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A Generalisation of Clifford Algebras

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Introduction

Let K be a field containing a primitive n-th root of unity ω and let $C_m^{(n)}$ be the algebra generated over K by elements $e_1, e_2, \ldots, e_m$ subject to the relations

$$e_i^n = 1, \qquad e_i e_j = \omega e_j e_i, \qquad i, j = 1, 2, \ldots, m$$

Morris [2] gave a complete description of the algebras $C_m^{(n)}$ for odd values of n and for even values of n, under the assumption that ω has a square root in K which is primitive.

In this paper, we describe the algebras $C_m^{(n)}$ when ω does not have a square root in K. For $n=2$, the algebras $C_m^{(2)}$ are Clifford algebras and the proof reduces to the proof given by Atiyah, Bott and Shapiro in [1].

We also give an interpretation of the algebra $C_m^{(n)}$ as an m-fold $(\mathbb{Z}/n\mathbb{Z})$-graded tensor product of the group algebra $K(\mathbb{Z}/n\mathbb{Z})$. In this way, we are able to extend our results to m-fold graded tensor products of the group algebra of a finite abelian group.

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I. G-Graded Algebras

Let K be a commutative field. All algebras will be associative algebras with unit over K and all tensor products are taken over K. Let A be a K-algebra. We shall write A^n for the direct sum of n copies of A.

Let G be a finite abelian group, written additively, and not zero. A *G-graded algebra A* is an algebra A, which as a K-module can be expressed as a direct sum,

$$A = \bigoplus_{g \in G} A_g$$

in such a way that the algebra multiplication maps $A_g \times A_h$ into A_{g+h}, $g \in G$, $h \in G$. The coefficient field K is clearly contained in A_0.

If A and B be two G-graded algebras, a G-graded map $\varphi: A \to B$ will always be a map of degree zero.

(1.1) *Example.* The group algebra KG of G over K is a simple nontrivial example of a G-graded algebra.

Suppose that the field K is G-cyclic. This means that if G has an element of order r, K contains a primitive r-th root of unity. Note that the characteristic of K does not divide the order of G.

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Write the group G as a direct product

$$G \cong \mathbb{Z}/P_1^{r_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/P_n^{r_n}\mathbb{Z} \tag{1.2}$$

where the P_i's are prime numbers such that $P_1 \leqq P_2 \leqq \cdots \leqq P_n$ and the r_i's are positive integers such that $r_k \geqq r_{k+1} \geqq \cdots \geqq r_m > 0$, if $P_k = P_{k+1} = \cdots = P_m$. For each $i = 1, \ldots, n$, let ω_i be a primitive $P_i^{r_i}$-th root of unity in K.

Let $g = (g_1, g_2, \ldots, g_n)$ and $h = (h_1, h_2, \ldots, h_n)$ be two elements of G in the decomposition (1.2). The map

$$(g, h) \mapsto \prod_{i=1}^{n} \omega_i^{g_i h_i} \tag{1.3}$$

defines a pairing $\varphi\colon G \times G \to K^*$ of G into the multiplicative group K^* of the non-zero elements of K. Clearly,

$$\varphi(g_1 + g_2, h) = \varphi(g_1, h)\,\varphi(g_2, h) \quad \text{and} \quad \varphi(g, h_1 + h_2) = \varphi(g, h_1)\,\varphi(g, h_2),\ g_i, h_i \in G.$$

The following property of φ will be used later. Let

$$G = H \times H' \tag{1.4}$$

be a direct product decomposition of G such that $H = \mathbb{Z}/P_1^{r_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/P_l^{r_l}\mathbb{Z}$ and $H' = \mathbb{Z}/P_{l+1}^{r_{l+1}}\mathbb{Z} \times \cdots \times \mathbb{Z}/P_n^{r_n}\mathbb{Z}$. Then,

$$\varphi(h, h') = 1 \qquad \text{for } h \in H \text{ and } h' \in H'.$$

(1.5) *Example.* Let $(\times_g)_{g \in G}$ be the canonical basis of the group algebra KG and let φ be a pairing (1.3). The multiplication given by

$$\times_g \cdot \times_h = \varphi(g, h) \times_{g+h}, \qquad g \in G,\ h \in G$$

defines another G-graded algebra structure on the K-module KG. We shall denote the algebra thus obtained by $K(G, \varphi)$. Observe that φ is a 2-cocycle in $Z^2(G, K^*)$, where G operates trivially on K^*. Hence $K(G, \varphi)$ is a special case of an algebra extension of K by G.

II. Tensor Products of G-Graded Algebras

Let G be a finite abelian group and K a G-cyclic field. A G-graded tensor product $A \hat{\otimes} B$ of two G-graded algebras A and B is the usual tensor product of the K-modules A and B with the grading

$$(A \hat{\otimes} B)_g = \bigoplus_{p+q=g} A_p \otimes B_q \tag{2.1}$$

and the multiplication

$$(a \hat{\otimes} b)(a' \hat{\otimes} b') = \varphi(g, h)\, a a' \hat{\otimes} b b', \qquad b \in B_g \text{ and } a' \in A_h, \tag{2.2}$$

where φ is a pairing $G \times G \to K^*$. In the following, φ will always be a fixed pairing of the form (1.3). One verifies easily that $A \hat{\otimes} B$ is a G-graded algebra. The usual tensor product of algebras will be denoted by $A \otimes B$.

(2.3) **Lemma.** *Let $G=H\times H'$ be a direct product decomposition of the form* (1.4). *Let A_1 and B_1 (resp. A_2 and B_2) be H-(resp. H'-) graded algebra. The (ordinary) tensor products $A_1\otimes A_2$ and $B_1\otimes B_2$ are G-graded and*

$$(A_1\otimes A_2)\hat{\otimes}(B_1\otimes B_2)\cong(A_1\hat{\otimes}B_1)\otimes(A_2\hat{\otimes}B_2);$$

the graded tensor products on the right side are H-(resp. H'-) graded.

Proof. The grading of $A_1\otimes A_2$ is defined by $(A_1\otimes A_2)_g=(A_1)_h\otimes(A_2)_{h'}$ where $g=(h,h')$ is the decomposition of g in $H\times H'$. The two modules are clearly isomorphic. Using $\varphi(h,h')=1$ for $h\in H$ and $h'\in H'$ one easily shows that the two multiplications coincide.

III. Tensor Products of Group Algebras

Let $C_n(G)$ (resp. $C_m(G,\varphi)$) be the m-fold G-graded tensor product of the group algebra KG (resp. the algebra $K(G,\varphi)$) with itself. The map φ is the fixed pairing chosen for the graded tensor product.

(3.1) *Example.* If G is the cyclic group $\mathbb{Z}/n\mathbb{Z}$ and ω a n-th primitive root of unity associated with the decomposition (1.2) of G, the algebra $C_m(\mathbb{Z}/n\mathbb{Z})=C_m^{(n)}$ is the (non-commutative) polynomial algebra generated by elements $e_1,e_2,\ldots,e_m$ subject to the relations

$$e_i^n=1,\qquad e_i\,e_j=\omega\,e_j\,e_i,\qquad i<j,\ i,j=1,2,\ldots,m.$$

The algebra $C_m(\mathbb{Z}/n\mathbb{Z},\varphi)=C_m^{(n)}(\varphi)$ is generated by elements $f_1,f_2,\ldots,f_m$ subject to the relations

$$f_i^n=\omega^{\frac{n(n-1)}{2}},\qquad f_i\,f_j=\omega\,f_j\,f_i,\qquad i<j,\ i,j=1,2,\ldots,m.$$

(3.2) **Lemma.** *The decomposition $G=H\times H'$ given in* (1.4) *induces isomorphisms*

$$C_m(G)\cong C_m(H)\otimes C_m(H'),$$
$$C_m(G,\varphi)\cong C_m(H,\varphi)\otimes C_m(H',\varphi).$$

The proof is immediate by Lemma (2.3) and induction.

(3.3) **Lemma.** *There exist isomorphisms of G-graded algebras*

$$C_{m+2}(G)\cong C_m(G,\varphi)\otimes C_2(G),$$
$$C_{m+2}(G,\varphi)\cong C_m(G)\otimes C_2(G,\varphi).$$

Proof. By Lemma (3.2), we can assume that $G=\mathbb{Z}/n\mathbb{Z}$. We shall use the notations introduced in (3.1). Let $v=e_1^{n-1}e_2$. One verifies that $v^n=\omega^{\frac{-n(n-1)}{2}}$ and that $e_i\,v=\omega\,v\,e_i$, $i=1,2$. Define $\rho\colon C_{m+2}(G)\to C_m(G,\varphi)\otimes C_2(G)$ by

$$\rho(e_i)=\begin{cases}1\otimes e_i, & i=1,2\\ f_{i-2}\otimes v, & i\geqq 3.\end{cases}$$

Using the defining relations of $C_m(G)$ and $C_m(G, \varphi)$, it is easily seen that ρ is a homomorphism of G-graded algebras. Basis elements are mapped onto basis elements and both algebras have the same dimension. Thus ρ is an isomorphism.

The proof of the second isomorphism is similar.

The determination of the algebras $C_m(G)$ and $C_m(G, \varphi)$ is thus reduced to the cases $m=1$ and $m=2$. Before describing these algebras, we shall note that in some cases KG and $K(G, \varphi)$ are isomorphic as G-graded algebras.

(3.4) **Lemma.** *If the order n of G is odd or if the order of G is even and the primitive 2^{r_1}-th root of unity ω_1, associated with the decomposition* (1.2) *of G, has a square root in K, then the algebras KG and $K(G, \varphi)$ are isomorphic as G-graded algebras.*

Proof. Let ω_i, $i=1, 2, \ldots, k$, be the primitive 2^{r_i}-th roots of unity associated with the decomposition (1.2) of G. If ω_1 has a square root in K, then the same is true for ω_i, $i=2, 3, \ldots, k$, since $r_i \leqq r_1$, for $i=2, 3, \ldots, k$. Hence it is sufficient to prove the lemma for $G=\mathbb{Z}/P^r\mathbb{Z}$, P a prime. Let ω be a primitive P^r-th root of unity. By assumption, ω has a square root ξ in K if $P=2$. The map $\psi: KG \to K(G, \varphi)$ given by

$$\psi(e_1)=\begin{cases} \omega^{\frac{n-1}{2}} f_1 & \text{if } P \text{ is odd} \\ \omega^{\frac{n}{2}} \xi f_1 & \text{if } P \text{ is even} \end{cases}$$

defines an isomorphism of G-graded algebras.

(3.5) **Proposition.** *Let n be the order of the group G. The algebra $C_1(G)=KG$ is isomorphic to the direct sum K^n of n copies of K and $C_2(G)$ is isomorphic to the algebra $K(n)$ of $n \times n$-matrices over K.*

Proof. By Lemma (3.2) we may assume that G is cyclic. Proposition (3.5) then reduces to Lemmas 1 and 2 of Morris [2].

Combining (3.4) and (3.5) we obtain a description of $C_1(G, \varphi)$ and $C_2(G, \varphi)$ if the order n of G is odd or if n is even and the primitive root ω_1 has a a square root in K. If ω_1 does not have a square root in K, the structure of $C_1(G, \varphi)$ and $C_2(G, \varphi)$ is more complicated. We shall first study the case $G=\mathbb{Z}/2^r\mathbb{Z}$.

Let ω be the 2^r-th primitive root of unity associated with G and let K_C be the extension $K(\xi)$ of K such that $\xi^2=\omega$. Observe that the characteristic of K is not 2, since K is G-cyclic. Let K_H be the algebra generated over K by elements ξ and η, subject to the relations

$$\xi^2=\omega, \eta^2=\omega \quad \text{and} \quad \xi\eta=-\eta\xi.$$

The algebra K_H is an immediate generalisation of the quaternions.

The following well known isomorphisms will be used later,

$$
\begin{aligned}
&K_C \otimes K_C \cong K_C \oplus K_C,\\
&K_C \otimes K_H \cong K_C(2),\\
&K_H \otimes K_H \cong K(4),\\
&K(n) \otimes K(m) \cong K(n\,m)
\end{aligned}
\tag{3.6}
$$

$K(n)$ is the algebra of $n \times m$-matrices over K.

(3.7) **Proposition.** *Let $G = \mathbb{Z}/2^r\mathbb{Z}$ and suppose that the 2^r-th primitive root of unity ω associated with G does not have a square root in K. Then $C_1(G, \varphi)$ is isomorphic to the direct sum of 2^{r-1} copies of K_C and $C_2(G, \varphi)$ is isomorphic to the matrix algebra $K_H(2^{r-1})$.*

Proof. We shall use the notations of (3.1) and write q for 2^{r-1}. The elements

$$u_j = \frac{1}{q} \sum_{n=0}^{q-1} \omega^{n(2j-1)} f_1^{2n}, \qquad j = 1, 2, \ldots, q,$$

are orthogonal idempotents of $C_1(G, \varphi)$ such that $\sum_{j=1}^{q} u_j = 1$.

Put

$$v_j = \frac{\omega^j}{q} \sum_{n=0}^{q-1} \omega^{n(2j-1)} f_1^{2n+1}, \qquad j = 1, 2, \ldots, q.$$

The following relations are easily verified:

$$u_i v_j = \delta_{ij} v_j \quad \text{and} \quad v_i v_j = \omega\, \delta_{ij} u_j, \qquad i, j = 1, 2, \ldots, q, \tag{3.8}$$

hence the first part of the proposition.

To prove the second assertion, we show that $C_2(G, \varphi) \cong K_H \otimes K(q)$.

Using (3.1), one shows that

$$f_2^i u_j = u_{j-i} f_2^i \quad \text{and} \quad f_2^i v_j = v_{j-i} f_2^i, \qquad \text{if } i < j,\ i, j = 1, 2, \ldots, q, \tag{3.9a}$$

$$f_2^i u_j = u_{q+j-i} f_2^i \quad \text{and} \quad f_2^i v_j = -v_{q+j-i} f_2^i, \qquad \text{if } i \geqq j,\ i, j = 1, 2, \ldots, q. \tag{3.9b}$$

It then follows easily that the elements $\xi = \sum_{j=1}^{q} v_j$ and $\eta = \xi f_2^q$ define an algebra isomorphic to K_H.

Put now

$$E_{ij} = \begin{cases} u_i f_2^{j-i}, & \text{if } i \leqq j,\ i, j = 1, 2, \ldots, q,\\ u_i f_2^{4q+j-i}, & \text{if } i > j,\ i, j = 1, 2, \ldots, q. \end{cases}$$

Using (3.9) one computes easily that $E_{ij} E_{mn} = \delta_{jm} E_{in}$. Furthermore ξ and η commute with the E_{ij}'s. Let S be the set $\{E_{ij}, \xi E_{ij}, \eta E_{ij}, \xi \eta E_{ij} | i, j = 1, 2, \ldots, q\}$. By reason of dimension the proof will be complete if we show that the elements

of S are linearly independent. Consider the subsets $S_r = \{x \in S | j-i=r\}$ of S. Then $S = \bigcup_{r=1-q}^{q}$ and $S_r \cap S_t = \emptyset$ if $r \neq t$. The set S_r is the union of all elements $u_j f_2^r$ and $v_j f_2^r$, $j=1, 2, \ldots, q$, which are linearly independent. Hence the elements of S are linearly independent.

Suppose now that G is a group of even order $n=2q$ and that the primitive 2^{r_1}-th root ω_1 does not have a square root in K. Suppose that G has l direct summands of order 2^{r_1}. If the group G has direct summands of order 2^{r_i}, $r_i < r_1$, the corresponding roots of unity ω_i will have square roots in K. Hence, using (3.5), (3.6) and (3.7), we obtain the following result.

(3.10) **Proposition.** *Under the above hypotheses, $C_1(K, \varphi)$ is isomorphic to a direct sum of q copies of K_C, and $C_2(K, \varphi)$ is isomorphic to $K_H(q)$ if l is odd and to $K(n)$ if l is even.*

Lemma (3.3) can now be used to determine all algebras $C_m(G)$ and $C_m(G, \varphi)$ recursively. The computation is straightforward and we shall not give the details. We shall only remark that three different cases appear:

A) The order n of the group G is odd or n is even and the primitive 2^{r_1}-th root of unity ω_1 has a square root in K. The algebras $C_m(G)$ and $C_m(G, \varphi)$ are isomorphic and $C_{m+2}(G) \cong C_m(G) \otimes K(n)$.

B) The order n of the group G is even, $n=2q$, the primitive 2^{r_1}-th root ω_1 does not have a square root in K and the number l of direct summands of G of order 2^{r_1} is even. The algebras $C_m(G)$ and $C_m(G, \varphi)$ have a periodicity four, $C_{m+4}(G) \cong C_m(G) \otimes K(n^2)$ and $C_{m+4}(G, \varphi) \cong C_m(G, \varphi) \otimes K(n^2)$.

C) The order of G is even, $n=2q$, ω_1 does not have a square root in K and the number l of direct summands of G of order 2^{r_1} is odd. The periodicity is eight, $C_{m+8}(G) = C_m(G) \otimes K(n^4)$ and $C_{m+8}(G, \varphi) \cong C_m(G, \varphi) \otimes K(n^4)$.

The periodicity in A) and C) is the same as for Clifford algebras (see [1]).

Added in Proof: After completion of this paper, a continuation of [2] appeared in Quart. J. Math. Oxford Ser. **19**, 289–299. It contains the case not dealt with in [2]. However the presentation of the results is different.

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