Werk

Titel: The Group of Conformal Transformations of a Compact Riemannian Manifold.

Autor: Hsiung, Chuan-Chih; Liu, Jong-Diing

Jahr: 1968

PURL: https://resolver.sub.uni-goettingen.de/purl?266833020_0105|log52

Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

Math. Zeitschr. 105, 307–312 (1968)

The Group of Conformal Transformations of a Compact Riemannian Manifold

CHUAN-CHIH HSIUNG* and JONG-DIING LIU

1. Introduction

Let $C(M^n)$, $I(M^n)$ be respectively the groups of conformal transformations and isometries of a Riemannian manifold M^n of dimension $n>2$, $C_0(M^n)$, $I_0(M^n)$ the connected components of the identities of $C(M^n)$, $I(M^n)$, and R_{ijkl} and R_{ij} the Riemann and Ricci tensors of the manifold M^n, respectively.

Recently Barbance [1] obtained the following interesting

Theorem 1. *For a compact Riemannian manifold M^n $(n>2)$ with positive constant scalar curvature R and constant $R^{ij}R_{ij}$, condition*

$$C(M^n)\neq I(M^n) \tag{1.1}$$

implies that M^n is isometric to a sphere.

Theorem 1 is due to Lichnerowicz [5] when the condition (1.1) is replaced by

$$C_0(M^n)\neq I_0(M^n). \tag{1.2}$$

The purpose of this paper is to generalize Theorem 1 to

Theorem 2. *Let M^n $(n>2)$ be a compact Riemannian manifold with positive constant R, and suppose that*

$$a^2P+b(2a+nb)Q=\text{const.}, \tag{1.3}$$

where

$$P=R^{ijkl}R_{ijkl}, \quad Q=R^{ij}R_{ij}, \tag{1.4}$$

and a, b are constant such that

$$a+nb\neq 0. \tag{1.5}$$

Then condition (1.1) implies that M^n is isometric to a sphere.

Theorem 2 is reduced to Theorem 1 for $a=0$, and is due to Hsiung [3] for $b=0$ and the condition (1.1) replaced by (1.2).

We need the following theorem of Obata [6] in our proof of Theorem 2.

Theorem 3. *A compact Einstein manifold M^n of constant R admits a non-constant function ρ such that*

$$\Delta\rho=nR\rho, \tag{1.6}$$

if and only if the manifold M^n is isometric to an n-sphere of radius $1/\sqrt{R}$, where Δ is the Laplace-Beltrami operator.

* The work of this author was partially supported by the National Science Foundation grant GP-4222.

2. Notations and Formulas

Let M^n be a Riemannian manifold of dimension $n>2$ and class C^∞, $\|g_{ij}\|$ the symmetric matrix of the positive definite metric tensor g of M^n, and $\|g^{ij}\|$ the inverse matrix of $\|g_{ij}\|$. Throughout this paper all indices take the values $1, \ldots, n$, and we shall follow the usual tensor convention that indices can be raised and lowered by using g^{ij} and g_{ij} respectively, and that repeated indices imply summation. In this section we shall list some well-known formulas for our latter use; for the details of their derivation see for example [2, pp. 89–90], where the Riemann tensor differs from ours in sign.

Let f be a conformal transformation of M^n. Then the metric tensor g and its transformed tensor g^* are related by

$$f^*(g_{ij})=g^*_{ij}=e^{2\sigma}g_{ij}, \tag{2.1}$$

where σ is any function on the manifold M^n. From Eq.(2.1) we have

$$f^*(g^{ij})=g^{*ij}=e^{-2\sigma}g^{ij}. \tag{2.2}$$

For any tensor on M^n, the corresponding tensor defined by the canonical Riemannian connection corresponding to g^* will be denoted by the same letter with a star. Then

$$e^{-2\sigma}R^*_{ijkl}=R_{ijkl}-g_{il}\sigma_{jk}-g_{jk}\sigma_{il}+g_{ik}\sigma_{jl}+g_{jl}\sigma_{ik}-(g_{il}g_{jk}-g_{ik}g_{jl})\Delta_1\sigma, \tag{2.3}$$

where we have put

$$\sigma_{ij}=\nabla_j\nabla_i\sigma-\nabla_i\sigma\nabla_j\sigma, \qquad \Delta_1\sigma=\nabla^i\sigma\nabla_i\sigma, \tag{2.4}$$

∇ being the operator of covariant derivation of M^n with respect to g.

By means of Eqs.(2.2), (2.3) we can have

$$R^*_{jk}=R_{jk}+(2-n)\sigma_{jk}+g_{jk}[\Delta\sigma+(2-n)\Delta_1\sigma], \tag{2.5}$$

$$R^*=e^{-2\sigma}[R+2(n-1)\Delta\sigma+(n-1)(2-n)\Delta_1\sigma], \tag{2.6}$$

where Δ is given by

$$\Delta\sigma=-\nabla^i\nabla_i\sigma. \tag{2.7}$$

3. Proof of Theorem 2

Due to condition (1.1) we may assume that there is a nonisometric conformal transformation f of the manifold M^n so that §2 can be used here. On the manifold M^n consider the covariant tensor field of order 4 obtained by Hsiung in [4]:

$$W_{ijkl}=a\left[R_{ijkl}-\frac{R}{n(n-1)}(g_{il}g_{jk}-g_{ik}g_{jl})\right]+b\,g_{il}\left(R_{jk}-\frac{R}{n}g_{jk}\right), \tag{3.1}$$

where a and b are constants satisfying (1.5). Substituting Eqs.(2.3), (2.5), (2.6) in the equation corresponding to Eq.(3.1) for $f^*(W_{ijkl})$, we obtain, after an elementary simplification,

$$\begin{aligned} e^{-2\sigma}W^*_{ijkl}=W_{ijkl}+a\Big[&-g_{il}\sigma_{jk}-g_{jk}\sigma_{il}+g_{ik}\sigma_{jl}\\ &+g_{jl}\sigma_{ik}-\frac{2}{n}(g_{il}g_{jk}-g_{ik}g_{jl})(\Delta\sigma+\Delta_1\sigma)\Big]\\ &+b\,g_{il}\Big[(2-n)\sigma_{jk}+\frac{1}{n}(2-n)(\Delta\sigma+\Delta_1\sigma)g_{jk}\Big]. \end{aligned} \tag{3.2}$$

By putting $u=e^{-\sigma}$, *we have*

$$\sigma=-\ln u. \tag{3.3}$$

Substituting Eq.(3.3) in Eqs.(2.4), (2.7) yields immediately

$$\sigma_{jk}=-(\nabla_k\nabla_j u)/u, \quad \Delta_1\sigma=(\Delta_1 u)/u^2, \quad \Delta\sigma=-(\Delta u)/u-(\Delta_1 u)/u^2. \tag{3.4}$$

Therefore Eq.(3.2) becomes, in consequence of Eqs.(3.4),

$$W^*_{ijkl}=u^{-2}\left[W_{ijkl}+\frac{1}{u}t_{ijkl}(u)\right], \tag{3.5}$$

where we have placed

$$\begin{aligned} t_{ijkl}(u)=a\Big[&g_{il}\nabla_k\nabla_j u+g_{jk}\nabla_l\nabla_i u-g_{ik}\nabla_l\nabla_j u-g_{jl}\nabla_k\nabla_i u+\frac{2\Delta u}{n}(g_{il}g_{jk}-g_{ik}g_{jl})\Big] \\ &+b\,g_{il}(n-2)\left(\nabla_k\nabla_j u+\frac{\Delta u}{n}g_{jk}\right). \end{aligned} \tag{3.6}$$

On the other hand, from Eq.(3.1), we have

$$W_{ijkl}W^{ijkl}=a^2P+b(2a+nb)Q-\frac{1}{n}\left[\frac{2a^2}{n-1}+b(2a+nb)\right]R^2, \tag{3.7}$$

which is equal to a positive constant A^2 by the assumption of Theorem 2. Therefore

$$f^*(A^2)=A^{*2}=A^2. \tag{3.8}$$

But from Eq.(3.5) we obtain, by use of Eq.(2.2),

$$A^{*2}=W^*_{ijkl}W^{*ijkl}=u^4\left[A^2+\frac{2}{u}t_{ijkl}(u)W^{ijkl}+\frac{1}{u^2}t_{ijkl}(u)t^{ijkl}(u)\right]. \tag{3.9}$$

Combining Eq.(3.8) with Eq.(3.9) thus gives

$$u^{-3}A^2=uA^2+2t_{ijkl}(u)W^{ijkl}+\frac{1}{u}t_{ijkl}(u)t^{ijkl}(u). \tag{3.10}$$

Now we divide our discussion into two cases.

First Case. $A=0$. In this case, since $\|g_{ij}\|$ and $\|g^*_{ij}\|$ are positive definite, we have

$$W^*_{ijkl}=W_{ijkl}=0, \tag{3.11}$$

which and Eq.(3.5) imply

$$t_{ijkl}(u)=0. \tag{3.12}$$

Multiplying Eqs.(3.1), (3.6) by g^{il}, summing for i, l, and using Eqs.(3.11), (3.12) and condition (1.5), we obtain

$$R_{jk}=\frac{R}{n}g_{jk}, \quad \nabla_k\nabla_j u+\frac{\Delta u}{n}g_{jk}=0, \tag{3.13}$$

the first of which showing that M^n is an Einstein manifold. Since the conformal transformation f is nonisometric, u can not be reduced to a constant.

By applying Ricci identity to $\nabla_i u$, multiplying the resulting equation by g^{ik}, and summing for i, k we have

$$\nabla^i \nabla_j \nabla_i u - \nabla_j \nabla^i \nabla_i u = R_j{}^i \nabla_i u. \tag{3.14}$$

From Eqs.(3.13) it follows that

$$R_j{}^i = g^{ik} R_{jk} = \frac{R}{n} g^{ik} g_{jk} = \delta_j^i \frac{R}{n}, \tag{3.15}$$

$$\nabla^i \nabla_j \nabla_i u = -\nabla^i \frac{\Delta u}{n} g_{ij} = -\nabla_j \frac{\Delta u}{n},$$

where $\delta_j^i = 1$ for $i=j$, and $=0$ for $i \neq j$.

Substituting Eqs.(3.15) in Eq.(3.14) and noticing, due to Eq.(2.7),

$$-\nabla_j \nabla^i \nabla_i u = \nabla_j \Delta u,$$

we obtain

$$\frac{R}{n} \nabla_j u = \frac{n-1}{n} \nabla_j \Delta u,$$

or

$$\nabla_j \left(\frac{R}{n-1} u - \Delta u \right) = 0,$$

since R is constant. Therefore there exists a constant u_0 such that

$$\Delta u = \frac{R}{n-1} (u - u_0), \tag{3.16}$$

so that there is a nonconstant function $u - u_0$ satisfying

$$\Delta (u - u_0) = \frac{R}{n-1} (u - u_0), \tag{3.17}$$

which shows, by Theorem 3, that M^n is isometric to a sphere.

Second Case. $A \neq 0$. We shall show that in this case $u = 1$, so that every conformal transformation f of the manifold M^n is isometric, which contradicts condition (1.1). Thus this case will never occur, and Theorem 2 is proved.

Since R is constant and $n > 2$, it is known that

$$\nabla_i R_{ij} = 0,$$

from which and Eq.(3.1) it is easily seen

$$g_{il} g_{jk} W^{ijkl} = 0, \quad g_{il} \nabla_j W^{ijkl} = 0, \tag{3.18}$$

so that

$$g_{il} (\nabla_k \nabla_j u) W^{ijkl} = \nabla_k [\nabla_j (u g_{il} W^{ijkl})]. \tag{3.19}$$

By means of the well-known theorem of Green that for any vector field ξ on a compact orientable Riemannian manifold M^n,

$$\int_{M^n} \nabla_i \xi^i \, dA_n = 0, \tag{3.20}$$

dA_n being the element of area of the manifold M^n at a point with respect to the tensor g, from Eq.(3.19) we thus obtain

$$\int_{M^n} g_{il}(V_k V_j u) W^{ijkl} dA_n = 0. \tag{3.21}$$

Using Eq.(3.21) and the first equation of (3.18), from Eq.(3.6) hence follows

$$\int_{M^n} t_{ijkl}(u) W^{ijkl} dA_n = 0. \tag{3.22}$$

Since $u>0$, integration of (3.10) over M^n and use of Eq.(3.22) yield immediately

$$\int_{M^n} u^{-3} dA_n \geqq \int_{M^n} u\, dA_n. \tag{3.23}$$

Remark. The equality in (3.23) holds only when

$$\int_{M^n} \frac{1}{u} t_{ijkl}(u)\, t^{ijkl}(u)\, dA_n = 0. \tag{3.24}$$

Eq.(3.24) implies

$$t_{ijkl}(u) = 0, \tag{3.25}$$

which and Eq.(3.10) give $u^{-3}=u$ and therefore $u=1$.

Substituting Eqs.(3.4) in Eq.(2.6) we have

$$R^* = u^2 \left[R + 2(1-n)\frac{\Delta u}{u} + n(1-n)\frac{\Delta_1 u}{u^2} \right]. \tag{3.26}$$

Let p be a real number. Since

$$V^i(u^{p+1} V_i u) = -u^{p+1} \Delta u + (p+1) u^p \Delta_1 u,$$

an application of Green's formula (3.20) yields

$$\int_{M^n} u^{p+1} \Delta u\, dA_n = (p+1) \int_{M^n} u^p \Delta_1 u\, dA_n. \tag{3.27}$$

Multiplying both sides of Eq.(3.26) by u^p, integrating them over M^n, and using Eq.(3.27), we obtain

$$\int_{M^n} R\, u^p\, dA_n = \int_{M^n} R\, u^{p+2}\, dA_n + (2p+2+n)(1-n) \int_{M^n} u^p \Delta_1 u\, dA_n. \tag{3.28}$$

Since $\Delta_1 u \geqq 0$ and R is constant, we have

$$\int_{M^n} u^p\, dA_n \leqq \int_{M^n} u^{p+2}\, dA_n \qquad \text{for } 2p+2+n \geqq 0. \tag{3.29}$$

For $n \geqq 4$, substitution of $p=-1$ and $p=-3$ in Eq.(3.29) gives

$$\int_{M^n} u^{-3}\, dA_n \leqq \int_{M^n} u^{-1}\, dA_n \leqq \int_{M^n} u\, dA_n. \tag{3.30}$$

By comparing (3.30) with (3.23) we obtain

$$\int_{M^n} u^{-3}\, dA_n = \int_{M^n} u\, dA_n, \tag{3.31}$$

and therefore $u=1$ by the above remark.

For $n=3$, putting $p=-1$ in Eq.(3.29) we have

$$\int_{M^n} u^{-1}\, dA_n \leqq \int_{M^n} u\, dA_n. \tag{3.32}$$

By assuming the area of the manifold M^n to be 1, an application of Schwarz inequality to $u^{-\frac{1}{2}}$ and $u^{\frac{1}{2}}$ yields

$$\int_{M^n} u^{-1}\, dA_n \int_{M^n} u\, dA_n \geqq \int_{M^n} dA_n = 1. \tag{3.33}$$

Comparing (3.33) with (3.32) we obtain

$$\int_{M^n} u\, dA_n \geqq 1. \tag{3.34}$$

Since $n=3$, from Eq.(2.1) it follows that

$$|g_{ij}^*| = u^{-6}\,|g_{ij}|,$$

and therefore that

$$dA_n^* = u^{-3}\, dA_n,$$

where dA_n^* denotes the element of area of the manifold M^n at a point with respect to the tensor g^*. Thus

$$\int_{M^n} dA_n^* = \int_{M^n} u^{-3}\, dA_n = \int_{M^n} dA_n = 1, \tag{3.35}$$

since the area of the manifold M^n is fixed. Substitution of Eq.(3.35) in (3.34) gives

$$\int_{M^n} u\, dA_n \geqq \int_{M^n} u^{-3}\, dA_n. \tag{3.36}$$

By comparing (3.36) with (3.23) we obtain Eq.(3.31) and therefore $u=1$ by the above remark. Hence the proof of Theorem 2 is complete.

Bibliography

1. Barbance, C.: Transformations conformes d'une variété riemanniene compacte. C.R. Acad. Sci. Paris **260**, 1547–1549 (1965).
2. Eisenhart, L. P.: Riemannian geometry, 2nd edition. Princeton: Princeton University Press 1949.
3. Hsiung, C. C.: On the group of conformal transformations of a compact Riemannian manifold. Proc. Nat. Acad. Sci. U.S.A. **54**, 1509–1513 (1965).
4. – On the group of conformal transformations of a compact Riemannian manifold. II. Duke Math. J. **34**, 337–341 (1967).
5. Lichnerowicz, A.: Sur les transformations conformes d'une variété riemannienne compacte. C.R. Acad. Sci. Paris **259**, 697–700 (1964).
6. Obata, M.: Certain conditions for a Riemannian manifold to be isometric with a sphere. J. Math. Soc. Japan **14**, 333–340 (1962).

Professor Chuan-Chih Hsiung
Department of Mathematics and Astronomy
Lehigh University
Bethlehem, Pa. 18015, USA

(Received October 19, 1967)