

We consider α-differential field Ω with α-differential extension fields K_1 and K_2 of Ω. An α-differential mapping π of K_1 *onto* K_2 is a Relative-α-differential mapping of K_1 onto K_2 over Ω iff there exists an α-differential field L containing K_1 and K_2 such that π leaves Ω elementwise fixed, i.e. $\pi(\omega)=\omega(\forall\,\omega\in\Omega)$.

For any subset $\Sigma_\alpha\subset\mathscr{S}_\alpha(L/\Omega)$, the subset Σ_α is an α-abundant set of α-differential mappings of L over Ω if and only if for every α-differential subfield K of L and each $q\in L\cap K'$ there exists one (at least one) $\sigma\in\Sigma_\alpha$ such that $\sigma(q)\neq q$.

Kolchin [4] has proved the following: For any differential extension field L of any differential field Ω, a necessary and sufficient condition that any $q\in L$ belongs to Ω is that for every $\sigma\in\mathscr{S}(L/\Omega)$ we have $\sigma(q)=q$. Since $\mathscr{S}_\alpha(L/\Omega)\supset\mathscr{S}(L/\Omega)$, we have that if every $\pi\in\mathscr{S}_\alpha(L/\Omega)$ leaves any $q\in L$ invariant then every $\sigma\in\mathscr{S}(L/\Omega)$ also leaves q invariant whence $q\in\Omega$, and since $q\in\Omega$ and $\sigma\in\mathscr{S}_\alpha(L/\Omega)$ imply $\pi(q)=q$, we have the following existence theorem:

Theorem 2. *A set of α-abundant α-differential mappings of L over Ω for any α-differential extension field L of an α-differential field Ω always exists.*

For any α-differential field Ω_0 with a chain of α-differential extension fields

$$\{0\}\subset\Omega_0\subset\Omega_1\subset\cdots\subset\Omega_j\subset\Omega_{j+1}\subset\cdots\subset K$$

we have

$$\begin{array}{ccccccccccc}
\mathscr{S}_\alpha(K)\supset & \mathscr{S}_\alpha(K/\Omega_0)\supset & \mathscr{S}_\alpha(K/\Omega_1)\supset\cdots\supset & \mathscr{S}_\alpha(K/\Omega_j)\supset & \mathscr{S}_\alpha(K/\Omega_{j+1})\supset\cdots\supset & \mathscr{S}_\alpha(K/K) \\
\| & & & & & \| \\
\cup \quad & \cup \quad & \cup \quad & \cup \quad & \cup \quad & \text{identity} \\
\| & & & & & \| \\
\mathscr{S}(K)\supset & \mathscr{S}(K/\Omega_0)\supset & \mathscr{S}(K/\Omega_1)\supset\cdots\supset & \mathscr{S}(K/\Omega_j)\supset & \mathscr{S}(K/\Omega_{j+1})\supset\cdots\supset & \mathscr{S}(K/K)
\end{array}$$

with

$$\mathscr{S}(K/\Omega_j)=\mathscr{S}(K)\cap\mathscr{S}_\alpha(K/\Omega_j). \tag{9}$$

Our aim now is to try to seek answers to the following questions:

(i) When is $\mathscr{S}_\alpha(K/\Omega_{j+1})$ normal in $\mathscr{S}_\alpha(K/\Omega_j)$.

(ii) If $\mathscr{S}_\alpha(K/\Omega_{j+1})$ is normal in $\mathscr{S}_\alpha(K/\Omega_j)$, when is $\mathscr{S}_\alpha(K/\Omega_{j+1})/\mathscr{S}_\alpha(K/\Omega_j)$ isomorphic to $\mathscr{S}(\Omega_j/\Omega_{j+1})$.

(iii) Under what conditions can we set up a Galois type correspondence between normal subgroups of $\mathscr{S}_\alpha(K)$ and α-differential subfields of K.

Theorem 3.

α-differential field K with α-differential subfield Ω_0.

Σ_{α_0} any α-abundant subgroup of $\mathscr{S}_\alpha(K/\Omega_0)$.

Ω_j any α-differential subfield of K such that $\Omega_0 \subset \Omega_j \subset K$.

$G_j = \{\sigma \in \Sigma_{\alpha_0} | \sigma(w_j) = w_j, \forall\, w_j \in \Omega_j\}$.

$\Phi_j = \{k \in K | g_j(k) = k, \forall\, g_j \in G_j\}$.

$\Rightarrow \Phi_j = \Omega_j$.

Proof. Evidently $\Omega_j \subset \Phi_j$. If $\Omega_j \neq \Phi_j$, then $\exists\, q \in \Phi_j \cap \Omega_j'$, Σ_{α_j} α-abundant $\Rightarrow \exists\, \sigma \in G_j \ni \sigma(q) \neq q$ whence $q \notin \Phi_j$ contradicting $q \in \Phi_j \cap \Omega_j'$. So $\Omega_j = \Phi_j$.

Thus the above theorem sets up a one-one correspondence between α-abundant subgroups G_j of $\mathscr{S}_\alpha(K/\Omega_0)$ and α-differential subfields Ω_j of K.

Corollary. $\{\Omega_j\}$ a family of α-differential subfields of K, then

$$\mathscr{S}_\alpha\Big(\Omega_0\Big(\bigcup_j \Omega_j\Big)\Big) = \bigcap \mathscr{S}_\alpha(\Omega_j),$$

$\mathscr{S}_\alpha(\bigcap \Omega_j)$ = smallest subgroup in the set of all subgroups of $\mathscr{S}_\alpha(K)$ containing $\bigcup_j \mathscr{S}_\alpha(\Omega_j)$.

Theorem 4. *α-differential field K with α-differential subfields Ω_0 and Ω_{j+1} such that $\Omega_0 \subset \Omega_{j+1} \subset K$.*

(i) $\mathscr{S}_\alpha(K/\Omega_{j+1})$ is normal in $\mathscr{S}_\alpha(K/\Omega_0)$ if and only if Ω_{j+1} is invariant under every $\pi \in \mathscr{S}_\alpha(K/\Omega_0)$, i.e. $\pi\,\Omega_{j+1} \subset \Omega_{j+1}$.

(ii) $\mathscr{S}_\alpha(K/\Omega_{j+1})$ is normal in $\mathscr{S}_\alpha(K/\Omega_0) \Rightarrow \mathscr{S}_\alpha(K/\Omega_0)/\mathscr{S}_\alpha(K/\Omega_{j+1})$ is isomorphic to the α-abundant subgroup (not necessarily proper subgroup) of $\mathscr{S}_\alpha(\Omega_{j+1}/\Omega_0)$.

Proof. (i) Only if

$$\pi\,\Omega_j \subset \Omega_j \;\forall\, \pi \in \mathscr{S}_\alpha(K/\Omega_0),$$

$$\pi_1 \in \mathscr{S}_\alpha(K/\Omega_{j+1}) \Rightarrow \pi^{-1}\pi_1\pi(\omega_{j+1}) = \pi^{-1}\pi_1(\omega'_{j+1}) = \pi^{-1}(\omega'_{j+1}) = \omega_{j+1},$$

$$\pi_1^{-1}\pi\pi_1 \in \mathscr{S}_\alpha(K/\Omega_{j+1}).$$

Hence, $\mathscr{S}_\alpha(K/\Omega_{j+1})$ is normal in $\mathscr{S}_\alpha(K/\Omega_0)$.

If

$\mathscr{S}_\alpha(K/\Omega_{j+1})$ a normal subgroup of $\mathscr{S}_\alpha(K/\Omega_0)$.

$\Rightarrow \forall\, \pi \in \mathscr{S}_\alpha(K/\Omega_0)$ and $\pi_1 \in \mathscr{S}_\alpha(K/\Omega_{j+1})$, we have $\pi^{-1}\pi_1\pi \in \mathscr{S}_\alpha(K/\Omega_{j+1})$.

i.e. $\pi^{-1}\pi_1\pi(\omega_{j+1}) = \omega_{j+1} \in \Omega_{j+1}$.

i.e. $\pi_1\pi(\omega_{j+1}) = \pi(\omega_{j+1})$

i.e. $\pi(\omega_{j+1})$ is invariant under $\pi_1 \in \mathscr{S}_\alpha(K/\Omega_{j+1})$, i.e. $\pi(\omega_{j+1}) \in \Omega_{j+1}$.

(ii) Every $\pi \in \mathscr{S}_\alpha(K/\Omega_0)$ restricted to Ω_{j+1} as domain instead of K determines a π_{j+1} which is an element of $\mathscr{S}_\alpha(\Omega_{j+1}/\Omega_0)$. $\pi \in \mathscr{S}_\alpha(K/\Omega_0)$ and $\pi' \in \mathscr{S}_\alpha(K/\Omega_0)$ restricted to Ω_{j+1} as domain determines the same element $\pi_{j+1} \equiv \pi'_{j+1}$ of $\mathscr{S}_\alpha(\Omega_{j+1}/\Omega_0)$ if and only if $\pi_{j+1}^{-1}\pi'_{j+1}$ belongs to $\mathscr{S}_\alpha(K/\Omega_{j+1})$; i.e. iff π_{j+1} and π'_{j+1} belong to the same coset of $\mathscr{S}_\alpha(K/\Omega_{j+1})$ in $\mathscr{S}_\alpha(K/\Omega_0)$. Thus every distinct $\pi_{j+1} \in \mathscr{S}_\alpha(\Omega_{j+1}/\Omega_0)$ determines a distinct element of $\mathscr{S}_\alpha(K/\Omega_0)/\mathscr{S}_\alpha(K/\Omega_{j+1})$. The isomorphism and α-abundance are now trivial.

Definition. α-differential field Ω_0 with α-differential extension field K. An α-differential subfield Ω_N of K containing Ω_0 is called an *α-normal α-differential extension of* Ω_0 iff the group $\mathscr{S}_\alpha(K/\Omega_N)$ is normal in $\mathscr{S}_\alpha(K/\Omega_0)$ and the factor group is isomorphic to an α-abundant subgroup of $\mathscr{S}_\alpha(\Omega_N/\Omega_0)$.

§ 3. α-Liouvillian Extensions

Ω an α-differential field throughout unless otherwise specified. The i-integral of an element p_i of an α-differential field Ω is any solution ψ of $\delta_i(\psi) = p_i$ $(i = 1, 2, 3)$. For any i-integral ψ of $p_i \in \Omega$ such that $\psi \notin \Omega$ and if the α-differential extension field $\Omega(\psi)$ and Ω have the same algebraically closed subfield of constants C, it can be easily shown that [4] α is transcendental over Ω and there is no differential field (hence no α-differential field) properly between Ω and $\Omega(\psi)$. Similarly, the i-exponential of an i-integral of an element $p_i \in \Omega$ is any solution of $\delta_i(\psi) = p_i\psi$. The complex i-exponential of an i-integral of an element $p_i \in \Omega$ is any solution of $\delta_i(\psi) = \sqrt{-1}\, p_i \psi$. For any $p_i \in \Omega$, any two i-integrals ψ_1 and ψ_2 differ by a constant and the two i-exponentials of two i-integrals of p_i differ by a constant multiple, i.e. $\psi_1 = c\psi_2$. Therefore, for any α-differential extension field $\Omega(\psi)$ of Ω by an i-exponential of an i-integral of an element $p_i \in \Omega$, we have that $\sigma(\psi)$ is also an i-exponential of an i-integral of p_i for all $\sigma \in \mathscr{S}(\Omega(\psi)/\Omega)$, but $\pi(\psi)$ is not necessarily an i-exponential of an i-integral of p_i for all $\pi \in \mathscr{S}_\alpha(\Omega(\psi)/\Omega)$. Thus we have $\sigma(\psi) = c\psi$ while $\pi(\psi)$ is not necessarily a multiple of ψ since $\pi(\psi)$ is not necessarily an i-exponential of an i-integral of p_i. However, if the element $p_i \in \Omega$ is such that

$$\begin{aligned} \pi(p_i \times \psi) &= p_i \times \pi(\psi) \qquad \forall\, \pi \in \mathscr{S}_\alpha(\Omega(\psi)/\Omega) \\ &= \pi(p_i) \times \pi(\psi) \end{aligned} \tag{10}$$

then for such a choice of p_i, an i-exponential ψ_0 of an i-integral of this p_i, $\pi(\psi_0)$, will also be an i-exponential of an i-integral of this p_i. So in this case, $\pi(\psi_0) = c\psi_0$ and π has a simple scalar representation. Evidently the set $\Omega_{0_i} \equiv \{p_i \in \Omega \mid i\text{-exponential } \psi_0 \text{ of an } i\text{-integral of this } p_i \text{ satisfies (10)}\ \forall\, \pi \in \mathscr{S}_\alpha(\Omega(\psi_0)/\Omega)\}$ is such that

$$\mathscr{S}\left(\frac{\Omega(\psi_0)}{\Omega_{0_i}}\right) = \mathscr{S}_\alpha\left(\frac{\Omega(\psi_0)}{\Omega}\right) \supset \mathscr{S}\left(\frac{\Omega(\psi_0)}{\Omega}\right). \tag{11}$$

An i-exponential ψ_0 of an i-integral of a $p_i \in \Omega$ satisfying (10) for all $\pi \in \mathscr{S}_\alpha(\Omega(\psi_0)/\Omega)$ may be such that ψ_0 may be algebraic over Ω or ψ_0 may be transcendental over Ω.

If ψ_0 is algebraic over Ω, then the group $\mathscr{S}_\alpha(\Omega(\psi_0)/\Omega)$ is such that if $\pi\in\mathscr{S}_\alpha(\Omega(\psi_0)/\Omega)$ then $\pi(\psi_0)$ is

$$\pi(\delta_i(\psi_0))=\pi(p_i\psi_0), \tag{12i}$$

$$\delta_i(\pi(\psi_0))=p_i(\pi(\psi_0)) \qquad \text{for } p_i\in\Omega_{0_i}\subset\Omega \tag{12ii}$$

whence $\pi(\psi_0)$ is also an i-exponential of an i-integral p_i. Therefore,

$$\pi(\psi_0)=c\,\psi_0 \qquad c\in C. \tag{12iii}$$

Thus, for every $\pi\in\mathscr{S}(\Omega(\psi_0)/\Omega_{0_i})=\mathscr{S}_\alpha(\Omega(\psi_0)/\Omega)$, we have that π is of finite order N and ψ_0^N belongs to Ω_{0_i}.

If ψ_0 is transcendental over Ω, then the group $\mathscr{S}(\Omega(\psi_0)/\Omega_{0_i})=\mathscr{S}_\alpha(\Omega(\psi_0)/\Omega)$ is isomorphic to the multiplicative group of the field of constants C. Corresponding to the subgroups of C, one can now identify the corresponding field extensions using the Galois correspondence established in the previous section.

α-differential field Ω_0 with algebraically closed field of constants C.

α-differential extension field $\Omega_m\equiv\Omega_0(p_m, q_m, r_m)$.

An α-differential field K_{ψ_m} is an α-Liouvillian extension of Ω_m if and only if K_{ψ_m} is an extension of Ω_m by elements $\{\psi_m\}_{m=1,2,\ldots,n}$ which are

(i) i-integrals of elements of Ω_m, and/or

(ii) i-exponential of an i-integral of elements of Ω_m and/or

(iii) complex i-exponential of an i-integral of elements of Ω_m, and/or

(iv) algebraic over Ω_m,

and the field of constants of K_{ψ_m} is the same as the field of constants of Ω_m (i.e. every constant in K_{ψ_m} belongs to Ω_m).

§ 4. ω-Simple-α-Differential Extensions

α-differential field Ω_0, $\omega\in\Omega_0$. An α-differential extension field K_{ψ_m} of Ω_0 is an ω-simple-α-differential extension iff there exists

(i) an intermediate α-differential field Ω_m between K_{ψ_m} and Ω_0 such that $\Omega_m\equiv\Omega_0(p_m, q_m, r_m)$ and $\alpha p_m^2+q_m^2+r_m^2=\omega$ and $\alpha\delta_1(p_m)+\delta_2(q_m)+\delta_3(r_m)=0$,

(ii) K_{ψ_m} is an α-Liouvillian extension of Ω_m.

Theorem 5. *K_{ψ_m} an ω-simple α-differential extension of Ω_0. Then ψ_m is a solution of the partial differential equation $\Delta_\alpha\psi_m=0$.*

Proof.

$$\left.\begin{aligned}&\delta_1(\psi_m)=p_m\\&\delta_2(\psi_m)=q_m\\&\delta_3(\psi_m)=r_m\\&\alpha\,\delta_1(p_m)+\delta_2(q_m)+\delta_3(r_m)=0\end{aligned}\right\}\Rightarrow\alpha\,\delta_1\,\delta_1(\psi_m)+\delta_2\,\delta_2(\psi_m)+\delta_3\,\delta_3(\psi_m)=0.$$

We conclude with an example of an ω-simple-α-differential extension field. We look at the case where the family of partial differential Eq. (1) is soluble by separation of variables. In this case,

$$\psi(x, y, z) = \phi_1(x)\,\phi_2(y)\,\phi_3(z)$$

and the separation constants p_m, q_m, r_m lead us to the ordinary system of differential equations

$$\delta_3^2\,\phi_3 + r_m^2\,\phi_3 \equiv \frac{\partial^2 \phi_3}{\partial z^2} + r_m^2\,\phi_3 = 0, \tag{13i}$$

$$\delta_2^2\,\phi_2 + q_m^2\,\phi_2 \equiv \frac{\partial^2 \phi_2}{\partial y^2} + q_m^2\,\phi_2 = 0, \tag{13ii}$$

$$\delta_1^2\,\phi_1 + p_m^2\,\phi_1 \equiv \frac{\partial^2 \phi_1}{\partial x^2} + p_m^2\,\phi_1 = 0 \tag{13iii}$$

with

$$\alpha\,p_m^2 + q_m^2 + r_m^2 = 0 \qquad \text{(in this case } \omega = 0\text{)}. \tag{14}$$

Evidently the solutions of (13) which are of the form

$$\phi_3 \equiv \exp(\pm\sqrt{-1}\,r_m z)$$

satisfy the condition of being a complex i-exponential of an i-integral of r_m, i.e. $\delta_3\,\phi_3 = \sqrt{-1}\,r_m\,\phi_3$. In the solution of (13) the p_m, q_m, r_m are determined by the boundary conditions, and consequently the intermediate α-differential extension field $\Omega_m = \Omega_0(p_m, q_m, r_m)$ is determined by the quadratic decomposition (14) of $\omega = 0$ and the boundary conditions. An α-Liouvillian α-differential extension field $\Omega_m(\exp[\pm\sqrt{-1}\,\{p_m x + q_m y + r_m z\}])$ of the α-differential field Ω_m is determined by the complex i-exponential of the integrals

$$\psi(x, y, z) \equiv \phi_1(x)\,\phi_2(y)\,\phi_3(z) = \exp[\pm\sqrt{-1}\,\{p_m x + q_m y + r_m z\}]. \tag{15}$$

On the basis of the results above, we are able to conclude the following: Any elementary solution ψ_m of partial differential equation $\Delta_\alpha \psi = 0$ is classified by an element ω of an α-differential field Ω_0 and a quadratic expression $\omega = \alpha\,p_m^2 + q_m^2 + r_m^2$ for ω such that $\alpha\,\delta_1(p_m) + \delta_2(q_m) + \delta_3(r_m) = 0$. The extension field $\Omega_0(p_m, q_m, r_m) \equiv \Omega_m$ obtained by such a quadratic decomposition of ω is then extended by an α-Liouvillian extension of Ω_m.

This gives the algebraic classification of the elementary solutions of the family $\Delta_\alpha \psi = 0$. Evidently the freedom in the choice of ω and the set (p_m, q_m, r_m) is the algebraic criterion for the boundary conditions necessary for the uniqueness of the solution of $\Delta_\alpha \psi = 0$.

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Isomorphism Invariants for Quotient Categories of Abelian Groups

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In this paper some results of Ensey [1] and Hart [4] are generalized. In [1] Ensey gives necessary and sufficient conditions on the Ulm invariants of two direct sums of countable reduced p-groups for them to be isomorphic in the category of Abelian groups modulo bounded groups. Hart [4] extended Ensey's result to the class of totally projective p-groups.

In Section 1 the original definition of quotient category given by Grothendieck was generalized. The appropriate generalization for our purposes is to let $\hat{\mathscr{S}}$ and $\hat{\mathscr{T}}$ be two Serre classes in an Abelian Category $\hat{\mathscr{C}}$, and to define the category $\hat{\mathscr{C}}(\hat{\mathscr{S}}, \hat{\mathscr{T}})$ to have objects those of $\hat{\mathscr{C}}$ and maps given by

$$\hat{\mathscr{C}}/(\hat{\mathscr{S}}, \hat{\mathscr{T}})(A, B) = \varinjlim_{A/A' \in \hat{\mathscr{S}},\ B' \in \hat{\mathscr{T}}} C(A', B/B').$$

If $\hat{\mathscr{B}}$ is the Serre class of all bounded Abelian groups, and $\hat{\mathscr{A}}$ is the category of Abelian groups then the category $\hat{\mathscr{A}}/(\hat{\mathscr{B}}, \hat{\mathscr{O}})$ provides a natural categorical setting where quasi-isomorphism in $\hat{\mathscr{A}}$ and isomorphism in this category are the same for all Abelian groups.

In Section II we give necessary and sufficient conditions on the Ulm invariants of two totally projective p-groups for them to be isomorphic in the category $\hat{\mathscr{A}}/(\hat{\mathscr{O}}, \hat{\mathscr{B}})$, and use these conditions to show that two totally projective p-groups are isomorphic in $\hat{\mathscr{A}}/(\hat{\mathscr{O}}, \hat{\mathscr{B}})$ if and only if they are isomorphic in $\hat{\mathscr{A}}/(\hat{\mathscr{B}}, \hat{\mathscr{B}}) = \hat{\mathscr{A}}/\hat{\mathscr{B}}$.

If we let the class of bounded p-groups of rank less than ξ for some infinite cardinal ξ be denoted by $\hat{\mathscr{B}}(\xi)$ then in Sections 3 and 4 we prove the theorem.

Theorem. *If G and H are totally projective p-groups and ξ is an infinite cardinal, then $G \approx H$ in $\hat{\mathscr{A}}/(\hat{\mathscr{O}}, \hat{\mathscr{B}}(\xi))$ (or $\hat{\mathscr{A}}/\hat{\mathscr{B}}(\xi)$) if and only if*

(i) *there exists a set I of ordinal numbers and a cardinal number $\xi_1 < \xi$, such that $|I| < \xi$, $f_G(\alpha) \leqq \xi_1 + f_H(\alpha)$ and $f_H(\alpha) \leqq \xi_1 + f_G(\alpha)$ for all $\alpha \in I$, and $f_G(\alpha) = f_H(\alpha)$ for all $\alpha \notin I$, and*

(ii) *there exists an integer $k \geqq 0$ such that for all integers $r \geqq 0$ and ordinal numbers α,*

$$\sum_{j=0}^{r} f_G(\alpha + k + j) \leqq \sum_{j=0}^{r+2k} f_H(\alpha + j) \quad \text{and} \quad \sum_{j=0}^{r} f_H(\alpha + k + j) \leqq \sum_{j=0}^{r+2k} f_G(\alpha + j).$$

A little stronger result is proven in Section 3 for $\xi = \aleph_0$.

I. In this section we will generalize Grothendieck's definition of quotient category [3]. Some of the basic properties of these categories will be stated and we will show that these generalized quotient categories provide a natural setting for the concept of quasi-isomorphism of Abelian groups.

Definition I.1 Let $\hat{\mathscr{C}}$ be an Abelian category and $\hat{\mathscr{S}}$ and $\hat{\mathscr{T}}$ Serre classes of $\hat{\mathscr{C}}$. Then $\hat{\mathscr{C}}/(\hat{\mathscr{S}}, \hat{\mathscr{T}})$ is the category given by the following:

(i) The objects of $\hat{\mathscr{C}}/(\hat{\mathscr{S}}, \hat{\mathscr{T}})$ are the objects of $\hat{\mathscr{C}}$.

(ii) $\hat{\mathscr{C}}/(\hat{\mathscr{S}}, \hat{\mathscr{T}})(A, B) = \varinjlim_{A/A' \in \hat{\mathscr{S}},\, B' \in \hat{\mathscr{T}}} \hat{\mathscr{C}}(A', B/B')$,

where the limit is taken with respect to the ordering $\hat{\mathscr{C}}(A', B/B') \leqq \hat{\mathscr{C}}(A'', B/B'')$ if $A'' \subset A'$ and $B' \subset B''$.

(iii) Composition is defined as follows. Let

$$\bar{f} \in \hat{\mathscr{C}}/(\hat{\mathscr{S}}, \hat{\mathscr{T}})(A, B), \qquad \bar{g} \in \hat{\mathscr{C}}/(\hat{\mathscr{S}}, \hat{\mathscr{T}})(B, C).$$

Then there exist subgroups $A' \subset A$, $B' \subset B$, $B'' \subset B$ and $C' \subset C$ such that $\{A/A', B/B''\} \subset \hat{\mathscr{S}}$, $\{B', C'\} \subset \hat{\mathscr{T}}$ and mappings $f\colon A' \to B/B'$ and $g\colon B'' \to C/C'$ such that f and g represent $\bar{f}$ and $\bar{g}$ respectively. Let $A'' = f^{-1}\left(\frac{B'+B''}{B'}\right)$, and let C'' be given by $C''/C' = \mathrm{Im}(B'' \cap B')$. Then $A/A'' \in \hat{\mathscr{S}}$ and $C'' \in \hat{\mathscr{T}}$. Define $\bar{g} \circ \bar{f}$ to be the mapping induced by the composition

$$A'' \xrightarrow{f'} \frac{B'+B''}{B''} \approx \frac{B''}{B'' \cap B'} \xrightarrow{g'} C/C'',$$

where f' and g' are induced by f and g, respectively.

An easy calculation shows that we have indeed defined a category, and the next proposition shows that some of the structure of $\hat{\mathscr{C}}$ is carried over to $\hat{\mathscr{C}}/(\hat{\mathscr{S}}, \hat{\mathscr{T}})$.

Proposition I.2. *If $\hat{\mathscr{C}}$ is an Abelian category and $\hat{\mathscr{S}}$, $\hat{\mathscr{T}}$ are Serre classes, then $\hat{\mathscr{C}}/(\hat{\mathscr{S}}, \hat{\mathscr{T}})$ is an additive category and has finite direct sums and finite direct products.*

Definition I.3. If $\hat{\mathscr{S}}$ and $\hat{\mathscr{T}}$ are Serre classes of objects of $\hat{\mathscr{C}}$ then $\mathrm{Ext}(\hat{\mathscr{S}}, \hat{\mathscr{T}})$ is the class of objects E of $\hat{\mathscr{C}}$ such that there exists an exact sequence $0 \to T \to E \to S \to 0$ with $S \in \hat{\mathscr{S}}$ and $T \in \hat{\mathscr{T}}$.

The following are readily verified.

Lemma I.4. *If $\hat{\mathscr{T}}$ is a Serre class and $\hat{\mathscr{S}}$ is closed under isomorphisms, then* $\mathrm{Ext}(\mathrm{Ext}(\hat{\mathscr{S}}, \hat{\mathscr{T}}), \hat{\mathscr{T}}) = \mathrm{Ext}(\hat{\mathscr{S}}, \hat{\mathscr{T}})$.

Lemma I.5. *If $\hat{\mathscr{S}}$ is a Serre class and $\hat{\mathscr{T}}$ is closed under isomorphisms, then* $\mathrm{Ext}(\hat{\mathscr{S}}, \mathrm{Ext}(\hat{\mathscr{S}}, \hat{\mathscr{T}})) = \mathrm{Ext}(\hat{\mathscr{S}}, \hat{\mathscr{T}})$.

Lemma I.6. *If $\hat{\mathscr{S}}$ and $\hat{\mathscr{T}}$ are Serre classes, then* $\mathrm{Ext}(\hat{\mathscr{S}}, \hat{\mathscr{T}})$ *is closed under images and subobjects.*

The proofs of Theorems I.7 and I.8 will be omitted as they are easy generalizations of results in [2].

Theorem I.7. *If $\hat{\mathscr{S}}$ and $\hat{\mathscr{T}}$ are Serre classes, $A/A' \in \hat{\mathscr{S}}$, $B' \in \hat{\mathscr{T}}$ and $f: A' \to B/B'$ represents $\bar{f} \in \hat{\mathscr{C}}/(\hat{\mathscr{S}}, \hat{\mathscr{T}})(A, B)$ then*

(i) *$\bar{f} = 0$ if and only if $\operatorname{Im} f \in \operatorname{Ext}(\hat{\mathscr{S}}, \hat{\mathscr{T}})$,*

(ii) *$\bar{f}$ is monic if and only if $\operatorname{Ker} f \in \operatorname{Ext}(\hat{\mathscr{S}}, \hat{\mathscr{T}})$, and*

(iii) *$\bar{f}$ is epic if and only if $\operatorname{Coker} f \in \operatorname{Ext}(\hat{\mathscr{S}}, \hat{\mathscr{T}})$.*

Theorem I.8. *If $\hat{\mathscr{S}}$ and $\hat{\mathscr{T}}$ are Serre classes, A, B objects of $\hat{\mathscr{C}}/(\hat{\mathscr{S}}, \hat{\mathscr{T}})$, and $f: A' \to B/B''$ represents an element $\bar{f} \in \hat{\mathscr{C}}/(\hat{\mathscr{S}}, \hat{\mathscr{T}})(A, B)$ then $\bar{f}$ is an isomorphism if and only if there exist objects A'', B'' of $\hat{\mathscr{C}}/(\hat{\mathscr{S}}, \hat{\mathscr{T}})$ such that $A/A'' \in \hat{\mathscr{S}}$, $B'' \in \hat{\mathscr{T}}$, $(\operatorname{Ker} f \cap A'') \in \hat{\mathscr{T}}$, and if P is a pushout for the diagram*

$$\begin{array}{ccc} B/B' & \twoheadrightarrow & \operatorname{Coker} f \\ \downarrow & & \\ B/B'' & & \end{array}$$

then $P \in \hat{\mathscr{S}}$.

These results then allow us to prove the following theorem.

Theorem I.9. *Let $\hat{\mathscr{A}}$ be the category of Abelian groups, $\hat{\mathscr{B}}$ the Serre class of bounded groups, and A and B be objects of $\hat{\mathscr{A}}$. Then A is quasi-isomorphic to B if and only if $A \approx B$ in $\hat{\mathscr{A}}/(\hat{\mathscr{B}}, \hat{\mathscr{O}})$.*

Proof. Suppose A is quasi-isomorphic to B. Then there exist groups $A' \subset A$, $B' \subset B$, such that $\{A/A', B/B'\} \subset \hat{\mathscr{B}}$ and $A' \approx B'$ in $\hat{\mathscr{A}}$. But $A \approx A' \approx B' \approx B$ in $\hat{\mathscr{A}}/(\hat{\mathscr{B}}, \hat{\mathscr{O}})$.

Suppose $\bar{f} \in \hat{\mathscr{A}}/(\hat{\mathscr{B}}, \hat{\mathscr{O}})(A, B)$ is an isomorphism, and is represented by $f: A' \to B$ where $A/A' \in \hat{\mathscr{B}}$. By Theorem I.8, there exists a group A'' such that $A/A'' \in \hat{\mathscr{B}}$, $\operatorname{Ker} f \cap A'' = 0$ and $\operatorname{Coker} f \in \hat{\mathscr{B}}$. The sequence $A'/A'' > \to B/f(A'') \twoheadrightarrow B/f(A')$ is exact, and since $A'/A'' \in \hat{\mathscr{B}}$ and $B/f(A') \in \hat{\mathscr{B}}$, we have that $B/f(A'') \in \hat{\mathscr{B}}$. Therefore $A'' \approx f(A'')$ is a quasi-isomorphism between A and B.

Notice that quasi-endomorphism, quasi-decomposition, etc., as defined by Reid [10] are just endomorphisms, decompositions, etc., in the category $\hat{\mathscr{A}}/(\hat{\mathscr{B}}, \hat{\mathscr{O}})$. Thus the decomposition theory of torsion-free groups in $\hat{\mathscr{A}}/(\hat{\mathscr{B}}, \hat{\mathscr{O}})$ is equivalent to the quasi-decomposition theory of torsion-free in $\hat{\mathscr{A}}$. The advantage of the setting $\hat{\mathscr{A}}/(\hat{\mathscr{B}}, \hat{\mathscr{O}})$ is that category theory in general is available for application.

The following example shows that the category $\hat{\mathscr{C}}/(\hat{\mathscr{S}}, \hat{\mathscr{T}})$ is not in general an Abelian category.

Example I.10. Let $\hat{\mathscr{C}} = \hat{\mathscr{A}}$, $\hat{\mathscr{S}} = \hat{\mathscr{B}}$ and $\hat{\mathscr{T}} = \hat{\mathscr{O}}$. If A is an unbounded p-group, and $f: A \to pA: a \to pa$, then f is epic with kernel $A[p]$. By Theorem I.7, $\bar{f}$ is both monic and epic. But if $A' \subset A$ such that $A' \neq 0$, then $A' \cap A[p] \neq 0$. Therefore by Theorem I.8, $\bar{f}$ is not an isomorphism. However in an Abelian category, maps that are monic and epic are isomorphisms. Thus $\hat{\mathscr{A}}/(\hat{\mathscr{B}}, \hat{\mathscr{O}})$ is not an Abelian category.

II. All groups considered in this section will be p-groups for a fixed prime p. If G is such a group and α is an ordinal number, then $f_G(\alpha)$ denotes the α^{th} Ulm invariant of G, defined as the dimension of the vector space $(p^\alpha G)[p]/(p^{\alpha+1} G)[p]$ over the field of integers modulo p. Also $G^\alpha = p^{\omega\alpha} G$, and G_α is the α^{th} Ulm factor, $G^\alpha/G^{\alpha+1}$ of G.

In this section we will find an equivalence relation on the Ulm invariants of totally projective p-groups such that two such groups are isomorphic in $\hat{\mathscr{A}}/(\hat{\mathscr{O}}, \hat{\mathscr{B}})$ if and only if their Ulm invariants are equivalent. Since this equivalence relation is the same as that given by Hart in [4], we have that two totally projective p-groups are isomorphic in $\hat{\mathscr{A}}/\hat{\mathscr{B}}$ if and only if they are isomorphic in $\hat{\mathscr{A}}/(\hat{\mathscr{O}}, \hat{\mathscr{B}})$.

If α is an ordinal number, an Abelian p-group G is *p^α-projective* if and only if $p^\alpha \operatorname{Ext}(G, H) = 0$ for all groups H. An Abelian p-group is *totally projective* if and only if it is reduced and $G/p^\alpha G$ is p^α-projective for every ordinal α. Nunke [7] showed that the class of totally projective groups is closed under the operations of forming arbitrary direct sums and taking direct summands, that for each ordinal α, G is totally projective if and only if $p^\alpha G$ and $G/p^\alpha G$ are totally projective, and that the class of direct sums of countable reduced p-groups is precisely the class of totally projective groups of length less than or equal to Ω, the first uncountable ordinal. These results imply that if G is totally projective then the Ulm factors G_α of G are direct sums of cyclic groups.

The following conditions on groups G and H will be used throughout the rest of this paper.

(I) There is an integer $k \geqq 0$ such that for all integers $n \geqq 0$ and $r \geqq 0$,

$$\sum_{j=0}^{r} f_G(n+k+j) \leqq \sum_{j=0}^{r+2k} f_H(n+j)$$

and

$$\sum_{j=0}^{r} f_H(n+k+j) \leqq \sum_{j=0}^{r+2k} f_G(n+j).$$

(II) For all $\alpha \geqq \omega$, $f_G(\alpha) = f_H(\alpha)$

(III) There is an integer $k \geqq 0$ such that for all integers $r \geqq 0$ and ordinal numbers α,

$$\sum_{j=0}^{r} f_G(\alpha+k+j) \leqq \sum_{j=0}^{r+2k} f_H(\alpha+j)$$

and

$$\sum_{j=0}^{r} f_H(\alpha+k+j) \leqq \sum_{j=0}^{r+2k} f_G(\alpha+j).$$

While the following result does not appear in [1], the proof is essentially Ensey's.

Lemma II.1. *Let G and H be direct sums of cyclic groups which satisfy I. Then $G = G_0 \oplus \cdots \oplus G_{2k}$ and $H = H_0 \oplus \cdots \oplus H_{2k}$ where $p^k G_0 \approx H_0$, $p^{k-1} G_1 \approx H_1, \ldots, G_k \approx H_k$, $G_{k+1} \approx p H_{k+1}, \ldots, G_{2k} \approx p^k H_{2k}$.*

Proof. Let G^* and H^* be such that $pG^* = G$, $pH^* = H$, and

$$f_{G^*}(0) = |G^*||H^*|\aleph_0 = f_{H^*}(0).$$

For $r \geqq 0$,

$$\sum_{j=0}^{r} f_{G^*}(k+j) \leqq |G^*|\aleph_0 \leqq f_{H^*}(0) \leqq \sum_{j=0}^{r+2k} f_{H^*}(j).$$

If $n > 0$, then

$$\sum_{j=0}^{r} f_{G^*}(n+k+j) = \sum_{j=0}^{r} f_{pG^*}((n-1)+k+j) \leqq \sum_{j=0}^{r+2k} f_{pH^*}((n-1)+j) = \sum_{j=0}^{r+2k} f_{H^*}(n+j).$$

Similarly,

$$\sum_{j=0}^{r} f_{H^*}(n+k+j) \leqq \sum_{j=0}^{r+2k} f_{G^*}(n+j)$$

for all $n \geqq 0$ and $r \geqq 0$. Now by Lemma 4.1 in [1], $G^* = G_0^* \oplus \cdots \oplus G_{2k}^*$ and $H^* = H_0^* \oplus \cdots \oplus H_{2k}^*$ such that $p^k G_0^* \approx H_0^*$, $p^{k-1} G_1^* \approx H_1^*, \ldots, G_k^* \approx H_k^*$, $G_{k+1}^* \approx pH_{k+1}^*, \ldots, G_{2k}^* \approx p^k H_{2k}^*$.

Multiplication by p yields the desired result.

Even though the following corollary and Lemma II.3 are immediate results of Theorem II.4 they will be proven in order to simplify the proof of II.4.

Corollary II.2. *If G and H are direct sums of cyclic groups satisfying* (I), *then $G \approx H$ in $\hat{\mathscr{A}}/(\hat{\mathscr{C}}, \hat{\mathscr{B}})$.*

Proof. By Lemma II.1 we may write $G = G_0 \oplus \cdots \oplus G_{2k}$ and $H = H_0 \oplus \cdots \oplus H_{2k}$ such that $p^k G_0 \approx H_0, \ldots, G_k \approx H_k, \ldots, G_{2k} \approx p^k H_{2k}$. Hence

$$G_0/G_0[p^k] \approx H_0, \quad G_1/G_1[p^{k-1}] \approx H, \ldots, G_k \approx H_k,$$

$$G_{k+1} \approx H_{k+1}/H_{k+1}[p], \ldots, G_{2k} \approx H_{2k}/H_{2k}[p^k].$$

Now if we let

$$B = \sum_{i=0}^{k-1} G_i[p^{k-i}] \quad \text{and} \quad C = \sum_{i=1}^{k} H_{k+1}[p^i],$$

then

$$G/B \approx G_0/G_0[p^k] \oplus \cdots \oplus G_{k-1}/G_{k-1}[p] \oplus \sum_{i=k}^{2k} G_i$$

$$\approx \sum_{i=0}^{k} H_i \oplus H_{k+1}/H_{k+1}[p] \oplus \cdots \oplus H_{2k}/H_{2k}[p^k] \approx H/C.$$

Since B and C are bounded by p^k, $G \approx H$ in $\hat{\mathscr{A}}/(\hat{\mathscr{C}}, \hat{\mathscr{B}})$.

Lemma II.3. *Suppose G and H are totally projective p-groups which satisfy* (I) *and* (II). *Then $G \approx H$ in $\hat{\mathscr{A}}/(\hat{\mathscr{C}}, \hat{\mathscr{B}})$.*

Proof. The first Ulm factors, G_0 and H_0 of G and H respectively, satisfy the hypothesis of Lemma II.1. Therefore we may write $G_0 = G_{0,0} \oplus \cdots \oplus G_{0,2k}$ and $H_0 = H_{0,0} \oplus \cdots \oplus H_{0,2k}$ such that $p^k G_{0,0} \approx H_{0,0}, \ldots, G_{0,k} \approx H_{0,k}, \ldots, G_{0,2k} \approx$

$p^k H_{0,2k}$. Therefore the final ranks of $G_{0,i}$ and $H_{0,i}$ are equal for $i\in\{0,\ldots,2k\}$. Now write

$$G_\alpha=\sum_{i=0}^{2k} S_{\alpha,i} \quad \text{and} \quad H_\alpha=\sum_{i=0}^{2k} T_{\alpha,i} \quad \text{for } \alpha>0,$$

such that for each $i\in\{0,\ldots,2k\}$ the sequences $G_{0,i}$, $S_{1,i},\ldots,S_{\alpha,i},\ldots$ and $H_{0,i}$, $T_{1,i},\ldots,T_{\alpha,i},\ldots$ satisfy Hill's existance theorem [5] and $S_{\alpha,i}\approx T_{\alpha,i}$ for all $\alpha>0$. Let K_i be a totally projective p-group with Ulm factors $(K_i)_0=G_{0,i}$ and $(K_i)_\alpha=S_{\alpha,i}$ for $\alpha>0$, $i\in\{0,\ldots,2k\}$. Similarly let L_i be a totally projective p-group with Ulm factors $(L_i)_0=H_{0,i}$, and $(L_i)_\alpha=T_{\alpha,i}$ for $\alpha>0$ and $i\in\{0,\ldots,2k\}$. This can be done by [5]. By [6]

$$\sum_{i=0}^{2k} K_i\approx G \quad \text{and} \quad \sum_{i=0}^{2k} L_i\approx H.$$

Corollary 4.19 of [1] implies that $(K_i/K_i[p^{k-i}])_\alpha\approx(K_i)_\alpha$ for $\alpha>0$, and

$$(K_i/K_i[p^{k-i}])_0\approx p^{k-i}K_i/p^\omega K_i$$

for $i\in\{0,\ldots,k\}$. Similarly, $(L_{i+k}/L_{i+k}[p^i])_\alpha\approx(L_i)_\alpha$ for $\alpha>0$, and

$$(L_{i+k}/L_{i+k}[p^i])_0\approx p^i L_{i+k}/p^\omega L_{i+k}$$

for $i\in\{0,\ldots,k\}$. Therefore $(K_i/K_i[p^{k-i}])_\alpha\approx(L_i)$ and

$$(L_{i+k}/L_{i+k}[p^i])_\alpha\approx(K_{i+k})_\alpha$$

for $i\in\{0,\ldots,k\}$ and for all α. By [6], $K_i/K_i[p^{k-i}]\approx L_i$ and $L_{i+k}/L_{i+k}[p^i]\approx K_{i+k}$ for $i\in\{0,\ldots,k\}$. Therefore $K_i\approx L_i$ in $\hat{\mathscr{A}}/(\hat{\mathscr{O}},\hat{\mathscr{B}})$ for $i\in\{0,\ldots,2k\}$, and hence $G\approx H$ in $\hat{\mathscr{A}}/(\hat{\mathscr{O}},\hat{\mathscr{B}})$.

Theorem II.4. *If G and H are totally projective p-groups, then they satisfy condition* (III) *if and only if $G\approx H$ in $\hat{\mathscr{A}}/(\hat{\mathscr{O}},\hat{\mathscr{B}})$.*

Proof. Assume G and H satisfy condition (III). By [4] $G\approx H$ in $\hat{\mathscr{A}}/\hat{\mathscr{B}}$. Therefore there exist groups X, Y, A, B such that $A\subset X\subset G$, $B\subset Y\subset H$, G/X, H/Y, A and B are bounded, and $X/A\approx Y/B$. Now $p^\omega X=p^\omega G$ and $X/p^\omega G\subset G/p^\omega G$ is a direct sum of cyclics. Therefore $p^\omega X$ and $X/p^\omega X$ are totally projective. This implies that X is totally projective. Similarly $Y$is totally projective. By [1] G and X satisfy the hypothesis of Lemma II.3 and therefore $G\approx X$ in $\hat{\mathscr{A}}/(\hat{\mathscr{O}},\hat{\mathscr{B}})$ by Lemma II.3. Similarly $H\approx Y$in $\hat{\mathscr{A}}/(\hat{\mathscr{O}},\hat{\mathscr{B}})$. But X is clearly isomorphic to Yin $\hat{\mathscr{A}}/(\hat{\mathscr{O}},\hat{\mathscr{B}})$. Therefore $H\approx G$ in $\hat{\mathscr{A}}/(\hat{\mathscr{O}},\hat{\mathscr{B}})$.

Since $G\approx H$ in $\hat{\mathscr{A}}/(\hat{\mathscr{O}},\hat{\mathscr{B}})$ implies $G\approx H$ in $\hat{\mathscr{A}}/\hat{\mathscr{B}}$, [1]. Theorem 4.13 gives the converse and completes the proof of the theorem.

Corollary II.5. *In the class of totally projective p-groups, isomorphism in $\hat{\mathscr{A}}/(\hat{\mathscr{O}},\hat{\mathscr{B}})$ and isomorphism in $\hat{\mathscr{A}}/\hat{\mathscr{B}}$ are equivalent.*

Whether or not Corollary II.5 holds with the class of totally projective p-groups replaced by the class of p-groups is not known. However Example I.10 shows that a map $f\colon G\to H$ in $\hat{\mathscr{A}}$ can represent an isomorphism in $\hat{\mathscr{A}}/\hat{\mathscr{B}}$ but not in $\hat{\mathscr{A}}/(\hat{\mathscr{O}},\hat{\mathscr{B}})$.

III. Two reduced p-groups G and H are said to be *Ulm equivalent* if $f_G(\alpha)=f_H(\alpha)$ for all α; that is, if G and H have the same Ulm invariants. To extent this notion to the quotient categories $\hat{\mathscr{A}}/(\hat{\mathscr{S}},\hat{\mathscr{T}})$, we say that G and H are *Ulm equivalent* in $\hat{\mathscr{A}}/(\hat{\mathscr{S}},\hat{\mathscr{T}})$ if G and H are isomorphic in $\hat{\mathscr{A}}/(\hat{\mathscr{S}},\hat{\mathscr{T}})$ to groups which are Ulm equivalent.

Now Ensey's theorem 4.13 [1] readily implies that if two reduced p-groups are Ulm equivalent in $\hat{\mathscr{A}}/\hat{\mathscr{B}}$, then they satisfy (III). The converse is true for totally projective groups, which is a consequence of Hart's result [4]. It is not known whether or not the converse is true in general but we rather suspect that it is. However, we can prove corresponding results for some of our other quotient categories, namely in $\hat{\mathscr{A}}/\hat{\mathscr{B}}(\aleph_0)$ and $\hat{\mathscr{A}}/(\hat{\mathscr{O}},\hat{\mathscr{B}}(\aleph_0))$. These results will in turn yield an Ulm's theorem for totally projective groups in these categories.

Let ξ be a cardinal number. The following condition (IV ξ) on the p-groups G and H will play an important role throughout the rest of this paper.

(IV ξ) There is a set I of ordinals and a cardinal $\xi_1<\xi$ such that, $|I|<\xi$, $f_G(\alpha)=f_H(\alpha)$ for all $\alpha\notin I$, and $f_G(\alpha)\leqq\xi_1+f_H(\alpha)$ and $f_H(\alpha)\leqq\xi_1+f_G(\alpha)$ for all $\alpha\in I$.

The condition (IV ξ) just says that the Ulm invariants of G and H differ at fewer than ξ ordinals, and at those, differ by an amount bounded away from ξ. Our principle result will be that two reduced p-groups G and H are Ulm equivalent in $\hat{\mathscr{A}}/\hat{\mathscr{B}}(\aleph_0)$ (and in $\hat{\mathscr{A}}/(\hat{\mathscr{O}},\hat{\mathscr{B}}(\aleph_0))$ if and only if they satisfy (IV $\aleph_0$). Before proceeding to this it is convenient to prove the following theorem, which is perhaps of interest in itself.

Theorem III.1 *Let $\hat{\mathscr{A}}_p$ be the category of p-groups, $\hat{\mathscr{F}}$ the class of finite p-groups, and $\hat{\mathscr{S}}$ any nonzero Serre class of p-groups. Then the categories $\hat{\mathscr{A}}_p/(\hat{\mathscr{F}},\hat{\mathscr{S}})$ and $\hat{\mathscr{A}}_p/(\hat{\mathscr{O}},\hat{\mathscr{S}})$ are equivalent.*

Proof. Let $\varphi_{A,B}\colon \hat{\mathscr{A}}_p/(\hat{\mathscr{O}},\hat{\mathscr{S}})(A,B)\to\hat{\mathscr{A}}_p/(\hat{\mathscr{F}},\hat{\mathscr{S}})(A,B)$ for any pair of elements (A,B) of $\hat{\mathscr{A}}_p$, be given by the definition of the direct limit. If $\bar{f}\in\hat{\mathscr{A}}_p/(\hat{\mathscr{O}},\hat{\mathscr{S}})(A,B)$ and represented by $f\colon A\to B/B'$ then $\varphi_{A,B}(\bar{f})$ is the map represented by f in $\mathscr{A}/(\hat{\mathscr{F}},\hat{\mathscr{S}})(A,B)$. Before defining the functors that give the equivalence of the categories, we will show that $\varphi_{A,B}$ is an isomorphism of groups for each pair of objects (A,B) of $\hat{\mathscr{A}}_p$. First $\varphi_{A,B}$ is a monomorphism. Suppose $\varphi_{A,B}(\bar{f})=0$, where $\bar{f}$ is represented by $f\colon A\to B/B'$. Then there exist subgroups A' of A and B'' of B such that $(A'\xrightarrow{f}B/B'\to B/B'')=0$, where $A/A'\in\hat{\mathscr{F}}$, $B''\in\hat{\mathscr{S}}$. But by Lemma 16.5 of [9] there exists $A''\subset A'$ such that $A/A''\in\hat{\mathscr{F}}$ and $A''\oplus A_1=A$ for some $A_1\subset A$. Define $f_1\colon A\to B/B''$ by $f_1(A_1)=0$ and $f_1(A'')=f\downarrow A''$. Clearly $f_1=0$. Let $B'''/B''=f(A_1)$, then the diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & B/B' & \longrightarrow & B/B'' \\ & {\scriptstyle f_1}\searrow & & \nearrow & \\ & & B/B'' & & \end{array}$$

commutes. Therefore $\bar{f}=\bar{f}_1$ in $\hat{\mathscr{A}}_p/(\hat{\mathscr{O}},\hat{\mathscr{S}})$ and hence $\varphi_{A,B}$ is monomorphism.

To show $\varphi_{A,B}$ is an epimorphism, let $\bar{f}\in\hat{\mathscr{A}}_p/(\hat{\mathscr{F}},\hat{\mathscr{P}})(A,B)$ and be represented by $f\colon A'\to B/B'$. Define f_1 as above taking $A\to B/B'$. Clearly $(\bar{f}_1)=\bar{f}$. Therefore $\varphi_{A,B}$ is an epimorphism.

Let $S\colon \hat{\mathscr{A}}_p/(\hat{\mathscr{O}},\hat{\mathscr{P}})\to\hat{\mathscr{A}}_p/(\hat{\mathscr{F}},\hat{\mathscr{P}})$ be defined by: $S(A)=A$ for $A\in\hat{\mathscr{A}}_p/(\hat{\mathscr{O}},\hat{\mathscr{P}})$ and if $\bar{f}\in\hat{\mathscr{A}}_p/(\hat{\mathscr{O}},\hat{\mathscr{P}})(A,B)$ then $S(\bar{f})=\varphi_{A,B}(\bar{f})$. Let $T\colon \hat{\mathscr{A}}_p/(\hat{\mathscr{F}},\hat{\mathscr{P}})\to\hat{\mathscr{A}}_p/(\hat{\mathscr{O}},\hat{\mathscr{P}})$ be defined by: $T(A)=A$ for $A\in\hat{\mathscr{A}}_p(\hat{\mathscr{F}},\hat{\mathscr{P}})$ and if $\bar{f}\in\hat{\mathscr{A}}_p/(\hat{\mathscr{F}},\hat{\mathscr{P}})(A,B)$ then $T(\bar{f})=\varphi^{-1}_{A,B}(\bar{f})$. It is clear that $ST=I$ and $TS=I$. Therefore $\hat{\mathscr{A}}_p/(\hat{\mathscr{O}},\hat{\mathscr{P}})$ is equivalent to $\hat{\mathscr{A}}_p/(\hat{\mathscr{F}},\hat{\mathscr{P}})$.

Theorem III.2. *If G and H are reduced p-groups then G and H satisfy condition* (IV $\aleph_0$) *if and only if G and H are Ulm equivalent in $\hat{\mathscr{A}}/(\hat{\mathscr{O}},\hat{\mathscr{F}})\approx\hat{\mathscr{A}}/\hat{\mathscr{F}}$.*

Proof. Suppose G and H satisfy condition (IV $\aleph_0$). Let α be the smallest element of I and β be such that $\omega\beta\leqq\alpha<\omega(\beta+1)$. Suppose $f_G(\alpha)>f_H(\alpha)$, let $B=\sum_{n=1}^{\infty}B_n$ be basic in G^β, where B_n is a direct sum of cyclics of order p^n. Now $\alpha=\omega\beta+k$, therefore let X be the direct sum of $f_G(\alpha)-f_H(\alpha)$ cyclic summands of B_{k+1}. Then $f_{G/X}(\gamma)=f_H(\gamma)$ for $\gamma\in I-\{\alpha\}$, and $f_{G/X}(\gamma)=f_G(\gamma)$ if $\gamma\neq\alpha$. Therefore by an easy induction on the cardinality of I we have that G is Ulm equivalent to H in $\hat{\mathscr{A}}/(\hat{\mathscr{O}},\hat{\mathscr{F}})$.

Suppose G and H are Ulm equivalent in $\hat{\mathscr{A}}/(\hat{\mathscr{O}},\hat{\mathscr{F}})$. By an easy induction it suffices to show that G and G/X satisfy condition (IV $\aleph_0$) where X is cyclic of order p. If $X\subset(G^\beta-G^{\beta+1})$ then X is contained in a finite summand of G^β. Therefore we may as well assume $\beta=0$ and G is a direct sum of cyclic groups. But in this case it is clear that G and G/X satisfy condition (IV $\aleph_0$) and this completes the proof of the theorem.

The first part of the proof of Theorem III.2 can be used to prove the following result.

Theorem III.3. *If G and H are reduced p-groups which satisfy condition* (IV ξ) *and $|I|<\aleph_0$ then G and H are Ulm equivalent in $\hat{\mathscr{A}}/(\hat{\mathscr{O}},\hat{\mathscr{B}}(\xi))$.*

Corollary III.4. *Suppose G and H are totally projective p-groups the $G\approx H$ in $\hat{\mathscr{A}}/\hat{\mathscr{F}}$ if and only if G and H satisfy condition* (IV $\aleph_0$).

Proof. This corollary follows immediately from Theorem III.2 and [6] if we show that G/A is totally projective when G is totally projective and A is finite. Again by an easy induction if suffices to assume A is cyclic of order p. Suppose $A\subset p^\alpha G-(p^{\alpha+1}G)$ then $p^\alpha G=A\oplus L$ and $p^\alpha(G/A)=p^\alpha G/A\approx L$ and therefore $p^\alpha(G/A)$ is totally projective. Now $(G/A)/p^\alpha(G/A)\approx G/p^\alpha G$ and hence is totally projective. Therefore G/A is totally projective and this completes the proof of the corollary.

IV. In this chapter ξ will always denote an infinite cardinal number. The main result is a generalization of Corollary III.5. It should be noted that condition (IV $\aleph_0$) implies condition (III). However condition (IV ξ) for $\xi>\aleph_0$ does not imply condition (III). Therefore, necessary and sufficient conditions for two totally projective p-groups to be isomorphic in $\hat{\mathscr{A}}/(\hat{\mathscr{O}},\hat{\mathscr{B}}(\xi))$ and $\hat{\mathscr{A}}/\hat{\mathscr{B}}(\xi)$,

have to contain condition (III) as well as condition (IV ξ). In order to prove this we will need the following generalization of Lemma II.1.

Lemma IV.1. *Let G and H be direct sums of cyclic groups which satisfy condition* (I) *and* (IV ξ) *for some cardinal ξ with ξ_1 infinite. Then $G=G_0\oplus\cdots\oplus G_{2k}$ and $H=H_0\oplus\cdots\oplus H_{2k}$ with $|G_i|\leqq\xi_1|I|$ and $|H_i|\leqq\xi_1|I|$ for $i\neq k$, and*

$$p^k G_0\approx H_0, p^{k-1}G_1\approx H_1, \ldots, G_k\approx H_k, \ldots, G_{2k}\approx p^k H_{2k}.$$

Proof. By Lemma II.1 we may write $G=G_0'\oplus\cdots\oplus G_{2k}'$ and $H=H_0'\oplus\cdots\oplus H_{2k}'$ with

$$p^k G_0'\approx H_0', \ldots, G_k'\approx H_k', \ldots, G_{2k}'\approx p^k H_{2k}'.$$

For each $i\in\{0,1,\ldots,2k\}$, let $I_i=\{\alpha\,|\,f_{G_i'}(\alpha)\leqq\xi_1\}$, $G_i''=\sum_{\alpha\in I_i}\big(\sum_{f_{G_i'}(\alpha)}C(p^\alpha)\big)$, and

$$\bar{G}_i=\sum_{\alpha\notin I_i}\Big(\sum_{f_{G_i'}(\alpha)}C(p^\alpha)\Big).$$

Define I_i', H_i'', and $\bar{H}_i$ similarly for each $i\in\{0,\ldots,2k\}$. The α^{th} Ulm invariant of $\sum_{i=0}^{2k}\bar{G}_i$ is the same as the α^{th} Ulm invariant of G if $f_G(\alpha)>\xi_1$, and similarly $f_{\sum_{i=0}^{2k}\bar{H}_i}(\alpha)=f_H(\alpha)$ if $f_H(\alpha)>\xi_1$. If $f_G(\alpha)\leqq\xi_1$ then $f_{\sum_{i=0}^{2k}\bar{G}_i}(\alpha)=0$ and similarly for $\sum_{i=0}^{2k}\bar{H}_i$. But if $f_G(\alpha)>\xi_1$ then $f_H(\alpha)=f_G(\alpha)$, and by Ulm's Theorem $\sum_{i=0}^{2k}\bar{H}_i\approx\sum_{i=0}^{2k}\bar{G}_i$. If $\alpha+(k-i)\in I_i$ for $i\in\{0,\ldots,k\}$ then $\alpha\in I_i'$, and if $\alpha+(i-k)\in I_i'$ for $i\in\{k,\ldots,2k\}$ then $\alpha\in I_i$. Therefore it is easy to see that

$$p^k G_0''\approx H_0'', p^{k-i}G_1''\approx H_1'', \ldots, G_k''\approx H_k'', \ldots, G_{2k}''\approx p^k H_{2k}''$$

and that $|G_i''|\leqq\xi_1|I|$ and $|H_i''|\leqq\xi_1|I|$. Now by letting

$$H_k=H_k''\oplus\sum_{i=0}^{2k}\bar{H}_i,\qquad G_k=G_k''\oplus\sum_{i=0}^{2k}\bar{G}_i,\qquad H_i=H_i''$$

and $G_i=G_i''$ for $i\neq k$, we get the statement of the lemma.

Theorem IV.2. *If G and H are totally projective p-groups which satisfy conditions* (III) *and* (IV ξ) *for some cardinal ξ with ξ_1 infinite, then $G\approx H$ in $\mathscr{A}/(\hat{\mathscr{O}},\hat{\mathscr{B}}(\xi))$.*

Proof. Let $\alpha=\min\{\beta\,|\,\mathrm{fr}(G_\beta)<\xi\}$. Then $\alpha=\min\{\beta\,|\,\mathrm{fr}(H_\beta)<\xi\}$ by condition (IV ξ). For each $\beta<\alpha$ we have that $\mathrm{fr}(G_\beta)=\mathrm{fr}(H_\beta)$, and hence we may write $G_\beta=G_\beta'\oplus G_\beta''$ and $H_\beta=H_\beta'\oplus H_\beta''$ with

$$\mathrm{fr}\,G_\beta'=\mathrm{fr}\,G_\beta''=\mathrm{fr}\,G_\beta=\mathrm{fr}\,H_\beta=\mathrm{fr}\,H_\beta''=\mathrm{fr}\,H_\beta',\ H_\beta'\approx G_\beta'$$

satisfying conditions (III) and (IV ξ). This can be done since G_β and H_β are direct sums of cyclic groups. Since G_α is a direct sum of cyclic groups, we may write $G_\alpha=\sum_{i=1}^{\infty}S_i$, where S_i is a direct sum of cyclic groups of order p^i. Let $S_i'=S_i$ if

$|S_i| \leqq \xi_1$ and $S_i' = \sum_{\xi_1} C(p^i)$ if $|S_i| > \xi_1$, and let S_i'' be a complementary summand of S_i' in S_i. If we let $G_\alpha' = \sum_{i=1}^{\infty} S_i'$, $G_\alpha'' = \sum_{i=1}^{\infty} S_i''$, and H_α' and H_α'' be defined similarly then we have,

$$G_\alpha = G_\alpha' \oplus G_\alpha'', H_\alpha = H_\alpha' \oplus H_\alpha'', \text{fr}(G_\alpha') = \text{fr}(G_\alpha) = |G_\alpha'|, \text{fr}(H_\alpha) = \text{fr}(H_\alpha') = |H_\alpha'|,$$

H_α' and G_α' satisfy conditions (III) and (IV ξ), and $H_\alpha'' \approx G_\alpha''$. By [5], there exists totally projective p-groups K_G, L_G, K_H, and L_H such that

$$(K_G)_\beta = \begin{cases} G'' & \text{if } \beta \leqq \alpha \\ 0 & \text{if } \beta > \alpha, \end{cases} \qquad (K_H)_\beta = \begin{cases} H_\beta'' & \text{if } \beta \leqq \alpha \\ 0 & \text{if } \beta > \alpha, \end{cases}$$

$$(L_G)_\beta = \begin{cases} G_\beta' & \text{if } \beta \leqq \alpha \\ G_\beta & \text{if } \beta > \alpha, \end{cases} \qquad (L_H)_\beta = \begin{cases} H_\beta' & \text{if } \beta \leqq \alpha \\ H_\beta & \text{if } \beta > \alpha. \end{cases}$$

By [6] $K_G \oplus L_G \approx G$ and $K_H \oplus L_H \approx H$. Now $(L_G)^\alpha$ and $(L_H)^\alpha$ satisfy condition (III), $|(L_G)^\alpha| < \xi$, $|(L_H)^\alpha| < \xi$, and $(L_G)^\alpha$ and $(L_H)^\alpha$ are totally projective. Therefore there exist bounded subgroups B and C of $(L_G)^\alpha$ and $(L_H)^\alpha$ respectively such that $(L_G)^\alpha/B \approx (L_H)^\alpha/C$. This implies by [6] that $L_G/B \approx L_H/C$. Therefore we may assume without loss of generality that $\lambda(G) = \lambda(H)$ and that $\text{fr}\, G_\beta \geqq \xi$ and $\text{fr}\, H_\beta \geqq \xi$ for all $\beta \leqq \lambda(G) = \lambda(H)$.

By Lemma IV.1 we may write

$$G_\alpha = G_{\alpha,0} \oplus \cdots \oplus G_{\alpha,2k} \quad \text{and} \quad H_\alpha = H_{\alpha,0} \oplus \cdots \oplus H_{\alpha,2k}$$

with

$$P^k G_{\alpha,0} \approx H_{\alpha,0}, \ldots, G_{\alpha,k} \approx H_{\alpha,k}, \ldots, G_{\alpha,2k} \approx p^k H_{\alpha,2k},$$

and $|G_{\alpha,i}| \leqq \xi_1 |I|$ and $|H_{\alpha,i}| \leqq \xi_1 |I|$ for $i \neq k$ and $\alpha <$ type (G). Now for each $\alpha <$ type (G) we may write

$$G_{\alpha,k} = \sum_{i \in I'} G_\alpha^i, \quad H_{\alpha,k} = \sum_{i \in I'} H_\alpha^i$$

where

$$H_\alpha^i \approx G_\alpha^i, \text{fr}\, H_\alpha^i = \text{fr}\, G_\alpha^i = \text{fr}\, G_{\alpha,k} = \text{fr}\, H_{\alpha,k} = \text{fr}\, G_\alpha$$

and

$$I' = \{\alpha | I \cap \{\beta | \omega\alpha \leqq \beta < \omega(\alpha+1)\} \neq \emptyset\}$$

where I is as in condition (IV ξ). By [5] there exists totally projective p-groups L_∞, K_∞, L_i, and K_i for each $i \in I'$ given by

$$(L_i)_\alpha = \begin{cases} H_\alpha^i & \text{if } \alpha < i \\ 0 & \text{if } \alpha > i \\ H_{i,0} \oplus \cdots \oplus H_{i,k-1} \oplus H_{i,k+1} \oplus \cdots \oplus H_{i,2k} & \text{if } \alpha = i, \end{cases}$$

$$(K_i)_\alpha = \begin{cases} G_\alpha^i & \text{if } \alpha < i \\ 0 & \text{if } \alpha > i \\ G_{i,0} \oplus \cdots \oplus G_{i,k-1} \oplus G_{i,k+1} \oplus \cdots \oplus G_{i,2k} & \text{if } \alpha = i, \end{cases}$$

$$(L_\infty)_\alpha = \sum_{i \leqq \alpha} H_\alpha^i, \text{ and } (K_\infty)_\alpha = \sum_{i \leqq \alpha} G_\alpha^i.$$

Now

$$\Big(\sum_{i\in I'\cup\infty} L_i\Big)=\sum_{\alpha<i} H^i_\alpha\oplus\sum_{\alpha\geqq i} H^i_\alpha\oplus H_{\alpha,0}\oplus\cdots\oplus H_{\alpha,k-1}\oplus H_{\alpha,k+1}\oplus\cdots\oplus H_{\alpha,2k}\approx H_\alpha$$

and similarly $\Big(\sum_{i\in I'\cup\infty} K_i\Big)_\alpha\approx G_\alpha$. Therefore by [6] $H\approx\sum_{i\in I'\cup\infty} L_i$ and $G\approx\sum_{i\in I'\cup\infty} K_i$. For each $i\in I$ let $C_i=\sum_{j=k}^{2k} H_{i,j}[p^{j-k}]$ and $B_i=\sum_{j=0}^{k} G_{i,j}[p^{k-j}]$. Let $C=\sum_{i\in I'} C_i$ and $B=\sum_{i\in I'} B_i$. Then

$$G/B\approx\Big(\sum_{i\in I'} K_i\Big)/\Big(\sum_{i\in I'} B_i\Big)\approx\sum_{i\in I'}(K_i/B_i)\approx\sum_{i\in I'}(L_i/G_i)\approx H/C,\quad |B|<\xi,$$

and $|C|<\xi$. Therefore $G\approx H$ in $\hat{\mathscr{A}}/(\hat{\mathscr{O}},\hat{\mathscr{B}}(\xi))$.

This then completes the proof of the sufficiency of conditions (III) and (IV ξ) for two groups to be isomorphic in $\hat{\mathscr{A}}/(\hat{\mathscr{O}},\hat{\mathscr{B}}(\xi))$. Condition (III) is necessary by Theorem II.4. Showing that condition (IV ξ) is necessary will be done in two parts. Theorem IV.5 shows that $f_G(\alpha)\leqq\xi_1+f_H(\alpha)$ and $f_H(\alpha)\leqq\xi_1+f_G(\alpha)$ for all $\alpha<\lambda(G)$ and Corollary IV.7 shows that $|I|<\xi$. The following proposition is needed in the proof of Theorem IV.5.

Proposition IV.3. *Let G be a totally projective p-group, B an infinite subgroup of G, and C be the cokernel of the map $(p^\alpha G)[p]\to(p^\alpha(G/B))[p]$, induced by the natural map $G\to G/B$. Then $|B|\geqq|C|$.*

Proof. Assume that if γ is an ordinal number, G, B and C are as in the hypothesis of the proposition, and $\lambda(G)<\gamma$ then $|B|\geqq|C|$.

Case I. Assume γ is a limit ordinal and $\lambda(G)=\gamma$. Since G is totally projective $G=\sum G_i$ where $\lambda(G_i)<\gamma$. Let B_i be the image of B under the projection of G onto G_i, let C' be the cokernel of the map $(p^\alpha G)[p]\to p^\alpha(G/\sum B_i)[p]$, and suppose $|C'|\leqq|B|$. The following diagram commutes, where f is induced by the mapping $G/B\twoheadrightarrow G/\sum B_i$ and g is given by the definition of the cokernel.

$$\begin{array}{ccccc}
(P^\alpha G)[p] & \longrightarrow & p^\alpha(G/\textstyle\sum B_i)[p] & \twoheadrightarrow & C' \\
\| & & \uparrow f & & \uparrow g \\
P^\alpha G[p] & \longrightarrow & p^\alpha(G/B)[p] & \twoheadrightarrow & C
\end{array}$$

Therefore there exists a map $\operatorname{Ker} f\to\operatorname{Ker} g$ and by the snake diagram this map is an epimorphism. However $\operatorname{Ker} f\approx\sum B_i/(B\cap p^\alpha(G/B)[p])\subset\sum B_i/B$. Therefore

$$|\operatorname{Ker} f|\leqq|\textstyle\sum B_i/B|\leqq|B|\quad\text{and}\quad|C|\leqq|C'|\cdot|\operatorname{Ker} g|\leqq|B|\cdot|B|=|B|.$$

Therefore we may assume without loss of generality that $B=\sum B_i$. Hence $C=\sum C_i$, where C_i is the cokernel of the map

$$(p^\alpha G_i)[p]\to p^\alpha(G_i/B_i)[p].$$

But by assumption, $|C_i| \leqq |B_i| \leqq |B|$, and therefore

$$|C| \leqq |\sum C_i| \leqq |B| \cdot |B| = |B|.$$

Case II. Assume $\gamma = \sigma + 1$ and $\lambda(G) = \gamma$. Since G is totally projective, $G = \sum G_i$ with $p^\sigma G_i \approx C(p)$. Let B_i be the image under the projection of G onto G_i of B, and $B_i' = B_i + p^\sigma G_i$. As in Case I we may assume without loss of generality that $B = \sum B_i'$. If C_i' is the cokernel of the map

$$p^\alpha(G_i/p^\sigma G_i)[p] \to p^\alpha(G_i/B_i)[p],$$

then by assumption $|C_i'| \leqq |B_i'| \leqq |B|$. But by the assumption, $C \approx \sum C_i'$ and hence

$$|C| \leqq |\sum C_i'| \leqq |B| \cdot |B| = |B|.$$

Therefore by induction, this completes the proof of the proposition.

The following proposition is proven in the same way as above, and since the result is not needed in this paper, the proof will be left to the reader.

Proposition IV.4. *Let G be a totally projective p-group, B be an infinite subgroup of G, and C be the cokernel of the map $p^\alpha G \to p^\alpha(G/B)$, induced by the natural map $G \to G/B$, then $|B| \geqq |C|$.*

Theorem IV.5. *Let G be a totally projective p-group and B an infinite subgroup of G. Then $f_{G/B}(\alpha) \leqq f_G(\alpha) + |B|$ and $f_G(\alpha) \leqq f_{G/B}(\alpha) + |B|$ for all $\alpha < \lambda(G)$.*

Proof. The following diagram is commutative with exact rows.

$$\begin{array}{ccccc} p^{\alpha+1}G[p] & \rightarrowtail & p^\alpha G[p] & \twoheadrightarrow & p^\alpha G[p]/p^{\alpha+1}G[p] \\ \downarrow f & & \downarrow g & & \downarrow h \\ p^{\alpha+1}(G/B)[p] & \rightarrowtail & p^\alpha(G/B)[p] & \twoheadrightarrow & p^\alpha(G/B)[p]/p^{\alpha+1}(G/B)[p] \end{array}$$

The maps f and g are induced by the natural map $G \to G/B$ and h is given by the definition of the cokernel. This implies by the snake diagram that the sequence

$$\operatorname{Ker} f \rightarrowtail \operatorname{Ker} g \to \operatorname{Ker} h \to \operatorname{Coker} f \to \operatorname{Coker} g \twoheadrightarrow \operatorname{Coker} h$$

is exact. Now $\operatorname{Ker} f \approx (p^{\alpha+1}G \cap B)[p]$ and $\operatorname{Ker} g \approx (P^\alpha G \cap B)[p]$. This implies that $|\operatorname{Ker} f| \leqq |B|$ and $|\operatorname{Ker} g| \leqq |B|$. By Proposition IV.3, $|\operatorname{Coker} f| \leqq |B|$ and $|\operatorname{Coker} g| \leqq |B|$. Therefore $|\operatorname{Ker} h| \leqq |B|$ and $|\operatorname{Coker} h| \leqq |B|$, and hence $f_{G/B}(\alpha) \leqq f_G(\alpha) + |B|$ and $f_G(\alpha) \leqq f_{G/B}(\alpha) + |B|$.

Lemma IV.6. *If $pB = 0$ and $x + B \in p^\alpha(G/B)$, then $px \in p^\alpha G$.*

Proof. If $\alpha = 0$, the lemma is true. If $\alpha > 0$ and $x + B \in p^\alpha(G/B)$, then by induction $px \in p^\beta G$ for all $\beta < \alpha$. If α is a limit, then $px \in p^\alpha G$. If $\alpha = \gamma + 1$, then $x + B = p(y + B)$ with $y + B \in p^\gamma(G/B)$. Thus $py \in p^\gamma G$, $x = py + b$ with $b \in B$, hence $px = p^2 y \in p(p^\gamma G) = p^\alpha G$.

Lemma IV.7. *If G is a reduced p-group, B an infinite subgroup of G such that $pB = 0$, and $g_\alpha + B \in p^\alpha(G/B)$ for all $\alpha < \lambda(G)$, then $|\{\alpha \mid \text{for all } b \in B,\ g_\alpha + b \notin p^\alpha G\}| \leqq |B|$.*

Proof. Suppose that if $x+B\in p^{\alpha}(G/B)$ then $x+b\in p^{\alpha}G$ for some $b\in B$. Let $x+B\in p^{\alpha+1}(G/B)$ then $x+B=py+B$ with $y+B\in p^{\alpha}(G/B)$. Therefore $y+b\in p^{\alpha}G$ for some $b\in B$. But $py=x+b'$ for some $b'\in B$ and therefore $x+b'\in p^{\alpha+1}G$. If we let $J=\{\alpha\mid \alpha$ is a limit ordinal and for all $b\in B$, $g_{\alpha}+b\notin p^{\alpha}G\}$ then it suffices to show $|J|\leqq|B|$. Suppose $|J|>|B|$, and let $J'=\{\alpha\mid \alpha\in J$ and α is not the limit of elements of $J\}$. Clearly $|J'|=|J|$. Therefore for each $\alpha\in J'$ there exists $\beta_{\alpha}<\alpha$ such that $\{\gamma\mid\beta_{\alpha}\leqq\gamma<\alpha\}\cap J'=\emptyset$. Let $g_{\alpha}+B=px_{\beta_{\alpha}}+B$ where $x_{\beta_{\alpha}}+B\in p^{\beta_{\alpha}}(G/B)$. Therefore we may assume without loss of generality that $H(g_{\alpha})\neq H(g_{\gamma})$ for α, $\gamma\in J'$, and that $H(G_{\alpha})=\beta_{\alpha}$. Let $y_{\alpha}+B\in p^{\beta_{\alpha}+1}(G/B)$ such that $g_{\alpha}+B=py_{\alpha}+B$. By Lemma IV.6, $H(py_{\alpha})\geqq\beta_{\alpha}+1$. But $g_{\alpha}-py_{\alpha}=b_{\alpha}\in B$, and therefore B contains elements of height β_{α} for each $\alpha\in J'$. This contradicts $|J'|>|B|$ and therefore $|J|\leqq|B|$ and this completes the proof of the Lemma.

Theorem IV.8. *If G is a reduced p-group, B a subgroup of G such that $pB=0$, and*

$$h_{\alpha}\in\mathrm{Hom}(p^{\alpha}G[p]/p^{\alpha+1}G[p],\ p^{\alpha}(G/B)[p]/p^{\alpha+1}(G/B)[p])$$

given by

$$h_{\alpha}(g+p^{\alpha+1}G[p]=(g+B)+p^{\alpha+1}(G/B)[p]),$$

then $|\{\alpha\mid h_{\alpha}$ *is not an isomorphism*$\}|\leqq|B|$.

Proof. Suppose it is not true that h_{α} is an epimorphism for all but $|B|$ α's. Then there is a set J of ordinals and a family $\{g_{\alpha}\}_{\alpha\in J}$ of elements of G such that $|J|>|B|$, and for all $\alpha\in J$,

$$(g_{\alpha}+B)+p^{\alpha+1}(G/B)[p]\in p^{\alpha}(G/B)[p]/p^{\alpha+1}(G/B)[p],$$

and $(g_{\alpha}+B)+p^{\alpha+1}(G/B)[p]$ is not in the image of h_{α}. Since $g_{\alpha}+B\in p^{\alpha}(G/B)[p]$ we may assume by Lemma IV.7 that $H(g_{\alpha})=\alpha$ for all but $|B|$ α's, and hence for all $\alpha\in J$. Let $I_{\gamma}=\{\alpha\in J\mid pg_{\alpha}=pg_{\gamma}\}$. Since $pg_{\alpha}\in B$ for all α, there exists a γ such that $|I_{\gamma}|>|B|$. Now $H(pg_{\alpha})>\alpha+1$ for all $\alpha\in I_{\gamma}$ except possibly the last one, if such exists, and therefore $pg_{\gamma}=py_{\alpha}$ with $H(y_{\alpha})\geqq\alpha+1$ for all these α. Now $g_{\alpha}=y_{\alpha}+s_{\alpha}$ where $s_{\alpha}\in G[p]$, and thus $H(g_{\alpha})=H(s_{\alpha})$. But

$$\begin{aligned}h_{\alpha}(s_{\alpha}+p^{\alpha+1}G[p])&=(s_{\alpha}+B)+p^{\alpha+1}(G/B)[p]\\&=(g_{\alpha}-y_{\alpha}+B)+p^{\alpha+1}(G/B)[p]\\&=(g_{\alpha}+B)+p^{\alpha+1}(G/B)[p],\end{aligned}$$

and this contradicts the way $\{g_{\alpha}+p^{\alpha+1}(G/B)[p]\}_{\alpha\in J}$ was picked. Therefore h_{α} is an epimorphism for all but $|B|$ α's.

To show h_{α} is a monomorphism for all but $|B|$ α's, suppose there exists a set J of ordinals and elements g_{α} of $p^{\alpha}G[p]-p^{\alpha+1}G[p]$ such that $h_{\alpha}(g_{\alpha}+p^{\alpha+1}G[p])=0$ for each $\alpha\in J$, and $|J|>|B|$. Therefore $g_{\alpha}+B\in p^{\alpha+1}(G/B)[p]$ for all $\alpha\in J$. Let $J'=\{\alpha\in J\mid h_{\alpha+1}$ is an epimorphism$\}$. Now $|J'|>|B|$ by the first part of this proof. Hence there exists $g'_{\alpha}\in p^{\alpha+1}G[p]$ such that

$$(g'_{\alpha}+B)+p^{\alpha+2}(G/B)[p]=(g_{\alpha}+B)+p^{\alpha+2}(G/B)[p]$$

for all $\alpha \in J'$. Thus $g_\alpha + B = (g'_\alpha + B) + (g''_\alpha + B)$ where $g''_\alpha + B \in p^{\alpha+2}(G/B)[p]$ and $g_\alpha = g'_\alpha + g''_\alpha + b_\alpha$ where $b_\alpha \in B$. By Lemma IV.7, we may take $H(g''_\alpha) \geqq \alpha + 2$ for all but $|B|$ α's. Therefore $H(b_\alpha) = \alpha$ for all but $|B|$ elements of J', and this contradicts $|J'| > |B|$. Therefore h_α is a monomorphism for all but $|B|$ α's.

Theorem IV.9. *Let G and H be totally projective p-groups. Then G and H satisfy conditions* (III) *and* (IV ξ) *for some cardinal ξ with ξ_1 infinite if and only if $G \approx H$ in $\hat{\mathscr{A}}/(\hat{\mathscr{O}}, \hat{\mathscr{B}}(\xi))$.*

Proof. Theorem IV.2 shows that if conditions (III) and (IV ξ) hold, then $G \approx H$ in $\hat{\mathscr{A}}/(\hat{\mathscr{O}}, \hat{\mathscr{B}}(\xi))$, and Theorem IV.5 shows that if $G \approx H$ in $\hat{\mathscr{A}}/(\hat{\mathscr{O}}, \hat{\mathscr{B}}(\xi))$, then $f_H(\alpha) \leqq f_G(\alpha) + \xi_1$ and $f_G(\alpha) \leqq f_H(\alpha) + \xi_1$, where ξ_1 is given by the cardinality of the element of $\hat{\mathscr{B}}(\xi)$ given by the definition of isomorphism in $\hat{\mathscr{A}}/(\hat{\mathscr{O}}, \hat{\mathscr{B}}(\xi))$. Hart [4] has shown that if $G \approx H$ in $\hat{\mathscr{A}}/\hat{\mathscr{B}}$ then condition (III) holds. Therefore it suffices to show that if $G \approx H$ in $\hat{\mathscr{A}}/(\hat{\mathscr{O}}, \hat{\mathscr{B}}(\xi))$, then $|I| < \xi$, where $I = \{\alpha \mid f_G(\alpha) \neq f_H(\alpha)\}$. To show this, let B be a bounded subgroup of G such that $|B| < \xi$. By an easy induction on the result of Theorem IV.8,

$$|\{\alpha \mid f_{G/B}(\alpha) \neq f_G(\alpha)\}| < \xi.$$

But $G/B \approx H/C$ for suitable bounded groups B and C of cardinal $< \xi$. Therefore

$$|\{\alpha \mid f_G(\alpha) \neq f_H(\alpha)\}| < \xi,$$

and this completes the proof of the theorem.

The following lemma is useful in proving that for totally projective groups, isomorphism in $\hat{\mathscr{A}}/\hat{\mathscr{B}}(\xi)$ and in $\hat{\mathscr{A}}/(\hat{\mathscr{O}}, \hat{\mathscr{B}}(\xi))$ are equivalent, and that conditions (I), (II) and (IV ξ) are necessary and sufficient to determine totally projective groups up to isomorphism in $\hat{\mathscr{A}}/(\hat{\mathscr{B}}(\xi), \hat{\mathscr{O}})$.

Lemma IV.10. *If $G \approx H$ in $\hat{\mathscr{A}}/(\hat{\mathscr{B}}(\xi), \hat{\mathscr{O}})$ for some cardinal $\xi \geqq \aleph_0$ then G and H satisfy conditions* (I), (II) *and* (IV ξ).

Proof. Since $G \approx H$ in $\hat{\mathscr{A}}/(\hat{\mathscr{B}}(\xi), \hat{\mathscr{O}})$, there exist subgroups G' of G and H' of H such that $G' \approx H'$ and G/G' and H/H' are in $\hat{\mathscr{B}}(\xi)$. Therefore it suffices to show that G and G' satisfy conditions (I), (II) and (IV ξ). It is well known, [1], that G and G' satisfy conditions (I), and (II). If $\xi = \aleph_0$ then by Lemma 16.5 of [9] we may assume that G' and H' are summands of G and H respectively and hence

$$|\{\alpha \mid f_G(\alpha) \neq f_{G'}(\alpha)\}| < \aleph_0 = \xi.$$

Also in this case $f_G(n) = f_{G'}(n) + f_{G/G'}(n)$ and condition (IV ξ) holds with ξ, equal the maximum of the set $\{f_{G/G'}(n) \mid n \in Z^+\}$. If $\xi > \aleph_0$ then condition (II) implies that $|\{\alpha \mid f_G(\alpha) \neq f_{G'}(\alpha)\}| < \xi$. Before showing the other half of condition (IV ξ) in this case we will show that $|p^n G[p]/p_n G'[p]| < \xi$ for all integers $n \geqq 0$.

It sufficies to show that $|p^n G/p^n G'| < \xi$ for all $n \geqq 0$. For $n = 0$, this is true since $G/G' \in \hat{\mathscr{B}}(\xi)$. Suppose $|p^n G/p^n G'| \leqq \xi_1 < \xi$, with ξ_1 infinite, and define the mapping

$$h_n\colon p^{n+1} G/p^{n+1} G' \to p^n G/p^n G'$$

by $h_n(a+p^{n+1}G')=a+p^nG'$. It suffices to show that $|\operatorname{Ker} h_n|\leqq\xi_1$. Now

$$\operatorname{Ker} h_n=(p^nG'\cap p^{n+1}G)/p^{n+1}G'.$$

Let $\{x_i+p^{n+1}G'\}_{i\in I}$ be a set of distinct elements of $\operatorname{Ker} h_n$. Then there exist $a_i\in p^nG$ such that $pa_i=x_i$. Now $a_i-a_j\notin p^nG'$ for $i\neq j$. If $i_0\in I$ then $(a_{i_0}-a_i)+p^nG'\neq(a_{i_0}+a_j)+p^nG'$ for $i,j\in J$ such that $i\neq j$. But since

$$(a_{i_0}-a_i)+p^nG'\in p^nG/p^nG',\qquad |I|\leqq\xi_1.$$

Therefore by induction we have that $|p^nG[p]/p^nG'[p]|\leqq\xi_1$ for all integers $n>0$.

The following sequence is exact for all integers n.

$$(p^{n+1}G[p]\cap p^nG'[p])/p^{n+1}G'[p]\rightarrowtail p^nG'[p]/p^{n+1}G'[p]\to p^nG[p]/p^{n+1}G[p]$$
$$\twoheadrightarrow p^nG[p]/(p^nG'[p]+p^{n+1}G[p]).$$

However

$$|(p^{n+1}G[p]\cap p^nG'[p])/p^{n+1}G'[p]|\leqq|p^{n+1}G[p]/p^{n+1}G'[p]|\leqq\xi_1$$

and

$$|p^nG[p]/(p^nG'[p]+p^{n+1}G[p])|\leqq|p^nG[p]/p^nG'[p]|\leqq\xi_1$$

and this implies that $f_G(n)\leqq f_{G'}(n)+\xi_1$ and $f_{G'}(n)\leqq f_G(n)+\xi_1$ for all $n\geqq0$.

Theorem IV.11. *If G and H are totally projective p-groups, then $G\approx H$ in $\mathscr{A}/\mathscr{B}(\xi)$ if and only if $G\approx H$ in $\hat{\mathscr{A}}/(\hat{\mathscr{O}},\hat{\mathscr{B}}(\xi))$.*

Proof. By Theorem III.1 we may assume without loss of generality that $\xi>\aleph_0$. If $G\approx H$ in $\hat{\mathscr{A}}/(\hat{\mathscr{O}},\hat{\mathscr{B}}(\xi))$ then clearly $G\approx H$ in $\mathscr{A}/\mathscr{B}(\xi)$. Conversely, if $G\approx H$ in $\mathscr{A}/\mathscr{B}(\xi)$ then there exist subgroups G' and H' of G and H respectively such that G/G', $H/H'\in\mathscr{B}(\xi)$ and $G'\approx H'$ in $\hat{\mathscr{A}}/(\hat{\mathscr{O}},\hat{\mathscr{B}}(\xi))$. Therefore it suffices to show that $G\approx G'$ in $\hat{\mathscr{A}}/(\hat{\mathscr{O}},\hat{\mathscr{A}}(\xi))$. But $G\approx G'$ in $\mathscr{A}/(\hat{\mathscr{A}}(\xi),\hat{\mathscr{O}})$ which implies by Lemma IV.10 that G and G' satisfy conditions (I) and (IV ξ). Therefore G and G' satisfy conditions (III) and (IV ξ), and by Theorem IV.9 $G\approx G'$ in $\hat{\mathscr{A}}/(\hat{\mathscr{O}},\hat{\mathscr{B}}(\xi))$. By the transitivity of isomorphism $G\approx H$ in $\hat{\mathscr{A}}/(\hat{\mathscr{O}},\hat{\mathscr{B}}(\xi))$.

Lemma IV.12. *Let G and H be direct sums of cyclic groups which satisfy conditions* (I) *and* (IV $\aleph_0$). *Then $G=G_0\oplus\cdots\oplus G_{2k}$ and $H=H_0\oplus\cdots\oplus H_{2k}$ with G_i and H_i finite for $i\neq k$, and $p^kG_0\approx H_0$, $p^{k-1}G_1\approx H_1,\ldots,G_k\approx H_k,\ldots,G_{2k}\approx p^kH_{2k}$.*

The proof of this Lemma is similar to the proof of Lemma IV.1 and hence it will be omitted.

Theorem IV.13. *If G and H are totally projective p-groups then they satisfy conditions* (I), (II) *and* (IV ξ) *for some infinite cardinal ξ if and only if $G\approx H$ in $\hat{\mathscr{A}}/(\hat{\mathscr{B}}(\xi),\hat{\mathscr{O}})$.*

Proof. By Lemma IV.10 we have that if $G\approx H$ in $\hat{\mathscr{A}}/(\hat{\mathscr{B}}(\xi),\hat{\mathscr{O}})$ then conditions (I), (II) and (IV ξ) are satisfied. Conversely, suppose conditions (I), (II) and (IV ξ) are satisfied by G and H. If G and H are the direct sum of cyclic groups then an easy application of Lemmas IV.1 and IV.12 shows that $G\approx H$ in $\hat{\mathscr{A}}/(\hat{\mathscr{B}}(\xi),\hat{\mathscr{O}})$.

If G and H are totally projective then $G/p^\omega G$ and $H/p^\omega H$ are direct sums of cyclic groups which satisfy (I), (II) and (IV ξ). Therefore there exist subgroups G_1 of G and H_1 of H such that $G_1/p^\omega G \approx H_1/p^\omega H$ and $\{G/G_1, H/H_1\} \subset \hat{\mathscr{B}}(\xi)$. But $p^\omega G \approx p^\omega G_1$, $p^\omega H \approx p^\omega H_1$, and by [6], $p^\omega G \approx p^\omega H$. Therefore by [8], $G_1 \approx H_1$ and $G \approx H$ in $\hat{\mathscr{A}}/(\hat{\mathscr{B}}(\xi), \hat{\mathscr{O}})$.

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Eine Charakterisierung der fast schwach modularen Verbände

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Die fast schwach modularen Verbände wurden in der Arbeit [5] des Autors eingeführt, um die Klasse aller Verbände L zu beschreiben, deren Kongruenzrelationenverband $\Theta(L)$ ein Stonescher Verband ist. Die fast schwach modularen Verbände stellen eine Verallgemeinerung der schwach modularen Verbände dar (s. Grätzer und Schmidt [2]), die zur Kennzeichnung der Klasse aller Verbände L für die $\Theta(L)$ ein Boolescher Verband ist, geeignet sind (s. Birkhoff [1, Problem 39]). In der vorliegenden Arbeit werden die Kongruenzrelationen in den fast schwach modularen Verbänden untersucht. Dabei erhält man eine Beschreibung dieser Verbände durch spezielle Kongruenzrelationen. Für schwach modulare Verbände kann man aus allgemeineren Sätzen zwei Charakterisierungen beweisen, unter denen auch die schon früher bewiesene Kennzeichnung ist (s. Iqbalunissa [3], Karolinskaja und Stelleckij [4]).

1. Einleitende Bemerkungen

Ein Verband L heißt *Brouwersch*, wenn es zu je zwei Elementen $a, b \in L$ ein Element $c \in L$ gibt, so daß für alle $x \in L$ gilt:

(1) $$a \cap x \leqq b \quad \text{genau dann, wenn} \quad x \leqq c.$$

Das Element c, das durch das Paar (a, b) eindeutig bestimmt ist, wird mit $a * b$ bezeichnet. Falls L ein kleinstes Element 0 besitzt, so schreibt man für $a * 0$ kürzer a^*.

Ein Paar (a, b) von Elementen eines Verbandes, für das $a \geqq b$ gilt, nennt man einen *Quotienten*, und bezeichnet diesen mit a/b. Ein Quotient a/b heißt *trivial*, wenn $a = b$ ist. c/d ist ein *Teilquotient* von a/b, wenn $a \geqq c \geqq d \geqq b$ gilt. Zwei Quotienten heißen *transponiert*, wenn sie sich in der Form $a \cup b/a$ und $b/a \cap b$ darstellen lassen. Ein Quotient a/b heißt *schwach projektiv* zum Quotienten c/d (Bezeichnung: $a/b \to c/d$), wenn es eine endliche Folge von Quotienten $a/b = a_0/b_0, \ldots, a_i/b_i, \ldots, a_n/b_n = c/d$ $(1 \leqq i \leqq n)$ gibt, so daß a_{i+1}/b_{i+1} und ein Teilquotient von a_i/b_i $(1 \leqq i \leqq n-1)$ transponiert sind. Es ist klar, daß „$\to$" eine transitive Relation ist.

Eine Äquivalenzrelation Θ eines Verbandes L heißt eine *Kongruenzrelation*, wenn sie die folgenden Bedingungen erfüllt:

wenn $a \equiv a_1(\Theta)$ und $b \equiv b_1(\Theta)$, so $a \cup b \equiv a_1 \cup b_1(\Theta)$ und $a \cap b \equiv a_1 \cap b_1(\Theta)$.

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Wenn für einen Quotienten a/b $a \equiv b(\Theta)$ gilt, so sagt man: a/b wird durch Θ *annuliert*, oder a/b *ist* Θ*-trivial*.

Es ist bekannt, daß für eine Kongruenzrelation Θ eines Verbandes L stets gilt:

$$x \equiv y(\Theta) \quad \text{genau dann, wenn} \quad x \cup y \equiv x \cap y(\Theta).$$

Das zeigt uns, daß jede Kongruenzrelation eines Verbandes bereits durch die Menge der durch sie annulierten Quotienten bestimmt ist. Genaueres sagt uns darüber der folgende Satz, den wir [7, Kap. I, Satz 4.2] entnehmen. Von diesem Satz machen wir im folgenden häufig Gebrauch, ohne es zu erwähnen. Zunächst führen wir noch einen Hilfsbegriff ein.

Erfüllt eine Menge N von Quotienten eines Verbandes L die folgenden Bedingungen (i) bis (iii), so nennt man N ein *Quotientenideal* von L:

(i) $a/a \in N$ *für beliebige* $a \in L$;

(ii) *mit einem Quotienten gehören auch die zu ihm schwach projektiven Quotienten zu* N;

(iii) *wenn* $a/b, b/c \in N$, *so auch* $a/c \in N$ [1].

1.1. *In einem Verband L sei eine Kongruenzrelation Θ gegeben. Dann ist die Gesamtheit N der durch Θ annulierten Quotienten ein Quotientenideal von L. Ist umgekehrt N ein Quotientenideal von L, so existiert genau eine Kongruenzrelation Θ, bei der N die Gesamtheit der durch Θ annullierten Quotienten ist.*

$\Theta(L)$ bezeichne die Gesamtheit der Kongruenzrelationen in einem Verband L. $\Theta(L)$ bildet bezüglich der Halbordnung

(2) $\Theta \leqq \Phi$ *genau dann, wenn aus* $x \equiv y(\Theta)$ *stets* $x \equiv y(\Phi)$ *folgt*

einen vollständigen Verband. Für $\{\Theta_\alpha; \alpha \in A\} \subseteq \Theta(L)$ sind dann die verbandstheoretischen Operationen folgendermaßen erklärt:

(3) $a \equiv b(\bigvee (\Theta_\alpha; \alpha \in A))$ *genau dann, wenn es eine endliche Teilmenge* $a = x_0, \ldots, x_n = b$ [2] *von L gibt, so daß für geeignete* $\alpha_i \in A$ $(i = 1, \ldots, n)$ $x_{i-1} \equiv x_i(\Theta_{\alpha_i})$ *gilt.*

(4) $a \equiv b(\bigwedge (\Theta_\alpha; \alpha \in A))$ *genau dann, wenn für alle* $\alpha \in A$ $a \equiv b(\Theta_\alpha)$ *gilt.*

Die Identität ist die kleinste Kongruenzrelation; wir bezeichnen sie mit ω.

Man betrachtet oft die kleinste Kongruenzrelation, durch die ein Quotient a/b eines Verbandes annulliert wird. Diese Kongruenzrelation bezeichnen wir mit $\Theta_{a/b}$. Für solche Kongruenzrelationen gilt der bekannte Satz (s. [2]).

1.2 (Dilworth). *Ein Quotient c/d eines Verbandes L wird genau dann durch $\Theta_{a/b}$ annulliert, wenn es eine endliche Kette* $d = y_0 \leqq \cdots \leqq y_n = c$ *von Elementen aus L gibt, so daß* $a/b \to y_{i+1}/y_i$ *für alle* $i = 0, \ldots, n-1$ *gilt.*

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[1] Die Bedingungen (i) – (iii) sind zu den Bedingungen (1°) – (4°) aus [7, Kap. I, Definition 4.1] äquivalent.

[2] In [2, Lemma 5] wurde bewiesen, daß man zu $a \leqq b$ immer eine Kette $a = x_0 \leqq \cdots \leqq x_n = b$ finden kann. Danach kann man (3) leicht für die Quotienten umformulieren.

Daraus läßt sich leicht der folgende Satz ableiten.

1.3. *Für eine beliebige Kongruenzrelation Θ eines Verbandes gilt*

$$\Theta = \bigvee (\Theta_{a/b};\ \Theta \text{ annulliert } a/b).$$

Der Kongruenzrelationenverband $\Theta(L)$ eines Verbandes L hat eine weitere interessante Eigenschaft, er ist nämlich unendlich $\vee$-distributiv, d. h. in $\Theta(L)$ gilt

$$\Phi \cap \bigvee(\Theta_\alpha;\ \alpha \in A) = \bigvee(\Phi \cap \Theta_\alpha;\ \alpha \in A).$$

Demnach (s. [1, VI, Theorem 9]) bildet $\Theta(L)$ einen Brouwerschen Verband.

Mit Hilfe der schwachen Projektivität kann man nun auch die relativen Pseudokomplemente in $\Theta(L)$ beschreiben.

1.4. *In einem Verband L seien die Kongruenzrelationen Θ und Φ gegeben. Dann annulliert $\Theta * \Phi$ genau dann den Quotienten x/y, wenn aus $x/y \to z/t$ und $z \equiv t(\Theta)$ stets $z \equiv t(\Phi)$ folgt.*

Beweis. Annulliert $\Theta * \Phi$ einen Quotienten x/y, dann hat man $\Theta_{x/y} \leqq \Theta * \Phi$, was mit $\Theta \cap \Theta_{x/y} \leqq \Phi$ äquivalent ist (s. (1)). Aus $x/y \to z/t$ und $z \equiv t(\Theta)$ folgt dann $\Theta_{z/t} \leqq \Theta \cap \Theta_{x/y} \leqq \Phi$, d. h. $z \equiv t(\Phi)$. Umgekehrt, sei nun vorausgesetzt, daß $x/y \to z/t$ und $z \equiv t(\Theta)$ stets $z \equiv t(\Phi)$ nach sich zieht. Annulliert $\Theta \cap \Theta_{x/y}$ den Quotienten c/d, dann gilt $c \equiv d(\Theta)$ und $c \equiv d(\Theta_{x/y})$. Aus der Voraussetzung und 1.2 erhält man $c \equiv d(\Phi)$. Demnach gilt $\Theta \cap \Theta_{x/y} \leqq \Phi$, was mit $\Theta_{x/y} \leqq \Theta * \Phi$ äquivalent ist. Die letzte Aussage besagt, daß x/y durch $\Theta * \Phi$ annulliert wird.

1.5. *Seien Θ und Φ Kongruenzrelationen eines Verbandes L. Dann annulliert die Kongruenzrelation $(\Theta * \Phi) * \Phi$ genau dann den Quotienten x/y, wenn es zu jedem nicht Φ-trivialen Quotienten z/t mit $x/y \to z/t$ einen nicht Φ-trivialen Quotienten r/s mit $z/t \to r/s$ und $r \equiv s(\Theta)$ gibt.*

Beweis. Der Quotient x/y werde durch $(\Theta * \Phi) * \Phi$ annulliert. Ferner gelte $x/y \to z/t$ für einen nicht Φ-trivialen Quotienten z/t. Der Quotient z/t ist offenbar nicht $\Theta * \Phi$-trivial. Aus $z \not\equiv t(\Theta * \Phi)$ und 1.4 ergibt sich die Existenz eines nicht Φ-trivialen Quotienten r/s, für den $z/t \to r/s$ und $r \equiv s(\Theta)$ gilt.

Umgekehrt nehmen wir an: für alle nicht Φ-trivialen Quotienten z/t mit $x/y \to z/t$ gibt es einen nicht Φ-trivialen Quotienten r/s mit $z/t \to r/s$ und $r \equiv s(\Theta)$. Es gelte nun $x/y \to z/t$ und $z \equiv t(\Theta * \Phi)$. Falls $z \not\equiv t(\Phi)$ ist, dann gibt es nach Voraussetzung einen nicht Φ-trivialen Quotienten r/s mit $z/t \to r/s$ und $r \equiv s(\Theta)$. Wegen $z \equiv t(\Theta * \Phi)$, $z/t \to r/s$ und $r \equiv s(\Theta)$ gilt nach (1) $r \equiv s(\Phi)$, was zum Widerspruch führt. Also bleibt nur $z \equiv t(\Phi)$ übrig. Aus 1.4 erhält man $x \equiv y((\Theta * \Phi) * \Phi)$, was zu beweisen war.

2. Φ-fast schwach modulare Verbände

2.1. Definition. L sei ein Verband und $\Phi \in \Theta(L)$. L heißt *Φ-fast schwach modular*, wenn für einen nicht Φ-trivialen Quotienten u/v aus $a/b \to u/v$ die Existenz eines nicht Φ-trivialen Teilquotienten a_1/b_1 von a/b folgt, so daß es zu jedem nicht Φ-trivialen Quotienten r/s mit $a_1/b_1 \to r/s$ einen nicht Φ-trivialen Quotienten z/t mit $u/v \to z/t$ und $r/s \to z/t$ gibt.

Bemerkung. Es läßt sich unschwer zeigen, daß der Begriff „ω-fast schwach modular" für die identische Kongruenzrelation ω mit dem Begriff „fast schwach modular" äquivalent ist, der in [5] eingeführt wurde. Deshalb benutzen wir weiterhin an Stelle von „ω-fast schwach modular" nur „fast schwach modular". Darüber hinaus kann man leicht einsehen, daß ein modularer Verband stets ein schwach modularer Verband (s. die Definition 3.1), und ein solcher auch immer fast schwach modular ist.

Bei einer Kennzeichnung der Φ-fast schwach modularen Verbänden spielt die folgende Relation Θ_Φ eine wesentliche Rolle.

2.2. Definition. Sei L ein Verband und Θ, Φ Kongruenzrelationen in L. Mit Θ_Φ bezeichnen wir eine binäre Relation in L, die die folgenden beiden Bedingungen erfüllt:

(i) $a \equiv b(\Theta_\Phi)$ genau dann, wenn $a \cup b \equiv a \cap b(\Theta_\Phi)$;

(ii) $a \equiv b(\Theta_\Phi)$ für einen Quotienten a/b genau dann, wenn es zu jedem nicht Φ-trivialen Teilquotienten a_1/b_1 von a/b einen nicht Φ-trivialen Quotienten r/s mit $a_1/b_1 \to r/s$ und $r \equiv s(\Theta)$ gibt.

2.3. *Seien Θ und Φ Kongruenzrelationen eines Verbandes L. Die binäre Relation Θ_Φ ist genau dann eine Kongruenzrelation in L, wenn $\Theta_\Phi = (\Theta * \Phi) * \Phi$ gilt.*

Beweis. Gilt $\Theta_\Phi = (\Theta * \Phi) * \Phi$, so ist Θ_Φ natürlich eine Kongruenzrelation in L. Setzen wir umgekehrt voraus, daß $\Theta_\Phi \in \Theta(L)$ gilt. Die Beziehung $\Theta_\Phi \geqq (\Theta * \Phi) * \Phi$ folgt dann aus 1.5, weil jeder Quotient zu seinen Teilquotienten schwach projektiv ist. Annulliere schließlich Θ_Φ den Quotienten a/b. Wird a/b zusätzlich durch Φ annulliert, dann auch durch $(\Theta * \Phi) * \Phi$. Die andere Möglichkeit ist: a/b ist nicht Φ-trivial. Ist z/t ein nicht Φ-trivialer Quotient mit $a/b \to z/t$, dann wird z/t durch Θ_Φ annulliert, weil Θ_Φ eine Kongruenzrelation ist. Nach der Definition von Θ_Φ gibt es einen nicht Φ-trivialen Quotienten r/s mit $z/t \to r/s$ und $r \equiv s(\Theta)$. Aus 1.5 folgt $a \equiv b(\Theta * \Phi) * \Phi$. Somit haben wir $\Theta_\Phi \leqq (\Theta * \Phi) * \Phi$ bewiesen, was zusammen mit der oben bewiesenen Ungleichung $\Theta_\Phi \geqq (\Theta * \Phi) * \Phi$ $\Theta_\Phi = (\Theta * \Phi) * \Phi$ ergibt.

Der folgende Satz kennzeichnet die Φ-fast schwach modularen Verbände.

2.4. Satz. *Sei eine Kongruenzrelation Φ eines Verbandes L gegeben. L ist genau dann Φ-fast schwach modular, wenn für jede Kongruenzrelation Θ die binäre Relation Θ_Φ eine Kongruenzrelation in L ist.*

Beweis. Sei L Φ-fast schwach modular und $\Theta \in \Theta(L)$. Wegen 2.3 liegt es nahe, $\Theta_\Phi = (\Theta * \Phi) * \Phi$ nachzuweisen. Für $\Theta \leqq \Phi$ gilt offenbar $\Theta_\Phi = \Phi = (\Theta * \Phi) * \Phi$. Also kann im folgenden $\Theta \not\leqq \Phi$ vorausgesetzt werden. Annulliere nun Θ_Φ den Quotienten a/b. Wir können außerdem $a/b \to u/v$ mit $u \equiv v(\Theta * \Phi)$ annehmen. Ist der Quotient u/v nicht Φ-trivial, dann gibt es einen nicht Φ-trivialen Teilquotienten a_1/b_1 von a/b (L ist Φ-fast schwach modular), so daß es zu jedem nicht Φ-trivialen Quotienten r/s mit $a_1/b_1 \to r/s$ wieder einen nicht Φ-trivialen Quotienten z/t mit $r/s \to z/t$ und $u/v \to z/t$ gibt. Wegen $a \equiv b(\Theta_\Phi)$ können wir

unter den Quotienten r/s einen Quotienten $\bar{r}/\bar{s}$ mit $\bar{r} \equiv \bar{s}(\Theta)$ finden. Bezeichne $\bar{z}/\bar{t}$ den zu $\bar{r}/\bar{s}$ existierenden Quotienten. Dann haben wir: $a_1/b_1 \to \bar{r}/\bar{s}$, $\bar{r}/\bar{s} \to \bar{z}/\bar{t}$ und $u/v \to \bar{z}/\bar{t}$. Wegen $\bar{r} \equiv \bar{s}(\Theta)$ und $u \equiv v(\Theta * \Phi)$ gilt $\bar{z} \equiv \bar{t}(\Theta \cap \Theta * \Phi)$, was mit $\bar{z} \equiv \bar{t}(\Phi)$ äquivalent ist. Die letzte Aussage führt zum Widerspruch, weil alle Quotienten z/t nicht Φ-trivial sind. Also bleibt nur $u \equiv v(\Phi)$ übrig. Demnach folgt aus 1.4 $a \equiv b((\Theta * \Phi) * \Phi)$, d.h. $\Theta_\Phi \leqq (\Theta * \Phi) * \Phi$. Die Beziehung $\Theta_\Phi \geqq (\Theta * \Phi) * \Phi$ erhält man aus 1.5, weil jeder Quotient zu seinem Teilquotienten schwach projektiv ist. Aus diesen beiden Beziehungen ergibt sich $\Theta_\Phi = (\Theta * \Phi) * \Phi$.

Umgekehrt, sei Θ_Φ für alle $\Theta \in \Theta(L)$ eine Kongruenzrelation in L. Angenommen, L ist kein Φ-fast schwach modularer Verband. Dann gibt es zwei nicht Φ-trivialen Quotienten a/b und u/v mit $a/b \to u/v$, die die Definition 2.1 nicht erfüllen. Genauer, zu jedem nicht Φ-trivialen Teilquotienten a_1/b_1 von a/b gibt es einen nicht Φ-trivialen Quotienten $\bar{r}/\bar{s}$ mit $a_1/b_1 \to \bar{r}/\bar{s}$, so daß für alle Quotienten z/t aus $\bar{r}/\bar{s} \to z/t$ und $u/v \to z/t$ stets $z \equiv t(\Phi)$ folgt. Bezeichne $N(a/b)$ die Gesamtheit der nicht Φ-trivialen Quotienten r/s, zu denen es einen nicht Φ-trivialen Teilquotienten a_1/b_1 von a/b mit $a_1/b_1 \to r/s$ gibt, so daß aus $r/s \to z/t$ und $u/v \to z/t$ schon $z \equiv t(\Phi)$ folgt. Offenbar ist $N(a/b)$ nicht leer. Setze $\Theta := \vee(\Theta_{r/s}; r/s \in N(a/b))$. Zunächst beweisen wir $u \equiv v(\Theta * \Phi)$. Gilt nämlich für einen Quotienten x/y $u/v \to x/y$ und $x \equiv y(\Theta)$, dann gibt es eine Kette $y = t_0 \leqq \cdots \leqq t_n = x$ und Quotienten $r_i/s_i \in N(a/b)$ $(1 \leqq i \leqq n)$, so daß $r_i/s_i \to t_i/t_{i-1}$ für $i = 1, \ldots, n$ gilt (s. (3) und 1.2). Wegen $u/v \to x/y$ und $x/y \to t_i/t_{i-1}$ $(1 \leqq i \leqq n)$ erhält man $u/v \to t_i/t_{i-1}$. Die Kongruenzrelation Θ ist so gewählt, daß aus $r_i/s_i \to t_i/t_{i-1}$ und $u/v \to t_i/t_{i-1}$ schon die Annullierung der Quotienten t_i/t_{i-1} $(1 \leqq i \leqq n)$ durch Φ folgt. Demnach gilt $x \equiv y(\Phi)$. Wegen 1.4 ist $u \equiv v(\Theta * \Phi)$. Aus der Wahl der Kongruenzrelation Θ folgt sofort $a \equiv b(\Theta_\Phi)$. Wir haben angenommen, daß Θ_Φ eine Kongruenzrelation in L ist. Daraus ergibt sich nach 2.3 $\Theta_\Phi = (\Theta * \Phi) * \Phi$. Wegen $a/b \to u/v$ muß dann $(\Theta * \Phi) * \Phi$ den Quotienten u/v annullieren. Oben haben wir gezeigt, daß u/v auch durch $\Theta * \Phi$ annulliert wird. Aus beiden Aussagen folgt dann $u \equiv v(\Phi)$. Das ist ein Widerspruch dazu, daß der Quotient u/v als nicht Φ-trivial vorausgesetzt wurde. Damit haben wir bewiesen, daß L ein Φ-fast schwach modularer Verband ist.

Eine andere Kennzeichnung der Φ-fast schwach modularen Verbände erhalten wir durch die folgende Relation.

2.5. Definition. Es sei L ein Verband, Φ eine Kongruenzrelation in L und a/b ein Quotient von L. Mit $\Theta_{a/b}(\Phi)$ bezeichnen wir eine binäre Relation in L, die die beiden folgenden Bedingungen erfüllt:

(i) $x \equiv y(\Theta_{a/b}(\Phi))$ genau dann, wenn $x \cup y \equiv x \cap y(\Theta_{a/b}(\Phi))$;

(ii) $x \equiv y(\Theta_{a/b}(\Phi))$ für einen Quotienten x/y, wenn es zu jedem nicht Φ-trivialen Teilquotienten x_1/y_1 von x/y einen nicht Φ-trivialen Quotienten z/t mit $x_1/y_1 \to z/t$ und $z \equiv t(\Theta_{a/b} * \Phi)$ gibt.

2.6. *In einem Verband L sei die Kongruenzrelation Φ und der Quotient a/b gegeben. Die binäre Relation $\Theta_{a/b}(\Phi)$ ist genau dann eine Kongruenzrelation in L, wenn $\Theta_{a/b}(\Phi) = \Theta_{a/b} * \Phi$ gilt.*

Beweis. Gilt $\Theta_{a/b}(\Phi)=\Theta_{a/b}*\Phi$, dann ist natürlich $\Theta_{a/b}(\Phi)$ eine Kongruenzrelation. Sei umgekehrt $\Theta_{a/b}(\Phi)\in\Theta(L)$. Es ist leicht zu sehen, daß $\Theta_{a/b}*\Phi\leqq\Theta_{a/b}(\Phi)$ gilt. Sei nun $x\equiv y(\Theta_{a/b}(\Phi))$ für einen Quotienten x/y und $x/y\to z/t$ mit $z\equiv t(\Theta_{a/b})$. Wir nehmen an, daß der Quotient z/t nicht Φ-trivial ist. Dann gibt es einen nicht Φ-trivialen Quotienten u/v mit $z/t\to u/v$ und $u\equiv v(\Theta_{a/b}*\Phi)$. Wegen $z\equiv t(\Theta_{a/b})$ gilt dann $u\equiv v(\Theta_{a/b}\cap\Theta_{a/b}*\Phi)$, d.h. $u\equiv v(\Phi)$, was zu einem Widerspruch führt. Deshalb wird der Quotient z/t durch die Kongruenzrelation Φ annulliert. Aus 1.4 folgt dann $x\equiv y(\Theta_{a/b}*\Phi)$, und die Beziehung $\Theta_{a/b}*\Phi\geqq\Theta_{a/b}(\Phi)$ ist bewiesen. Daraus und aus $\Theta_{a/b}*\Phi\leqq\Theta_{a/b}(\Phi)$ folgt $\Theta_{a/b}*\Phi=\Theta_{a/b}(\Phi)$.

2.7 Satz. *In einem Verband L sei eine Kongruenzrelation Φ gegeben. L ist genau dann ein Φ-fast schwach modularer Verband, wenn $\Theta_{a/b}(\Phi)$ für alle Quotienten a/b von L eine Kongruenzrelation in L bildet.*

Beweis. Sei L ein Φ-fast schwach modularer Verband. Wegen 2.6 liegt es nahe, $\Theta_{a/b}(\Phi)=\Theta_{a/b}*\Phi$ für alle Quotienten a/b von L nachzuweisen. Betrachten wir $\Theta_{a/b}(\Phi)$ für einen Quotienten a/b von L. Sei nun $x\equiv y(\Theta_{a/b}(\Phi))$ für einen Quotienten x/y und $x/y\to z/t$ mit $z\equiv t(\Theta_{a/b})$. Sei z/t ein nicht Φ-trivialer Quotient von L. Da L Φ-fast schwach modular ist, existiert dann ein nicht Φ-trivialer Teilquotient x_1/y_1 von x/y, so daß es zu jedem nicht Φ-trivialen Quotienten r/s mit $x_1/y_1\to r/s$ stets einen nicht Φ-trivialen Quotienten p/q mit $r/s\to p/q$ und $z/t\to p/q$ gibt. Wegen $x\equiv y(\Theta_{a/b}(\Phi))$ kann man unter den Quotienten r/s einen nicht Φ-trivialen Quotienten $\bar r/\bar s$ mit $x_1/y_1\to\bar r/\bar s$ und $\bar r\equiv\bar s(\Theta_{a/b}*\Phi)$ finden. $\bar p/\bar q$ sei der entsprechende nicht Φ-triviale Quotient mit $\bar r/\bar s\to\bar p/\bar q$ und $z/t\to\bar p/\bar q$. Daraus ergibt sich $\bar p\equiv\bar q(\Theta_{a/b}\cap\Theta_{a/b}*\Phi)$, d.h. $\bar p/\bar q$ wird durch Φ annulliert, was ein Widerspruch ist. Also muß $z\equiv t(\Phi)$ gelten. Aus 1.4 ergibt sich $x\equiv y(\Theta_{a/b}*\Phi)$. Somit haben wir $\Theta_{a/b}(\Phi)\leqq\Theta_{a/b}*\Phi$ im Verband aller binären Relationen des Verbandes L bewiesen. Die Beziehung $\Theta_{a/b}(\Phi)\geqq\Theta_{a/b}*\Phi$ ist klar. Aus den beiden letzten Beziehungen folgt schließlich $\Theta_{a/b}(\Phi)=\Theta_{a/b}*\Phi$.

Umgekehrt sei $\Theta_{a/b}(\Phi)$ eine Kongruenzrelation in L für alle Quotienten a/b von L. Aus 2.6 folgt dann $\Theta_{a/b}(\Phi)=\Theta_{a/b}*\Phi$. Nehmen wir an, daß es zwei nicht Φ-trivialen Quotienten a/b und u/v von L mit $a/b\to u/v$ gibt, die die Bedingungen der Φ-fast schwachen Modularität nicht erfüllen (s. die Definition 2.1). Genauer gesagt, gibt es zu jedem nicht Φ-trivialen Teilquotienten a_1/b_1 von a/b einen nicht Φ-trivialen Quotienten r/s von L mit $a_1/b_1\to r/s$, so daß $r/s\to z/t$ und $u/v\to z/t$ stets $z\equiv t(\Phi)$ impliziert. Aus 1.4 und 1.2 erhält man $r\equiv s(\Theta_{u/v}*\Phi)$. Die letzte Aussage ergibt mit 2.6 und der Annahme $a\equiv b(\Theta_{u/v}*\Phi)$. Wegen $a/b\to u/v$ annulliert $\Theta_{u/v}*\Phi$ auch den Quotienten u/v. Somit haben wir auch $u\equiv v(\Theta_{u/v}\cap\Theta_{u/v}*\Phi)$, d.h. $u\equiv v(\Phi)$, was zu einem Widerspruch führt, weil u/v als ein nicht Φ-trivialer Quotient angenommen wurde. Also L ist Φ-fast schwach modular.

3. Schwach modulare Verbände

Einen Spezialfall der fast schwach modularen Verbände stellen die schwach modularen Verbände im Sinne von [2] dar. Wir geben noch einmal eine dazu äquivalente Definition dieser Verbände an.

3.1. Definition. Ein Verband L heißt schwach modular, wenn es zu je zwei nichttrivialen Quotienten a/b und u/v von L mit $a/b \to u/v$ immer einen nichttrivialen Teilquotienten a_1/b_1 von a/b mit $u/v \to a_1/b_1$ gibt.

Man kann nachweisen (s. [2]), daß jeder modulare Verband schwach modular ist. Aufgrund der Definitionen 2.1 und 3.1 zeigt man leicht, daß jeder schwach modulare Verband fast schwach modular ist.

Aus einem Vergleich der Definitionen 2.1 und 3.1 bekommt man

3.2. *Ein fast schwach modularer Verband L ist genau dann schwach modular, wenn der Quotient z/t aus der Definition* 2.1 *ein Teilquotient von r/s ist.*

In [3] und [4] wurden die schwach modularen Verbände mit Hilfe der folgenden binären Relation $\Theta^+_{a/b}$ gekennzeichnet, deren Definition wir in äquivalenter Form wiedergeben.

3.3. Definition. In einem Verband L sei ein Quotient a/b gegeben. Mit $\Theta^+_{a/b}$ bezeichnen wir eine binäre Relation in L, die die beiden folgenden Bedingungen erfüllt:

(i) $x \equiv y(\Theta^+_{a/b})$ genau dann, wenn $x \cup y \equiv x \cap y(\Theta^+_{a/b})$;

(ii) $x \equiv y(\Theta^+_{a/b})$ für einen Quotienten x/y von L genau dann, wenn aus $u \equiv v(\Theta_{a/b})$ für einen Teilquotienten u/v von x/y stets $u = v$ folgt.

3.4. *Sei a/b ein Quotient eines Verbandes L. Die binäre Relation $\Theta^+_{a/b}$ ist genau dann eine Kongruenzrelation in L, wenn $\Theta^+_{a/b} = \Theta^*_{a/b}$ gilt.*

Beweis. Gilt $\Theta^+_{a/b} = \Theta^*_{a/b}$, dann ist natürlich $\Theta^+_{a/b} \in \Theta(L)$. Umgekehrt, sei $\Theta^+_{a/b}$ eine Kongruenzrelation in L. Angenommen, es gilt $x \equiv y(\Theta^+_{a/b})$ für einen Quotienten von L. Ist außerdem für einen Quotienten z/t $x/y \to z/t$ mit $z \equiv t(\Theta_{a/b})$ erfüllt, dann erhalten wir $z \equiv t(\Theta^+_{a/b})$ (wegen $\Theta^+_{a/b} \in \Theta(L)$). Demnach folgt $z = t$. 1.4 ergibt $x \equiv y(\Theta^*_{a/b})$, woraus $\Theta^+_{a/b} \leqq \Theta^*_{a/b}$ folgt. Die Beziehung $\Theta^*_{a/b} \leqq \Theta^+_{a/b}$ ist klar. Aus den letzten Aussagen bekommt man $\Theta^+_{a/b} = \Theta^*_{a/b}$.

In [3, Theorem 4] und [4, Theorem 1] wurde der folgende Satz bewiesen. Wir beweisen diesen Satz mit einer anderen Methode.

3.5. Satz. *Ein Verband L ist genau dann schwach modular, wenn für alle Quotienten a/b von L die binäre Relation $\Theta^+_{a/b}$ eine Kongruenzrelation in L ist.*

Beweis. Sei L schwach modular und a/b ein Quotient von L. Wir zeigen $\Theta^+_{a/b} = \Theta^*_{a/b}$. Sei nun $x \equiv y(\Theta^+_{a/b})$ für einen Quotienten x/y von L und außerdem gelte $x/y \to z/t$ mit $z \equiv t(\Theta_{a/b})$. Ist $z \neq t$, dann gibt es einen nichttrivialen Teilquotienten x_1/y_1 von x/y mit $z/t \to x_1/y_1$. Offenbar wird x_1/y_1 durch $\Theta_{a/b}$ annulliert. Daher bekommt man aus $x \equiv y(\Theta^+_{a/b})$ $x_1 = y_1$, was ein Widerspruch ist. Also bleibt die andere Möglichkeit: $z = t$. Nach 1.4 annulliert dann $\Theta^*_{a/b}$ den Quotienten x/y. Somit haben wir $\Theta^+_{a/b} \leqq \Theta^*_{a/b}$ bewiesen. Die Ungleichung $\Theta^+_{a/b} \geqq \Theta^*_{a/b}$ kann man leicht einsehen. Die beiden letzten Beziehungen liefern uns $\Theta^+_{a/b} = \Theta^*_{a/b}$.

Umgekehrt, sei für alle Quotienten a/b von L die binäre Relation $\Theta^+_{a/b}$ eine Kongruenzrelation in L. Es gilt dann $\Theta^+_{a/b} = \Theta^*_{a/b}$ (s. 3.4). Darüber hinaus kann

man noch $\Theta^+_{a/b} = \Theta_{a/b}(\omega)$ für die identische Kongruenzrelation ω zeigen (vgl. die Definition 2.5). Nach 2.7 bildet L einen fast schwach modularen Verband. Angenommen man hat $a/b \to u/v$ für nichttriviale Quotienten a/b und u/v. Dann gibt es einen nichttrivialen Teilquotienten a_1/b_1 von a/b, so daß für alle nichttrivialen Quotienten r/s mit $a_1/b_1 \to r/s$ ein nichttrivialer Quotient z/t existiert mit $r/s \to z/t$ und $u/v \to z/t$. Wir zeigen zunächst, daß man z/t immer als Teilquotienten von r/s wählen kann. Gibt es nämlich keinen Teilquotienten r_1/s_1 von r/s mit $u/v \to r_1/s_1$, dann gilt $r \equiv s(\Theta^+_{u/v})$ (s. 1.2). Wegen $\Theta^+_{u/v} = \Theta^*_{u/v}$ und $r/s \to z/t$ erhält man $z \equiv t(\Theta_{u/v} \cap \Theta^*_{u/v})$, d.h. $z = t$, was zu einem Widerspruch führt. Also kann man z/t als Teilquotienten von r/s wählen. Nach 3.2 bildet dann L einen schwach modularen Verband.

Aufgrund von 2.4 kann man die schwach modularen Verbände noch anders charakterisieren. Zunächst führen wir eine neue binäre Relation ein.

3.6. Definition. In einem Verband L sei die Kongruenzrelation Θ gegeben. Mit Θ^{++} bezeichnen wir eine binäre Relation in L, die die beiden folgenden Bedingungen erfüllt:

(i) $x \equiv y(\Theta^{++})$ genau dann, wenn $x \cup y \equiv x \cap y(\Theta^{++})$;

(ii) $x \equiv y(\Theta^{++})$ für einen Quotienten x/y von L genau dann, wenn es zu jedem nichttrivialen Teilquotienten x_1/y_1 von x/y einen nichttrivialen Teilquotienten x_2/y_2 von x_1/y_1 gibt mit $x_2 \equiv y_2(\Theta)$.

3.7. *In einem Verband L sei eine Kongruenzrelation Θ gegeben. Hat die binäre Relation Θ^{++} die folgende Eigenschaft:*

$$x/y \to z/t \text{ und } x \equiv y(\Theta^{++}) \text{ impliziert } z \equiv t(\Theta^{++}),$$

dann gilt im Verband aller binären Relationen in L $\Theta^{++} \leqq \Theta^{**} \leqq \Theta_\omega$.

Der Beweis folgt aus 1.5, 2.2 und 3.6.

3.8. Satz. *Ein Verband L ist genau dann schwach modular, wenn für alle Kongruenzrelationen Θ in L $\Theta^{++} = \Theta_\omega$ und $\Theta^{++} \in \Theta(L)$ gilt.*

Beweis. Sei L schwach modular und $\Theta \in \Theta(L)$. Dann ist L natürlich fast schwach modular. Nach 2.4 ist Θ_ω eine Kongruenzrelation in L. Aus 2.3 folgt dann $\Theta_\omega = \Theta^{**}$. Es bleibt noch $\Theta^{++} = \Theta^{**}$ zu zeigen. Sei nun $a \equiv b(\Theta^{**})$ für einen Quotienten a/b. Wegen $\Theta_\omega = \Theta^{**}$ gibt es zu jedem nichttrivialen Teilquotienten a_1/b_1 von a/b einen nichttrivialen Quotienten r/s mit $a_1/b_1 \to r/s$ und $r \equiv s(\Theta)$. Weiterhin gibt es einen nichttrivialen Teilquotienten a_2/b_2 von a_1/b_1, so daß $r/s \to a_2/b_2$ gilt (L ist schwach modular). $r \equiv s(\Theta)$ liefert uns $a_2 \equiv b_2(\Theta)$. Demnach gilt $a \equiv b(\Theta^{++})$. Somit haben wir $\Theta^{**} \leqq \Theta^{++}$ bewiesen. $\Theta^{++} \leqq \Theta^{**}$ folgt aus 3.7. Aus beiden Beziehungen erhält man $\Theta^{++} = \Theta^{**}$.

Umgekehrt sei $\Theta^{++} \in \Theta(L)$ und $\Theta^{++} = \Theta_\omega$ für alle Kongruenzrelationen in L. Nach 3.7 gilt $\Theta^{++} = \Theta^{**} = \Theta_\omega$. Aus 2.4 erhält man, daß L fast schwach modular ist. Angenommen L erfüllt die Bedingungen der schwachen Modularität nicht. Dann gibt es zwei nichttriviale Quotienten a/b und u/v von L, so daß $a/b \to u/v$ und $u/v \to a_1/b_1$ für einen Teilquotienten von a/b genau dann

gilt, wenn $a_1 = b_1$ ist. N bezeichne die Gesamtheit der nichttrivialen Quotienten r/s von L mit der Eigenschaft: Aus $a/b \to r/s$ und $r/s \to a_1/b_1$ für einen Teilquotienten von a/b folgt $a_1 = b_1$. Offenbar ist N nicht leer, weil $u/v \in N$. Wir wissen aber, daß L fast schwach modular ist. Dann gibt es zu jedem nichttrivialen Quotienten x/y mit $a/b \to x/y$ einen nichttrivialen Teilquotienten a_1/b_1 von a/b, so daß es zu jedem nichttrivialen Quotienten r/s mit $a_1/b_1 \to r/s$ einen nichttrivialen Quotienten z/t gibt mit $r/s \to z/t$ und $x/y \to z/t$. Die Gesamtheit aller nichttrivialen Teilquotienten a_1/b_1 von a/b, die zu x/y bestimmt sind, bezeichnen wir mit $T(x/y)$. Sie brauchen nicht eindeutig bestimmt zu sein. Mit M bezeichnen wir die Gesamtheit der nichttrivialen Quotienten x/y mit $a/b \to x/y$, zu denen es mindestens einen nichttrivialen Teilquotienten a_1/b_1 von a/b mit $x/y \to a_1/b_1$ gibt. Schließlich bezeichne E die Gesamtheit der Quotienten $p/q \in M$ mit der folgenden Eigenschaft:

Ist a_1/b_1 ein nichttrivialer Teilquotient von a/b mit $p/q \to a_1/b_1$, und ist $c/d \in T(x/y)$ für einen Quotienten $x/y \in N$, dann ist der mengentheoretische Durchschnitt der Intervalle $[b_1, a_1]$ und $[d, c]$ höchstens einelementig.

Setzen wir nun $\Theta := \vee(\Theta_{r/s}; r/s \in N \bigcup E)$[3]. Offenbar ist Θ eine Kongruenzrelation in L. Zunächst beweisen wir, daß der Quotient a/b durch Θ^{**} annulliert wird. Dazu genügt es nach 2.4 und 2.3, nur $a \equiv b(\Theta_\omega)$ nachzuweisen. Genauer ist zu zeigen, daß es zu jedem nichttrivialen Teilquotienten a_1/b_1 von a/b einen nichttrivialen Quotienten r/s gibt mit $a_1/b_1 \to r/s$ und $r \equiv s(\Theta)$. Sei a_1/b_1 ein nichttrivialer Teilquotient von a/b und gelte $a_1/b_1 \to r/s$ für einen nichttrivialen Quotienten r/s von L. Wegen $a/b \to a_1/b_1$ gilt auch $a/b \to r/s$. Demnach gilt $r/s \in M \bigcup N$. Es gibt zwei Möglichkeiten:

(a) Es gibt einen Quotienten r/s mit $a_1/b_1 \to r/s$ und $r/s \in N \bigcup E$;

(b) Kein nichttrivialer Quotient r/s mit $a_1/b_1 \to r/s$ liegt in $N \bigcup E$.

Im ersten Fall wird r/s durch Θ annulliert. Im zweiten Fall liegt dann jeder Quotient r/s in $M - (N \bigcup E)$. Zu r/s gibt es einen nichttrivialen Teilquotienten c/d von a/b mit $r/s \to c/d$. Ferner gibt es einen nichttrivialen Quotienten $p/q \in N$ mit einem nichttrivialen Teilquotienten a_2/b_2 von a/b und $a_2/b_2 \in T(p/q)$, so daß der mengentheoretische Durchschnitt von Intervallen $[b_2, a_2]$ und $[d, c]$ mindestens zweielementig ist. Also enthalten die Quotienten a_2/b_2 und c/d einen nichttrivialen Teilquotienten a_3/b_3. Nach der Definition der Menge $T(p/q)$ gibt es auch noch einen nichttrivialen Quotienten z/t mit $a_3/b_3 \to z/t$ und $p/q \to z/t$. Wegen $p/q \in N$ gilt $z \equiv t(\Theta)$. Aus $a_1/b_1 \to r/s, r/s \to c/d, c/d \to a_3/b_3$ und $a_3/b_3 \to z/t$ folgt $a_1/b_1 \to z/t$. Damit haben wir bewiesen, daß der Quotient a/b durch Θ^{**} annulliert wird.

Wir haben angenommen, daß $u/v \in N$ ist. Dann gibt es einen nichttrivialen Teilquotienten a_1/b_1 von a/b mit $a_1/b_1 \in T(u/v)$. Natürlich gilt auch $a_1 \equiv b_1(\Theta^{**})$. Wegen $\Theta^{++} = \Theta^{**}$ gibt es einen nichttrivialen Teilquotienten a_2/b_2 von a_1/b_1 mit $a_2 \equiv b_2(\Theta)$. Nach (3) und 1.2 gibt es eine Kette $b_2 = y_0 \leqq \cdots \leqq y_n = a_2$ und Quotienten $r_i/s_i \in N \bigcup E$ $(1 \leqq i \leqq n)$ mit $r_i/s_i \to y_i/y_{i-1}$ für jedes $i = 1, \ldots, n$. Wegen

[3] „$\bigcup$" bezeichnet die mengentheoretische Vereinigung.

$r_i/s_i \in N \bigcup E$ $(1 \leqq i \leqq n)$ muß für alle $i=1, \ldots, n$ $y_i = y_{i-1}$ gelten. Daraus folgt $a_2 = b_2$, was einen Widerspruch bedeutet, weil $a_2 \neq b_2$ war. Damit haben wir bewiesen, daß L schwach modular ist.

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On Analytic Contractions in $L^p(G)$

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Throughout this paper, G will be a locally compact abelian group with dual Γ. We say that Γ is *algebraically ordered* if there exists a semi-group $P \subset \Gamma$ such that $P \cup -P = \Gamma$, $P \cap -P = \{0\}$. We do not assume that P is closed in Γ, so that, e.g. R^k, $k > 1$, is algebraically ordered. If P is closed in Γ, then Γ is called *ordered* (Rudin [6]). If Γ is discrete the two notions of ordered and algebraically ordered coincide.

Let Γ be discrete and ordered, let $u \in L^p(G)$ and let $\hat{u}$ be its Fourier transform. Put

$$\hat{v}(\gamma) = \begin{cases} \hat{u}(\gamma) & \text{if } \gamma \in P \\ 0 & \text{if } \gamma \notin P. \end{cases}$$

If $\hat{v}$, defined as above, is the transform of some $v \in L^p(G)$, then v is called the analytic contraction of u.

A famous theorem of Riesz [5], generalized by Helson [4] is the following: Let G be compact, abelian with ordered dual Γ. Suppose $1 < p < \infty$. Then every $u \in L^p(G)$ has an analytic contraction $v \in L^p(G)$ and there exists a constant A_p (independent of G) such that

$$\|v\|_p \leqq A_p \|u\|_p \quad \text{for every } u \in L^p(G).$$

The statement becomes false if $p = 1$ or ∞.

Assume now that Γ is not necessarily discrete and that Γ is algebraically ordered. Let $u \in L^p(G)$, $1 < p \leqq 2$, and let $\hat{u}$ be its Fourier transform. If P is not measurable, the function $\hat{v}$, defined as above, in general, will not be measurable, and therefore, an analogue of the Marcel Riesz theorem cannot even be stated. We adopt the following definition:

$$\hat{w}(\gamma) = \begin{cases} \hat{u}(\gamma) & \text{if } \gamma \in \bar{P} \\ 0 & \text{if } \gamma \notin \bar{P} \end{cases}$$

where $\bar{P}$ is the closure of P. If $\hat{w}$, defined as above, is the transform of some $w \in L^p(G)$, then again w will be called the *analytic contraction* of u.

Observe that when Γ is discrete the two functions $\hat{v}$, $\hat{w}$ coincide.

In case Γ is not discrete and the ordering semi-group P is measurable (with respect to Haar measure m in Γ), it is natural to consider the older definition:

$$\hat{v}(\gamma) = \hat{u}(\gamma) \quad \text{if } \gamma \in P, \quad \hat{v}(\gamma) = 0 \quad \text{if } \gamma \notin P.$$

We shall show that *when P is measurable then* $\hat{w} = \hat{v}$ almost everywhere and therefore *the two definitions are equivalent.* In fact, we shall prove that *P is measurable if and only if P differs from its closure $\bar{P}$ by a locally null set A*, i.e.

a set A such that $m(K \cap A)=0$ for every compact set K in Γ. Thus, when P is measurable, the function $\hat{w}-\hat{v}$ is locally null, and since $\hat{w}, \hat{v} \in L^q(\Gamma)$, ($q$ conjugate of p), then $\hat{w}-\hat{v}=0$ almost everywhere.

Next, our aim in this paper, is to generalize the Riesz-Helson theorem to the case of any locally compact abelian group with algebraically ordered dual. Specifically:

For $1<p\leqq 2$, there is a constant A_p (not depending on G) such that every element u of $L^p(G)$ has an analytic contraction w satisfying $\|w\|_p \leqq A_p \|u\|_p$.

We restrict p to the interval $1<p\leqq 2$, for if $u \in L^p(G)$, $p>2$, then its transform $\hat{u}$ is not in general defined. But if $p>2$ and if $u \in L^p \cap L^2$ so as to ensure the existence of $\hat{u}$, then the analytic contraction w exists and satisfies

$$\|w\|_p \leqq A_p \|u\|_p, \quad \text{with} \quad A_p = A_q, \quad \frac{1}{q}+\frac{1}{p}=1.$$

We do not use Banach algebra techniques as Helson, but we avail ourselves of standard methods at steps where applicable. The two new points are (a) the introduction of a "nice set", dense in all L^p, so that one can push the theorem through, for this nice set, (b) the replacement, in case P is not measurable, of every $u \in L^p(G)$ by another function $u-u_0 \in L^p(G)$ whose transform vanishes on an open subgroup Γ_0 of Γ.

Lemma 1. *Let P be an algebraic order for Γ. Put $\Gamma_0 = \bar{P} \cap (-\bar{P})$, where $\bar{P}$ denotes the closure of P. Then Γ_0 is a closed subgroup of Γ.*

The proof is routine and is omitted.

Theorem 1. *Let P be an algebraic order for the locally compact abelian group Γ. Then P is measurable if and only if P differs from its closure $\bar{P}$ by a locally null set.*

Proof. We may suppose that Γ is not discrete, otherwise there is nothing to prove. If P differs from $\bar{P}$ (closed hence measurable) by a locally null, hence measurable set A, then P is measurable.

Conversely, we shall prove that if P is measurable, then the closed subgroup $\Gamma_0 = \bar{P} \cap (-\bar{P})$ is locally null. Since

$$\bar{P} = (\bar{P} \cap P) \cup (\bar{P} \cap (-P)) \subset P \cup \Gamma_0,$$

this will show that $\bar{P}$ differs from P by a locally null set and the proposition will be proved.

So let P be measurable, and assume that Γ_0 is not a locally null set. By a well known theorem (of Steinhaus) the set $\Gamma_0 - \Gamma_0 = \Gamma_0$ contains a neighborhood V_0 of 0 in Γ:

$$V_0 \subset \Gamma_0 \subset (-\bar{P}). \tag{1}$$

Choose a symmetric compact neighborhood V_1 of 0 such that $V_1 + V_1 \subset V_0$. Put $P_1 = V_1 \cap P$. The set P_1 has positive measure, otherwise the set $V_1 \subset P_1 \cup (-P_1)$ would have zero measure. By a known theorem, (see e.g. [3] 20, 17), the set $P_1 + P_1$ contains a non void open set. Hence, there is $\gamma_0 \in \Gamma$ and a symmetric

neighborhood V of 0 in Γ such that

$$\gamma_0 + V \subset P_1 + P_1 \subset P. \tag{2}$$

In particular

$$\gamma_0 \in P_1 + P_1 \subset V_1 + V_1 \subset V_0. \tag{3}$$

We have $\gamma_0 \neq 0$ otherwise $V \subset P$, $V \subset -P$, $V = \{0\}$ and Γ would be discrete. Hence we may assume $-\gamma_0 \notin V$, i.e. $0 \notin \gamma_0 + V$. Then, by (2)

$$\gamma_0 + V \text{ does not meet } -P. \tag{4}$$

By (3) and (1) we have $\gamma_0 \in (\overline{-P})$. Hence $\gamma_0 + V$ meets $-P$, against (4). This contradiction shows that Γ_0 must be locally null and the theorem is proved.

Lemma 2. *Let Γ_0 be an open subgroup of Γ. Then the function φ, equal to 1 on Γ_0 and equal to 0 outside Γ_0, is positive definite.*

Proof. Let $a_1, \ldots, a_N$ be complex numbers, $\gamma_1, \ldots, \gamma_N$ be elements of Γ.

We partition the set $\{\gamma_1, \ldots, \gamma_N\}$ into disjoint subsets of equivalence classes mod Γ_0

$$\{\gamma_1, \ldots, \gamma_N\} = \{\gamma_1^1, \ldots\} \cup \{\gamma_1^2, \ldots\} \cup \cdots \cup \{\gamma_1^K, \ldots\}.$$

If $\gamma_i^k = \gamma_n$ we put $a_i^k = a_n$. Then

$$\sum_{m,n=1}^{N} a_m \bar{a}_n \varphi(\gamma_m - \gamma_n) = \sum_{k=1}^{K} \sum_{i,j} a_i^k \overline{a_j^k} = \sum_{k=1}^{K} \left|\sum_i a_i^k\right|^2 \geqq 0.$$

On the other hand, since Γ_0 is open and closed, the function φ is continuous. Thus φ is positive definite.

Remark. We conclude, by Bochner's theorem that $\varphi(\gamma) = \hat{\mu}(\gamma)$, where μ is some positive measure on G. Moreover since μ is positive, $\|\mu\| = \hat{\mu}(0) = \varphi(0) = 1$.

Lemma 3 (Compare Calderon [1]). *Let q be an integer of the form $2(2k+1)$. Then there exists a $\beta = \beta_q$ such that*

$$1 \leqq -2\cos q\theta + \beta^q (\cos\theta)^q$$

for every θ.

Proof. Since cos is even and has period 2π it is sufficient to consider the case $0 \leqq \theta \leqq \pi$.

If $0 \leqq \theta \leqq \frac{\pi}{2} - \frac{\pi}{4q}$ or $\frac{\pi}{2} + \frac{\pi}{4q} \leqq \theta \leqq \pi$ then

$$|\cos\theta| \geqq \cos\left(\frac{\pi}{2} - \frac{\pi}{4q}\right) = A \quad (say), \qquad A > 0.$$

Choose $\beta > 0$ such that $\beta^q A^q - 2 \geqq 1$. Then

$$\beta^q (\cos\theta)^q - 2\cos q\theta \geqq \beta^q A^q - 2 \geqq 1.$$

If $\frac{\pi}{2} - \frac{\pi}{4q} \leqq \theta \leqq \frac{\pi}{2} + \frac{\pi}{4q}$ then

$$(2k+1)\pi - \frac{\pi}{4} \leqq q\theta \leqq (2k+1)\pi + \frac{\pi}{4}.$$

Hence

$$\cos q\theta < -\tfrac{1}{2}$$

and therefore

$$1 \leqq -2\cos q\theta \leqq -2\cos q\theta + \beta^q(\cos\theta)^q.$$

This completes the proof of the lemma.

In the sequel, G will be a locally compact, *non-compact*, abelian group, whose dual Γ is algebraically ordered by P.

Lemma 4. *Let $f\in L^2(G)$ be such that $\hat{f}\in L^1(\Gamma)$ and $\hat{f}(\gamma)=0$ a.e. for $\gamma<0$. If $q>1$ is an integer then $f^q\in L^1(G)$ and $(f^q)\hat{}\,(0)=0$.*

Proof. Denote by $D(G)$ the class of functions $f\in L^2(G)$ such that $\hat{f}\in L^1(\Gamma)$ and $\hat{f}(\gamma)=0$ a.e. for $\gamma<0$. We shall show that if $f, g\in D(G)$, then $fg\in D(G)\cap L^1(G)$ and $(fg)\hat{}\,(0)=0$. This will prove the lemma.

So let $f, g\in D(G)$. Evidently $fg\in L^1(G)$. We have

$$(fg)\hat{}\,(\gamma_0)=\int_\Gamma \hat{f}(\gamma)\hat{g}(\gamma_0-\gamma)\,d\gamma=\hat{f}*\hat{g}(\gamma_0).$$

From $\hat{f}, \hat{g}\in L^1(\Gamma)$ we deduce $\hat{f}*\hat{g}\in L^1(\Gamma)$, i.e. $(fg)\hat{}\in L^1(\Gamma)$. Also, since $\hat{f}, \hat{g}\in L^2(\Gamma)$ we have $\hat{f}*\hat{g}\in L^\infty(\Gamma)$, i.e. $(fg)\hat{}\in L^\infty(\Gamma)$. This, combined with $(fg)\hat{}\in L^1(\Gamma)$ yields $(fg)\hat{}\in L^2(\Gamma)$, i.e. $fg\in L^2(G)$. Finally, we may suppose that $\hat{f}, \hat{g}$ vanish everywhere outside P. Then $\hat{f}*\hat{g}$ vanishes outside $P+P=P$, i.e.

$$(fg)\hat{}\,(\gamma)=0 \qquad \text{for } \gamma<0. \tag{1}$$

Now Γ is not discrete and every symmetric neighborhood of 0 in Γ meets $-P$ in some point $\gamma<0$. By continuity we get from (1): $(fg)\hat{}\,(0)=0$ and the lemma is proved.

Lemma 5. *Let $u\in L^1(G)\cap L^q(G)$, $q=2(2k+1)$, be a real function and let $\hat{u}\in L^1(\Gamma)$. Then the analytic contraction w exists and satisfies*

$$\|w\|_q \leqq \tfrac{1}{2} B_q \|u\|_q$$

where B_q is a constant independent of the group G and the ordering P.

Proof. Since $u\in L^2(G)$, then $\hat{u}\in L^2(\Gamma)$ and therefore $\hat{w}\in L^2(\Gamma)$, so that $\hat{w}$ is the Plancherel transform of some $w\in L^2(G)$.

By Lemma 2, there is a positive measure μ_0 (we take $\mu_0=0$ in case Γ_0 is not open, and hence locally null in Γ) such that $\|\mu_0\|\leqq 1$, $\hat{\mu}_0(\gamma)=1$ locally a.e. in Γ_0, $\hat{\mu}_0(\gamma)=0$ outside Γ_0. Put

$$u_0=\mu_0*u.$$

Then u_0 is real, $u_0\in L^q(G)$, $\|u_0\|_q\leqq\|\mu_0\|\,\|u\|_q\leqq\|u\|_q$. We want to show that $2\,\mathrm{Re}\,w=u+u_0$ a.e., that is

$$w+\bar{w}=u+u_0 \quad \text{a.e.} \tag{1}$$

For, let $\gamma\in\Gamma_0$. Then since $\Gamma_0\subset\bar{P}$ we have $\hat{w}(\gamma)=\hat{u}(\gamma)$. Also

$$(\overline{w})^{\wedge}(\gamma)=\overline{\hat{w}(-\gamma)}=\overline{\hat{u}(-\gamma)} \qquad (\text{since } -\gamma\in\Gamma_0\subset\bar{P})=\hat{u}(\gamma)$$

(since u is real)$=\hat{u}_0(\gamma)$ locally a.e. in Γ_0. Thus $\hat{w}(\gamma)+(\overline{w})^{\wedge}(\gamma)=\hat{u}(\gamma)+\hat{u}_0(\gamma)$ locally a.e. in Γ_0. Let now $\gamma\notin\Gamma_0$.

If (a): $\gamma\in P$, then $-\gamma\notin\bar{P}$, otherwise $\gamma\in\overline{(-P)}$ which would imply $\gamma\in\Gamma_0$. Hence $\hat{w}(\gamma)=\hat{u}(\gamma)$, $(\overline{w})^{\wedge}(\gamma)=\overline{\hat{w}(-\gamma)}=0$. Thus

$$\hat{w}(\gamma)+(\overline{w})^{\wedge}(\gamma)=\hat{u}(\gamma)=\hat{u}(\gamma)+\hat{u}_0(\gamma).$$

If (b): $\gamma\notin\bar{P}$, then $-\gamma\in\bar{P}$, for $\gamma\notin P$, $\gamma\in -P$, $-\gamma\in P\subset\bar{P}$. Hence $\hat{w}(\gamma)=0$, $(\overline{w})^{\wedge}(\gamma)=\overline{\hat{w}(-\gamma)}=\overline{\hat{u}(-\gamma)}=\hat{u}(\gamma)$ and therefore $\hat{w}(\gamma)+(\overline{w})^{\wedge}(\gamma)=\hat{u}(\gamma)=\hat{u}(\gamma)+\hat{u}_0(\gamma)$. Thus $\hat{w}+\overline{w}^{\wedge}=\hat{u}+\hat{u}_0$ locally a.e., hence a.e. and (1) is proved.

It is easy to check that

$$(\hat{w}-\hat{u}_0)=0 \qquad \text{a.e. for } \gamma<0. \tag{2}$$

For let $\gamma\in -P$. If $\gamma\notin\bar{P}$ then $\hat{w}(\gamma)=\hat{u}_0(\gamma)=0$. If $\gamma\in\bar{P}$, then $\gamma\in\Gamma_0$ and therefore $\hat{w}(\gamma)=\hat{u}(\gamma)=\hat{u}_0(\gamma)$ locally a.e. Thus (2) is proved.

We shall show that $w-u_0\in L^q(G)$ and that

$$\|w-u_0\|_q\leqq\frac{\beta}{2}\|u-u_0\|_q$$

where β is the constant in Lemma 3. In fact, since u_0 is real we have

$$2\,\mathrm{Re}(w-u_0)=u+u_0-2u_0=u-u_0.$$

Putting

$$w-u_0=|w-u_0|e^{i\theta}, \qquad \frac{u-u_0}{2}=\mathrm{Re}(w-u_0)=|w-u_0|\cos\theta$$

we have, by Lemma 3

$$\int_G|w-u_0|^q\leqq -2\int|w-u_0|^q\cos q\theta+\beta^q\int|w-u_0|^q(\cos\theta)^q$$

$$=-2\,\mathrm{Re}\int(w-u_0)^q+\beta^q\int\left(\frac{u-u_0}{2}\right)^q.$$

By Lemma 4, since $w-u_0\in L^2(G)$, $\hat{w}-\hat{u}_0\in L^1(\Gamma)$ and $(\hat{w}-\hat{u}_0)(\gamma)=0$ a.e. for $\gamma<0$, we have $\int_G(w-u_0)^q=0$. Therefore

$$\int|w-u_0|^q\leqq\left(\frac{\beta}{2}\right)^q\int(u-u_0)^q$$

that is $\|w-u_0\|_q\leqq\frac{\beta}{2}\|u-u_0\|_q$. This gives

$$\|w\|_q\leqq(\beta+1)\|u\|_q=\tfrac{1}{2}B_q\|u\|_q \qquad (\text{say})$$

and the lemma is proved.

Lemma 6. *Let* $q=2(2k+1)$, $\frac{1}{p}+\frac{1}{q}=1$. *Let* f *be any element of*

$$L^1(G)\cap L^p(G)\cap L^2(G).$$

Then the analytic contraction g *of* f *exists and satisfies*

$$\|g\|_p\leqq B_q\,\|f\|_p.$$

Proof. Let $M=\{u\in L^1(G)\cap L^q(G):\hat{u}\in L^1(\Gamma),\ u \text{ real}\}$. If $u\in M$, and w is the analytic contraction of u, then, by Lemma 5, $w\in L^q(G)$ and $\|w\|_q\leqq\frac{1}{2}B_q\,\|u\|_q$. Put $\hat{g}(\gamma)=\hat{f}(\gamma)$ for $\gamma\in\bar{P}$, $\hat{g}(\gamma)=0$ for $\gamma\notin\bar{P}$. Then $\hat{g}\in L^2(\Gamma)$ and therefore $\hat{g}$ is the transform of some $g\in L^2(G)$. Now $u\in L^1(G)$, hence $g*u\in L^2(G)$. Similarly $f*w\in L^2(G)$. Since $\hat{g}\,\hat{u}=\hat{f}\hat{w}$, then $g*u=f*w$ a.e. On the other hand, $g*u$ is continuous, since $g\in L^2$, $u\in L^2$. Similarly, $f*w$ is continuous. Hence $g*u=f*w$ everywhere, in particular at 0. Thus, for $u\in M$,

$$\left|\int_G g(x)\,u(-x)\,dx\right|=\left|\int_G f(x)\,w(-x)\,dx\right|\leqq\|f\|_p\,\|w\|_q\leqq\tfrac{1}{2}B_q\,\|f\|_p\,\|u\|_q.$$

If $u\in M+iM$, then

$$\left|\int_G g(x)\,u(-x)\,dx\right|\leqq B_q\,\|f\|_p\,\|u\|_q. \tag{1}$$

We want to use the converse of Hölder's inequality (see e.g. [2], 13.1.5). Since $g\in L^2(G)$, all integrations appearing in (1) are performed on a σ-compact set. Let v be any simple function such that $\|v\|_q\leqq 1$. Then $v\in L^q\cap L^2$. Using approximate identities whose transforms have compact support, we can easily find $u_n\in M+iM$ such that

$$\|u_n-v\|_2\to 0,\qquad \|u_n-v\|_q\to 0\quad\text{as}\quad n\to\infty.$$

Then, since $g\in L^2$, we have, by (1)

$$\left|\int g\,v\right|=\lim_{n\to\infty}\left|\int g\,u_n\right|\leqq B_q\,\|f\|_p\lim_{n\to\infty}\|u_n\|_q\leqq B_q\,\|f\|_p.$$

This yields $g\in L^p$ and $\|g\|_p\leqq B_q\,\|f\|_p$. The lemma is now proved.

Theorem 2. *Let* G *be a locally compact abelian group whose dual* Γ *is algebraically ordered. Let* $1<p\leqq 2$. *There is a constant* A_p *(independent of* G *and the order of* Γ*) such that every element* f *of* $L^p(G)$ *has an analytic contraction* g *satisfying the relation*

$$\|g\|_p\leqq A_p\,\|f\|_p.$$

Proof. Choose q_1 of the form $2(2k+1)$ in such a way that its conjugate p_1, defined by $\frac{1}{p_1}+\frac{1}{2(2k+1)}=1$, is less than p. By Lemma 6, the map

$$T: L^1\cap L^{p_1}\cap L^2\to L^{p_1}\cap L^2$$

defined by the analytic contraction $Tf=g$ is bounded as a map from L^{p_1} to L^{p_1}:

$$\|Tf\|_{p_1}\leqq B_{q_1}\,\|f\|_{p_1}\qquad\text{for } f\in L^1\cap L^{p_1}\cap L^2.$$

T is also bounded as a map from L^2 to L^2:

$$\|Tf\|_2 \leqq \|f\|_2 \qquad \text{for } f \in L^1 \cap L^{p_1} \cap L^2.$$

Put

$$p = \frac{2p_1}{2\alpha + (1-\alpha)p_1}.$$

Then $0 \leqq \alpha < 1$ and by the Riesz-Thorin theorem, T is bounded from L^p to L^p:

$$\|Tf\|_p \leqq A_p \|f\|_p \qquad \text{for } f \in L^1 \cap L^{p_1} \cap L^2 \tag{1}$$

with $A_p = (B_{q_1})^{1-\alpha}$.

Now $L^1 \cap L^{p_1} \cap L^2$ is dense in L^p. Hence T can be extended uniquely to a mapping of the whole of L^p into L^p, still satisfying (1). To check that the extended map is precisely the analytic contraction, assume $f \in L^p$. Let $f_n \in L^1 \cap L^{p_1} \cap L^2$ be such that $\|f_n - f\|_p \to 0$. By (1), $Tf_n = g_n$ is a Cauchy sequence in L^p which therefore converges to $Tf = g$. Then (q conjugage of p):

$$\|\hat{g} - \hat{g}_n\|_q \leqq \|g - g_n\|_p \to 0 \quad \text{as} \quad n \to \infty.$$

Since all the $\hat{g}_n(\gamma)$ vanish for $\gamma \notin \bar{P}$, then $\hat{g}(\gamma) = 0$ a.e. for $\gamma \notin \bar{P}$. On the other hand

$$\left[\int_{\hat{P}} |\hat{f} - \hat{g}|^q\right]^{1/q} \leqq \|\hat{f} - \hat{f}_n\|_q + \left[\int_{\hat{P}} |\hat{f}_n - \hat{g}_n|^q\right]^{1/q} + \|\hat{g}_n - \hat{g}\|_q.$$

In the r.h.s. all terms are zero or tend to zero when $n \to \infty$. Therefore $\hat{f}(\gamma) = \hat{g}(\gamma)$ a.e. for $\gamma \in \bar{P}$. Thus $Tf = g$ is the analytic contraction of f.

The proof is now complete.

Corollary. *Suppose $q > 2$. Then for every $f \in L^2(G) \cap L^q(G)$ the analytic contraction belongs to $L^q(G)$ and we have*

$$\|g\|_q \leqq A_p \|f\|_q, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

Proof. Use the converse of Hölder's inequality as in the proof of Lemma 6.

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Die Betti-Reihe und die Abweichungen eines lokalen Rings

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Einleitung

Unter den *Betti-Zahlen* eines lokalen Ringes R mit Restklassenkörper K versteht man die Dimensionen $b_p(R)$ der Vektorräume $\mathrm{Tor}_p^R(K,K)$. Die Nützlichkeit dieses Begriffs hat Serre gezeigt, als er mit seiner Hilfe in [13] die homologische Dimension der lokalen Ringe bestimmte und in [14] Vielfachheiten lokaler Ringe untersuchte. Die Betti-Zahlen wurden später weiter verwendet, um besondere Klassen von lokalen Ringen (insbesondere reguläre lokale Ringe und vollständige Durchschnitte) zu charakterisieren ([1, 7, 16] und [18]).

Die Bestimmung der Betti-Zahlen war zunächst schwierig. Gleichungen erhielt man nur unter speziellen Voraussetzungen über R, für den allgemeinen Fall leitete man Ungleichungen ab ([1] und [12–17]). Seit Tate [16] und Scheja [12] hat es sich als zweckmäßig erwiesen, die Betti-Zahlen zu der formalen Potenzreihe

$$\mathscr{B}(R) := \sum_{p\in\mathbb{N}} b_p(R)\, Z^p,$$

der *Betti-Reihe* von R, zusammenzufassen. Scheja [12] hat $\mathscr{B}(R)$ für Ringe R einer homologischen Kodimension $\geqq b_1(R)-2$ als rationale Funktion dargestellt; vgl. auch [6, 15] und [18]. Die formalen Potenzreihen in einer Unbestimmten mit ganzzahligen Koeffizienten und dem konstanten Glied 1 lassen sich eindeutig darstellen als formale Produkte

$$\prod_{i\in\mathbb{N}} (1-(-Z)^{i+1})^{(-1)^i \varepsilon_i}$$

mit $\varepsilon_i \in \mathbb{Z}$ und umgekehrt. Wählt man diese Darstellung für $\mathscr{B}(R)$, so heißen die ε_i die *Abweichungen* $\varepsilon_i(R)$. Tate hatte auf die Wichtigkeit des Problems hingewiesen, ob es stets sog. minimale R-Algebren-Auflösungen von K gibt. Mit der Bejahung dieser Frage durch Gulliksen [5] (der Beweis in [11] verwendet seinerseits [5]) war ein wichtiger Fortschritt erreicht. Es folgt nämlich unmittelbar, daß $\varepsilon_i(R)$ die minimale Erzeugeranzahl für einen gewissen R-Modul (und insbesondere nichtnegativ) ist (vgl. auch [17]). Die Berechnung und weitere Untersuchung der Betti-Zahlen blieb trotzdem schwierig, weil man eine R-freie Auflösung finden muß, was bedeutet, daß lineare Gleichungssysteme in einem beliebigen lokalen Ring gelöst werden müssen.

Wir nehmen in dieser Arbeit an, R enthält einen Körper und ist Restklassenring eines regulären lokalen Ringes $\tilde{R}$. Unter dieser Voraussetzung hat Wiebe [18] $\varepsilon_1(R)$ und $\varepsilon_2(R)$ bestimmt ($\varepsilon_0(R)$ ist, wie man sofort sieht, die minimale Erzeugeranzahl des maximalen Ideals von R). Wir werden hier in Verallgemeinerung der Ergebnisse von Wiebe alle höheren Abweichungen eines lokalen Ringes deuten und damit gleichzeitig eine Methode zur Berechnung der Betti-Zahlen angeben. A sei eine spezielle $\tilde{R}$-freie Auflösung von R (vgl. § 2). In A ist dann eine Multiplikation $*$ definiert, für die gilt:

$$A_j = \sum_{i=1}^{j-1} A_i * A_{j-i} \oplus E_j$$

mit einem freien $\tilde{R}$-Modul E_j. Der Rang von E_j ist gerade $\varepsilon_j(R)$ $(j>0)$.

In Anwendungen ist ein lokaler Ring häufig als Restklassenring eines regulären lokalen Rings $\tilde{R}$ gegeben. Eine $\tilde{R}$-freie Auflösung von R ist, weil $\tilde{R}$ regulär ist, verhältnismäßig leicht zu finden. Aus dieser läßt sich eine spezielle $\tilde{R}$-freie Auflösung und damit die Betti-Reihe von R berechnen. Unsere Methode scheint daher sehr nützlich zu sein, sowohl für die Bestimmung der Betti-Zahlen spezieller lokaler Ringe als auch für das Auffinden von Zusammenhängen mit anderen Invarianten lokaler Ringe.

Wir benötigen die Ergebnisse der oben aufgeführten Arbeiten nicht; insbesondere verwenden wir keine minimale R-Algebra-Auflösung von K, sondern wir konstruieren eine i.a. „größere" R-freie Auflösung von K.

§ 1. Einige Begriffe und Sätze der homologischen Algebra

Für diese ganze Arbeit gelten folgende Konventionen: Ein Ring ist ein Noetherscher kommutativer und assoziativer Ring mit Einselement. R ist ein Ring, K ein kommutativer Körper. Ein Vektorraum ist ein Vektorraum über K. Die Zeichen dim, $\otimes$ und $\bigotimes$ (ohne Index) in Verbindung mit Zeichen für Vektorräume bezeichnen die Dimension über K bzw. das Tensorprodukt über K. Ist M ein R-Modul, so identifizieren wir M in der üblichen Weise mit $M \otimes_R R$ und $R \otimes_R M$.

Wir werden wiederholt folgende Bezeichnung verwenden: Ist $(x_j)_{j\in J}$ Basis eines freien R-Moduls L und $R \to R'$ ein Homomorphismus von Ringen, so nennen wir die Elemente der entsprechenden Basis des freien R'-Moduls $L \otimes_R R'$ statt $x_j \otimes 1$ ebenfalls x_j.

Ein *(positiv) graduierter R-Modul* E ist eine Familie $(E_i)_{i\in\mathbb{Z}}$ von R-Moduln mit $E_i=0$ für $i<0$ (bzw. für $i\leqq 0$); manchmal bezeichnen wir der Kürze halber auch den R-Modul $\bigoplus_{i\in\mathbb{Z}} E_i$ als graduierten R-Modul E. Ein *(positiver) Komplex* von R-Moduln ist ein Paar (E, χ), wo E ein (positiv) graduierter R-Modul und χ eine Familie von R-linearen Abbildungen $\chi_i\colon E_i \to E_{i-1}$ mit $\chi_i \circ \chi_{i+1}=0$ ist. Einen graduierten R-Modul E betrachten wir auch als Komplex $(E, 0)$.

Die *äußere* und die *symmetrische Algebra* eines R-Moduls M (vgl. [3]) bezeichnen wir mit $\Lambda_R M$ und $\mathsf{S}_R M$, die Verknüpfung darin mit $\wedge$ bzw. s.

Statt $(\wedge_R M)_k$ und $(\mathbf{S}_R M)_k$ schreiben wir $\overset{k}{\wedge}_R M$ und $\overset{k}{\mathbf{S}}_R M$. Der Index R wird weggelassen, wenn $R=K$ ist.

Definition 1. (Vgl. Tate [16].) Eine *R-Algebra* $(A, \chi, *)$ ist gegeben durch

1. einen Komplex (A, χ) von $\mathbb{Z}$-Moduln,
2. $\mathbb{Z}$-bilineare Abbildungen $A_i \times A_j \to A_{i+j}$, $(x, y) \mapsto x * y$ $(i, j \in \mathbb{Z})$,

wenn gilt:

Für $x \in A_i$, $y \in A_j$, $z \in A_k$ ist $\chi(x*y)=\chi(x)*y+(-1)^i x*\chi(y)$, $x*(y*z)-(x*y)*z$, $x*y=(-1)^{ij} y*x$, und falls i ungerade ist, $x*x=0$. $(A_0, *)$ ist ein Restklassenring von R.

Ist eine R-Algebra A wie oben gegeben, so betrachten wir $\bigoplus_{i\in\mathbb{Z}} A_i$ vermöge $*$ als einen R-Modul, den wir dann ebenfalls mit A bezeichnen.

Unter einem *Homomorphismus zwischen zwei R-Algebren* versteht man einen Homomorphismus der zugrundeliegenden Komplexe von R-Moduln, der mit der Multiplikation verträglich ist und das Einselement in das Einselement überführt.

Beispiel. E sei ein positiver Komplex von R-Moduln, die wir der Einfachheit halber als paarweise fremd annehmen. Wir wollen eine R-Algebra $\ast E$ definieren. Für $0<j\in\mathbb{N}$ sei

$$E^{(j)}\colon \cdots \to 0 \to E_j \to 0 \to \cdots \qquad (E_j \text{ an der Stelle } j).$$

Für $i\in\mathbb{Z}$ sei

$$(\ast E^{(j)})_i := \begin{cases} 0, & \text{falls } i \notin j\mathbb{N}, \\ \overset{k}{\wedge}_R E_j, & \text{falls } i=jk,\ k\in\mathbb{N},\ j\notin 2\mathbb{N}, \\ \overset{k}{\mathbf{S}}_R E_j, & \text{falls } i=jk,\ k\in\mathbb{N},\ j\in 2\mathbb{N}. \end{cases}$$

Das kanonische Bild von $(x_1, \ldots, x_k)\in E_j^{\,k}$ in diesem Modul nennen wir $x_1 * \cdots * x_k$. Wir setzen

$$(\ast E)_j := \bigoplus_{i_1+\cdots+i_j=j} \overset{j}{\otimes}_R (\ast E^{(k)})_{i_k}.$$

Die Multiplikation $*$ auf $\ast E := ((\ast E)_j)_{j\in\mathbb{Z}}$ definiert man durch

$$(x_1\otimes x_2\otimes\cdots)*(y_1\otimes y_2\otimes\cdots) := (-1)^{j_1 i_3+(j_1+j_3)i_5+(j_1+j_3+j_5)i_7+\cdots} x_1 * y_1 \otimes x_2 * y_2 \otimes \cdots,$$

falls $x_k\in(\ast E^{(k)})_{i_k}$, $y_k\in(\ast E^{(k)})_{j_k}$.

Wir indentifizieren E_j, $j>0$, in natürlicher Weise mit dem Untermodul $R\otimes_R R\otimes_R\cdots\otimes_R E_j$ von $(\ast E)_j$. Dann sind alle Elemente von $(\ast E)_j$ Summen von Produkten $x_1 * x_2 * \cdots * x_j$ $(x_i\in E_i)$. Wir definieren die Differentiation auf $\ast E$ durch die Forderung, daß $\ast E$ eine R-Algebra und E ein Unterkomplex von $\ast E$ sein soll.

$\ast E$ heißt die *freie R-Algebra über E* (vgl. auch [8], VI.7).

Einer R-Algebra $(A, \chi, *)$ mit $\chi_1 = 0$ ordnen wir eine weitere R-Algebra $B(A)$ zu (vgl. Eilenberg-MacLane [4] und Cartan [2]; siehe auch [8], X). Wir definieren

$$B(A)_k := \bigoplus_{p \in \mathbb{N}} \bigoplus_{\substack{j_1, \ldots, j_p \geq 2 \\ j_1 + \cdots + j_p = k}} A_{j_1 - 1} \otimes_R \cdots \otimes_R A_{j_p - 1}.$$

Es gilt also $B(A)_0 = R$, $B(A)_1 = 0$, $B(A)_2 = A_1$. Die Differentiation ψ von $B(A)$ wird erklärt durch

$$\psi(x_1 \otimes \cdots \otimes x_p) := \sum_{\substack{\mu = 1 \\ j_\mu > 2}}^{p} (-1)^{j_1 + \cdots + j_{\mu - 1}} x_1 \otimes \cdots \otimes x_{\mu - 1} \otimes \chi(x_\mu) \otimes x_{\mu + 1} \otimes \cdots \otimes x_p$$

$$+ \sum_{\mu = 2}^{p} (-1)^{j_1 + \cdots + j_{\mu - 1}} x_1 \otimes \cdots \otimes x_{\mu - 1} * x_\mu \otimes \cdots \otimes x_p,$$

wenn $x_1 \in A_{j_1 - 1}, \ldots, x_p \in A_{j_p - 1}$. Die Multiplikation $\wedge$ ist das sog. *shuffle-Produkt* ([4]), das rekursiv folgendermaßen erklärt werden kann (die Produkte mit einem Faktor in $B(A)_0 = R$ sind ja bereits durch die R-Modul-Struktur von $B(A)$ gegeben): Für $x_1 \in A_{j_1 - 1}, \ldots, x_p \in A_{j_p - 1}$, $y_1 \in A_{i_1 - 1}, \ldots, y_q \in A_{i_q - 1}$ sei

$$x_1 \otimes \cdots \otimes x_p \wedge y_1 \otimes \cdots \otimes y_q := x_1 \otimes (x_2 \otimes \cdots \otimes x_p \wedge y_1 \otimes \cdots \otimes y_q)$$

$$+ (-1)^{i_1 (j_1 + \cdots + j_p)} y_1 \otimes (x_1 \otimes \cdots \otimes x_p \wedge y_2 \otimes \cdots \otimes y_q).$$

Es ist einfach, nachzurechnen, daß $B(A)$ eine R-Algebra ist.

Ist A eine R-Algebra, so trägt in natürlicher Weise $R' := H_0(A)$ die Struktur eines Ringes und $H(A)$ die einer R'-Algebra mit der Differentiation 0.

Ist E ein Komplex von endlich-dimensionalen Vektorräumen über K, so heißen die Zahlen $b_p(E) := \dim H_p(E)$ die *Betti-Zahlen* und die formale Potenzreihe $\mathscr{B}(E) := \sum_{p \in \mathbb{N}} b_p(E) Z^p$ die *Betti-Reihe von E*. Ist E' ein weiterer Komplex von Vektorräumen, so ist

$$\mathscr{B}(E \otimes E') = \mathscr{B}(E) \mathscr{B}(E').$$

Satz 1. *E', E'' seien positive Komplexe von freien R-Moduln, $E = E' \oplus E''$. Die Differentiation von E sei 0. Dann ist*

$$H(B(*E)) \cong H(B(*E') \otimes_R B(*E'')).$$

Ist E ein Komplex von endlich-dimensionalen Vektorräumen, so ist diese Behauptung offenbar äquivalent zu

$$\mathscr{B}(B(*E)) = \mathscr{B}(B(*E')) \mathscr{B}(B(*E'')).$$

Beweis. I sei die lexikographisch wohlgeordnete Menge

$$\{(i_0, i_1, i_2) \in \mathbb{N}^3 : i_1 < i_2 < i_0\}.$$

Es sei $X_{(2,0,1)} := B(*E)$; für $i=(i_0, i_1, i_2)\in I$ sei i^+ der Nachfolger von i und Y_i der vom kanonischen Bild von

$$B(*E)_{i_0} \cap \big(B(*E')_{i_1} \otimes_R B(*E'')_{i_0-i_2} \otimes_R \bigoplus_{k\neq 0} (*E')_k \otimes_R B(*E)\big)$$

in X_i erzeugte Unterkomplex und $X_{i^+} := X_i/Y_i$. Da Y_i: $\cdots\to 0\to (Y_i)_{i_0}\to (Y_i)_{i_0-1}\to 0\to\cdots$, wie man leicht sieht, exakt ist, ist $H(X_i)\cong H(X_{i^+})$, also $H\big(B(*E)\big)\cong H(\varinjlim_{i\in I} X_i)\cong H\big(B(*E')\otimes_R B(*E'')\big)$, q.e.d.

Korollar 1. *E sei ein positiv graduierter Vektorraum,* $\dim E_i=\varepsilon_i<\infty$. *Dann ist*

$$\mathscr{B}\big(B(*E)\big)=\prod_{i=1}^{\infty}\big(1-(-Z)^{i+1}\big)^{(-1)^i\varepsilon_i}.$$

Beweis. Es genügt, die Behauptung für einen abbrechenden Komplex zu beweisen. Durch vollständige Induktion kann man Satz 1 auf beliebige endliche Summen $E=E'\oplus E''\oplus\cdots$ ausdehnen. Für $i=1, 2, \ldots$ sei $F^{(i)}$ ein graduierter Vektorraum mit $\dim(F^{(i)})_j=\delta_{ij}$. Es ist $E\cong\bigoplus_i \bigoplus_1^{\varepsilon_i} F^{(i)}$. Es gilt

$$\mathscr{B}\big(B(*F^{(i)})\big)=\begin{cases}1+Z^{i+1}, & \text{falls } i\in 2\mathbb{Z} \text{ (folgt aus dem untenstehenden Lemma),}\\ 1+Z^{i+1}+Z^{2(i+1)}+\cdots & \text{sonst (trivial nachzurechnen),}\end{cases}$$

also in jedem Fall $\mathscr{B}\big(B(*F^{(i)})\big)=\big(1-(-Z)^{i+1}\big)^{(-1)^i}$. Es folgt $\mathscr{B}\big(B(*E)\big)=\prod_i \mathscr{B}\big(B(*F^{(i)})\big)^{\varepsilon_i}=\prod_i\big(1-(-Z)^{i+1}\big)^{(-1)^i\varepsilon_i}$, q.e.d.

Lemma. *F sei ein 1-dimensionaler Vektorraum,* $1<m\in\mathbb{N}$. *Für* $s_1, \ldots, s_k\in \mathsf{S}F$ *mit* $s_1\otimes\cdots\otimes s_k\in(\otimes \mathsf{S}F)_m$ *sei*

$$\chi_m(s_1\otimes\cdots\otimes s_k) := (s_1 \,\mathsf{s}\, s_2)\otimes\cdots\otimes s_k - s_1\otimes(s_2 \,\mathsf{s}\, s_3)\otimes\cdots\otimes s_k+\cdots$$
$$+(-1)^k s_1\otimes\cdots\otimes(s_{k-1}\,\mathsf{s}\, s_k).$$

Dann ist $\operatorname{im}\chi_m=\ker\chi_m$.

Beweis durch vollständige Induktion nach m. Für $m=2$ ist die Behauptung klar. – Induktionsschritt: Die K-linearen Abbildungen der exakten Sequenz (es sei $0\neq e\in F$)

$$0\to(\otimes \mathsf{S}F)_m \to (\otimes \mathsf{S}F)_{m+1} \to (\otimes \mathsf{S}F)_m\to 0$$
$$s_1\otimes\cdots\otimes s_k\mapsto s_1\otimes\cdots\otimes(s_k\,\mathsf{s}\, e)\mapsto 0$$
$$s_1\otimes\cdots\otimes s_k\otimes e\mapsto s_1\otimes\cdots\otimes s_k$$

sind mit χ vertauschbar.

Korollar 2. *E sei ein positiv graduierter Vektorraum,* $\dim E_i<\infty$. $(e_\lambda)_{\lambda\in\Lambda}$ *sei eine homogene Basis von E; Λ sei totalgeordnet. Die aus den* e_λ *mit Hilfe von* $\otimes$*-Zeichen und wenigstens einem* $*$*-Zeichen zusammengesetzten Produkte erzeugen zusammen mit den Tensorprodukten* $e_{\lambda(1)}\otimes\cdots\otimes e_{\lambda(k)}$, *bei denen* $\lambda^{(\kappa)}\geqq\lambda^{(\kappa-1)}$ *und*

$e_{\lambda^{(\kappa)}} \in E_{2j}$ oder $\lambda^{(\kappa)} > \lambda^{(\kappa-1)}$ für wenigstens ein κ, einen Vektorraum X, der einen Unterkomplex von $B(*E)$ bildet. Die Ausdrücke $e_{\lambda^{(1)}}^{(m_1)} \wedge \cdots \wedge e_{\lambda^{(k)}}^{(m_k)}$ mit $\lambda^{(1)} < \cdots < \lambda^{(k)}$ und $m_i = 1$, falls $e_{\lambda^{(i)}} \in E_{2j}$, (wobei $e^{(m)} := e \otimes \cdots \otimes e$) bilden die Basis eines Unterkomplexes Y von $B(*E)$. Es gilt: $B(*E) = X \oplus Y$, X ist exakt; die Differentiation von Y ist null.

Beweis. Alle Behauptungen bis auf die Exaktheit von X sind trivial. Die letztere folgt daraus, daß nach Korollar 1 die Homologie von $B(*E)$ isomorph zu Y ist.

§ 2. Der Hauptsatz

R sei ein lokaler Ring, $\tilde{R}$ ein regulärer lokaler Ring mit dem maximalen Ideal $\tilde{\mathfrak{m}}$.

Definition 2. $(A, \chi, *, \pi)$ heißt eine *spezielle $\tilde{R}$-freie Auflösung* von R, wenn gilt:

1. $(A, \chi, *)$ ist eine $\tilde{R}$-Algebra;
2. $\pi\colon \tilde{R} \to R$ ist ein Epimorphismus von Ringen;
3. (A, χ, π) ist eine $\tilde{R}$-freie Auflösung von R;
4. $(A, 0, *)$ ist isomorph zu der freien $\tilde{R}$-Algebra über einem positiven Komplex E von freien $\tilde{R}$-Moduln;
5. $\chi(A) \subset \sum_{i,j>0} A_i * A_j + \tilde{\mathfrak{m}} A$.

Nach 4. gibt es also freie $\tilde{R}$-Moduln $E_1, E_2, \ldots$, derart, daß

$$A_1 = E_1,$$

$$A_2 \cong \overset{2}{\wedge}_{\tilde{R}} E_1 \oplus E_2,$$

$$A_3 \cong \overset{3}{\wedge}_{\tilde{R}} E_1 \oplus E_1 \otimes_{\tilde{R}} E_2 \oplus E_3,$$

$$A_4 \cong \overset{4}{\wedge}_{\tilde{R}} E_1 \oplus \overset{2}{\wedge}_{\tilde{R}} E_1 \otimes_{\tilde{R}} E_2 \oplus E_1 \otimes_{\tilde{R}} E_3 \oplus \overset{2}{\mathfrak{S}}_{\tilde{R}} E_2 \oplus E_4$$

usw., wobei die Multiplikation in naheliegender Weise durchgeführt wird.

Bemerkung. Falls $\tilde{R}/\tilde{\mathfrak{m}}$ die Charakteristik 0 hat, erhält man eine spezielle $\tilde{R}$-freie Auflösung gerade durch das von Tate [16] angegebene Verfahren. Im allgemeinen kann man die durch die Methode von Tate konstruierten $\tilde{R}$-Algebren charakterisieren, wenn man 4. durch folgende Bedingung ersetzt:

4'. *$(A, 0, *)$ ist isomorph zu der Algebra der symmetrischen Tensoren über einem positiven Komplex E von freien $\tilde{R}$-Moduln.*

Diese Algebra ist die Unteralgebra des Komplexes $\otimes_{\tilde{R}} E = \bigoplus_{k \in \mathbb{N}} \overset{k}{\otimes}_{\tilde{R}} E$ mit der shuffle-Multiplikation, die von den symmetrischen Tensoren erzeugt wird. Ein Tensor in $\overset{k}{\otimes}_{\tilde{R}} E$ heißt symmetrisch, wenn er für alle $\sigma \in \mathfrak{S}_k$ unter der $\tilde{R}$-linearen Abbildung

$$x_1 \otimes \cdots \otimes x_k \mapsto (-1)^u x_{\sigma 1} \otimes \cdots \otimes x_{\sigma k}$$

für $x_j \in E_{m_j}$ und $u = \text{Anzahl } \{(i,j)\colon i<j, \sigma i > \sigma j, m_{\sigma i} m_{\sigma j} \text{ ungerade}\}$ invariant ist.

Definition 3. $(A, \chi, *, \pi)$ sei eine spezielle $\tilde{R}$-freie Auflösung von R. Die Dimension von $\tilde{R}$ heißt die 0-*te Abweichung* $\varepsilon_0(A)$, der Rang des freien $\tilde{R}$-Moduls $A_j \Big/ \sum_{i=1}^{j-1} A_{j-i} * A_i$ die *j-te Abweichung* $\varepsilon_j(A)$.

Man sieht leicht:

Satz 2. $\pi\colon \tilde{R} \to R$ *sei ein Epimorphismus von lokalen Ringen. Es gibt eine bis auf Isomorphie einzige spezielle* $\tilde{R}$*-freie Auflösung* $(A, \chi, *, \pi)$ *von* R.

Das Hauptziel dieser Arbeit ist der Beweis des folgenden Satzes.

Hauptsatz. *R sei ein lokaler Ring, der einen Körper enthält; $\tilde{R}$ sei ein regulärer lokaler Ring derselben Einbettungsdimension und A eine spezielle $\tilde{R}$-freie Auflösung von R. Dann sind die Abweichungen von R gleich denen von A:*

$$\varepsilon_i(R) = \varepsilon_i(A) \qquad \textit{für alle } i \in \mathbb{N}.$$

Der Hauptsatz kann auf einen gegebenen lokalen Ring R, der einen Körper enthält, nur angewendet werden, wenn man einen Epimorphismus eines regulären lokalen Rings auf R zur Verfügung hat. Ist das nicht der Fall, so kann man zur Vervollständigung $\hat{R}$ von R übergehen. $\hat{R}$ hat dieselben Betti-Zahlen wie R und kann als Restklassenring eines vollständigen, regulären Rings $\hat{\tilde{R}}$ realisiert werden. Man kann die Abweichungen von R also nach dem Hauptsatz als die Abweichungen einer speziellen $\hat{\tilde{R}}$-freien Auflösung von $\hat{R}$ berechnen.

Eine ähnliche Überlegung zeigt, daß es genügt, den Hauptsatz für vollständiges $\tilde{R}$ zu beweisen: Setzen wir voraus, daß der Hauptsatz für vollständiges $\tilde{R}$ gilt! Die Voraussetzungen des Hauptsatzes mögen (mit einem nicht notwendig vollständigen $\tilde{R}$) gelten. $\hat{\tilde{R}}$ sei eine Vervollständigung von $\tilde{R}$. Dann sind $\hat{A} := A \otimes_{\tilde{R}} \hat{\tilde{R}}$ und $\hat{R} := R \otimes_{\tilde{R}} \hat{\tilde{R}}$ Vervollständigungen von A und von R und $\hat{A}$ ist eine spezielle $\hat{\tilde{R}}$-freie Auflösung von $\hat{R}$. Es gilt: $\varepsilon_i(A) = \varepsilon_i(\hat{A}) = \varepsilon_i(\hat{R}) = \varepsilon_i(R)$, q.e.d.

Der folgende Beweis, in dem wir daher $\tilde{R}$ als vollständig annehmen, besteht aus zwei Teilen. In § 3 wird (unter Verwendung von A) eine R-freie Auflösung des Restklassenkörpers von R konstruiert; in § 4 wird die Betti-Reihe von R berechnet.

§ 3. Konstruktion einer R-freien Auflösung des Restklassenkörpers von R

Die Voraussetzungen des Hauptsatzes seien erfüllt. $(A, \chi, *, \pi)$ sei eine spezielle $\tilde{R}$-freie Auflösung von R. $\mathfrak{m}$ und $\tilde{\mathfrak{m}}$ seien die maximalen Ideale von R und $\tilde{R}$. $(z_1, \ldots, z_n)$ sei ein minimales Erzeugersystem von $\tilde{\mathfrak{m}}$.

Für alle $i \in \mathbb{N}$ sei $(e_{\lambda_i})_{\lambda_i \in \Lambda_i}$ eine Basis des $\tilde{R}$-Moduls A_i, $e_{\lambda_0} = 1$. Es gilt

$$\chi\, e_{\lambda_i} = \sum_{\lambda_{i-1}} a_{\lambda_i \lambda_{i-1}}\, e_{\lambda_{i-1}}$$

und

$$e_{\lambda_i} * e_{\lambda'_j} = \sum_{\lambda_{i+j}} b_{\lambda_i \lambda'_j \lambda_{i+j}} e_{\lambda_{i+j}} \qquad \text{mit } a_{\lambda_i \lambda_{i-1}}, b_{\lambda_i \lambda'_j \lambda_{i+j}} \in \tilde{R}.$$

Wir benötigen in diesem Paragraphen statt 4. in Definition 2 nur, daß man durch passende Wahl der Basen erreichen kann

$$b_{\lambda_i \lambda'_j \lambda_{i+j}} \in K \qquad \text{für } i+j<n.$$

(A ist dann nicht mehr einzig und kann so gewählt werden, daß $A_i=0$ ist für $i>n$.)

K sei ein Koeffizientenkörper von $\tilde{R}$, den wir in kanonischer Weise mit $\tilde{R}/\mathfrak{m}\tilde{R}$ und $R/\mathfrak{m}$ identifizieren. Jedes $x \in \tilde{R}$ gestattet eine einzige Entwicklung als formale Potenzreihe in $z_1, \ldots, z_n$ mit Koeffizienten in K. Für $\nu=0,1,\ldots,n$ betrachten wir die Endomorphismen τ_ν: $\tilde{R} \to \tilde{R}$, $\tau_\nu z_1 = \cdots = \tau_\nu z_\nu = 0$, $\tau_\nu | K[\![z_{\nu+1}, \ldots, z_n]\!] = \mathrm{Id}$. Für $\nu=1,\ldots,n$ sei $\partial_\nu := \dfrac{1}{z_\nu}(\tau_{\nu-1}-\tau_\nu)$. Dann ist $\tau_0 x = x$, $\tau_n x$ das konstante Glied von x und

$$x = \tau_n x + \sum_{\nu=1}^{n} z_\nu \partial_\nu x.$$

Weiter gilt:

$$\partial_\nu(xy) = \tau_{\nu-1} x \, \partial_\nu y + \partial_\nu x \, \tau_\nu y.$$

Beweis.

$$z_\nu(\tau_{\nu-1} x \, \partial_\nu y + \partial_\nu x \, \tau_\nu y) = \tau_{\nu-1} x(\tau_{\nu-1}-\tau_\nu) y + (\tau_{\nu-1}-\tau_\nu) x \, \tau_\nu y$$
$$= \tau_{\nu-1} x \, \tau_{\nu-1} y - \tau_\nu x \, \tau_\nu y = \tau_{\nu-1}(xy) - \tau_\nu(xy) = z_\nu \partial_\nu(xy)$$

und z_ν ist kein Nullteiler.

Es ist leicht, eine $\tilde{R}$-freie Auflösung von K anzugeben: $\tilde{R} f_\nu$ sei ein freier $\tilde{R}$-Modul mit einer Basis (f_ν) $(\nu=1,\ldots,n)$. $\tilde{M}$ sei das Tensorprodukt der n $\tilde{R}$-Algebren $\cdots \to 0 \to \tilde{R} f_\nu \to \tilde{R} \to 0 \to \cdots$, $f_\nu \mapsto z_\nu$. Die Multiplikation in $\tilde{M}$ werde mit $\wedge$ bezeichnet, die Differentiation mit $\tilde{\varphi}$. Es ist bekannt, daß $\tilde{M}$ eine $\tilde{R}$-freie Auflösung von K ist. Es sei $M := R \otimes_{\tilde{R}} \tilde{M}$, $\bar{A} := A \otimes_{\tilde{R}} K$; die Differentiation in M sei φ, die in $\bar{A}$ $\bar{\chi}$.

Wir definieren nun γ: $\bar{A} \to M$ als die K-lineare Abbildung mit

$$\gamma(e_{\lambda_i}) = (-1)^i \sum_{\nu_1 < \cdots < \nu_i} \sum_{\lambda_0, \ldots, \lambda_{i-1}} \partial_{\nu_1} a_{\lambda_1 \lambda_0} \partial_{\nu_2} a_{\lambda_2 \lambda_1} \cdots \partial_{\nu_i} a_{\lambda_i \lambda_{i-1}} f_{\nu_i} \wedge \cdots \wedge f_{\nu_1}.$$

Behauptung. γ: $\bar{A} \to M$ ist ein Homomorphismus von R-Algebren.

Beweis. Zunächst zeigen wir die Kommutativität des Diagramms

$$\begin{array}{ccc} \bar{A}_{i+1} & \xrightarrow{\bar{\chi}} & \bar{A}_i \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\gamma} \\ {\scriptstyle i+1} & & {\scriptstyle i} \\ \wedge M_1 & \xrightarrow{\varphi} & \wedge M_1. \end{array}$$

$$\varphi\circ\gamma(e_{\lambda_{i+1}})=(-1)^{i+1}\sum_{t=1}^{i+1}\sum_{\nu_1<\cdots<\nu_{i+1}}\sum_{\lambda_0,\ldots,\lambda_i}(-1)^{i+1-t}\,\partial_{\nu_1}a_{\lambda_1\lambda_0}$$
$$\ldots\partial_{\nu_{t-1}}a_{\lambda_{t-1}\lambda_{t-2}}(\partial_{\nu_t}a_{\lambda_t\lambda_{t-1}})\,z_{\nu_t}\,\partial_{\nu_{t+1}}a_{\lambda_{t+1}\lambda_t}$$
$$\ldots\partial_{\nu_{i+1}}a_{\lambda_{i+1}\lambda_i}f_{\nu_{i+1}}\wedge\cdots\wedge\hat{f}_{\nu_t}\wedge\cdots\wedge f_{\nu_1}$$
$$=\sum_{\nu_1<\cdots<\nu_i}\sum_{\lambda_0,\ldots,\lambda_i}\big(-(\tau_0-\tau_{\nu_1-1})\,a_{\lambda_1\lambda_0}\,\partial_{\nu_1}a_{\lambda_2\lambda_1}\ldots\partial_{\nu_i}a_{\lambda_{i+1}\lambda_i}$$
$$+\sum_{t=2}^{i}(-1)^t\,\partial_{\nu_1}a_{\lambda_1\lambda_0}\ldots\partial_{\nu_{t-1}}a_{\lambda_{t-1}\lambda_{t-2}}(\tau_{\nu_{t-1}}-\tau_{\nu_{t-1}-1})\,a_{\lambda_t\lambda_{t-1}}\,\partial_{\nu_t}a_{\lambda_{t+1}\lambda_t}$$
$$\ldots\partial_{\nu_i}a_{\lambda_{i+1}\lambda_i}$$
$$+(-1)^{i+1}\,\partial_{\nu_1}a_{\lambda_1\lambda_0}\ldots\partial_{\nu_i}a_{\lambda_i\lambda_{i-1}}(\tau_{\nu_i}-\tau_n)\,a_{\lambda_{i+1}\lambda_i}\big)f_{\nu_i}\wedge\cdots\wedge f_{\nu_1}$$
$$=\sum_{\nu_1<\cdots<\nu_i}\sum_{\lambda_0,\ldots,\lambda_i}\big(-\tau_0\,a_{\lambda_1\lambda_0}\,\partial_{\nu_1}a_{\lambda_2\lambda_1}\ldots\partial_{\nu_i}a_{\lambda_{i+1}\lambda_i}$$
$$+\sum_{t=1}^{i}(-1)^{t-1}(\partial_{\nu_1}a_{\lambda_1\lambda_0}\ldots\partial_{\nu_{t-1}}a_{\lambda_{t-1}\lambda_{t-2}})\,\partial_{\nu_t}(a_{\lambda_t\lambda_{t-1}}a_{\lambda_{t+1}\lambda_t})(\partial_{\nu_{t+1}}a_{\lambda_{t+2}\lambda_{t+1}}$$
$$\ldots\partial_{\nu_i}a_{\lambda_{i+1}\lambda_i})+(-1)^i\,\partial_{\nu_1}a_{\lambda_1\lambda_0}\ldots\partial_{\nu_i}a_{\lambda_i\lambda_{i-1}}\tau_n\,a_{\lambda_{i+1}\lambda_i}\big)f_{\nu_i}\wedge\cdots\wedge f_{\nu_1}$$
$$=\sum_{\lambda_i}\tau_n\,a_{\lambda_{i+1}\lambda_i}\gamma\,e_{\lambda_i}=\gamma\sum_{\lambda_i}\tau_n\,a_{\lambda_{i+1}\lambda_i}e_{\lambda_i}=\gamma\circ\bar{\chi}\,e_{\lambda_{i+1}}.$$

Weiter haben wir zu zeigen: $\gamma(e_{\lambda_i'}*e_{\lambda_j''})=\gamma\,e_{\lambda_i'}\wedge\gamma\,e_{\lambda_j''}$. Die Gleichung $\chi(e_{\lambda_i'}*e_{\lambda_j''})=\chi(e_{\lambda_i'})*e_{\lambda_j''}+(-1)^i e_{\lambda_i'}*\chi(e_{\lambda_j''})$ bedeutet: Für alle $\lambda_{i+j-1}\in\Lambda_{i+j-1}$ ist

$$\sum_{\lambda_{i+j}}b_{\lambda_i'\lambda_j''\lambda_{i+j}}a_{\lambda_{i+j}\lambda_{i+j-1}}=\sum_{\lambda_{i-1}'}b_{\lambda_{i-1}'\lambda_j''\lambda_{i+j-1}}a_{\lambda_i'\lambda_{i-1}'}$$
$$+(-1)^i\sum_{\lambda_{j-1}''}b_{\lambda_i'\lambda_{j-1}''\lambda_{i+j-1}}a_{\lambda_j''\lambda_{j-1}''}.\tag{1}$$

Ferner gilt

$$(-1)^{i+j}\gamma(e_{\lambda_i'}*e_{\lambda_j''})=\sum_{\nu_1<\cdots<\nu_{i+j}}\sum_{\lambda_0,\ldots,\lambda_{i+j}}\tau_n\,b_{\lambda_i'\lambda_j''\lambda_{i+j}}\,\partial_{\nu_1}a_{\lambda_1\lambda_0}$$
$$\ldots\partial_{\nu_{i+j}}a_{\lambda_{i+j}\lambda_{i+j-1}}f_{\nu_{i+j}}\wedge\cdots\wedge f_{\nu_1}$$
$$=\sum_{\nu_1<\cdots<\nu_{i+j}}\sum_{\lambda_0,\ldots,\lambda_{i+j}}\partial_{\nu_1}a_{\lambda_1\lambda_0}\tag{2}$$
$$\ldots\partial_{\nu_{i+j}}(a_{\lambda_{i+j}\lambda_{i+j-1}}b_{\lambda_i'\lambda_j''\lambda_{i+j}})f_{\nu_{i+j}}\wedge\cdots\wedge f_{\nu_1};$$

denn die Differenz der beiden letzten Ausdrücke ist

$$\varphi\sum_{\nu_1<\cdots<\nu_{i+j+1}}\sum_{\lambda_0,\ldots,\lambda_{i+j}}\partial_{\nu_1}a_{\lambda_1\lambda_0}$$
$$\ldots\partial_{\nu_{i+j}}a_{\lambda_{i+j}\lambda_{i+j-1}}\partial_{\nu_{i+j+1}}b_{\lambda_i'\lambda_j''\lambda_{i+j}}f_{\nu_{i+j+1}}\wedge\cdots\wedge f_{\nu_1}=0,$$

da $b_{\lambda_i'\lambda_j''\lambda_{i+j}}\in K$ oder $i+j+1>n$ ist. Wir beweisen $\gamma(e_{\lambda_i'}*e_{\lambda_j''})=\gamma(e_{\lambda_i'})\wedge\gamma(e_{\lambda_j''})$ nun durch vollständige Induktion nach $i+j$. Für $ij=0$ ist die Behauptung

klar. Sei also $i+j>0$, $ij\neq 0$. Man erhält unter Verwendung von (1) und (2)

$$\begin{aligned}(-1)^{i+j}\gamma(e_{\lambda_i'}*e_{\lambda_j'})=&\sum_{\nu_{i+j},\lambda_{i-1}'}\partial_{\nu_{i+j}}a_{\lambda_i'\lambda_{i-1}'}f_{\nu_{i+j}}\\&\wedge\Big(\sum_{\substack{\nu_1<\cdots<\nu_{i+j-1}\\ \text{alle}<\nu_{i+j}}}\sum_{\lambda_0,\ldots,\lambda_{i+j-1}}\tau_n b_{\lambda_{i-1}'\lambda_j'\lambda_{i+j-1}}\partial_{\nu_1}a_{\lambda_1\lambda_0}\\&\ldots\partial_{\nu_{i+j-1}}a_{\lambda_{i+j-1}\lambda_{i+j-2}}f_{\nu_{i+j-1}}\wedge\cdots\wedge f_{\nu_1}\Big)\\&+(-1)^i\sum_{\nu_{i+1},\lambda_{j-1}'}\partial_{\nu_{i+j}}a_{\lambda_j'\lambda_{j-1}'}f_{\nu_{i+j}}\\&\wedge\Big(\sum_{\substack{\nu_1<\cdots<\nu_{i+j-1}\\ \text{alle}<\nu_{i+j}}}\sum_{\lambda_0,\ldots,\lambda_{i+j-1}}\tau_n b_{\lambda_i'\lambda_{j-1}'\lambda_{i+j-1}}\partial_{\nu_1}a_{\lambda_1\lambda_0}\\&\ldots\partial_{\nu_{i+j-1}}a_{\lambda_{i+j-1}\lambda_{i+j-2}}f_{\nu_{i+j-1}}\wedge\cdots\wedge f_{\nu_1}\Big).\end{aligned}$$

Wendet man die Induktionsannahme auf jede der eingeklammerten Summen an, so folgt die Behauptung nach einer kurzen Umformung.

γ induziert einen Homomorphismus $H(\bar{A})\to H(M)$. Sowohl $H(\bar{A})$ als auch $H(M)$ ist in natürlicher Weise zu $\mathrm{Tor}^{\bar{R}}(R,K)$ isomorph. Wir haben also einen natürlichen Isomorphismus

$$H(\bar{A})\to H(M).$$

Diesen erhält man folgendermaßen: Die Augmentierungen $A\to R$ und $\tilde{M}\to K$ induzieren Homomorphismen von Komplexen von Vektorräumen

$$A\otimes_{\bar{R}}\tilde{M}\to R\otimes_{\bar{R}}\tilde{M}=M$$

und

$$A\otimes_{\bar{R}}\tilde{M}\to A\otimes_{\bar{R}}K=\bar{A},$$

die Isomorphismen der Homologie induzieren. Zum zuletzt genannten Homomorphismus wird ein rechtsinverser Homomorphismus gegeben durch

$$\begin{aligned}e_{\lambda_j}\mapsto\sum_{i=0}^{j}(-1)^{ij+i}&\sum_{\lambda_i,\ldots,\lambda_{j-1}}e_{\lambda_i}\otimes\sum_{\nu_{i+1}<\cdots<\nu_j}\partial_{\nu_{i+1}}a_{\lambda_{i+1}\lambda_i}\\&\ldots\partial_{\nu_j}a_{\lambda_j\lambda_{j-1}}f_{\nu_j}\wedge\cdots\wedge f_{\nu_{i+1}}.\end{aligned}$$

Mit $A\otimes_{\bar{R}}\tilde{M}\to M$ verknüpft ergibt das

$$e_{\lambda_j}\mapsto\sum_{\lambda_0,\ldots,\lambda_{j-1}}\sum_{\nu_1<\cdots<\nu_j}\partial_{\nu_1}a_{\lambda_1\lambda_0}\ldots\partial_{\nu_j}a_{\lambda_j\lambda_{j-1}}f_{\nu_j}\wedge\cdots\wedge f_{\nu_1}=(-1)^j\gamma(e_{\lambda_j}).$$

Folglich induziert auch γ einen Isomorphismus $H(\bar{A})\xrightarrow{\cong}H(M)$.

Wir definieren nun eine R-freie Auflösung

$$\begin{array}{ccccccccccc}N\colon & \cdots & \xrightarrow{\delta} & N_2 & \xrightarrow{\delta} & N_1 & \xrightarrow{\delta} & N_0 & \xrightarrow{\delta} & 0 & \to\cdots\\ & & & & & & & \downarrow & & & \\ & & & & & & & K & & & \end{array}$$

von K. Es sei

$$N_k := \bigoplus_{\kappa=0}^{k} B(\bar{A})_{k-\kappa} \otimes M_\kappa .$$

Für $y \in B(\bar{A})_l$ und $z \in M$ sei

$$y * z := \begin{cases} 0, & \text{falls } l=0, \\ y' \otimes (\gamma(y'') \wedge z), & \text{falls } y = y' \otimes y'',\ y'' \in \bar{A}_i \quad (i \neq 0). \end{cases}$$

Die Differentiation in $B(\bar{A})$ bezeichnen wir mit ψ. Für $y \in B(\bar{A})_l$ und $z \in M_k$ sei

$$\delta(y \otimes z) := \psi(y) \otimes z + (-1)^l y * z + (-1)^l y \otimes \varphi(z).$$

Satz. *Mit δ als Differentiation und der kanonischen Projektion $R \to K$ als Augmentation ist N eine R-freie Auflösung von K.*

Beweis. Zunächst ist klar, daß alle N_k freie R-Moduln sind. $\delta^2 = 0$: (1) $\delta^2(B(\bar{A})_0 \otimes M) = 0$ ist trivial. (2) Für y und z wie oben ist

$$\delta^2(y \otimes z) = (-1)^l \big(\delta(y * z) - \psi(y) * z + (-1)^l y * \varphi(z)\big).$$

Man hat also zu zeigen, daß

$$\delta(y * z) = \psi(y) * z + (-1)^{l+1} y * \varphi(z)$$

ist. Diese Gleichung kann man für die beiden Fälle, in denen $y * z$ definiert ist, leicht einzeln nachrechnen.

Wir beweisen die Exaktheit von

$$\cdots \xrightarrow{\delta} N_2 \xrightarrow{\delta} N_1 \xrightarrow{\delta} N_0 \to K$$

durch vollständige Induktion.

Nehmen wir also an, $N_k \xrightarrow{\delta} N_{k-1} \xrightarrow{\delta} \cdots \xrightarrow{\delta} N_0 \to K$ sei exakt. Dann wollen wir die Exaktheit von $N_{k+1} \xrightarrow{\delta} N_k \xrightarrow{\delta} N_{k-1}$ beweisen. Sei $x \in \ker(\delta_k : N_k \to N_{k-1})$; sei $x = \sum_{i=0}^{h} x_i (0 \leqq h < k)$ mit $x_0 \in M_k$ und $x_i \in \bar{A}_{k-i} \otimes N_{i-1}$ für $i = 1, \ldots, h$. Durch vollständige Induktion nach h zeigen wir, daß $x \in \delta(N_{k+1})$ ist.

$h = 0$. $x = x_0 \in \ker \varphi_k$ vertritt ein Element von $\mathrm{Tor}_k^{\bar{R}}(R, K)$. Dasselbe Element von $\mathrm{Tor}_k^{\bar{R}}(R, K)$ werde von $y \in \ker \bar{\chi}_k$ repräsentiert. Dann ist $-\delta(y \otimes 1) = (-1)^k \gamma(y) \equiv x \bmod \varphi(M_{k+1})$, also $x \in \delta\big(-y + B(\bar{A})_0 \otimes M_{k+1}\big) \subset \delta(N_{k+1})$.

$h = 1$. $(\varphi(m_i))_{i \in I}$ sei eine Basis des Vektorraums $\mathfrak{m}$, $0 \notin I$. $x_1 = y_0 \otimes 1 + \sum_I y_i \otimes \varphi(m_i)$ mit $y_0, y_i \in \bar{A}_{k-1}$. Es sei $x - \delta\big(\sum_I (-1)^k y_i \otimes m_i\big) = x_0' + x_1' + x_2'$ mit $x_0' \in M_k$, $x_1' \in \bar{A}_{k-1} \otimes M_0$, $x_2' \in \bar{A}_{k-2} \otimes M_1$. Es ist $0 = \delta(x_0 + x_1) = \delta(x_0) + (-1)^k \cdot \big(\sum_I \gamma(y_i)\varphi(m_i) + \gamma(y_0)\big) + \sum_I \chi(y_i) \otimes \varphi(m_i) + \chi(y_0) \otimes 1$ und daher $\chi(y_0) = \chi(y_i) = 0$ für alle i. Daraus folgt $x_2' = -(-1)^k \sum_I \chi(y_i) \otimes m_i = 0$. Weiter ist $x_1' = x_1 - \sum_I y_i \otimes \varphi(m_i) = y_0 \otimes 1$. Somit gilt $0 = \delta(x) = \delta(x_0' + x_1') = \varphi(x_0') + (-1)^k \gamma(y_0) + \chi(y_0) \otimes 1$, also $\gamma(y_0) \in \varphi(M_{k+1})$. $\gamma(y_0)$ vertritt also das Nullelement von $\mathrm{Tor}_k^{\bar{R}}(R, K)$,

y_0 folglich ebenso, d.h. es gibt ein $z\in\bar{A}_{k+1}$ mit $\bar{\chi}(z)=y_0$. Nun ist

$$x=(x_0'+(-1)^{k-1}\gamma(z))+\delta(z\otimes 1+\sum_I(-1)^k y_i\otimes m_i)\in\overset{k}{\wedge} N_1+\delta N_{k+1}.$$

Die Behauptung ist damit auf den Fall $h=0$ zurückgeführt.

Sei $1<h<k$. Wir nehmen an, die Behauptung sei für Elemente der Form $\sum_{i=0}^{h-1} x_i$ bewiesen. Vorgelegt sei $\sum_{i=0}^{h} x_i\in\ker(N_k\to N_{k-1})$ mit $x_0\in K\otimes M_k$ und $x_i\in\bar{A}_{k-i}\otimes N_{i-1}$ für $i=1,\ldots,h$.

$$\begin{aligned}
x_h&\in\ker(\bar{\chi}_{k-h}\otimes N_{h-1})\cap\ker(\bar{A}_{k-h}\otimes N_{h-1}\to\operatorname{coker}(\bar{\chi}_{k-h+1}\otimes N_{h-2}))\\
&=\ker(\bar{\chi}_{k-h})\otimes N_{h-1}\cap\ker(\bar{A}_{k-h}\otimes N_{h-1}\to\operatorname{coker}(\bar{\chi}_{k-h+1})\otimes N_{h-2})\\
&=\ker(\bar{\chi}_{k-h})\otimes N_{h-1}\cap(\bar{A}_{k-h}\otimes\ker(\delta_{h-1})+\ker(\bar{A}_{k-h}\to\operatorname{coker}(\bar{\chi}_{k-h+1}))\otimes N_{h-1})\\
&=\ker(\bar{\chi}_{k-h})\otimes N_{h-1}\cap(\bar{A}_{k-h}\otimes\operatorname{im}(\delta_h)+\operatorname{im}(\bar{\chi}_{k-h+1})\otimes N_{h-1})\\
&=\ker(\bar{\chi}_{k-h})\otimes N_{h-1}\cap\bar{A}_{k-h}\otimes\operatorname{im}(\delta_h)+\operatorname{im}(\bar{\chi}_{k-h+1})\otimes N_{h-1}\\
&=\ker(\bar{\chi}_{k-h})\otimes\operatorname{im}(\delta_h)+\operatorname{im}(\bar{\chi}_{k-h+1})\otimes N_{h-1}\\
&=\operatorname{im}(\ker(\bar{\chi}_{k-h})\otimes N_h\to\bar{A}_{k-h}\otimes N_{h-1})+\operatorname{im}(\bar{\chi}_{k-h+1}\otimes N_{h-1}).
\end{aligned}$$

Die Behauptung folgt jetzt aus der Induktionsannahme.

§ 4. Berechnung der Betti-Reihe von R

Sei $\bar{N}:=N\otimes_R K$ und $\bar{M}:=M\otimes_R K$. Die Differentiation in $\bar{M}$ ist null, die Differentiation $\bar{\delta}$ in $\bar{N}$ ist gegeben durch $\bar{\delta}(x\otimes y)=\psi(x)\otimes y$ für $x\in B(\bar{A})$ und $y\in\bar{M}$, da im $\gamma\in M_0+\mathfrak{m}M$. Daher ist $\bar{N}\cong B(\bar{A})\otimes\bar{M}$. Es folgt

$$\mathscr{B}(R)=\mathscr{B}(\bar{N})=\mathscr{B}(B(\bar{A}))\,\mathscr{B}(\bar{M})=(1+Z)^n\,\mathscr{B}(B(\bar{A})).$$

Um $\mathscr{B}(B(\bar{A}))$ zu bestimmen, nehmen wir an, A ist eine spezielle $\tilde{R}$-freie Auflösung von R. E habe dieselbe Bedeutung wie in Definition 2. Weiter sei $\varepsilon_i:=\varepsilon_i(A)$, $\bar{E}:=E\otimes_R K$ und ψ' die Differentiation in $B(*\bar{E})$.

Für $i\in\mathbb{Z}$ seien $B(\bar{A})^{(i)}$ und $B(*\bar{E})^{(i)}$ die Unterkomplexe von $B(\bar{A})$ und von $B(*\bar{E})$, die von den Vektorräumen

$$\sum_{k\in\mathbb{N}}\sum_{\substack{i_1,\ldots,i_k>0\\ i_1+\cdots+i_k\leqq i}}\bar{A}_{i_1}\otimes\cdots\otimes\bar{A}_{i_k}$$

bzw.

$$\sum_{k\in\mathbb{N}}\sum_{\substack{i_1,\ldots,i_k>0\\ i_1+\cdots+i_k\leqq i}}(*\bar{E})_{i_1}\otimes\cdots\otimes(*\bar{E})_{i_k}$$

gebildet werden.

Die kanonische Abbildung $H(B(\bar{A})^{(i)})\to H(B(\bar{A}))$ ist injektiv, wie wir durch vollständige Induktion nach i zeigen. Für $i\leqq 0$ ist die Behauptung trivial. Nehmen wir an, sie gilt für ein gewisses i. $x\in B(\bar{A})^{(i+1)}$ vertrete ein Element von

$\ker\left(H(B(\bar{A})^{(i+1)}) \to H(B(\bar{A}))\right)$. Dann ist $x \in \psi(B(\bar{A}))$. Sei $x = x_1 + x_2$ mit $x_1 \in B(\bar{A})^{(i)}$ und

$$x_2 \in V := \sum_{k \in \mathbb{N}} \sum_{\substack{i_1, \ldots, i_k > 0 \\ i_1 + \cdots + i_k = i+1}} \bar{A}_{i_1} \otimes \cdots \otimes \bar{A}_{i_k}.$$

Es gilt $0 = \psi x = \psi x_1 + (\psi - \psi')x_2 + \psi' x_2$. Die beiden ersten Summanden dieser Summe liegen in $B(\bar{A})^{(i)}$, der letzte in V. Daher ist $\psi' x_2 = 0$. Andererseits liegt mit x auch x_2 in X (s. Korollar 2 zu Satz 1), also $x_2 \in X \cap \ker \psi' = \operatorname{im} \psi'$. Es gibt ein $y \in V$ mit $\psi' y = x_2$, $x - \psi y = x_1 - (\psi - \psi') y \in B(\bar{A})^{(i)} \cap \psi(B(\bar{A})) \subset \psi(B(\bar{A})^{(i)})$ nach Induktionsannahme. Es folgt $x \in \psi(B(\bar{A})^{(i+1)})$.

Mit $H(B(\bar{A})^{(i)}) \to H(B(\bar{A}))$ ist auch $H(B(\bar{A}))^{(i)}) \to H(B(\bar{A})^{(i+1)})$ injektiv. Es folgt $\mathscr{B}(B(\bar{A})^{(i+1)}) = \mathscr{B}(B(\bar{A})^{(i)}) + \mathscr{B}(B(\bar{A})^{(i+1)}/B(\bar{A})^{(i)})$ und weiter $\mathscr{B}(B(\bar{A})) = \sum_{i \in \mathbb{N}} \mathscr{B}(B(\bar{A})^{(i+1)}/B(\bar{A})^{(i)})$; ebenso gilt $\mathscr{B}(B(*\bar{E})) = \sum_{i \in \mathbb{N}} \mathscr{B}(B(*\bar{E})^{(i+1)}/B(*\bar{E})^{(i)})$. Die rechten Seiten dieser beiden Gleichungen sind gleich, also auch die linken. Nach Korollar 1 zu Satz 1 folgt nun:

$$\mathscr{B}(R) = (1+Z)^{\varepsilon_0}\, \mathscr{B}(B(\bar{A})) = (1+Z)^{\varepsilon_0}\, \mathscr{B}(B(*\bar{E}))$$
$$= \prod_{i \in \mathbb{N}} (1-(-Z)^{i+1})^{(-1)^i \varepsilon_i}, \qquad \text{q.e.d.}$$

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