Werk

Titel: II. Finite Unitary Groups Generated by Reflections.

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In the next chapter we will study the cases given by Coxeter, Shephard, and Todd.

Essential to our study are the orders of the groups. Let $|G|_r$ denote the order of the Sylow r-subgroup of G. Let e be the least positive integer for which r divides q^e-1.

$$|GL(n,q)|=q^{\frac{n(n-1)}{2}}\cdot(q^n-1)(q^{n-1}-1)\ldots(q-1) \quad \text{[25, p. 534]},$$

$$|GL(n,q)|_r=|q^e-1|_r^{[n/e]}\cdot|[n/e]!|_r \quad \text{[22, p. 262]},$$

$$|GL(n,q)|_2=|q-1|_2^n\cdot|q+1|_2^{[n/2]}\cdot 2^{[n/4]+[n/8]+\cdots} \quad \text{[22, p. 267]},$$

$$|\Sigma_n|_r=|n!|_r=r^M \quad \text{where} \quad M=[n/r]+[n/r^2]+[n/r^3]+\cdots \quad \text{[14, p. 81]}.$$

II. Finite Unitary Groups Generated by Reflections

In the text which follows we will use N rather than N^* for the finite unitary groups generated by reflections. Coxeter and Moser list and characterize the ten real cases [9, p. 141]. Coxeter describes six non-real cases in a separate paper [8, p. 270]. Finally we will refer to Shephard and Todd's complete listing [21, p. 301].

Section 1. The Real Coxeter Groups

In considering the questions of which of the Coxeter groups determine Hall subgroups of $GL(n,q)$, we ask whether each Coxeter group N is a normal subgroup of a Hall non-solvable subgroup H of $GL(n,q)$. If $C_H(N)$ denotes the centralizer of N in H, then for $GL(n,q)\supset H\rhd N$ we have $|H|=|C_H(N)|\cdot|H/C_H(N)|$ and $H/C_H(N)\cong \mathrm{Inn}_H(N)$ where $\mathrm{Inn}_H(N)$ denotes the group of automorphisms of N induced by inner automorphisms of H. Since we assume N to be irreducible, Schur's lemma gives $C_H(N)$ to be the set of scalar matrices, so $|C_H(N)|=q-1$. The normality of N in H gives $|\mathrm{Inn}_H(N)|\leqq|\mathrm{Aut}\,N|$, so that we will use the relation $|H|\leqq(q-1)\cdot|\mathrm{Aut}\,N|$ as it applies to successive cases of the Coxeter groups.

Case 1. $N=S_{k+1}$ of order $(k+1)!$ and degree k. If $k\neq 5$ all the automorphisms are inner, so

$$|\mathrm{Inn}_H(N)|=|\mathrm{Inn}\,N|=|N/Z|=(k+1)!$$

Then

$$|H|_2=|q-1|_2\cdot|S_{k+1}|_2$$

and

$$|H|_2=|q-1|_2\cdot 2^{[\frac{k+1}{2}]+[\frac{k+1}{4}]+\cdots}$$

where $k\neq 5$.

$$|GL(k,q)|_2\geqq|q-1|_2\cdot(2^3)^{[\frac{k}{2}]}\cdot 2^{[\frac{k-1}{2}]}\cdot 2^{[\frac{k}{4}]+[\frac{k}{8}]+\cdots}.$$

We will compare exponents and show that for $k>3$

$$(*)\quad 1+\left[\frac{k+1}{2}\right]+\left[\frac{k+1}{4}\right]+\cdots<3\left[\frac{k}{2}\right]+\left[\frac{k-1}{2}\right]+\left[\frac{k}{4}\right]+\left[\frac{k}{8}\right]+\cdots$$

so that H is not a Hall subgroup of $GL(n, q)$ if $k>3$, $k\neq 5$.

Proof of (*). Let $k>3$.

i) $\left[\frac{k+1}{2}\right]\leqq\frac{k+1}{2}<k-1\leqq 2\left[\frac{k}{2}\right]$.

ii) $1+\left[\frac{k+1}{4}\right]\leqq 1+\frac{k+1}{4}<k-1<\frac{3k-3}{2}\leqq\left[\frac{k-1}{2}\right]+\left[\frac{k}{2}\right]$.

iii) $\left[\frac{k+1}{2^m}\right]<\frac{k}{2^{m-1}}$ for $2^m\leqq k+1$ and $m>2$.

Since $2^{m-1}\leqq\frac{k+1}{2}<k-1$ we have $\frac{k+1}{2^m}<\frac{k-1}{2^{m-1}}<\frac{k}{2^{m-1}}$. Addition of the three inequalities gives (*).

If $k=5$, then $[\mathrm{Aut}\, S_6 : \mathrm{Inn}\, S_6]=2$, so $|\mathrm{Aut}\, S_6|=2\cdot 6!$ Then $|H|_2=|q-1|_2\cdot 2^4$ and $|GL(5, q)|_2=|q-1|_2^5\cdot|q+1|_2^2\cdot 2\geqq|q-1|_2\cdot 2^9$. So H is not Hall. It is clear from this proof that N itself is not Hall.

Burnside limits representations of S_k in $GL(n, q)$ to $k\leqq n+1$ for $k>1$ [2, p. 468]. A comparison of the orders of the Sylow 2-subgroups of $GL(n, q)$ shows that the subgroups H containing S_3 and S_2 as normal subgroups cannot have representations which are Hall subgroups of $GL(n, q)$ for $n>1$. $|GL(n, q)|_2\geqq|q-1|_2\cdot 2^3$, while $|H|_2\leqq|q-1|_2\cdot|S_2/Z|_2=|q-1|_2$ and $|H|_2\leqq|q-1|_2\cdot|S_3|_2=|q-1|_2\cdot 2$.

The only remaining case is for S_4. Since $|GL(3, q)|_2\geqq|q-1|_2\cdot 2^4$ and $|H|_2\leqq|q-1|_2\cdot|S_4|_2=|q-1|_2\cdot 2^3$, H is not a Hall subgroup of $GL(n, q)$ for $n\geqq 3$.

To show that S_4 has no representation in $GL(2, q)$ we observe that the commutator subgroup of S_4 contains the four-group. Then the representation of this four-group would be a subgroup of $SL(2, q)$, the commutator subgroup of $GL(2, q)$. Let $A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$. Then $A^2=1$ implies $a^2+bc=1$ and $c(a+d)=b(a+d)=0$. Also $|A|=1$ implies $ad-bc=1$. Now if $a+d=0$, then $-a^2-bc=1$ which is clearly impossible. Hence $b=c=0$ and $d=a^{-1}$. Thus $SL(2, q)$ has three distinct non-identity elements

$$\begin{pmatrix} s & 0\\ 0 & s^{-1}\end{pmatrix},\quad \begin{pmatrix} t & 0\\ 0 & t^{-1}\end{pmatrix}\quad\text{and}\quad \begin{pmatrix} st & 0\\ 0 & (st)^{-1}\end{pmatrix}$$

of order 2. Thus in $GF(q)$ we have $s^2=t^2=(st)^2=1$ which is impossible for $q>3$.

Thus H is not a Hall subgroup of $GL(n, q)$.

Note that we have proved

Lemma 2.1. *S_{k+1} is never faithfully represented as a Hall subgroup of $GL(n, q)$ for odd q.*

Case 2. $N=[3^{n-2}, 4]$ of order $2^n n!$ and degree $n \geqq 2$. We shall show that $[\mathrm{Aut}\, N : \mathrm{Inn}\, N] \leqq 2$. Before proving this relation we consider some properties of the conjugacy classes. According to Coxeter and Todd [10, p. 198] N has the n generators

$$\begin{aligned} R_1 &= (12)(1'\,2') \\ R_2 &= (23)(2'\,3') \\ &\vdots \\ R_{n-1} &= (n-1, n)(n-1', n') \\ R_n &= (n, n') \end{aligned}$$

where N is a permutation group on the $2n$ elements $\{1, 2, \ldots, n, 1', \ldots, n'\}$. Clearly then N also has generators

$$\begin{aligned} P_1 &= (1, 1') \\ P_2 &= (12)(1'\,2') \\ P_3 &= (13)(1'\,3') \\ &\vdots \\ P_n &= (1\,n)(1'\,n'). \end{aligned}$$

Let C_1 denote the congruence class $\{(11'), (22'), \ldots, (n\,n')\}$. Let C_2 denote the congruence class containing $P_2, \ldots, P_n$.

Lemma 2.2. *If $\phi \in \mathrm{Aut}\, N$ is such that $\phi\, C_1 = C_1$ and $\phi\, C_2 = C_2$, then $\phi \in \mathrm{Inn}\, N$.*

The proof is by induction on $t \leqq n$ that there exist conjugations $\beta_1, \beta_2, \ldots, \beta_t$ such that $\beta_t^{-1}\beta_{t-1}^{-1}\ldots\beta_1^{-1}\phi$ fixes $P_1, P_2, \ldots, P_t$. If $t=1$, suppose $\phi(11')=(a\,a')$. Let β_1 be conjugation by $(1\,a)(1'\,a')$. Then $\beta_1(11')=\phi(11')$, so $\beta_1^{-1}\phi$ fixes P_1. If $n=2$, then $a=2$ and $\beta_1^{-1}\phi(P_2)=(12)(1'\,2')$ or $(12')(1'\,2)$. In the first case let $\beta_2 = e$ the identity map. In the second case let β_2 be conjugation by $(11')$. Then $\beta_2^{-1}\beta_1^{-1}\phi(P_i)=P_i$ for $i=1, 2$. Now let $n>2$ and suppose $\gamma=\beta_t^{-1}\beta_{t-1}^{-1}\ldots\beta_1^{-1}\phi$ is given by the induction hypothesis so that γ fixes $P_1, \ldots, P_t$. Since γ preserves C_2, we have $\gamma(1, t+1)(1', t+1')=(h, k)(h', k')$.

Furthermore $\{1, t, 1', t'\} \cap \{h, k, h', k'\} \neq \phi$, for otherwise $\gamma(1, t)(1', t')(1, t+1)\cdot(1', t+1')=(1, t)(1', t')(h, k)(h', k')$ would have order 2 while $(1, t)(1', t')(1, t+1)\cdot(1', t+1')$ has order 3 contradicting the fact that γ preserves order.

Suppose for convenience that $\gamma(1, t+1)(1', t+1')=(1, k)(1', k')$. We must have $k>t$ for otherwise $\gamma(1, k)(1', k')=\gamma(1, t+1)(1', t+1')$ contradicting the fact that γ is one-to-one.

Define $\beta_{t+1}\colon N \to N$ to be conjugation by $(k, t+1)(k', t+1')$ which is an element of N. Now β_{t+1} fixes $P_1, \ldots, P_t$ and $(1\,k)(1'\,k')=\beta_{t+1}(1, t+1)(1', t+1')=$

$\gamma(1, t+1)(1', t+1')$. Thus $\beta_{t+1}^{-1}\beta_t^{-1}\ldots\beta_1^{-1}\phi$ fixes $P_1, \ldots, P_{t+1}$. Then $\beta_n^{-1}\ldots\beta_1^{-1}\phi$ fixes $P_1, \ldots, P_n$. Since these permutations generate all of N, $\beta_n^{-1}\ldots\beta_1^{-1}\phi$ is the identity and so $\phi=\beta_1\beta_2\ldots\beta_n\in\operatorname{Inn} N$. So the lemma is proved.

Let C_1 be the conjugacy class of elements $\{(11'), (22'), \ldots, (n\,n')\}$. Let C_2 be the conjugacy class of elements

$$\{(a\,b)(a'\,b'), (a\,b')(a'\,b) \mid 1 \leqq a < b \leqq n\}.$$

Lemma 2.3. *If* $\alpha\in\operatorname{Aut} N$, *then* $\alpha^2 C_1 = C_1$; $\alpha C_2 = C_2$ *unless* $n=3$ *or* 8. *If* $n=3$, $\alpha^2 C_2 = C_2$ *and if* $n=8$, $\alpha^6 C_2 = C_2$.

Proof. An automorphism maps a conjugacy class of m elements of order r to a class of m elements of order r. C_1 has order n and C_2 is of order $n(n-1)=\binom{n}{2}\cdot 2$. Let C_k denote any class whose elements are k disjoint permutations for $k\geqq 3$. If $n\geqq 4$, the distinct elements of C_2 will be combined with other transpositions so that $|C_k| > |C_2|$ unless the transpositions in C_k are of the type $(a\,a')(b\,b')\ldots(x\,x')$. In this case $|C_k| = \binom{n}{k}$. Now $\binom{n}{k} = n$ if and only if $k=1$ or $k=n-1$. Thus $\alpha C_1 = C_1$ or C_{n-1} and so $\alpha^2 C_1 = C_1$. But $\binom{n}{k} = n(n-1)$ is satisfied only if $n=8$ and $k=3$ or 5. Thus if $n\neq 8$, $\alpha C_2 = C_2$ and if $n=8$ then $\alpha^6 C_2 = C_2$. Now if $n=3$, computation of the congruence classes shows there is only one other congruence class of the same order as C_1 and one other of the same order as C_2. Thus $\alpha^2 C_1 = C_1$ and $\alpha^2 C_2 = C_2$.

Lemma 2.4. $[\operatorname{Aut} N : \operatorname{Inn} N] \leqq 2$ *unless* $n=8$. *In that case* $[\operatorname{Aut} N : \operatorname{Inn} N]$ *divides* 6.

To prove the lemma, first assume $n\neq 3, 8$. Suppose there is a ϕ in $\operatorname{Aut} N$ for which $\phi C_1 \neq C_1$. Then if α is any automorphism of N which is not inner and $\alpha C_1 \neq C_1$ then $\alpha C_1 = \phi C_1$ so that $\phi^{-1}\alpha$ fixes C_1 and C_2 and so $\phi^{-1}\alpha\in \operatorname{Inn} N$. Thus there are just two cosets of $\operatorname{Inn} N$ in $\operatorname{Aut} N$. For $n=3$, $|C_1| = |D_1|$ where $D_1 = \{P_1, P_2, P_1 P_2\}$. Computation of $\alpha(11')(22')$ shows that $\alpha C_1 \neq D_1$. So if $\alpha C_2 \neq C_2$ any other automorphism ϕ which moves C_2 satisfies $\alpha C_2 = \phi C_2$. Again $[\operatorname{Aut} N : \operatorname{Inn} N] \leqq 2$. Finally if $n=8$, there may be α_1 in $\operatorname{Aut} N$ so that $\alpha_1 C_1 \neq C_1$, but $\alpha_1^2 C_1 = C_1$. There may also be α_2 so that $\alpha_2 C_2 \neq C_2$. Then either $\alpha_2^2 C_2 = C_2$ or $\alpha_2^3 C_2 = C_2$. Since $\alpha_i C_1 \neq C_2$ for any $\alpha_i \in \operatorname{Aut} N$, then $\alpha_1\alpha_2 C_i = \alpha_2\alpha_1 C_i$ for $i=1, 2$. Thus $(\alpha_1\alpha_2)^6 C_i = \alpha_1^6 \alpha_2^6 C_i$. So if ϕ is any automorphism $\phi^j C_i = C_i$ for $i=1, 2$ and some j which divides six. Clearly there are at most six cosets for $\operatorname{Inn} N$.

We can establish a bound on the order of the Sylow 2-subgroup of H. Coxeter gives the order of the center of N to be 2. Unless $n=8$,

$$|H|_2 \leqq |q-1|_2 \cdot 2\,|\operatorname{Inn} N|_2 = |q-1|_2 \cdot |n!|_2 \cdot 2^n.$$

So

$$|H|_2 \leqq |q-1|_2 \cdot 2^{n+[\frac{n}{2}]+[\frac{n}{4}]+\cdots}.$$

Since

$$|GL(n,q)| \geqq 2^{3\left[\frac{n}{2}\right]} \cdot |q-1|_2^{\left[\frac{n-1}{2}\right]} \cdot 2^{\left[\frac{n}{4}\right]+\left[\frac{n}{8}\right]+\cdots}$$

we show that

$$1+n+\left[\frac{n}{2}\right]+\left[\frac{n}{4}\right]+\cdots<3\left[\frac{n}{2}\right]+\left[\frac{n-1}{2}\right]+\left[\frac{n}{4}\right]+\left[\frac{n}{8}\right]+\cdots$$

and thus that H is not a Hall subgroup. If $n>11$,

$$1+n+\left[\frac{n}{2}\right] \leqq \frac{3n+2}{2} < \frac{(4n-9)}{2} = \frac{3(n-2)}{2} + \frac{(n-3)}{2} \leqq 3\left[\frac{n}{2}\right]+\left[\frac{n-1}{2}\right].$$

The inequality

$$1+n+\left[\frac{n}{2}\right]<3\left[\frac{n}{2}\right]+\left[\frac{n-1}{2}\right]$$

can be shown directly for $6 \leqq n \leqq 11$. If $n=2$, 3 or 4, H is solvable. If $n=5$ or 8, comparison of the Sylow 3-groups shows $|H|_3<|GL(n,q)|_3$. So H is not a Hall non-solvable subgroup of $GL(n,q)$ for $n \geqq 2$.

Case 3. $N=[3^{n-3,1,1}]$ is a subgroup of index 2 in $[3^{n-2},4]$; $n \geqq 4$. Coxeter [5, p. 607] gives generators:

$$\begin{aligned} R_1 &= (12)(1'\,2') \\ R_2 &= (23)(2'\,3') \\ &\vdots \\ R_{n-1} &= (n-1,n)(n-1',n') \\ R_n &= (n-1,n')(n-1',n) \end{aligned}$$

where N is a permutation group of $2n$ letters. Since $R_{n-1}R_n=(n-1,n-1')\cdot(n,n')$ we can use as generators:

$$\begin{aligned} P_1 &= (11')(22') \\ P_2 &= (12)(1'\,2') \\ P_3 &= (13)(1'\,3') \\ &\vdots \\ P_n &= (1\,n)(1'\,n'). \end{aligned}$$

Let C_1 denote the congruence class $\{a\,a')(b\,b')|1 \leqq a<b \leqq n\}$. Let C_2 denote the congruence class containing $P_2, \ldots, P_n$.

Lemma 2.5. *If $\phi \in \mathrm{Aut}\,N$ is such that $\phi\,C_1=C_1$ and $\phi\,C_2=C_2$, then $\phi \in \mathrm{Inn}\,N$.*

Proof by induction on t as defined in $N=[3^{n-2},4]$.

1) $t=1$. Suppose $\phi(11')(22')=(a\,a')(b\,b')$. Let β_1 be conjugation by $(1\,a)\cdot(1'\,a')(2\,b)(2'\,b')$. Then $\beta_1^{-1}\phi$ fixes P_1.

2) $n>t>1$. Suppose $\gamma=\beta_t^{-1}\beta_{t-1}^{-1}\ldots\beta_1^{-1}\phi$ is given by induction hypothesis so that γ fixes $P_1,\ldots,P_t$. Since γ preserves C_2 we proceed by an argument identical to that for $N=[3^{n-2},4]$.

Lemma 2.6. *If $\phi\in\mathrm{Aut}\,N$ and $n>4$, then $\phi^2(C_1)=C_1$ and $\phi(C_2)=C_2$. If $n=4$, $\phi(C_1)=C_1$ and $\phi^i(C_2)=C_2$ for $i\leqq 3$.*

Proof. As in the case of $N=[3^{n-2},4]$ we compare the orders of the conjugacy classes, since the classes containing the generators map to conjugacy classes of products of disjoint transpositions.

$|C_1|=\binom{n}{2}$ and $|C_2|=n(n-1)$. If a conjugacy class of products of disjoint transpositions combines elements from both C_1 and C_2, then its order will exceed $|C_1|$ and $|C_2|$. But there is a class C_1' of order $|C_1|$ consisting of $\binom{n}{n-2}$ elements which are the product of $2n-2$ disjoint transpositions from C_1 and thus $|C_1|=|C_1'|$. Such a complement class does not exist for C_2 because of the possible permutations as shown in the case for $n=4$. Thus $\phi^2(C_1)=C_1$ and $\phi(C_2)=C_2$.

In case $n=4$, we have three distinct conjugate classes with the same order as C_2, namely $\{(a\,b)(a'\,b')(c\,c')(d\,d')\}$ and $\{(a\,b)(a'\,b')(c\,d)(c'\,d')\}$. But $\phi^i C_2=C_2$ for at most $i=3$, since ϕ is one-to-one. Also C_1 is the only class with six elements of order 2.

Lemma 2.7. $[\mathrm{Aut}\,N:\mathrm{Inn}\,N]\leqq 2$ *if $n>4$ and at most 4 if $n=4$.*

Proof. If $n>4$, the result is a direct corollary to the previous lemma.

If $n=4$ and there exists $\theta\in\mathrm{Aut}\,N$ such that $\theta\in\mathrm{Inn}\,N$, then either $\theta^2 C_2=C_2$ or $\theta^3 C_2=C_2$. Then if ϕ is any automorphism which is not inner $\phi C_2\neq C_2$, so either $\phi C_2=\theta C_2$ or $\phi C_2=\theta^2 C_2$. Thus either $\theta^{-1}\phi$ or $(\theta^2)^{-1}\phi$ fixes C_2; thus there are at most four cosets of Inn N in Aut N. This completes the proof of the lemma.

Now we take up the question of the Hall property. If $n>4$, $|H|\leqq(q-1)\cdot|\mathrm{Aut}\,N|=(q-1)\cdot 2\cdot|\mathrm{Inn}\,N|$. In this case $|N|=2^{n-1}\cdot n!$, so $|H|_2\leqq|q-1|_2\cdot 2^n|n!|_2$. Comparison with the previous case shows H never contains a Sylow 2-subgroup of $GL(n,q)$.

If $n=4$, since the center of N has order 2, $|H|\leqq(q-1)\cdot 2^4\cdot(4!)$, so $|H|_2\leqq|q-1|_2\cdot 2^7$. But $|GL(4,q)|_2=|q-1|_2^4\cdot|q+1|_2^2\cdot 2\geqq|q-1|_2\cdot|q^2-1|_2^2\cdot|q-1|_2\cdot 2\geqq|q-1|_2\cdot 2^8$. So for the case $N=[3^{n-3,1,1}]$, neither H nor N is Hall.

Case 4. $N=[r]$, the dihedral group of order $2r$ and degree 2.

$$N=\langle a,b\,|\,a^2=1,b^r=1,a\,b\,a=b^{-1}\rangle.$$

N is solvable because it has an abelian (cyclic) subgroup $C_r=\langle b\rangle$ of index 2. If $r\neq 2$, C_r is invariant in N because every element of N has unique expression as $a^i b^j$, $i=0,1$ and $j=0,1,\ldots,r$ and $a\,b^j$ has order 2 for all j. Thus for $\phi\in\mathrm{Aut}\,N$, $\phi(b)\in C_r$.

Now suppose N is normal in a Hall subgroup H. We will show that H is solvable.

1) The centralizer $C_H(N)$ of N in H is, according to Schur's lemma, a group of scalar matrices and so $C_H(N)$ is a solvable group. We will prove that Aut N is solvable. Then $H/C_H(N) \cong \mathrm{Inn}_H(N)$ and is solvable.

2) Aut C_r is abelian because any two automorphisms ϕ_1, ϕ_2 on C_r are determined by their images on the generator of C_r.

$$\phi_1 \phi_2(b) = (b^i)^j = (b^j)^i = \phi_2 \phi_1(b).$$

3) Aut N maps homomorphically onto Aut C_r by the restriction of each automorphism in Aut N to C_r. Clearly such a map is well-defined and preserves operations. To show it is onto, let $\phi \in \mathrm{Aut}\ C_r$ and let $\phi_1(a) = a$, $\phi_1(b) = \phi(b) = b^i$, $\phi_1(a\,b^k) = a\,b^{ik}$.

To show $\phi_1 \in \mathrm{Aut}\ N$:

i) ϕ_1 is one-to-one. For if $\phi_1(a\,b^k) = a\,b^{ik} = a\,b^{im} = \phi_1(a\,b^m)$ then $b^{i(k-m)} = 1$ implies $i(k-m) \equiv 0 \bmod r$ for $k, m < r$. But since $\phi \in \mathrm{Aut}\ C_r$, $(i, r) = 1$, so $m = k$.

ii) ϕ_1 preserves operations. We need only consider elements outside C_r.

$$\begin{aligned}
\phi_1(b^j)\,\phi_1(a\,b^k) &= b^{ij} \cdot a\,b^{ik} = a(a\,b^{ij}\,a)\,b^{ik} = a\,b^{ik-ij} \\
&= a\,b^{(k-j)i} = \phi_1(a\,b^{k-j}) = \phi_1(b^j\,a\,b^k). \\
\phi_1(a\,b^k)\,\phi_1(b^j) &= a\,b^{ik}\,b^{ij} = a\,b^{(k+j)i} = \phi_1(a\,b^k\,b^j). \\
\phi_1(a\,b^k)\,\phi_1(a\,b^j) &= a\,b^{ik}\,a\,b^{ij} = b^{i(j-k)} = \phi_1(b^{j-k}) = \phi_1(b^{-(k-j)}) = \phi_1(a\,b^k\,a\,b^j).
\end{aligned}$$

Thus $\phi_1 \in \mathrm{Aut}\ N$.

4) The kernel of the mapping θ: Aut $N \to$ Aut C_r is abelian, and hence is solvable. To prove this we note that all automorphisms in the kernel have the form

$$\phi_i(b) = b, \qquad \phi_i(a) = a\,b^i, \qquad \phi_i(a\,b^j) = a\,b^i\,b^j.$$

Now all such automorphisms commute:

$$\phi_m \phi_k(a) = a\,b^m\,b^k = a\,b^k\,b^m = \phi_k \phi_m(a).$$

Thus Aut N is solvable because Aut $N/\mathrm{Ker}\,\theta$ is solvable and so is $\mathrm{Ker}\,\theta$.

If $r = 2$, Aut N has order 6 and so is solvable. Thus in all cases H is solvable.

Case 5. $N = [3, 5]$ of degree 3 and order 120 is described by Coxeter-Moser [7, p. 38] as a direct product; $[3, 5] \cong Z_2 \times A_5$ where Z_2 is the center of N. Both subgroups of N are fully invariant, so $|\mathrm{Aut}\ N| = |\mathrm{Inn}\ A_5| = 60$. Then $|H|_2 \leqq |q-1|_2 \cdot 2^2$. But $|GL(3, q)|_2 \geqq |q-1|_2 \cdot 2^4$, so H is not Hall.

Case 6. $N = [3, 4, 3]$ of order $2^7 \cdot 3^2$ and degree 4 is solvable by Burnside's condition. Suppose that H is any subgroup of $GL(n, q)$ which contains N as a normal subgroup. We will show that H is solvable using the relation $H/C_H(N) \cong$

$\mathrm{Inn}_H(N)$. It will be sufficient to show that Aut N is solvable. Consider the chain of fully invariant subgroups

$$C: N \supset N' \supset Z_2 \supset 1.$$

By Coxeter-Moser [9, p. 126], $[N:N']=4$ and $N'/Z_2 \cong A_4 \times A_4$. In order to apply the theory of stability groups to the chain C, we establish the following properties:

1) Aut N/N' is solvable.

2) Aut N'/Z_2 is solvable.

3) Aut Z_2 is solvable.

We will prove only 2), the others being easy.

Let $N'/Z_2 \cong A \times B$ where $A \cong A_4 \cong B$. Suppose $A=\langle(123),(142)\rangle$ and $B=\langle(1'\,2'\,3'),(1'\,4'\,2')\rangle$. Now for $\phi \in \mathrm{Aut}\, A \times B$ either $\phi(A)=A$ in which case $\phi \in \mathrm{Aut}\, A \times \mathrm{Aut}\, B = M$ or $\phi(A)=B$ in which case $\phi^2 \in M$. The reason for this is that the conjugacy class of the generators of A maps to itself or entirely to the conjugacy class of the generators of B. We note that $M \lhd \mathrm{Aut}\, A \times B$ and that for any ϕ, θ in Aut $A \times B$, $(\phi\,\theta)^2 \in M$ and so we conclude that $(\mathrm{Aut}\, A \times B)/M$ is an elementary 2-group and hence solvable. Scott [18, p. 315] shows that if $n>3$ and $n \neq 6$, $\mathrm{Aut}\, S_n \cong \mathrm{Aut}\, A_n \cong S_n$. Thus $|\mathrm{Aut}\, A_4| = 3 \cdot 2^3$; so M is then a direct product of solvable groups; hence, solvable. Since a subgroup and a factor group of Aut N'/Z_2 are solvable, so is the group.

Let θ: Aut $N \to$ Aut $N/N' \times$ Aut $N'/Z_2 \times$ Aut Z_2 be a mapping which is defined by $\phi \to \phi_N \times \phi_{N'} \times \phi_{Z_2}$ for $\phi \in$ Aut N where

$$\begin{aligned} \phi_N(aN') &= \phi(a)\,N' \quad \text{for } a \in N, \\ \phi_{N'}(aZ_2) &= \phi(a)\,Z_2 \quad \text{for } a \in N', \\ \phi_{Z_2}(a) &= \phi(a) \quad \text{for } a \in Z_2. \end{aligned}$$

We must show that ϕ_N, $\phi_{N'}$, ϕ_{Z_2} are automorphisms of the respective factor groups.

1) ϕ_N is well-defined, for $aN'=bN'$ implies $\phi(aN')=\phi(bN')$ so $\phi(a)N'=\phi(b)N'$. It is one-to-one, for if $\phi(a)N'=\phi(b)N'$ then $\phi(b^{-1}a)N'=N'$ and hence $\phi(b^{-1}a) \in N'$. But N' is invariant under ϕ, and its restriction to N' is an automorphism on N', so $b^{-1}a \in N'$ and thus $aN'=bN'$. Finally ϕ_N is onto; let $xN' \in N/N'$. For some y in N, $\phi(y)=x$. So $xN'=\phi(y)N'=\phi_N(yN')$.

2) Similar arguments show $\phi_{N'}$, and ϕ_{Z_2} to be automorphisms.

Philip Hall [15, p. 787] defines the stability group of a chain as the group of automorphisms of a group which act as the identity on the factor group of the chain. In our case Ker θ is seen to be the stability group for C of N. It also follows from Hall's paper and our conditions that this stability group of N is solvable; thus Ker θ is solvable, and so is Aut N. Hence we can conclude that H is solvable.

Case 7. $N=[3,3,5]$ of order 14400 and degree 4. According to Coxeter-Moser [9, p. 128], the commutator subgroup N' is of index 2 in N. The center Z of N is of order 2, is contained in N', and $N'/Z=A_5\times A_5$. We shall now show that

$$|\mathrm{Aut}\, N|\leqq 2\,|\mathrm{Aut}(N'/Z)|.$$

Lemma 2.8. *If* $\mathrm{Aut}\, N$ *is homomorphic to* $\mathrm{Aut}(N'/Z)$ *with kernel* K, *then* $|K|\leqq 2$.

Proof. Let $\alpha\in\mathrm{Aut}\, N$. Since N' and Z are characteristic subgroups, α restricted to N' and Z induces automorphisms on each subgroup. We define a homomorphism by the rule $\alpha\to\hat{\alpha}$ where $\hat{\alpha}(xZ)=z^{\alpha}Z$ for $x\in N'$.

1) First we show that $\hat{\alpha}\in\mathrm{Aut}(N'/Z)$. We pick $x, y\in N'$. $xZ=yZ$ if and only if $x^{-1}y\in Z$ if and only if $(x^{\alpha})^{-1}y^{\alpha}=(x^{-1}y)^{\alpha}\in Z$ if and only if $y^{\alpha}Z=x^{\alpha}Z$. Thus $\hat{\alpha}$ is well-defined and one-to-one on N'/Z. That $\hat{\alpha}$ preserves operations can be seen from

$$\hat{\alpha}(xZ\,yZ)=\hat{\alpha}(xy\,Z)=(xy)^{\alpha}Z=x^{\alpha}Z\,y^{\alpha}Z=\hat{\alpha}(xZ)\cdot\hat{\alpha}(yZ).$$

2) The mapping $\alpha\to\hat{\alpha}$ is well-defined for if $\alpha, \beta\in\mathrm{Aut}\, N$, then $\alpha=\beta$ implies $x^{\alpha}=x^{\beta}$ for all x. So $x^{\alpha}Z=x^{\beta}Z$. That the mapping is a homomorphism follows from the relations

$$\widehat{\alpha\beta}(xZ)=x^{\alpha\beta}Z=\hat{\alpha}(x^{\beta}Z)=\hat{\alpha}\hat{\beta}(xZ).$$

3) Let $\alpha\in K$, the kernel of the homomorphism of parts 1) and 2).

i) α acts trivially on N'/Z. For if $xZ\in N'/Z$, $\alpha(xZ)=x^{\alpha}Z=\hat{\alpha}(xZ)=xZ$.

ii) α acts trivially on N/Z. The computations are mod Z. Let $x\in N$ so that $x^{\alpha}\neq x$. Then by i) $x\notin N'$, so $x^{\alpha}=xy$ for $y\in N'$. Now for any $w\in N'$, $w^{x}=(w^{x})^{\alpha}=(w\alpha)^{x\alpha}=w^{xy}$ and $w^{xy}=w^{x}$ if and only if $w^{y}=w$ for all $w\in N'$. Thus $y\in$center (N'/Z) and since $N'/Z\cong A_5\times A_5$ its center is Z. Thus $x^{\alpha}=x$ in N/Z.

iii) α acts trivially on N'. Let $w\in N'$; $w^{\alpha}=wz$ for some $z\in Z$. Now the mapping ϕ: $wZ\to z=w^{-1}w^{\alpha}$ is a homomorphism of N'/Z to Z. First we show ϕ is well-defined; take $w_1Z=w_2Z$. Then $w_1^{-1}w_2\in Z$, so $(w_1^{-1}w_2)^{\alpha}=w_1^{-1}w_2$. Suppose $w_1^{-1}w_1^{\alpha}=z_1$. Then $(w_1)^{\alpha}=w_1z_1$ and $w_2^{\alpha}=(w_1)^{\alpha}(w_1^{-1}w_2)^{\alpha}=(w_1z_1)(w_1^{-1}w_2)$ $=w_2z_1$. So $w_2^{-1}w_2^{\alpha}=z_1=w_1^{-1}w_1^{\alpha}$ as we wished to show.

Operations are preserved; for $\phi(w_1Zw_2Z)=\phi(w_1w_2Z)=(w_1w_2)^{-1}(w_1w_2)^{\alpha}$ $=w_2^{-1}(w_1^{-1}w_1^{\alpha})w_2^{\alpha}=w_1^{-1}w_1^{\alpha}w_2^{-1}w_2^{\alpha}=\phi(w_1Z)\phi(w_2Z)$. We use the fact that $w_1^{-1}w_1^{\alpha}\in Z$.

Now the composition mapping of the "natural map" of N' to N'/Z with ϕ maps N' to Z homomorphically. But if such a map were on all of Z, the kernel of ϕ would provide a normal subgroup of N'/Z of index 2 contrary to our condition that N'/Z is isomorphic to $A_5\times A_5$. Thus for all $w\in N'$, $w^{-1}w^{\alpha}=1$ so $w^{\alpha}=w$.

iv) Let $x \notin N'$; since $N = N \cup xN'$ and α maps N' identically, α is determined by the image of x. But by ii) above $(xZ)^\alpha = x^\alpha Z = xZ$, so $x^\alpha = x \cdot z$ for $z \in Z$ and thus there are at most two automorphisms in K.

Lemma 2.9. $|\mathrm{Aut}\, A_5 \times A_5| \leqq 2\,|\mathrm{Inn}\, A_5 \times A_5|$.

Let $B \cong C \cong A_5$ where B acts on $\{1, 2, 3, 4, 5\}$ and C acts on $\{1', 2', 3', 4', 5'\}$. Since A_5 is generated by its three-cycles we consider the conjugacy classes of elements of order 3. In B, there are 20 distinct 3-cycles $(a\,b\,c)$ forming a single conjugacy class C_3; similarly for 3-cycles $(a'\,b'\,c')$ of C_3' in C. In $B \times C$ the class of pairs of 3-cycles has order 400. So if $\phi \in \mathrm{Aut}\, B \times C$, $\phi(C_3) = C_3$ or $\phi(C_3) = C_3'$. If $\phi\, C_3 = C_3$, then $\phi(C_3') = C_3'$ and since A_5 is a complete group ϕ induces inner automorphisms on B and C; thus $\phi \in \mathrm{Inn}\, B \times C$.

If $\phi(C_3) = C_3'$, then $\phi^2(C_3) = C_3$ and $\phi^2(C_3') = C_3'$ and so $\phi^2 \in \mathrm{Inn}\,(B \times C)$. If β is any other automorphism of $B \times C$ and not an inner automorphism then $\beta(C_3) = C_3' = \phi(C_3)$. Thus $\beta^{-1}\phi \in \mathrm{Inn}\, B \times C$ and so $[\mathrm{Aut}\, B \times C : \mathrm{Inn}\, B \times C] \leqq 2$.

Combining the above lemmas we have

$$|\mathrm{Aut}\, N| \leqq 2\,|\mathrm{Aut}\, B \times C| \leqq 4\,|\mathrm{Inn}\, B \times C| = 5^2 \cdot 3^2 \cdot 2^6.$$

To see whether H contains a Sylow 2-subgroup of $GL(4, q)$ we note that $|GL(4, q)|_2 = |q-1|_2^4 \cdot |q+1|_2^2 \cdot 2 = |q-1|_2^2 \cdot |q^2-1|^2 \cdot 2 \geqq |q-1|_2 \cdot 2^8$ for 8 divides q^2-1 for all odd q. However $|H|_2 \leqq |q-1|_2\,|\mathrm{Aut}\, N|_2 \leqq |q-1|_2 \cdot 2^6$. So H is not a Hall subgroup of $GL(4, q)$.

Case 8. $N = [3^{2,2,1}]$ of order $2^7 \cdot 3^4 \cdot 5$ and degree 6. Coxeter-Moser [9, p. 125] describe the commutator subgroup $N' = [3^{2,2,1}]^+$ as a simple group of index 2 in N, viz. Dickson's $HO(4, 2^2)$ [12, p. 309]. Hence N' is normal and complete in N and therefore a direct factor. Thus $N = Z_2 \times N'$. So $|\mathrm{Aut}\, N| = |\mathrm{Inn}\, N'| = \dfrac{|N|}{2} = (2^6 \cdot 3^4 \cdot 5)$ and thus $|H|_2 \leqq |q-1|_2 \cdot 2^6$. But $|GL(6, q)|_2 = |q-1|_2^6 \cdot |q+1|_2^3 \cdot 2 \geqq 2^{12}\,|q-1|_2$, so H is not a Hall subgroup.

Case 9. $N = [3^{3,2,1}]$ of order $2^{10} \cdot 3^4 \cdot 5 \cdot 7$ and degree 7. Coxeter-Moser [9, p. 127], describe N', the commutator subgroup of N, as the group of automorphisms of the 28 bitangents of the general quartic curve in the projective plane. Thus $N' \cong N/Z$ is a simple group of order $4 \cdot 9!$ isomorphic to Dickson's $A(6, 2)$ [12, p. 310]. So $|\mathrm{Aut}\, N| = |\mathrm{Inn}\, N'|$ and $|H|_2 \leqq |q-1|_2 \cdot 2^9$. But $|GL(7, q)|_2 \geqq |q-1|_2 \cdot 2^{13}$ so H is not a Hall subgroup of G.

Case 10. $N = [3^{4,2,1}]$ of order $7 \cdot 5^2 \cdot 3^5 \cdot 2^{14}$ and degree 8. As in Case 7, the commutator subgroup $N' = [3^{4,3,1}]^+$ of index 2 in N, and N' contains the center Z of N where Z has 2 elements. Coxeter-Moser [9, p. 129] describe $[3^{4,2,1}]^+/Z$ as a simple group, so $|\mathrm{Aut}\, N| = |\mathrm{Aut}\, N'/Z| = |\mathrm{Inn}\, N'/Z|$. Hence $|H| \leqq (q-1)\,|\mathrm{Aut}\, N| \leqq (q-1)\dfrac{|N|}{4}$ and $|H|_2 \leqq |q-1|_2 \cdot 2^{12}$. But $|GL(8, q)|_2 = |q-1|_2^8 \cdot |q+1|^4 \cdot 2^2 \geqq |q-1|_2 \cdot 2^{16}$. Thus H is not a Hall subgroup of G.

Section 2. Coxeter's Non Real Cases [6, p. 270]

Case 11. $N=[1,1,\ldots,1]^n$ in Coxeter's notation is described by Shephard [20, p. 374] as $[1,1;n-2]^m$ of degree n and order $m^{n-1}\cdot n!$ For generators $P, Q, R_1, R_2, \ldots, R_{n-2}$ we have the relations

$$P^2=Q^2=R_1^2=(QR)^3=(R_1P)^3=(PQ)^m=1,$$
$$(PQR_1)^2=(R_1PQ)^2,$$
$$R_i^2=(PR_i)^2=(QR_i)^2=(R_iR_{i-1})^3=1 \quad \text{for } i=2,3,\ldots,n-2$$

and $R_iR_j=R_jR_i$ for $i,j=1,2,\ldots,n-1$ and $|i-j|\geqq 2$. We will give a matrix description of this group and show that it has a normal abelian subgroup of order m^{n-1}. Consider $n\times n$ matrices over the field of complex numbers.

Let $C_m=$cyclic subgroup of all m-th roots of $1=\langle e^{2\pi i/m}\rangle$.

Let $D_0=$group of diagonal matrices

$$\begin{pmatrix} \delta_1 & & & 0 \\ & \ddots & & \\ & & \delta_i & \\ & & & \ddots \\ 0 & & & \delta_n \end{pmatrix}$$

with δ_i in C_m and $\prod_{i=1}^{n}\delta_i=1$. Note that D is the direct product of $n-1$ cyclic subgroups of order m.

Let $\Sigma=n\times n$ permutation matrices.

Then D_0 is normalized by Σ in $GL(n,C)$ and $D\cap\Sigma=1$.

Thus $S=D_0\Sigma$ is a group of order $m^{n-1}\cdot n!$ which we shall show to be isomorphic to N as given by Coxeter and Shephard's relations.

Let $Q, R_1, \ldots, R_{n-2}$ be the $n-1$ transposition matrices (12), (23), ..., $(n-1,n)$.

Let $\delta=e^{2\pi i/m}$ be a fixed primitive m-th root of 1.

Let

$$P=\begin{pmatrix} 0 & \delta^{-1} & & 0 \\ \delta & 0 & & \\ & & 1 & \\ 0 & & & \ddots \\ & & & & 1 \end{pmatrix}=Q'\begin{pmatrix} \delta^{-1} & & 0 \\ & \delta_1 & \\ & & \ddots \\ 0 & & 1 \end{pmatrix}.$$

These generators of S satisfy the Shephard relations and thus $S\cong N$. Furthermore by Theorem 1, since N is generated by reflections, for each prime p which does not divide the order of N, there is a finite field of order $q=p^k$ for which $GL(n,q)$ has a subgroup isomorphic to S. Thus it is possible to choose a basis for the underlying vector space so that D_0 is isomorphic to the group of

diagonal matrices of determinant 1 of $GL(n, q)$. Now let $D = \{\text{diagonal matrices of } GL(n, q)\}$. Then $H \cong \Sigma D$ contains $N \cong \Sigma D_0$ as a normal subgroup where H and N are regarded as splitting extensions of D and D_0 respectively by the permutation matrices Σ. Thus $|H| = m^n \cdot n!$ and $|N| = m^{n-1} \cdot n!$

Lemma 2.10. *The following conditions taken together are necessary and sufficient for H to be a Hall non-solvable subgroup of $GL(n, q)$:*

1) *$n \geqq 5$ and 4 divides $q-1$.*

2) *If r is a prime less than n, then r divides $q-1$.*

3) *If n is a prime such that n divides $q^e - 1$ then $e > n/2$ and n^2 does not divide $q^e - 1$.*

Proof. That such conditions can be satisfied, consider for example $GL(7, 61)$. The sufficiency of the general conditions can be seen by comparing the orders of the Sylow subgroups of H and $GL(n, q)$. To prove the necessity we compare the Sylow 2-subgroups and observe that $4 \nmid q+1$; thus $4 \mid q-1$.

$$|GL(n, q)|_2 = |q-1|_2^n \cdot |q+1|_2^m \cdot |m!|_2$$

where $m = \left[\frac{n}{2}\right]$ while $|H|_2 = |q-1|_2^n \cdot |n!|_2$.

Let r be an odd prime not greater than n,

$e =$ least integer for which $r \mid q^e - 1$,

$|q^e - 1|_r = r^a$ and

$f = \left[\frac{n}{e}\right]$.

Then $|GL(n, q)|_r = r^{fa} \cdot |f!|_r$ and $|H|_r = |q-1|_r^n \cdot |n!|_r$. Then we assume that $e > 1$ and show that this leads to a contradiction, thus establishing condition 2).

Let $n = ef + d$ where $0 \leqq d < e$.

Proposition 2.11. *$e \leqq r-1$ and $e \mid r-1$.*

To prove this we apply Fermat's theorem so that $r \nmid q$ implies $q^{t-1} \equiv 1 \bmod r$. But $q^e \equiv 1$. Thus $r-1 = es+t$ for $0 \leqq t < e$, and so $q^{r-1} = q^{es} \cdot q^t$. Hence $1 \equiv q^{r-1} \equiv q^t$ for $t < e$. But then $t = 0$ and so $e \mid r-1$.

We now assume the Hall property for H and use the above notation along with a formula for the order of the Sylow subgroups of Σ_n.

$$r^{fa} \cdot |f!|_r = |n!|_r$$

or equivalently

$$a\left[\frac{n}{e}\right] + \left[\frac{f}{r}\right] + \left[\frac{f}{r^2}\right] + \left[\frac{f}{r^3}\right] + \cdots = \left[\frac{n}{r}\right] + \left[\frac{n}{r^2}\right] + \left[\frac{n}{r^3}\right] + \cdots.$$

Proposition 2.12. *Under these conditions, $a = 1$.*

Proof. Assume $a>1$. Then following Weir [23, p. 530] and Spitznagel [20, p. 260] we construct a Sylow r-subgroup of $GL(n, q)$ as follows. $GL(e, q)$ has a cyclic subgroup of order q^e-1 whose (cyclic) Sylow r-subgroup has order $|q^e-1|_r=r^a$; call the r-subgroup X_e. Using e by e blocks down the diagonal we can imbed a direct product of f copies of X_e within $GL(n, q)$. Call the imbedded product X.

Let R_0 be a Sylow r-subgroup of the group of f by f permutation matrices. Define $\mathscr{R}=\{R\otimes I_e \mid R\in R_0\}$ where $R\otimes I_e$ is the Kronecker or tensor product of the f by f matrix R with the e by e identity matrix I_e. Then $\mathscr{R}$ is a subgroup of $GL(n, q)$, $\mathscr{R}$ normalizes X, $\mathscr{R}\cap X=1$ and $|\mathscr{R}|=|R_0|=|\Sigma_f|_r$. Hence $|\mathscr{R}X|_r=|GL(n, q)|_r$, so $\mathscr{R}X$ is a Sylow r-subgroup of $GL(n, q)$.

But we know [14, p. 81] that every Sylow r-subgroup of Σ_n has a unique structure different from this, namely a direct product of r-subgroups of increasing orders, each of which is the wreath product of elementary abelian subgroups. Since the smallest subgroup of the direct product is of order r, and since $a>1$ by hypothesis the cyclic subgroups of X have order $r^a>r$. This contradiction proves the proposition.

Thus Lemma 2.10 provides a family of Hall non-solvable subgroups of $GL(n, q)$, the so-called monomial matrices. We will show that this is the only such family by showing that H is the normalizer of N. Let $\mathscr{N}(N)$ denote the normalizer of N.

Proposition 2.13. $H=\mathscr{N}(N)=\mathscr{N}(D_0)$.

Proof. Since D_0 is a maximal abelian subgroup of N, D_0 is characteristic in N. Hence $H\subseteq\mathscr{N}(N)\subseteq\mathscr{N}(D_0)$. Suppose x normalizes D_0. Since D_0 is generated by elements of the form

$$\alpha=\begin{pmatrix} 1 & & & & & & & & 0 \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & \delta & & & & & \\ & & & & 1 & & & & \\ & & & & & \ddots & & & \\ & & & & & & 1 & & \\ & & & & & & & \delta^{-1} & \\ & & & & & & & & 1 \\ & & & & & & & & & \ddots \\ 0 & & & & & & & & & & 1 \end{pmatrix}$$

its conjugate is also diagonal. But since α is in Jordan form, the entries will be permuted. The only elements which so conjugate α are elements of H.

Case 12. $N=[1\,1\,1^4]^4$ is of degree 3 and order $336=7\cdot 3\cdot 2^4$. According to Coxeter [6, p. 252], $[1\,1\,1^4]^4\cong Z_2\times LF(2, 7)$ where Z_2 is the center and $LF(2, 7)$ is a simple group. Thus $|\mathrm{Aut}\,N|=|\mathrm{Inn}\,LF(2, 7)|=168$ and so $|H|_2\leqq$

$|q-1|_2 \cdot 2^3 < |q-1|_2 \cdot |q^2-1|_2 \, |q+1|_2 = |GL(3,q)|_2$. Thus H is non-solvable, but is not a Hall subgroup.

Case 13. $N=[1\,1\,1^5]^4$ is of degree 3 and order $5 \cdot 3^3 \cdot 2^4$ with center Z cyclic of order 6 [6, p. 255]. Since $N/Z \cong A_6$ we have $|\mathrm{Inn}_H N| = |\mathrm{Inn}_H(N/Z)| \leqq |\mathrm{Aut}(N/Z)| = |\mathrm{Aut}\, A_6| = 6!$ Thus $|H| = (q-1)$. $|\mathrm{Inn}_H N| \leqq (q-1)2^4 \cdot 3^2 \cdot 5$. Since 3 divides $q-1$, then 3 also divides q^2+q+1, so $|H|_3 \leqq |q-1|_3 \cdot 3^2 < |q-1|_3 \cdot 3^3 \leqq |q-1|_3^3 \cdot |q^2+q+1|_3 = |GL(3,q)|_3$.

Thus H is not a Hall subgroup of $GL(3,q)$.

Case 14. $N=[1\,1\,2]^4$ is of order $3 \cdot 5 \cdot 2^9$ and degree 4 with center Z_4. Coxeter gives defining relations for N/Z_4 [6, p. 256]. For generators R_i we have the relations $R_i^2 = (R_1 R_2)^3 = (R_2 R_3)^3 = (R_1 R_3)^3 = (R_1 R_2 R_3 R_2)^4 = (R_1 R_4)^2 = (R_2 R_4)^2 = (R_3 R_4)^3 = (R_1 R_2 R_3 R_4)^5 = E$.

Let $N/Z_4 = G$. Let ϕ be defined by $R_1 \to (12)$, $R_2 \to (23)$, $R_3 \to (24)$, $R_4 \to (45)$. Then ϕ defines a homomorphism from G onto S_5 because all the relations are satisfied in the images and because S_5 is generated by these transpositions. The kernel A of the homomorphism is a normal subgroup of G having order 16.

We will show that A is abelian and in fact an elementary abelian 2-group. Assume that A/K is a chief factor of G. Since the derived group of S_5 is A_5 and $(G/A)' = (G'/A)$ we have $[G:G'] = 2$. Now consider the chief series $G \supset G' \supset A \supset K \supset \cdots \supset 1$ where $[A:K] = 2^n$ and A/K is an elementary abelian 2-group. If $C_G(A/K)$ properly contains A, then $C_G(A/K) \geqq G'$ and so G/G' acts irreducibly on A/K because G' would be the kernel of the representations. But since A/K is a 2-group, G/G' acts trivially on A/K. Thus $\dim_{F_2}(A/K) = 1$ and so $|A/K| = 2$. Then $|A/K| \leqq Z(G/K)|$ and so A/K is a Schur multiplier of G/K. Thus G/K is isomorphic to a representation group of S_5.

Proposition 2.14. *If α and β are homomorphisms of G onto S_5 then $\theta: \beta(R_i) \to \alpha(R_i)$ defines an automorphism of S_5.*

Proof. According to the definition of β we see that θ is well defined. Clearly the mapping preserves operations: $\theta(\beta(R_i) \cdot \beta(R_j)) = \theta(\beta(R_i R_j)) = \alpha(R_i R_j)) = \alpha(R_i) \cdot \alpha(R_j) = \theta(\beta(R_i)) \cdot \theta(\beta(R_j))$. Since θ is onto S_5 it must be one-to-one. Thus the homomorphism is unique up to automorphisms.

If T is a representation group of S_5, we will show that there is no homomorphism $\theta: G \to T$ which is onto T. Suppose there is such $\theta: G \to T$ and $\phi: T \to S_5$, both homomorphisms. Then $\theta\phi$ is a homomorphism of G onto S_5 and according to Proposition 2.14 it is "known". Then θ is determined to within the kernel. If $x \in G$, then $\phi(\theta(x)) = \phi\,\theta(x)$, so $\theta(x) \in \phi^{-1}(\phi(\theta(x)))$ which has two elements; $\ker \phi = \pm E$.

Now

$$\begin{aligned} \phi\theta: R_1 &\to (12), \\ R_2 &\to (23), \\ R_3 &\to (24), \\ R_4 &\to (45). \end{aligned}$$

Suppose $T=SL^*(2,5)=\{A\in U(2,5)\,|\det A=\pm 1\}$ in Wong's notation [26, p. 54]. Then

$$T=\left\{\begin{pmatrix} a & b \\ -\bar{b} & \bar{a}\end{pmatrix},\ \begin{pmatrix} a & b \\ \bar{b} & -\bar{a}\end{pmatrix}\ \middle|\ \bar{x}=x^5\right\}.$$

Then

$$\phi^{-1}((12))=\left\{\pm\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right\},\qquad \phi^{-1}((23))=\left\{\pm\begin{pmatrix}2 & \varepsilon\\ \bar{\varepsilon} & -2\end{pmatrix}\right\},$$

$$\phi^{-1}((24))=\left\{\pm\begin{pmatrix}2 & -1\\ -1 & -2\end{pmatrix}\right\},\qquad \phi^{-1}((45))=\left\{\pm\begin{pmatrix}0 & 2\varepsilon+2\\ 2\bar{\varepsilon}+2 & 0\end{pmatrix}\right\}$$

where ε is a primitive generator of the cyclic subgroup of $GF(5^2)$ and satisfies $\varepsilon^2-\varepsilon+2=0$.

Now θ satisfies the following relations:

$$\theta(R_1)=\delta_1\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\qquad \text{for } \delta_1=\pm 1;$$

$$\theta(R_2)=\delta_2\begin{pmatrix}2 & \varepsilon\\ \bar{\varepsilon} & -2\end{pmatrix}\qquad \text{for } \delta_2=\pm 1;$$

$$\theta(R_3)=\delta_3\begin{pmatrix}2 & \varepsilon-1\\ \bar{\varepsilon}-1 & -2\end{pmatrix}\qquad \text{for } \delta_3=\pm 1;$$

$$\theta(R_1R_2)=\delta_1\delta_2\begin{pmatrix}2 & \varepsilon\\ -\bar{\varepsilon} & 2\end{pmatrix},\qquad \theta(R_1R_2)^3=\delta_1\delta_2E,\ \text{so } \delta_1=\delta_2;$$

$$\theta(R_1R_3)=\delta_1\delta_3\begin{pmatrix}2 & \varepsilon-1\\ -\bar{\varepsilon}+1 & 2\end{pmatrix},\qquad \theta(R_1R_3)^3=\delta_1\delta_3E,\ \text{so } \delta_1=\delta_3.$$

Finally $\theta(R_2R_3)^3=\delta_2\delta_3\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}=-\delta_2\delta_3E$ and so $\delta_2=-\delta_3$ contrary to the above results. Thus $SL^*(2,5)$ is not the representation group of S_5 for which N maps through T onto S_5. The only other possibility for a representation group of S_5 with a Schur multiplier of order 2 would occur if $G'/G''=2$, thus giving a normal subgroup of G/A of index 4. But clearly $(G'/A)'=G''/A=G'/A$ since G'/A is simple.

We have shown that $C_G(A/K)$ cannot properly contain A, and so $C_G(A/K)=A$. Hence G/A acts faithfully on A/K. So if $|A/K|=2^n$, then S_5 is isomorphic to a subgroup of $GL(n,5)$. A consideration of the orders makes $n\geqq 4$ and so $n=4$. Thus $K=1$ and so A is elementary abelian.

Let $H=$ the normalizer of G in $GL(4,q)$. Let $B=\{\beta\in\mathrm{Aut}\,G\,|\,\beta$ is induced on G by β' in $H\}$. Then B stabilizes A and acts on G/A because A is a unique maximal normal abelian subgroup.

Let $B_0=$ subgroup of B which acts trivially on G/A. Since every β in B_0 is induced by an element x in H, and x centralizes every element of an irreducible group S_5, by Schur's lemma, x is a scalar matrix and so centralizes A. Thus B stabilizes the chain $G\supset A\supset 1$.

Now we will show that B_0 is a 2-group. Let p divide $|B_0|$ and $g \in G$. For α in B we have $\alpha(g) = g\,a$ for some a in A. Also $\alpha^p(g) = g\,a^p = g$ implies $a^p = e$ and so $p = 2$.

But $B \cong B_0 \cdot \operatorname{Inn} G$ and so $|\operatorname{Aut} N/Z_4| = |\operatorname{Inn} N/Z| \cdot 2^k$ and so $|\operatorname{Aut} N| = 3 \cdot 5 \cdot 2^m$ for some finite m.

If N is normal in a subgroup H of $GL(4, q)$, then $|H|_3 \leqq |q-1|_3 \, |\operatorname{Aut} N|_3 = 3\,|q-1|_3$. But the Sylow 3-group of $GL(4, q)$ has order at least $|q^6(q-1)^4 \cdot (q+1)^2|_3 \geqq |q-1|_3 \cdot 3^2$. So although H is a non-solvable subgroup it is not a Hall subgroup of $SL(4, q)$.

Case 15. $N = [2\,1\,2]^3$ of degree 5 and order $72 \cdot 6! = 5 \cdot 3^4 \cdot 2^7$. The center Z is of order 2 and the central quotient group [6, p. 257] $[2\,1\,2]^3/Z \cong [2\,1\,2]^{3+}$ is simple and N is its direct product with Z.

Thus $|\operatorname{Aut} N|_2 = 1/2\,|N|_2 = 2^6$ and so $|H|_2 \leqq |q-1|_2 \cdot 2^6 < |q-1|_2 \cdot 2^9 \leqq |GL(5, q)|$. Although H is non-solvable, it is not a Hall subgroup.

Case 16. $N = [2\,1\,3]^3$ of degree 6 and order $108 \cdot 9! = 7 \cdot 5 \cdot 3^7 \cdot 2^9$. N has a center of 6 elements and a central quotient group whose commutator subgroup is the simple group of order $9 \cdot 9!$ [6, p. 256]; namely $[2\,1\,3]^{3+}/Z \cong HO(4, 3^2)$.

Since N/N' has order 2 we have $\operatorname{Inn}_H N = \operatorname{Inn}_H(N/Z) = \operatorname{Inn}_H(N'/Z) = \operatorname{Inn}(N'/Z) \cong \operatorname{Inn} N'/Z$ of order $9 \cdot 9!$

$$|H|_2 \leqq |q-1|_2 \cdot |\operatorname{Inn}_H N|_2 \leqq |q-1|_2 \cdot 2^8.$$

But

$$|GL(6, q)|_2 = |q-1|_2^6 \cdot |q+1|_2^3 \cdot 2 \geqq |q-1|_2 \cdot 2^{11}.$$

Again although H is non-solvable, it is not a Hall subgroup.

Section 3. The Remaining Groups of Shephard and Todd

We now turn to the groups given in Shephard and Todd's complete list [21, p. 301] of irreducible finite unitary groups generated by reflections. See the table.

Case 17. $N = [\ ]^m$, the cyclic group of order m and degree 1. The automorphisms are abelian and so $\mathcal{N}(N)$ is solvable.

Cases 18 to 36. N is of degree 2 and has a cyclic center Z whose elements are scalar matrices. Furthermore N/Z has generators s and t satisfying $s^2 = t^3 = (s\,t)^p = 1$ which defines A_4, S_4 or A_5 according as p is 3, 4, or 5. We will use the fact that $\operatorname{Inn}_H(N)$ will fix all elements of Z since they are scalars.

a) 18 to 29. The orders of these groups are the form $2^x\,3^y$ and so they are solvable. Furthermore N/Z is either A_4 or S_4 and so Aut N is solvable. Thus $\mathcal{N}(N)$ is solvable.

b) 30 to 36. Here $N/Z \cong A_5$. The orders of the center must divide $q-1$ and so if $\mathcal{N}(N)$ is to be a Hall subgroup of $GL(2, q)$ the only possible prime factors of $q-1$ are 2, 3, and 5. We compare orders for the chain $N \subseteq \mathcal{N}(N) \subseteq$

$GL(2, q)$ for each N, and see that $|N| \leqq |\mathcal{N}(N)| \leqq 5 \cdot 3 \cdot 2^2 \cdot (q-1) < (q-1)^2 \cdot (q+1)$. We conclude that neither $\mathcal{N}(N)$ nor N is a Hall subgroup of $GL(2, q)$ for these cases.

Case 37. N has order 648, degree 3 and center of order 3. The central quotient group is the Hessian group described by Coxeter [4, p. 165] as having an invariant subgroup N_1 of index nine. Furthermore N_1 the binary tetrahedral group has the quarternion group as its unique Sylow 2-group. Thus N as well as $\mathcal{N}(N)$ have orders divisible only by 2 and 3, thus making them solvable groups.

Case 38. N has order 1296, degree 3 and center Z_6. The central quotient group is the Hessian group and so as in Case 37, N and $\mathcal{N}(N)$ are solvable.

Case 39. N is of order $64 \cdot 6!$ and degree 4 with center Z_4. Shephard shows [19, p. 275] that this group is generated by five reflections P, Q, R, S, and T with relations

$$P^2 = Q^2 = R^2 = (QR)^3 = (PQ)^3 = (RP)^3 = E,$$

$$(QPRP)^4 = T^2 = (TPRP)^2 = (TR)^4 = P(TRQR)\,P(TRQR)^{-1} = E,$$

$$S^2 = (SP)^2 = (SR)^2 = (SQ)^3 = (ST)^3 = E.$$

Let ϕ be the mapping

$$\begin{aligned} \phi\colon\ & P \to (12), \\ & Q \to (13), \\ & R \to (14), \\ & S \to (35), \\ & T \to (36). \end{aligned}$$

Since Shephard's relations hold in the image set, and these transpositions generate S_6 we have a homomorphism of N onto S_6 with kernel A of order 64.

To show that A is invariant under automorphisms of N, note that for any other normal subgroup A_1 of order 64, AA_1 is a 2-group, $\dfrac{AA_1}{A}$ is normal in $\dfrac{N}{A} \cong S_6$. So AA_1 can only be A the unique normal subgroup of order 64.

We will show that A/Z_4 is elementary abelian. Assume otherwise. By a theorem on Frattini subgroups $\Phi(A/Z_4) \neq 1$ [13, p. 178]. Then $A/Z_4/\Phi(A/Z_4)$ is elementary abelian of order $\leqq 2^3$. Now let M be a normal subgroup of N so that A is a subgroup of M and $M/A \cong A_6$. Furthermore $\dfrac{M}{C(A/Z_4/\Phi(A/Z_4))}$ is isomorphic to a subgroup of $GL(3, 2)$ whose order is 168. Thus there is an element x in $C(A/Z_4/\Phi(A/Z_4))$ of order 5. According to a theorem of P. Hall [17, p. 275] x can only be the identity. This contradiction establishes our claim.

Now we propose to show that if $x \in \mathrm{Aut}(M/A)$ and $x \neq 1$, then $x \notin C_{\mathcal{N}}(A/Z_4)$ where $\mathcal{N}$ is the normalizer of N. For suppose it is. Since $A \subset C_{\mathcal{N}}(A/Z_4) \lhd \mathcal{N}$

we have

$$\frac{C_{\mathcal{N}}(A/Z_4)\cap M}{A} \lhd \frac{M}{A} \cong A_6.$$

Thus either $C_{\mathcal{N}}(A/Z_4)\cap M \subseteq A$ or $C_{\mathcal{N}}(A/Z_4) \supseteq M$. We see the first case to be impossible because $x \notin A$. The second case is shown to be impossible by noting that M has an element of order 5 which fixes the elements of A/Z_4 as an inner automorphism. By the theorem of P. Hall cited above, this is impossible. So we conclude that $x \notin C_{\mathcal{N}}(A/Z_4)$.

Next we claim that $\left|\frac{\mathcal{N}}{C_{\mathcal{N}}(M/Z)}\right| = |S_6|$. For if not, then $\mathcal{S} = \frac{\mathcal{N}}{C_{\mathcal{N}}(M/A)}$ is the full automorphism group of A_6 having order $4 \cdot |A_6|$. And $\mathcal{N}/C_{\mathcal{N}}(A/Z_4) \cong$ subgroup of $GL(4, 2)$. Consider the Sylow 3-subgroup $P_3 = \langle (123), (456)\rangle$ of S_6 which is elementary abelian of order 9. The normalizer of P_3 is $\langle (1524)(36), (12)\rangle \cdot P_3$ having order $8 \cdot 9$. Then if S_6 is a normal subgroup of K with index 2, $|N_K(P_3)| = 16 \cdot 9$ by the Sylow theorem on the number of Sylow subgroups.

Let $\mathcal{T}$ be a Sylow 3-subgroup of $\mathcal{S}$. If we regard $\mathcal{S}$ as acting as $PSL(4, 2) \cong A_8$, then we have $|N_{\mathcal{S}}(\mathcal{T})| \geqq 2^4 \cdot 3^2$ and $|A_8 : N_{\mathcal{S}}(\mathcal{T})| \leqq 2^2 \cdot 5 \cdot 7$. Considering all the possible divisors d of this index and the requirement that $d \equiv 1 \bmod 3$ we have $d = 1, 7, 2^2, 2 \cdot 5, 2^2 \cdot 7$, or $2 \cdot 5 \cdot 7$. We dismiss each as follows:

1) If 4 does not divide $|A_8 : N_{A_8}(\mathcal{T})|$, then 2^5 divides $|N_{A_8}(\mathcal{T})|$. Since $\frac{N_{A_8}(\mathcal{T})}{C_{A_8}(\mathcal{T})} \cong$ subgroup $GL(2, 3)$ of order 48, $C_{A_8}(\mathcal{T})$ must contain an involution i. But Wong has shown [27, p. 359] that $|C_{A_8}(i)| = 2^6 \cdot 3$ or $2^5 \cdot 3$. So this case cannot occur (i.e., $d \neq 1, 7, 2 \cdot 5, 2 \cdot 5 \cdot 7$).

2) If 7 does not divide the index then there is an element of order 7 which centralizes (123) and (456) and this is impossible in A_8.

3) If 5 does not divide the index, then there is an element of order 5 which centralizes P_3. Again this is impossible.

Thus our assumption that $|\mathcal{S}| = 2|S_6|$ leads to a contradiction so we conclude that $|\mathcal{S}| = |S_6|$.

Finally we apply this result to show that N is self-normalizing by showing that $C_{\mathcal{N}}\left(\frac{M}{A}\right) = A$. Let $H_1 = \frac{C_{\mathcal{N}}\left(\frac{M}{A}\right)}{A} \cap \frac{N}{A}$. Since $H_1 \lhd \frac{N}{A} \cong S_6$ either $H_1 = \frac{M}{A}$ or 1. But $C_{\mathcal{N}}\left(\frac{M}{A}\right) \neq \frac{M}{A}$ since A_6 is not abelian. Thus $H_1 = 1$. Next consider that $\left\langle M, C_{\mathcal{N}}\left(\frac{M}{A}\right)\right\rangle \subseteq N_G(A)$ where A is elementary abelian; we let

$$H_2 = \frac{C_{\mathcal{N}}\left(\frac{M}{A}\right)}{A} \times \frac{M}{A} \cong \text{subgroup of } GL(4, 2) \cong A_8.$$