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Titel: Two Classes of Metabelian p-Groups.

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Jahr: 1969

PURL: https://resolver.sub.uni-goettingen.de/purl?266833020_0112|log8

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Math. Z. 112, 5 – 12 (1969)

Two Classes of Metabelian p-Groups

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1.

Several investigations have been made of the relation between power structure and commutator structure in metabelian groups, especially metabelian p-groups; see for instance Meier-Wunderli [5] and Gupta and Newman [2]. In the present paper we undertake a more detailed study of two classes of nilpotent metabelian p-groups, denoted by $\mathfrak{C}_p$ and $\mathfrak{C}'_p$ respectively. It turns out that these classes have very special properties even though each generates the variety $\mathfrak{M}$ of all metabelian groups.

Let p be a fixed prime. If $p>2$ then we define $\mathfrak{C}_p$ to be the class of all regular metabelian p-groups. That $\mathfrak{C}_p$ generates $\mathfrak{M}$ is proved in [4]. The general resemblance between metabelian p-groups and regular p-groups, such as commutator relations that are adapted to calculation, gives some interest to the investigation of $\mathfrak{C}_p$ which we undertake in § 3. The main result, Theorem 3.1, is a criterion for a nilpotent metabelian p-group to be regular. Since regularity is defined in terms of two-generator subgroups, it is hardly surprising that our criterion is also in terms of two-generator subgroups.

Next we define $\mathfrak{C}'_p$ to be the class of all nilpotent groups of the form AB where A and B are both abelian p-groups. It is a theorem of Itô [3] that ensures that $\mathfrak{C}'_p$ consists of metabelian groups, and the fact that $\mathfrak{C}'_p$ generates $\mathfrak{M}$ is an easy consequence of the proof of the main result in [4]. Our attention was drawn to $\mathfrak{C}'_p$ by Kegel at the International Conference on the Theory of Groups, Canberra, 1965. He raised and has since answered affirmatively two questions about the group AB where A and B are periodic abelian:

(i) Is AB periodic?

(ii) Does A or B contain an element of given prime order if AB does?

It follows from the answer to (i) and by a standard argument with Cartesian products that AB has finite exponent if A and B have. In the case that A and B are abelian p-groups, we are able to give in Theorem 4.3 an explicit bound for the exponent of AB in terms of the exponents of A and B.

Our Corollary 4.2 and Theorem 4.3 below have been improved following a comment of Kegel, whom we thank for his help.

The classes $\mathfrak{C}_p$ and $\mathfrak{C}'_p$ are defined by pleasant properties and in a certain sense form large subclasses of $\mathfrak{M}$. It is therefore mildly surprising that we can

* The second author thanks the University of Dundee for its generosity in respect of finance and hospitality.

find a subvariety $\mathfrak{M}_{p^n}$ of $\mathfrak{M}$ which is not generated by $\mathfrak{C}_p \cap \mathfrak{M}_{p^n}$, and similarly for $\mathfrak{C}'_p$. No doubt such a situation exists for many varieties other than $\mathfrak{M}$ and for many of their subclasses, but to our knowledge none has been explicitly described.

2.

Commutator notation will be standard left-normed. Thus (x_1, x_2) means $x_1^{-1} x_2^{-1} x_1 x_2$, and $(x_1, \ldots, x_n)$ for $n>2$ is defined inductively as $((x_1, \ldots, x_{n-1}), x_n)$. We write $(x_1, (n-1)\, y)$ for $(x_1, \ldots, x_n)$ when $x_2 = \cdots = x_n = y$.

If H is a non-empty subset of the group G then $\langle H \rangle$ is the subgroup generated by H. We define $\gamma_n(G)$ and G^n for each $n \geqq 1$ to be $\langle (x_1, \ldots, x_n) : x_1 \in G, \ldots, x_n \in G \rangle$ and $\langle x^n : x \in G \rangle$ respectively, with the convention $\gamma_1(G) = G$. If $\gamma_2(G)$ is abelian then G is said to be metabelian, and if $\gamma_{n+1}(G) = 1$ then G is said to be nilpotent of class n. If A and B are subgroups of G then AB is the subset $\{ab : a \in A, b \in B\}$; in general AB is not a subgroup, but if A or B is normal in G then AB is a subgroup.

Now let G be a *nilpotent* p-group. We say that G is regular if for each pair x, y of elements of G there exists an element z in $\gamma_2(\langle x, y \rangle)$ for which

$$y^{-p} x^{-p} (xy)^p = z^p.$$

Next we list the standard commutator identities for the variety of metabelian groups.

$$(uv, xy) = (u, y)^v (v, y)\, (u, x)^{vy} (v, x)^y. \tag{2.1}$$

$$\begin{aligned} (x^n, y) &= \prod_{i=1}^{n} (x, y, (i-1)\, x)^{\binom{n}{i}}, \\ (x, y^n) &= \prod_{i=1}^{n} (x, i\, y)^{\binom{n}{i}}, \end{aligned} \tag{2.2}$$

for each $n>0$; here and later $\binom{n}{i}$ denotes the usual binomial coefficient with the convention $\binom{n}{i} = 0$ when $n<i$.

$$(c^m, x_1, \ldots, x_n) = (c, x_1, \ldots, x_n)^m \tag{2.3}$$

for any integer m and any element c in $\gamma_2(G)$.

$$(x, y, z)\, (y, z, x)\, (z, x, y) = 1. \tag{2.4}$$

A useful consequence of (2.4) with (2.3) is:

$$\text{if} \quad (y, z) = 1 \quad \text{then} \quad (x, y, z) = (x, z, y). \tag{2.5}$$

$$(x, y, u, v) = (x, y, v, u). \tag{2.6}$$

We now prove two lemmas which are essentially developments of (2.2).

Lemma 2.1. *Let $H=\langle a, b\rangle$ be a metabelian p-group and let α be such that a^{p^α} and b^{p^α} are central in H. Put $\Gamma(n)=\gamma_n(H)$. Then*

$$\Gamma(k+\alpha p^\alpha-(\alpha-1)\,p^{\alpha-1})\leqq\Gamma(k+1)^{p^\alpha}$$

for each $k\geqq p^\alpha$.

Proof. We have $(b, a^{p^\alpha})=1$, therefore (2.2) gives

$$(b, a)^{p^\alpha}\ldots(b, i\,a)^{\binom{p^\alpha}{i}}\ldots(b, p^\alpha\,a)=1$$

and similarly with a and b interchanged.

Use of (2.3) and elementary properties of binomial coefficients gives

$$\Gamma(k+p^\alpha)\leqq\Gamma(k+1)^{p^\alpha}\ldots\Gamma(k+p^i)^{p^{\alpha-i}}\ldots\Gamma(k+p^{\alpha-1})^p \tag{2.7}$$

for all $k\geqq p^\alpha$.

Next we prove that for $k\geqq p^\alpha$ and $1\leqq i\leqq\alpha$,

$$\Gamma(k+i\,p^\alpha-(i-1)\,p^{\alpha-1})\leqq\Gamma(k+1)^{p^{\alpha+i-1}}\ldots\Gamma(k+p^{\alpha-1})^{p^i}. \tag{2.8}$$

We use induction on i, noting that for $i=1$ (2.8) is just (2.7). Suppose then that (2.8) holds for some $i<\alpha$. Repeated use of (2.3) on (2.8) shows that $\Gamma(k+(i+1)p^\alpha-i\,p^{\alpha-1})$ lies in

$$\Gamma(k+1+p^\alpha-p^{\alpha-1})^{p^{\alpha+i-1}}\ldots\Gamma(k+p^{\alpha-2}+p^\alpha-p^{\alpha-1})^{p^{i+1}}\Gamma(k+p^\alpha)^{p^i}.$$

Since (2.7) implies

$$\Gamma(k+p^\alpha)^{p^i}\leqq\Gamma(k+1)^{p^{\alpha+i}}\ldots\Gamma(k+p^{\alpha-1})^{p^{i+1}},$$

we have

$$\Gamma(k+(i+1)\,p^\alpha-i\,p^{\alpha-1})\leqq\Gamma(k+1)^{p^{\alpha+i}}\ldots\Gamma(k+p^{\alpha-1})^{p^{i+1}}.$$

That proves (2.8) for $k\geqq p^\alpha$ and $1\leqq i\leqq\alpha$.

The case $i=\alpha$ now gives the lemma. //

The defining property of regular p-groups motivates a calculation of $y^{-k}x^{-k}(x\,y)^k$ by commutator collection in a metabelian group. The details follow.

Lemma 2.2. *Let k be a positive integer and let x, y be elements of a metabelian group. Then*

$$y^{-k}x^{-k}(x\,y)^k=\prod_{\substack{i\geqq 1\\ j\geqq 0}}(y, i\,x, j\,y)^{\alpha_{ij}}$$

where

$$\alpha_{ij}=\alpha_{ij}(k)=\sum_{s=1}^{k-1}\binom{k-s}{i}\binom{k-s}{j}.$$

Proof. We use induction on k; when $k=1$ the lemma certainly holds. In general we have $(x\,y)^k = x^k\,y^k\,\xi_k$ where the commutator collection process allows us to assume that ξ_k has the form $\prod_{\substack{i\geqq 1\\ j\geqq 0}} c_{ij}^{\alpha_{ij}}$ where $\alpha_{ij}=\alpha_{ij}(k)$ and $c_{ij}=(y, i\,x, j\,y)$ in the obvious notation. We can then calculate $(x\,y)^{k+1}$ by the commutator rules, especially (2.1):

$$(x\,y)^{k+1} = x^{k+1}\,y^{k+1}(y^k, x)\,(y^k, x, y)\,\xi_k(\xi_k, x)\,(\xi_k, y)\,(\xi_k, x, y).$$

Use of (2.2) allows us to express $\alpha_{ij}(k+1)$ in terms of $\alpha_{ij}(k)$ for $k\geqq 1$ as follows:

$$\begin{aligned}
&\alpha_{1\,0}(k+1)=\alpha_{1\,0}(k)+k,\\
&\alpha_{1\,j}(k+1)=\alpha_{1\,j}(k)+\binom{k}{j}+\binom{k}{j+1}+\alpha_{1,\,j-1}(k) && \text{for } j>0,\\
&\alpha_{i\,0}(k+1)=\alpha_{i\,0}(k)+\alpha_{i-1,\,0}(k) && \text{for } i>1,\\
&\alpha_{ij}(k+1)=\alpha_{ij}(k)+\alpha_{i-1,\,j}(k)+\alpha_{i,\,j-1}(k)+\alpha_{i-1,\,j-1}(k) && \text{for } i>1,\ j>0.
\end{aligned}$$

It is now an easy matter to verify the formula for $\alpha_{ij}(k)$ given in the statement of the lemma. //

Corollary 2.3. *For each prime p the metabelian variety has a law of the form*

$$y^{-p}\,x^{-p}(x\,y)^p = \prod_{i,\,j\geqq 1} (y, i\,x, (j-1)\,y)^{\beta_{ij}}$$

where $i<p$, $j\leqq p$, and

$$\beta_{ij}\equiv 0, \quad \beta_{ij}\equiv(-1)^i, \quad \beta_{ij}\not\equiv 0 \quad \textit{modulo } p$$

in the respective cases

$$i+j<p, \quad i+j=p, \quad p\leqq i+j<2p.$$

Proof. Because of the elementary fact that

$$\binom{p-s}{j-1}\equiv(-1)^{p+s+j+1}\binom{p-j}{s} \qquad \text{modulo } p$$

the formula of Lemma 2.2 gives

$$\beta_{ij}\equiv(-1)^{i+j+1}\sum_{s=1}^{p-1}\binom{p-i-1}{s}\binom{p-j}{s} \qquad \text{modulo } p.$$

Therefore a well known relation in binomial coefficients gives

$$\beta_{ij}\equiv(-1)^{i+j+1}\binom{2p-i-j-1}{p-j} \qquad \text{modulo } p,$$

and the rest is trivial. //

3.

In this section p will usually be an odd prime, though some of our results are trivially true when $p=2$.

Theorem 3.1. *A nilpotent metabelian p-group G is regular if and only if $\gamma_p(H)\leqq\gamma_2(H)^p$ for every two-generator subgroup H of G.*

Proof. If $\gamma_p(H)\leqq\gamma_2(H)^p$ for each H then G is clearly regular, in view of Corollary 2.3 for example. We therefore take a regular metabelian p-group G and put $H=\langle a, b\rangle$ where a, b are two fixed elements of G; and we have to show that $\Gamma(p)\leqq\Gamma(2)^p$ where we define $\Gamma(m)$ to be $\gamma_m(H)$.

If $x=a^\lambda$ where $\lambda\in\{1, \ldots, p-1\}$ and $y=b$ then we have by hypothesis $y^{-p}x^{-p}(xy)^p\in\Gamma(2)^p$. Define d_{ij} by

$$d_{ij}=(b, i\,a, (j-1)\,b).$$

Then by the properties of the β_{ij} specified in Corollary 2.3

$$\prod_{i=1}^{p-1} d_{i,p-i}^{(-1)^i\lambda^i}=1$$

modulo $\Gamma(2)^p\,\Gamma(p+1)$.

We now have $p-1$ equations in $p-1$ symbols $d_{1,p-1}, \ldots, d_{p-1,1}$ over the field of p elements, as is obvious when the equations are written in multiplicative notation. Well known properties of the determinant involved give

$$d_{i,p-i}=1$$

modulo $\Gamma(2)^p\,\Gamma(p+1)$ for $i=1, \ldots, p-1$; this is an easy case of a process described in detail in [1]. The usual identities now give $d_{ij}\in\Gamma(2)^p\,\Gamma(p+1)$ for $i+j\geqq p$, and it is evident that $\Gamma(p)\leqq\Gamma(2)^p\,\Gamma(p+1)$.

We now find that $\gamma_p(H)\leqq\gamma_{p+1}(H)$ if we work in H modulo $\gamma_2(H)^p$. Since H is nilpotent we have $\gamma_p(H)=1$ modulo $\gamma_2(H)^p$. This proves the theorem. //

(In fact each basic commutator of weight p in $\{a, b\}$ lies in the subgroup generated by the p-th. powers of all basic commutators of weights $2, \ldots, p-1$.)

Corollary 3.2. *If G is a regular metabelian p-group then $\gamma_{p+1}(G)\leqq\gamma_2(G)^p$.*

Proof. We work in G modulo $\gamma_2(G)^p$. Then the theorem states that every two-generator subgroup of G has class $p-1$. It is one of the consequences of the theorem of Gupta and Newman in [2] that G has class p, still modulo $\gamma_2(G)^p$. Therefore $\gamma_{p+1}(G)\leqq\gamma_2(G)^p$. //

The obvious converse of this corollary is false, because there is a non-regular metabelian p-group G of class p with elementary abelian commutator subgroup.

Now let G be a regular metabelian p-group. An easy induction based on Corollary 3.2 gives $\gamma_{m(p-1)+2}(G)\leqq\gamma_2(G)^{p^m}$ for $m=1, 2, \ldots$. Therefore if $\gamma_2(G)$ has exponent p^m then G is nilpotent of class $m(p-1)+1$. But it is well known

that the metabelian variety $\mathfrak{M}$ contains wreath products of elementary abelian p-groups which have exponent p^2 and arbitrarily large class. We have therefore proved:

Theorem 3.3. *If $\mathfrak{M}_{p^n}$ is the variety of all metabelian groups of exponent (dividing) p^n then $\mathfrak{C}_p \cap \mathfrak{M}_{p^n}$ does not generate $\mathfrak{M}_{p^n}$ for any $n>1$.* //

4.

We now consider the class $\mathfrak{C}'_p$, with p any prime. It is not at all obvious that if A and B are abelian groups of p-power exponent then AB has p-power exponent, but we shall prove this.

Lemma 4.1. *Let $G=AB$ where A and B are abelian groups of exponent (dividing) p^α. If $H=\langle a, b\rangle$ where $a\in A$ and $b\in B$ then $\gamma_m(H)^{p^\alpha}\leqq\gamma_{2m}(H)$ for each $m>1$.*

Proof. It is sufficient to prove that $(a, b, x_3, \ldots, x_m)^{p^\alpha}\in\gamma_{2m}(H)$ where each x_i is a or b. There are elements $a_m\in A$ and $b_m\in B$ for which

$$(a, b, x_3, \ldots, x_m)=a_m b_m.$$

Then we have

$$\begin{aligned}(a_m b_m, b_m)&=(a, b, x_3, \ldots, x_m, b_m)\\&=(a, b, b_m, x_3, \ldots, x_m)\\&=(a, b_m, b, x_3, \ldots, x_m)\\&=(a, a_m b_m, b, x_3, \ldots, x_m)\end{aligned}$$

by (2.6), (2.5), (2.1) respectively; and it follows that $(a_m b_m, b_m)\in\gamma_{2m}(H)$.

Next we show that $(a_m b_m, k b_m)\in\gamma_{2m}(H)$ for each $k\geqq 1$. We make the inductive hypothesis that for a certain $k\geqq 1$

$$(a_m b_m, k b_m)=(a, a_m b_m, y_1, \ldots, y_{km-1})^{\pm 1}$$

where each y_i is a or b. Then

$$\begin{aligned}(a_m b_m, (k+1) b_m)&=(a, a_m b_m, y_1, \ldots, y_{km-1}, b_m)^{\pm 1}\\&=(a, a_m b_m, b_m, y_1, \ldots, y_{km-1})^{\pm 1}\\&=((a_m b_m, b_m), a, y_1, \ldots, y_{km-1})^{\mp 1}\end{aligned}$$

which we obtain by using (2.3) and (2.6); note carefully that $a_m b_m\in\gamma_2(H)$. The required conclusion for the case $k+1$ follows when the known result for $k=1$ is applied to this. Hence $(a_m b_m, k b_m)\in\gamma_{2m}(H)$.

It is therefore clear from (2.2) that

$$(a_m b_m, b_m^k)\in\gamma_{2m}(H) \qquad \text{for } k=1, 2, \ldots .$$

We now calculate $(a_m b_m)^k$. Suppose inductively that $(a_m b_m)^k$ has the form $a_m^k b_m^k c$ where $c \in \gamma_{2m}(H)$. Then

$$\begin{aligned}(a_m b_m)^{k+1} &= a_m^k b_m^k c\, a_m b_m \\ &= a_m^k b_m^k a_m b_m c \\ &= a_m^{k+1} b_m^{k+1} (b_m^k, a_m b_m)\, c \\ &= a_m^{k+1} b_m^{k+1} c'\end{aligned}$$

where $c' \in \gamma_{2m}(H)$. So we have proved that $b_m^{-k} a_m^{-k} (a_m b_m)^k$ lies in $\gamma_{2m}(H)$ for each $k \geqq 1$.

Finally we have

$$(a, b, x_3, \ldots, x_m)^{p^\alpha} = (a_m b_m)^{p^\alpha} \in \gamma_{2m}(H),$$

and the lemma is proved. //

Corollary 4.2. The group H of the previous lemma is nilpotent of class $2m-1$ where $m = \alpha p^\alpha - (\alpha - 1) p^{\alpha-1}$, and $\gamma_m(H)^{p^\alpha} = 1$.

Proof. Lemma 2.1 gives $\gamma_{2m}(H) \leqq \gamma_{m+1}(H)^{p^\alpha}$ with $m = \alpha p^\alpha - (\alpha - 1) p^{\alpha-1}$, and Lemma 4.1 gives $\gamma_{m+1}(H)^{p^\alpha} \leqq \gamma_{2m+2}(H)$. Therefore $\gamma_{2m}(H) \leqq \gamma_{2m+2}(H)$. Since H is a finitely generated metabelian p-group H is nilpotent and this implies $\gamma_{2m}(H) = 1$.

Next we have $\gamma_m(H)^{p^\alpha} \leqq \gamma_{2m}(H)$ and $\gamma_{2m}(H) \leqq \gamma_{m+1}(H)^{p^\alpha}$ by the same lemmas. Therefore $\gamma_m(H)^{p^\alpha} \leqq \gamma_{m+1}(H)^{p^\alpha}$. In fact this equation holds for *any* $m \geqq \alpha p^\alpha - (\alpha - 1) p^{\alpha-1}$, and the nilpotence of H gives $\gamma_m(H)^{p^\alpha} = 1$ for the above value of m. //

Theorem 4.3. *Let $G = AB$ where A and B are abelian groups of exponent (dividing) p^α. Then G has exponent (dividing) $p^{k\alpha}$ where k is the minimal integer such that $2^{k-1} \geqq \alpha p^\alpha - (\alpha - 1) p^{\alpha-1}$.*

Proof. An inductive argument based on Lemma 4.1 shows easily that $\gamma_2(H)^{p^{(r-2)\alpha}} \leqq \gamma_{2^{r-1}}(H)$ for any $r > 1$ and $H = \langle a, b \rangle$. Corollary 4.2 then gives $\gamma_2(H)^{p^{(k-1)\alpha}} = 1$. Since $(ab)^{p^\alpha} \in \gamma_2(H)$ the theorem follows. //

Corollary 4.4. *Under the conditions of the theorem, G is locally nilpotent.*

Proof. We see that G is a metabelian group of finite exponent. //

Theorem 4.3 does not always give the best possible result for the exponent of AB. For instance, when $\alpha = 1$, Corollaries 2.3 and 4.2 give $(ab)^{p^2} \in \gamma_2(H)^{p^2}$, and expansion of $(a^p, b) = 1$ on the lines of the proof of Lemma 2.1 will give $(a, b)^{p^2} = 1$, and so $(ab)^{p^2} = 1$. The minimal exponent for AB *is* in fact p^2, as suitable examples show.

Finally we consider $\mathfrak{C}'_p$ in relation to subvarieties of $\mathfrak{M}$.

Theorem 4.5. *If $\mathfrak{M}_p$ is the variety of all metabelian groups of exponent p, with $p > 2$, then $\mathfrak{C}'_p \cap \mathfrak{M}_p$ does not generate $\mathfrak{M}_p$.*

Proof. The papers [5] and [2] give the well known facts that the members of $\mathfrak{M}_p$ have class p, the two-generator members have class $p-1$, and these results are best possible. Suppose the member G of $\mathfrak{M}_p$ has the form AB where A and B are abelian. In G we have the relation $(a, k\,b, (p-k-1)\,a)=1$ for each $a\in A$, $b\in B$ and $k\in\{1, \ldots, p-1\}$.

We now replace a by $a_1 \ldots a_{p-k}$, b by $b_1 \ldots b_k$, and expand the resulting commutator identity. The formal details of this process are similar to the proof of Lemma 2.2 in [2], but it should be noted that in place of the fact $a\in\gamma_2(G)$ or $b\in\gamma_2(G)$ we use (2.5) and the fact that A is abelian or B is abelian. Since G is a p-group we find that $(a_1, b_1, a_2, \ldots, a_{p-k}, b_2, \ldots, b_k)=1$ for any $a_1, \ldots, a_{p-k}\in A$ and $b_1, \ldots, b_k\in B$.

Now take $(x_1, x_2, \ldots, x_p)$ where each $x_i\in AB$. The usual identities allow us to expand this commutator, and the resulting commutators with elements in either A or B are trivial, as has just been proved. Therefore G has class $p-1$.

It follows that $\mathfrak{C}'_p \cap \mathfrak{M}_p$ generates a variety of class $p-1$ which is therefore a proper subvariety of $\mathfrak{M}_p$. //

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(Received February 28, 1969)