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Homeomorphs of Three Subspaces of $\beta\mathbf{N}\setminus\mathbf{N}$

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The theme is this: that each of the three subspaces in question – called $\beta\mathbf{N}\setminus\mathbf{N}$, Ω, and Λ_π – has many homeomorphs, some of them familiar. Some corollaries make the project worthwhile.

The use of the continuum hypothesis in the proof of a result is indicated by the symbol [CH] in its statement. We consider only completely regular Hausdorff spaces, and we denote the ring of real-valued continuous functions on X by the symbol $C(X)$. Other terms and symbols are used as in the Gillman-Jerison textbook [3].

§ 1. The Space $\beta\mathbf{N}\setminus\mathbf{N}$

1.1 Theorem. [CH] *Let X be a zero-dimensional space for which $|C(X)|=\mathfrak{c}$, and let Z be any nonempty zero-set in βX such that $Z\cap X=\emptyset$. Then Z is homeomorphic with $\beta\mathbf{N}\setminus\mathbf{N}$.*

For any such space Z is a compact zero-dimensional F-space without isolated points for which $|C(Z)|=\mathfrak{c}$, and each nonempty zero-set in such a space has nonempty interior. Parovičenko's [4] shows that such a space is homeomorphic with $\beta\mathbf{N}\setminus\mathbf{N}$.

§ 2. The Space Ω

For D the discrete set with $|D|=\aleph_1$, let Ω be the subset of $\beta D\setminus D$ consisting of all the elements in the closure (in βD) of some countable subset of D. (Ω is one of the spaces called E_0 in [1].)

2.1 Theorem. (a) *Ω can be expressed in the form $\Omega=\bigcup_{\alpha<\omega_1}\Omega_\alpha$, where each Ω_α is a homeomorph of $\beta\mathbf{N}\setminus\mathbf{N}$, open-and-closed in Ω, with $\Omega_\alpha\supsetneq\bigcup_{\gamma<\alpha}\Omega_\gamma$.*

(b) [CH] *Up to homeomorphism, Ω is the only such space.*

The proof of (b) follows from a theorem of M. E. Rudin in [5], based on [6], according to which each homeomorphism between two proper cozero-sets in $\beta\mathbf{N}\setminus\mathbf{N}$ extends to a homeomorphism of $\beta\mathbf{N}\setminus\mathbf{N}$ onto itself.

2.2 Corollary. [CH] *Let Z^0 denote the interior (in $\beta\mathbf{N}\setminus\mathbf{N}$) of any nonopen zero-set of $\beta\mathbf{N}\setminus\mathbf{N}$, and let p be a P-point of $\beta\mathbf{N}\setminus\mathbf{N}$. Then the spaces Z^0, $\beta\mathbf{N}\setminus\mathbf{N}\setminus\{p\}$, Ω are homeomorphic.*

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(That Z^0 can be expressed in the form given in 2.1(a) follows from the fact that $\beta\mathbf{N}\setminus\mathbf{N}$ is a zero-dimensional F-space with $|C(\beta\mathbf{N}\setminus\mathbf{N})|=\mathfrak{c}$.)

An application of this corollary is W. Rudin's theorem in [6] that if p and q are P-points of $\beta\mathbf{N}\setminus\mathbf{N}$, then there is a homeomorphism h of $\beta\mathbf{N}\setminus\mathbf{N}$ onto itself for which $h(p)=q$ and $h^2=$ identity. (This theorem follows easily from 2.2, but we cannot and do not claim to have given a new proof of Rudin's theorem: Our Theorem 2.1, after all, itself depends on the combinatorial Lemma 4.8 of [6].)

2.3 Theorem. [CH] *For each $p\in\beta\mathbf{N}\setminus\mathbf{N}$ there is an open-and-closed copy of Ω in $\beta\mathbf{N}\setminus\mathbf{N}\setminus\{p\}$.*

Proof. If p is a P-point, we take $\beta\mathbf{N}\setminus\mathbf{N}\setminus\{p\}$ itself as the desired copy of Ω. If p is not a P-point of $\beta\mathbf{N}\setminus\mathbf{N}$, then there is a zero-set Z of $\beta\mathbf{N}\setminus\mathbf{N}$ with p in the boundary BdZ of Z. Let $\{Z_\alpha\}_{\alpha<\omega_1}$ be a family of zero-sets of $\beta\mathbf{N}\setminus\mathbf{N}$ such that $Z_\alpha\supset Z_\gamma$ for $\alpha<\gamma$, $\bigcap_{\alpha<\lambda} Z_\alpha=Z_\lambda$ for every limit ordinal λ, and $\bigcap_{\alpha<\omega_1} Z_\alpha=\{p\}$. We choose for S_1 any nonempty open-and-closed subset of $Z^0\cap Z_1^0$, and inductively we suppose that we have defined, for each $\gamma<\alpha$, open-and-closed subsets S_γ of $\beta\mathbf{N}\setminus\mathbf{N}$ such that $\bigcup_{\delta<\gamma} S_\delta\subsetneq S_\gamma\subset Z^0$ and $S_\gamma\setminus\bigcup_{\delta<\gamma} S_\delta\subset Z_\gamma$. We now define S_α.

Case 1. $\alpha=\gamma+1$. Notice that the set $(Z^0\cap Z_\alpha^0)\setminus S_\gamma$ is an open subset of Z^0, nonempty since it contains p in its closure. Let T_α be any nonempty open-and-closed subset of $(Z^0\cap Z_\alpha^0)\setminus S_\gamma$, and let $S_\alpha=S_\gamma\cup T_\alpha$.

Case 2. α is a limit ordinal. Defining $\tilde{S}_\alpha=\bigcup_{\gamma<\alpha} S_\alpha$, we see that the relation

$$\bigcup_{\gamma<\alpha}(C_\gamma\setminus S_\gamma)=C_\alpha\setminus\tilde{S}_\alpha$$

expresses $C_\alpha\setminus\tilde{S}_\alpha$ as the countable union of cozero-sets in $\beta\mathbf{N}\setminus\mathbf{N}$. Thus, with $C=(\beta\mathbf{N}\setminus\mathbf{N})\setminus Z$, it follows that $(C_\alpha\setminus\tilde{S}_\alpha)\cup C$ and $\tilde{S}_\alpha$ are disjoint cozero-sets in $\beta\mathbf{N}\setminus\mathbf{N}$. We choose for S_α any open-and-closed subset of $\beta\mathbf{N}\setminus\mathbf{N}$ which contains $\tilde{S}_\alpha$ and which is disjoint from $(C_\alpha\setminus\tilde{S}_\alpha)\cup C$. It is routine to verify that the set $\bigcup_{\alpha<\omega_1} S_\alpha$, which is clearly a homeomorph of Ω, is open-and-closed in $\beta\mathbf{N}\setminus\mathbf{N}\setminus\{p\}$.

2.4 Corollary. (Fine and Gillman [1]; Gillman [2].) [CH] *If $p\in\beta\mathbf{N}\setminus\mathbf{N}$, then $\beta\mathbf{N}\setminus\mathbf{N}\setminus\{p\}$ is not C^*-embedded in $\beta\mathbf{N}\setminus\mathbf{N}$.*

Proof. Since Ω is dense in the space $\beta D\setminus D$ described above, there is a continuous map from $\beta\Omega$ onto $\beta D\setminus D$. Thus $|\beta\Omega\setminus\Omega|\geqq 2^{2^{\aleph_1}}$, so that surely $\beta\Omega\setminus\Omega$ is not a single point.

2.5 Corollary. [CH] *For each p in $\beta\mathbf{N}\setminus\mathbf{N}$ there is a copy of $\beta\mathbf{N}\setminus\mathbf{N}$ contained in $\beta\mathbf{N}\setminus\mathbf{N}$ relative to which p is a P-point.*

The corollary which follows has been communicated to us (implicitly in a letter of March 21, 1966, and explicitly in a colloquium at Wesleyan University on March 11, 1967) by Gillman. The proof, also, is essentially Gillman's, though it depends formally on 1.1 as well as on 2.4.

2.6 Corollary. (Gillman.) [CH] *If p is a non-P-point of $\beta\mathbf{N}\backslash\mathbf{N}$, then $\beta\mathbf{N}\backslash\mathbf{N}\backslash\{p\}$ is not normal.*

Proof. Let Z be a zero-set in $\beta\mathbf{N}\backslash\mathbf{N}$ with $p\in Bd\,Z$, and set $C=\beta\mathbf{N}\backslash\mathbf{N}\backslash Z$. It is well-known (see 14**0**.1 and Theorem 14.25 of [3], for example) that C is C^*-embedded in $\beta\mathbf{N}\backslash\mathbf{N}$, so from 1.1 above we have: $\beta C\backslash C$, which is just $Bd\,Z$, is homeomorphic to $\beta\mathbf{N}\backslash\mathbf{N}$. From the Fine-Gillman theorem (Corollary 2.4 above) it follows that $Bd\,Z\backslash\{p\}$ is not C^*-embedded in $Bd\,Z$.

Suppose now that $\beta\mathbf{N}\backslash\mathbf{N}\backslash\{p\}$ is normal, and let $f\in C^*(Bd\,Z\backslash\{p\})$. Then by Tietze's theorem f extends continuously to a function in $C^*(\beta\mathbf{N}\backslash\mathbf{N}\backslash\{p\})$, whose restriction to C extends continuously to p. Thus from the normality of $\beta\mathbf{N}\backslash\mathbf{N}\backslash\{p\}$ we have deduced that $Bd\,Z\backslash\{p\}$ is C^*-embedded in $Bd\,Z$, a contradiction.

Gillman has asked whether $\beta\mathbf{N}\backslash\mathbf{N}\backslash\{p\}$ is normal when p is a P-point of $\beta\mathbf{N}\backslash\mathbf{N}$; Fine and Gillman have asked implicitly in [1] whether E_0 is a normal space. Though we cannot answer either of these questions we observe that, assuming the continuum hypothesis, they are equivalent for the case $|D|=\aleph_1$, by virtue of Theorem 2.1.

§ 3. The Space Λ_π

The space Λ is 2^{ω_1}, the lexicographically ordered set of all $\{0,1\}$-sequences indexed by ω_1, endowed with the order topology. For any topological space X, we denote by $\mathbf{P}(X)$ the set of P-points of X in its induced topology, and by X_π the set X itself with coarsest P-space topology containing the original topology on X. (X_π has as a base the collection of subsets of X which are G_δ sets in the original topology.) For each X, the two spaces X and X_π induce the same topology on $\mathbf{P}(X)$. When $X=\Lambda$, the space $\mathbf{P}(X)$ is in the terminology of [4] the set of all non-kappa-points of Λ, where the kappa-points of Λ may be characterized as those sequences $x=(x_\alpha)$ in Λ for which there is an index γ for which $x_\alpha=x_\gamma$ whenever $\alpha>\gamma$ and the smallest such γ is a limit ordinal. It is shown in [4] that the spaces $\mathbf{P}(\Lambda)$, $\mathbf{P}(\beta\mathbf{N}\backslash\mathbf{N})$, $(\beta\mathbf{N}\backslash\mathbf{N})_\pi$ are homeomorphic.

3.1 Theorem. *Let X be a P-space without isolated points and suppose that X has a basis $\mathscr{B}=\bigcup_{\alpha<\omega_1}\mathscr{B}_\alpha$ such that*

(i) $|\mathscr{B}|\leqq\mathfrak{c}$;

(ii) *each $\mathscr{B}_\alpha$ is a cover of X by disjoint open-and-closed sets;*

(iii) *if for each $\alpha\in I$ a set B_α is chosen in $\mathscr{B}_\alpha$ so that $\{B_\alpha\colon \alpha\in I\}$ has the finite intersection property, where I is either the set of all countable ordinals or one of its countable subsets, then $\bigcap_{\alpha\in I}B_\alpha\neq\emptyset$.*

Then X is homeomorphic to Λ_π.

That Λ_π has such a basis is seen as in [4]. The proof of the theorem is achieved by showing that the covers may be chosen to satisfy also these conditions:

(iv) if $\alpha<\gamma<\omega_1$, then $\mathscr{B}_\gamma$ refines $\mathscr{B}_\alpha$;

(v) if $\alpha<\gamma<\omega_1$ and $B\in\mathscr{B}_\alpha$, then B contains $\mathfrak{c}$ distinct elements of $\mathscr{B}_\gamma$.

It can be verified without difficulty that conditions (i) through (iv) are satisfied by the family $\mathscr{A} = \bigcup_{\alpha<\omega_1} \mathscr{A}_\alpha$, where by definition

$$\mathscr{A}_\alpha = \{\bigcap_{\gamma<\alpha} B_\gamma : B_\gamma \in \mathscr{B}_\gamma \text{ and } \{B_\gamma\}_{\gamma<\alpha} \text{ has the finite intersection property}\}.$$

To construct from $\mathscr{A}$ a family satisfying (i) through (v) we use the following result, whose proof employs a tedious but familiar process involving dyadic sequences: If $\mathscr{A}$ satisfies (i) through (iv) and $A \in \mathscr{A}$, then there exists $\alpha<\omega_1$ such that A can be expressed as the union of exactly $\mathfrak{c}$ elements of $\mathscr{A}_\alpha$.

To complete the proof of the theorem we have to show that two spaces X and Y with bases $\mathscr{B}$ and $\mathscr{C}$ respectively, both satisfying conditions (i) through (v), are homeomorphic. This is achieved by defining a one-to-one order- and intersection-preserving set function ψ from $\mathscr{B}$ onto $\mathscr{C}$ for which $\psi(\mathscr{B}_\alpha)=\mathscr{C}_\alpha$ for each α, and defining the required homeomorphism by the rule

$$h(\bigcap_{\alpha<\omega_1} B_\alpha) = \bigcap_{\alpha<\omega_1} \psi(B_\alpha).$$

We note that the proof of 3.1 does not use the continuum hypothesis. Without the continuum hypothesis, however, we know of no familiar homeomorphs of Λ_π.

3.2 Theorem. [CH] *Let X be compact. Then X_π is homeomorphic to Λ_π if and only if $|C(X)|=\mathfrak{c}$ and no point of X is a G_δ.*

The implication "if" is achieved by applying the homeomorphism Theorem 3.1 to the covers $\mathscr{B}_\alpha=\{f_\alpha^{-1}(r): r \text{ is a real number}\}$, where $C(X)=\{f_\alpha\}_{\alpha<\omega_1}$. For the implication "only if" we show that the points of X are distinguished by a suitably defined collection of $\mathfrak{c}$ zero-sets, so that X embeds into the product $[0,1]^{\mathfrak{c}}$.

3.3 Corollary. [CH] *Let X be a locally compact, realcompact space that is not compact, and let $|C(X)|=\mathfrak{c}$. Then $(\beta X\setminus X)_\pi$ is homeomorphic to Λ_π.*

3.4 Theorem. [CH] *If X is a separable noncompact metric space, then $(\beta X\setminus X)_\pi$ is homeomorphic to Λ_π.*

Proof. Let Y be a metric compactification of X, and h the natural mapping from βX onto Y. Then $h(\beta X\setminus X)$ is $Y\setminus X$, a nonempty set of cardinality at most $\mathfrak{c}$. Thus $\beta X\setminus X$ can be expressed as the disjoint union of at most $\mathfrak{c}$ zero-sets of βX. The fact that Λ_π is homeomorphic with the discrete union of $\mathfrak{c}$ copies of Λ_π, which follows from 3.1, now yields the theorem.

The hypothesis of local compactness, absent in the theorem just given, cannot be omitted from the next result since $\beta\mathbf{Q}\setminus\mathbf{Q}$ (where $\mathbf{Q}$ denotes the space of rational numbers), for example, is a separable space without isolated points, so that $\mathbf{P}(\beta\mathbf{Q}\setminus\mathbf{Q})=\emptyset$.

3.5 Theorem. [CH] *Let X be a locally compact, noncompact space such that $|C(X)|=\mathfrak{c}$, and suppose either that X is σ-compact or that X is a realcompact F-space. Then $\mathbf{P}(\beta X\setminus X)$ is homeomorphic to Λ_π.*

Recalling from [6] the fact that $\mathbf{P}(\beta X\backslash X)$ is a dense subset of $\beta X\backslash X$ without isolated points, the reader will easily prove Theorem 3.5 on the basis of 3.1 and the following lemma.

3.6 Lemma. [CH] *If Z^0 is the interior of the zero-set Z in Y, where Y is a compact F-space for which $|C(Y)|=\mathfrak{c}$, then Z^0 is the union of pairwise disjoint zero-sets of Y.*

Proof. We write $Z^0=\bigcup_{\alpha<\omega_1} S_\alpha$, where each S_α is a cozero-set in Y and $S_\alpha=\emptyset$ for each limit ordinal α. Set $Z_0=\emptyset$, $C_1=S_1$. Since Y is an F-space, we have $cl_Y C_1\subset Z^0$. We choose for Z_1 any zero-set such that $C_1\subset Z_1\subset Z^0$, and inductively we suppose that for all $\gamma<\alpha$ we have defined cozero-sets C_γ and zero-sets Z_γ such that

(i) if $\delta<\gamma<\alpha$, then $C_\delta\subset Z_\delta\subset C_\gamma\subset Z_\gamma\subset Z^0$; and

(ii) if $\gamma<\alpha$, then $C_\gamma\supset S_\gamma$.

We now define C_α and Z_α.

Case 1. $\alpha=\gamma+1$. Because Y is an F-space we can find a cozero-set C_α for which $S_\gamma\cup Z_\gamma\subset C_\alpha\subset Z^0$. For Z_α we choose any zero-set such that $C_\alpha\subset Z_\alpha\subset Z^0$.

Case 2. α is a limit ordinal. Notice that $\bigcup_{\gamma<\alpha} C_\gamma=\bigcup_{\gamma<\alpha} Z_\gamma$, and set $C_\alpha=\bigcup_{\gamma<\alpha} C_\gamma$. Again we let Z_α be any zero-set for which $C_\alpha\subset Z_\alpha\subset Z^0$.

It can be verified now that Z^0 is the disjoint union of the sets $C_{\alpha+1}\backslash Z_\alpha$ (with $0\leqq\alpha<\omega_1$) together with the sets $Z_\alpha\backslash C_\alpha$ (with $1\leqq\alpha<\omega_1$). Each of these sets is either a cozero-set or a zero-set. To complete the argument we observe simply that a cozero-set is always the pairwise disjoint union of a family of zero-sets, since

$$\{y\in Y: f(y)\neq 0\}=\bigcup_{r\neq 0}\{y\in Y: f(y)=r\}.$$

A modification of the arguments used above yields the following more general result.

3.7 Theorem. [CH] *Let X be as in* 3.5, *and let A be a set of P-points in $\beta X\backslash X$ with $|A|\leqq\mathfrak{c}$. Then $\mathbf{P}(\beta X\backslash X)\backslash A$ is homeomorphic to Λ_π.*

Applying some of these results we find, for example, that each of the spaces $\mathbf{P}(\beta\mathbf{N}\backslash\mathbf{N})$, $(\beta\mathbf{N}\backslash\mathbf{N})_\pi$, $\mathbf{P}(\beta\mathbf{R}\backslash\mathbf{R})$, $(\beta\mathbf{R}\backslash\mathbf{R})_\pi$, $(\beta\mathbf{Q}\backslash\mathbf{Q})_\pi$ is homeomorphic to Λ_π. It also follows that Λ_π is homeomorphic to the totally ordered space $\mathbf{P}(\Lambda)$. Thus for each space X as in 3.6, the space $\beta X\backslash X$ is realized both as a compactification of the totally ordered space $\mathbf{P}(\Lambda)$ and as a one-to-one continuous image of $\mathbf{P}(\Lambda)$.

An examination of the hypotheses of Theorems 3.3 and 3.5 suggests the following question, which we have not been able to answer: Is every locally compact, realcompact space X for which $|C(X)|=\mathfrak{c}$ necessarily σ-compact? In case Lusin's hypothesis $2^{\aleph_0}=2^{\aleph_1}$ is valid, the discrete space of cardinality $\aleph_1$ furnishes a negative answer (see proof below). Our final result, strictly speaking not part of the subject proper of this paper, answers the analogous question for paracompact spaces.

3.8 Theorem. *The following are equivalent.*

(a) $2^{\aleph_0} < 2^{\aleph_1}$;

(b) *each locally compact, paracompact space X for which $|C(X)| = \mathfrak{c}$ is σ-compact.*

Proof. (a) $\Rightarrow$ (b). It is well-known that each locally compact, paracompact space X may be expressed in the form $X = \bigcup_{\alpha \in A} X_\alpha$, where the sets X_α are pairwise disjoint, σ-compact, open-and-closed subsets of X. If X is not itself σ-compact, so that $|A| \geqq \aleph_1$, then $\mathfrak{c} = |C(X)| \geqq \mathfrak{c}^{\aleph_1} = 2^{\aleph_1}$, contrary to (a).

(b) $\Rightarrow$ (a). With X the discrete space of cardinality $\aleph_1$ we have $|C(X)| = 2^{\aleph_1}$. If (a) fails, so that $\mathfrak{c} = 2^{\aleph_1}$, then X constitutes a counterexample to (b).

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