Werk

Label: Periodical issue

Jahr: 1968

PURL: https://resolver.sub.uni-goettingen.de/purl?266833020_0103|log19

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Mathematische Zeitschrift

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Wissenschaftlicher Beirat A. E. Ingham, Cambridge
H. Kneser, Tübingen
W. Magnus, New York
O. Perron, München
G. Pickert, Gießen

Band 103 · Heft 2 · 1968

Springer-Verlag · Berlin · Heidelberg · New York

Math. Z. 4. Januar 1968

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*

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Springer-Verlag

69 Heidelberg 1
Postfach 1780
Fernsprecher 49101
Fernschreibnummer 04-61723

1 Berlin 33
Heidelberger Platz 3
Fernsprecher 830301
Fernschreibnummer 01-83319

Springer-Verlag
New York Inc.
175 Fifth Avenue
New York, N.Y. 10010

THOMAS, E.
Math. Zeitschr. 103, 85–93 (1968)

Vector Fields on Low Dimensional Manifolds

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Received April 25, 1967

1. The Main Results

Let M be a smooth, closed, connected manifold. How many linearly independent tangent vector fields does M have? The maximal number is defined to be the *span* of M. In previous papers [17–19, 21] we have obtained some general results on span M for manifolds of arbitrary dimension. In this note we give an essentially complete computation of span M for orientable manifolds of dimension $\leqq 7$.

The only restrictions we place on M (other than dimension $\leqq 7$) are the following: for some results we assume that M is a *spin* manifold, i.e., that $w_2 M = 0$, where $w_i M \in H^i(M; Z_2)$, $i \geqq 0$, denotes the i^{th} Stiefel-Whitney class. For other results we will need to assume that M satisfies the following condition.

Condition (T). *$H^4(M; Z)$ has no element of order 2.*

By HOPF, span M is positive $\Leftrightarrow$ the Euler characteristic of M, $\chi(M)$, vanishes. Also, for an orientable m-manifold, span $M \geqq m-1 \Leftrightarrow$ span $M = m$, so in our results we omit the case span $M \geqq m-1$. (Notice that span $M = m$ means that M is parallelizable.) Finally we only consider manifolds with dimension $\geqq 3$.

Dimension 3

Theorem 1 (STIEFEL [12]). *Every orientable 3-manifold is parallelizable.*

The proof follows from the fact that $w_2 M = 0$ (WU [23]) and that

$$\pi_2 SO(3) = 0.$$

Dimension 4

From now on our results will frequently involve the following hypothesis.

Condition (P). *There is a class $X \in H^2(M; Z)$ such that*

$$X^2 = P_1 M, \quad X \bmod 2 = w_2 M.$$

Here $P_1 M$ denotes the Pontryagin class.

* Research supported by the National Science Foundation. The author is a Professor in the Miller Institute for Basic Research (Berkeley).

Theorem 2 (HIRZEBRUCH-HOPF [7], MASSEY [10]). *Let M be an orientable 4-manifold with $\chi(M)=0$.*

(a) Span $M \geqq 2 \Leftrightarrow$ *M satisfies Condition (P).*

(b) *M is parallelizable $\Leftrightarrow w_2 M=0$ and $P_1 M=0$.*

We give a short proof in § 3.

Dimension 5

Following KERVAIRE [9] we define

$$\hat{\chi}_2 M=\left(\sum_{i=0}^{2} \dim H_i(M; Z_2)\right) \bmod 2,$$

where $\dim M=5$. Thus $\hat{\chi}_2 M \in Z_2$.

Theorem 3. *Let M be an orientable 5-manifold.*

(a) *Suppose that M is a spin manifold. Then*

$$\text{Span } M \geqq 2 \Leftrightarrow \hat{\chi}_2 M=0.$$

(b) *Suppose that M satisfies Condition (T) and that* Span $M \geqq 2$ *(e.g., see* (a)*). Then*

$$\text{Span } M \geqq 3 \Leftrightarrow M \text{ satisfies Condition } (P).$$

(c) *Suppose that M satisfies Condition (T). Then, M is parallelizable $\Leftrightarrow w_2 M=0$, $P_1 M=0$, and $\hat{\chi}_2 M=0$.*

Dimension 6

Theorem 4. *Let M be an orientable 6-manifold such that $\chi(M)=0$.*

(a) *Every such manifold has* span $\geqq 2$.

(b) Span $M \geqq 3 \Leftrightarrow (w_2 M)^2=0$.

Suppose that M satisfies Condition (T). Then:

(c) Span $M \geqq 4 \Leftrightarrow$ *M satisfies Condition (P).*

(d) *M is parallelizable $\Leftrightarrow w_2 M=0$ and $P_1 M=0$.*

Dimension 7

Theorem 5. *Let M be an orientable 7-manifold.*

(a) *Every such manifold has* span $\geqq 2$.

Suppose that M satisfies Condition (T). Then:

(b) *If M is a spin manifold,* Span $M \geqq 4$.

(c) Span $M \geqq 5 \Leftrightarrow$ *M satisfies Condition (P).*

(d) *M is parallelizable $\Leftrightarrow w_2 M=0$ and $P_1 M=0$.*

2. Some Lemmas

The proofs of our theorems will follow from a succession of lemmas. In brief our method in each case will be to reduce the problem to a question about stable bundles. We will use the following notation. If ξ is a vector bundle over a complex X, we let

$$w_i \xi \in H^i(X; Z_2), \quad P_i \xi \in H^{4i}(K; Z), \quad i \geqq 0,$$

denote respectively the Stiefel-Whitney classes and Pontryagin classes of ξ, and we let (ξ) denote the stable bundle determined by ξ.

Lemma 1. *Let X be a complex of dimension $\leqq 7$ such that $H^4(X; Z)$ has no 2-torsion. Let ξ and η be orientable vector bundles over X. Then, $(\xi)=(\eta) \Leftrightarrow w_2 \xi = w_2 \eta$, $P_1 \xi = P_1 \eta$. In particular, a bundle ξ is stably trivial $\Leftrightarrow w_2 \xi = 0, P_1 \xi = 0$.*

The result is due to DOLD-WHITNEY [3]. (The proof follows by remarking that the stable classifying space B Spin is 3-connected, that $H^4(B \text{ Spin}; Z) \approx Z$ generated by $\frac{1}{2} P_1$, and that $\pi_i(B \text{ Spin}) = 0$ for $5 \leqq i \leqq 7$. Note ([14], § 4).)

Lemma 2. *Let X be a $2n$-complex, $n \geqq 1$, such that $H^{2n}(X; Z)$ has no 2-torsion. Let ξ and η be oriented $2n$-plane bundles over X such that $(\xi)=(\eta)$. Then, ξ and η are isomorphic $\Leftrightarrow \chi(\xi)=\chi(\eta)$, where χ denotes the Euler class.*

The lemma is well-known; we omit the proof.

For a vector bundle ξ we define "span ξ" to be the maximal number of linearly independent cross-sections in ξ.

Lemma 3. *Let X be a $(4k+1)$-complex, $k \geqq 1$, and let ξ be an orientable $(4k+1)$-plane bundle over X such that* span $\xi \geqq 2$. *Suppose that*

$$(*) \qquad Sq^2(u) + w_2(\xi) \cdot u = 0,$$

for all classes $u \in H^{4k-1}(X; Z_2)$. Let η be a $(4k+1)$-plane bundle such that $(\xi)=(\eta)$. Then ξ and η are isomorphic $\Leftrightarrow$ span $\eta \geqq 2$.

The proof is given in § 4.

We will need certain 2-plane and 3-plane bundles over complexes.

Lemma 4. *Let X be a complex. For each class $u \in H^2(X; Z)$ there is an oriented 2-plane bundle $\eta(u)$ such that*

$$P_1 \eta(u) = u^2, \quad w_2 \eta(u) = u \bmod 2.$$

The proof follows at once from the fact that the classifying space $BSO(2)$ is an Eilenberg-MacLane space $K(Z, 2)$.

Let Φ be the (unique) secondary cohomology associated with the relation

$$Sq^2 Sq^2 = 0, \quad \text{on integral classes.}$$

We prove:

Lemma 5. *Let X be a complex of dimension $\leqq 7$ and let $v \in H^4(X; Z)$ be a class such that $Sq^2 v = 0$, $0 \in \Phi(v)$. Then there is an oriented 3-plane bundle $\gamma(v)$ over X such that*

$$P_1 \gamma(v) = 4v, \quad w_2 \gamma(v) = 0.$$

Proof. By HOPF, there is a map f from the 5-skeleton of X to S^4 such that

$$f^* s_4 = v,$$

where s_4 generates $H^4(S^4; Z)$. By STEENROD-ADEM [1, 11], since $Sq^2 v = 0$ and $0 \in \Phi(v)$, f extends to a map $f' : X \to S^4$. But on S^4 there is a 3-plane bundle ω with $P_1 \omega = 4 s_4$, and so $\gamma = f'^* \omega$ gives the desired bundle over X.

3. Proof of Theorems 2—5

In each case the necessity of the given hypothesis is clear. Thus we prove only the sufficiency. (For completeness we give a proof for dim $M = 4$.)

We let ε denote generically the trivial line bundle; for two bundles η, ξ over a space X we let $\eta \oplus \xi$ denote the Whitney sum.

Proof of Theorem 2. (a) Suppose that M satisfies Condition (P). Then by Lemma 4 there is a 2-plane bundle η over M such that

$$P_1 \eta = P_1 M, \quad w_2 \eta = w_2 M,$$

and so by Lemma 1,

$$(\eta \oplus 2\varepsilon) = (\tau_M),$$

where τ_M denotes the tangent bundle of M. But

$$\chi(\eta \oplus 2\varepsilon) = 0, \quad \chi(\tau_M) = 0,$$

and so by Lemma 2, $\eta \oplus 2\varepsilon \approx \tau_M$, which means that Span $M \geqq 2$.

(b) By Lemma 1, τ_M is stably trivial and so by Lemma 2, τ_M is trivial since $\chi(\tau_M) = 0$.

Proof of Theorem 3. (a) This is given in Theorem 1.1 in [18].

(b) As above, there is a 2-plane bundle η over M such that

$$(\eta \oplus 3\varepsilon) = (\tau_M).$$

By hypothesis span $\tau_M \geqq 2$. Trivially span $(\eta \oplus 3\varepsilon) \geqq 2$ and so by Lemma 3,

$$\tau_M \approx \eta \oplus 3\varepsilon,$$

which implies Span $M \geqq 3$. (We need the theorem of WU [23] to show that τ_M satisfies (*) in Lemma 3.)

(c) By Lemma 1, τ_M is stably trivial, and so by Lemma 3 τ_M is trivial, since by Theorem 3(a) span $M \geqq 2$. (See also [2, 15].)

Proof of Theorem 4. (a) This is given in Theorem 1.1 in [17].

(b) This is in [20]. (For a 6-manifold, $w_4 M=(w_2 M)^2$.)

(c) Use Lemmas 1, 2, and 4, as above.

(d) Use Lemmas 1 and 2, as above.

Proof of Theorem 5. (a) This is given in Theorem 1.1 in [17].

(b) Since $w_2 M=0$, it follows by Wu [23] that $w_4 M=0$, and consequently, $P_1 M=4v$ for some class $v\in H^4(M; Z)$. (See [13].) Moreover, $Sq^2 v=0$, again because $w_2 M=0$. Finally, one can show that $0\in\Phi(v)$, by using Theorem 5.1 of [19]. Thus by Lemma 5 there is then a 3-plane bundle γ such that

$$P_1\gamma=P_1 M\,,\qquad w_2\gamma=w_2 M\,,$$

and so by Lemma 1,

$$(\gamma\oplus 4\varepsilon)=(\tau_M)\,.$$

But by ([8], Th. 1.7), this means that $\gamma\oplus 4\varepsilon\approx\tau_M$ (since dim $M=7$), and so Span $M\geqq 4$.

(c) Use Lemmas 1, 4, and ([8], Th. 1.7).

(d) Use Lemma 1 and ([8], Th. 1.7).

4. Proof of Lemma 3

We develop some preliminary material before proving the lemma. Let B_q, $q\geqq 2$, denote the classifying space for oriented q-plane bundles, and B the stable classifying space. Fix k (in Lemma 3) and set $n=4k+1$. Let $p\colon E\to B$ denote the principal fibration with classifying map $w_{n+1}\colon B\to K(Z_2, n+1)$. (Here w_i denotes generically the i^{th} Stiefel-Whitney class of B_n or B, $i\geqq 2$.) The inclusion $B_n\to B$ factors into $p\,q$, where $q\colon B_n\to E$ is a map whose "fiber" [5] is n-connected, and so if X is a complex of dimension $\leqq n$,

$$q_*\colon [X, B_n]\approx[X, E]\,.$$

(We let $[X, Y]$ denote the set of homotopy classes of maps from a space X to a space Y.) Thus, from the point of view of n-plane bundles over an n-complex, we can replace B_n by E. In particular, consider the following commutative diagram:

$$\begin{array}{ccccc} V_{n,2} & \xrightarrow{j} & B_{n-2} & \xrightarrow{\pi} & B_n \\ {\scriptstyle f}\downarrow & & \| & & \downarrow{\scriptstyle q} \\ V & \xrightarrow{l} & B_{n-2} & \xrightarrow{q\circ\pi} & E\,. \end{array}$$

In the diagram π is the natural map with fiber the Stiefel manifold $V_{n,2}$. V denotes the fiber of the composite map $q\circ\pi$. Because the right hand square commutes there is a canonical map f (cf. [20]), making the left hand square

commute. Moreover, one easily shows (by the 5-Lemma) that

$$f_*\colon \pi_i(V_{n,2}) \approx \pi_i(V), \qquad 0 \leqq i \leqq n-1.$$

Thus we have:

(4.1) Let X be a n-dimensional complex, and let ξ be an oriented n-plane bundle over X, thought of as a class in $[X, B_n]$. Then $\operatorname{span} \xi \geqq 2 \Leftrightarrow$ there is a class $\zeta \in [X, B_{n-2}]$ such that

$$(q \circ \pi)_* \zeta = q_* \xi .$$

By naturality the fundamental class of V transgresses to $p^* w_{n-1}$ in $H^n(E; Z_2)$. Let $s\colon Y \to E$ denote the principal fiber space with classifying map $p^* w_{n-1}\colon E \to K(Z_2, n-1)$. There is then a map $t\colon B_{n-2} \to Y$ such that $st = q\pi$, and such that the fiber of t, call it W, is $(n-2)$-connected. Moreover,

$$\pi_{n-1}(W) \approx \pi_{n-1}(V) \approx \pi_{n-1}(V_{n,2}) \approx Z_2 .$$

Let $k \in H^n(Y; Z_2)$ denote the transgression of the fundamental class of W. Thus, k is the Postnikov invariant in dimension n for the fibration $(q\pi)$. (See [6] and [16].)

Suppose now that X is a complex and that $\eta \in [X, E]$ is a class such that $\eta^* p^* w_{n-1} = 0$. Then η lifts to Y. Define

$$k(\eta) = \bigcup g^* k \subset H^n(X; Z_2),$$

where the union is taken over all maps $g\colon X \to Y$ such that $sg = \eta$. One then has the following facts (e.g., see [17]).

(4.2) $k(\eta)$ is a coset in $H^n(X; Z_2)$ of the subgroup consisting of all classes of the form

$$Sq^2(u) + (w_2 \eta) \cdot u ,$$

where u runs over all classes in $H^{n-2}(X; Z_2)$.

(4.3) Suppose that $\dim X \leqq n$. Then η lifts to B_{n-2} (and hence η has span $\geqq 2$) $\Leftrightarrow 0 \in k(\eta)$.

Let $w\colon B \to K(Z_2, n-1)$ be a map representing the class w_{n-1}, and let $v = w \circ p\colon E \to K(Z_2, n-1)$. Then v represents $p^* w_{n-1}$. Set

$$F = \Omega K(Z_2, n+1) \subset E.$$

By definition of E and v it is clear that

$$v(F) = * \in K(Z_2, n-1) .$$

(We take all spaces with basepoint $*$.) Let

$$\mu\colon F \times E \longrightarrow E$$

denote the action map for the fibration p. By definition one sees that

$$v(\mu(x, y))=v(y), \tag{4.4}$$

for all $x\in F$, $y\in E$.

Recall the space Y given above. This is defined to be the space of pairs (e, λ), where $e\in E$ and λ is a path on $K(Z_2, n-1)$ such that $\lambda(0)=*$, $\lambda(1)=v(e)$. Thus F is embedded in Y by the map $i\colon F\to Y$ given by $x\mapsto(x, *)$. Now define

$$\bar{\mu}\colon F\times Y\to Y$$

by

$$\bar{\mu}(x, (e, \lambda))=(\mu(x, e), \lambda).$$

Thus we obtain a commutative diagram

$$\begin{array}{ccc} F\times Y & \xrightarrow{\bar{\mu}} & Y \\ \downarrow{\scriptstyle 1\times s} & & \downarrow{\scriptstyle s} \\ F\times E & \xrightarrow{\mu} & E \end{array} \tag{4.5}$$

such that:

(4.6) The map $F\to Y$, given by $x\to\bar{\mu}(x, *)$, is homotopic to the embedding i given above.

(4.7) The map $Y\to Y$, given by $y\to\bar{\mu}(*, y)$, is homotopic to the identity.

The map $\bar{\mu}$ has the following key property:

$$\bar{\mu}^* k=\iota\otimes 1+1\otimes k, \tag{4.8}$$

where ι denotes the fundamental class of F.

Assuming this for the moment we give the proof of Lemma 3.

Proof of Lemma 3. Let ξ and η be the bundles in the lemma, thought of as classes in $[X, E]$. Since $(\xi)=(\eta)$ by hypothesis, we have $p_*\xi=p_*\eta$, and hence there is a class $u\in H^n(X; Z_2)$ (i.e., a class in $[X, F]$) such that

$$\eta=u\cdot\xi,$$

where the dot indicates the action induced by μ. Since $\operatorname{span}\xi\geqq 2$, $k(\xi)$ consists simply of zero, using (*) in Lemma 3 and (4.2). Let $\varphi\colon X\to Y$ be a map such that $s\varphi=\xi$. Then $\varphi^* k=0$. Let $\psi=u\cdot\varphi$, where the dot here indicates the action induced by $\bar{\mu}$. By diagram (4.5),

$$s\circ\psi=u\cdot\xi=\eta,$$

and so $\psi^* k$ is the sole element in $k(\eta)$. But by (4.8)

$$\psi^* k=u+\varphi^* k=u.$$

Therefore, $\operatorname{span}\eta\geqq 2\Leftrightarrow u=0$, and consequently, $\operatorname{span}\eta\geqq 2\Leftrightarrow\xi\approx\eta$. This completes the proof.

Proof of (4.8). Since F is $(n-1)$-connected we see by (4.6, 7) that we need simply show

$$(*) \qquad i^* k = \iota,$$

in order to prove (4.8).

Let τ: $S^n \to B_n$ denote the tangent bundle for S^n; set $\rho_n = q \circ \tau$: $S^n \to E$. Since τ is stably trivial there is a map f_n: $S^n \to F$ such that $j \circ f_n \simeq \rho_n$, where j: $F \to E$ is the fiber inclusion. Let $g = i \circ f_n$: $S^n \to Y$. Since $s \circ i = j$, we obtain

$$s \circ g = s \circ i \circ f_n = j \circ f_n \simeq \rho_n .$$

Now $n \equiv 1 \bmod 4$ and so Span $S^n = 1$ [4, 22]. Thus, $0 \notin k(\rho_n)$, and consequently, $g^* k \neq 0$. Thus $i^* k \neq 0$ and so $i^* k = \iota$, as claimed. This proves (*) and hence (4.8).

Remark. Since span $S^{4k+1} = 1$, Lemma 3 is complementary to Theorem 1 in [15]. Moreover the method of proof given here is essentially the method used originally to prove ([15], Th. 1). (We take this opportunity to make a correction in [15]. ROITBERG has pointed out that in Theorem 2 of [15], for the case n odd, we also need to assume that $H^n(X; Z) \approx Z$.)

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BORWEIN, D., and F. P. CASS
Math. Zeitschr. 103, 94–111 (1968)

Strong Nörlund Summability

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Received May 17, 1967

1. Introduction

In this paper we give a definition of strong Nörlund summability, and show that in the case of Cesàro summability our definition is equivalent to the standard definition of strong Cesàro summability. We answer such questions as: "If one Nörlund method of summability includes another, is the same true of the associated strong methods?". We establish relations between strong Nörlund, absolute Nörlund and Nörlund summability. For a certain class of Nörlund methods of summability, we construct associated one parameter families of Nörlund methods, the methods in each family increasing in strength as the parameter increases. A special case is the family of Cesàro methods of summability (C, α) with $\alpha > -1$. Finally we consider a method of summability (C^*, α), known to be equivalent to (C, α) for $\alpha > 0$, and use our theorems to show that the associated strong methods of summability are equivalent.

2. Generalities and Definitions

Throughout this paper, H, H_1 etc. will denote positive constants, which will not necessarily take the same value at different occurences.

Suppose throughout that

$$s_n = \sum_{r=0}^{n} a_r, \qquad \varepsilon_n^{\alpha} = \binom{n+\alpha}{n} = \frac{(\alpha+1)\dots(\alpha+n)}{n!}.$$

Given an arbitrary sequence $\{w_n\}$, we define

$$\Delta w_n = w_n - w_{n-1}, \qquad w_{-1} = 0.$$

Let $\{p_n\}$ and $\{q_n\}$ be arbitrary sequences of complex numbers, let

$$P_n = \sum_{r=0}^{n} p_r, \qquad Q_n = \sum_{r=0}^{n} q_r,$$

and assume throughout that P_n and Q_n are non-zero for all values of n. Also let

$$P_n^* = \sum_{r=0}^{n} |p_r|, \qquad Q_n^* = \sum_{r=0}^{n} |q_r|.$$

The n-th (N, p_n) and n-th (N, q_n) transforms of the sequence $\{s_n\}$ are respectively

$$t_n=\frac{1}{P_n}\sum_{r=0}^{n} p_r s_{n-r}=\frac{1}{P_n}\sum_{r=0}^{n} P_r a_{n-r} \tag{2.1}$$

and

$$u_n=\frac{1}{Q_n}\sum_{r=0}^{n} q_r s_{n-r}=\frac{1}{Q_n}\sum_{r=0}^{n} Q_r a_{n-r}. \tag{2.2}$$

The series

$$\sum_{r=0}^{\infty} a_r$$

is said to be summable (N, p_n) to s if $t_n \to s$ as $n \to \infty$. We denote this by

$$\sum_{r=0}^{\infty} a_r = s(N, p_n) \quad \text{or} \quad s_n \to s(N, p_n). \tag{2.3}$$

All limits in the sequel will be taken as the variable tends to infinity, unless otherwise specified.

We shall also have occasion, in the case that $p_n \neq 0$ and $q_n \neq 0$ for all values of n, to use the notation

$$t_n^{\Delta}=\frac{1}{p_n}\sum_{r=0}^{n} p_r a_{n-r} \tag{2.4}$$

and

$$u_n^{\Delta}=\frac{1}{q_n}\sum_{r=0}^{n} q_r a_{n-r}. \tag{2.5}$$

We note that (2.4) and (2.5) are respectively the n-th $(N, \Delta p_n)$ and n-th $(N, \Delta q_n)$ transforms of the sequence $\{s_n\}$.

A method of summability is *regular*, if it sums every convergent series to its ordinary sum.

If P and Q are methods of summability, Q is said to *include* P (written "$P \Rightarrow Q$") if every series summable by the method P is also summable by the method Q to the same sum. P and Q are said to be *equivalent* (written "$P \Leftrightarrow Q$") if each method includes the other.

Using standard results, [3, Theorem 2], we find that the Nörlund method (N, p_n) is regular if and only if

$$P_n^* = O(|P_n|) \tag{2.6}$$

and

$$p_n/P_n \to 0. \tag{2.7}$$

See also [4].

Thus for a regular Nörlund method (N, p_n), either,

$$\sum_{r=0}^{\infty} |p_r| < \infty \tag{2.8}$$

or

$$|P_n| \to \infty . \tag{2.9}$$

Since p_0 and q_0 are both non-zero, there exist sequences $\{k_n\}$, $\{l_n\}$ and $\{\gamma_n\}$ such that,

$$k_0 p_n + \cdots + k_n p_0 = q_n, \qquad n=0, 1, 2, \ldots, \tag{2.10}$$

$$l_0 q_n + \cdots + l_n q_0 = p_n, \qquad n=0, 1, 2, \ldots, \tag{2.11}$$

$$\gamma_0 p_n + \cdots + \gamma_n p_0 = 0, \qquad n=1, 2, 3, \ldots, \tag{2.12}$$

$$\gamma_0 = 1/p_0 . \tag{2.13}$$

Thus

$$t_n = \frac{1}{P_n} \sum_{r=0}^{n} p_r s_{n-r},$$

if and only if

$$s_n = \sum_{r=0}^{n} \gamma_{n-r} P_r t_r .$$

Therefore every Nörlund transformation is invertible.

The following propositions give necessary and sufficient conditions for inclusion or equivalence relations to hold between two Nörlund methods. The proofs are closely modelled on proofs given by HARDY [3, Theorems 19 and 21], but are applicable to a larger class of Nörlund methods than are considered by HARDY. In connection with the second proposition see also [4, Corollary 1].

Proposition 1. *For Nörlund methods (N, p_n) and (N, q_n), (not necessarily regular), necessary and sufficient conditions that $(N, p_n) \Rightarrow (N, q_n)$ are*

$$|k_0||P_n| + \cdots + |k_n||P_0| \leqq H|Q_n| \tag{2.14}$$

where H is independent of n, and

$$k_{n-r}/Q_n \to 0, \qquad \text{for each } r. \tag{2.15}$$

Proof. Referring to (2.2), we have

$$u_n = \sum_{r=0}^{\infty} c_{n,r} t_r, \tag{2.16}$$

with $c_{n,r} = k_{n-r} P_r / Q_n$ for $r \leqq n$ and $c_{n,r} = 0$ for $r > n$. cf. [3, Theorem 19].

Since a Nörlund method is invertible. $(N, p_n) \Rightarrow (N, q_n)$ if and only if the sequence to sequence transformation (2.16) is regular. A standard result now yields (2.14) and (2.15). See for example [3, Theorem 2].

Proposition 2. *For regular Nörlund methods (N, p_n) and (N, q_n), necessary and sufficient conditions that $(N, p_n) \Leftrightarrow (N, q_n)$ are*

$$\sum_{r=0}^{\infty} |k_r| < \infty \quad \text{and} \quad \sum_{r=0}^{\infty} |l_r| < \infty . \tag{2.17}$$

Proof. Necessity.

By Proposition 1 both $\{|P_n|/|Q_n|\}$ and $\{|Q_n|/|P_n|\}$ are bounded. Also by (2.14),

$$|k_0|+|k_1|\frac{|P_{n-1}|}{|P_n|}+\cdots+|k_r|\frac{|P_{n-r}|}{|P_n|}\leqq H|Q_n|/|P_n|$$

for $r\leqq n$. Now fixing r, letting n tend to infinity, and using the fact that for a regular Nörlund method (N, p_n), $P_{n-r}\sim P_n$ for each r, we have

$$|k_0|+\cdots+|k_r|\leqq H\limsup(|Q_n|/|P_n|)=H_1<\infty,$$

so that

$$\sum_{n=0}^{\infty}|k_n|<\infty.$$

Similarly

$$\sum_{n=0}^{\infty}|l_n|<\infty,$$

and, hence we have (2.17).

Sufficiency.

We now have $k_{n-r}\to 0$ for each r, and, because of the regularity of, (N, q_n), $|Q_n|\geqq HQ_n^*\geqq HQ_0^*>0$: it follows that (2.15) holds. Also, by (2.6) and its analogue for the method (N, q_n),

$$|P_n|\leqq|Q_0||l_n|+\cdots+|Q_n||l_0|\leqq H|Q_n|\sum_{r=0}^{\infty}|l_r|,$$

and

$$|k_0||P_n|+\cdots+|k_n||P_0|\leqq H_1|Q_n|\sum_{r=0}^{\infty}|k_r|\sum_{r=0}^{\infty}|l_r|.$$

Thus (2.14) holds, and hence $(N, p_n)\Rightarrow(N, q_n)$. Similarly, $(N, p_n)\Rightarrow(N, q_n)$, and the proof is complete.

The n-th $(\bar{N}, p_n)$ transform of the sequence $\{s_n\}$ is

$$\tau_n=\frac{1}{P_n}\sum_{r=0}^{n}p_r s_r. \tag{2.18}$$

Associated with this transform is a method of summability, $(\bar{N}, p_n)$, defined in the same way as in the case of (N, p_n).

Using a standard result [3, Theorem 2], we find that $(\bar{N}, p_n)$ is regular if and only if (2.6) and (2.9) hold.

Proposition 3. *If $(\bar{N}, p_n)$ is regular, $p_n>0$ for all n and either*

$$\{p_n\}\ \textit{is non-decreasing and}\quad n\leqq H_1\frac{P_n}{p_n}$$

or

$$\{p_n\}\ \textit{is non-increasing and}\quad P_n/p_n\leqq H_2(n+1)$$

then

$$(\bar{N}, p_n)\Leftrightarrow(\bar{N}, 1).$$

Proof. This result is an immediate consequence of a theorem given by HARDY [3, Theorem 14].

Definitions. 1. *Strong summability* $[N, p_n]_\lambda$, $\lambda>0$.

Let (N, p_n) be a Nörlund method with $p_n \neq 0$ for all values of n. We shall say that

$$\sum_{r=0}^{\infty} a_r$$

is strongly summable (N, p_n) *with index* λ *to* s, if

$$\frac{1}{|P_n|} \sum_{r=0}^{n} |p_r| \, |t_r^A - s|^\lambda = o(1). \tag{2.19}$$

We shall denote this by

$$\sum_{r=0}^{\infty} a_r = s[N, p_n]_\lambda \quad \text{or} \quad s_n \to s[N, p_n]_\lambda .$$

Remark. Whenever (2.6) holds,

$$\sum_{r=0}^{\infty} a_r = s[N, p_n]_\lambda$$

if and only if

$$\frac{1}{P_n^*} \sum_{r=0}^{n} |p_r| \, |t_r^A - s|^\lambda = o(1). \tag{2.20}$$

We shall take advantage of this result without further comment.

2. *Absolute summability* $|N, p_n|_\lambda$, $\lambda>0$.

We shall say that

$$\sum_{r=0}^{\infty} a_r$$

is absolutely summable (N, p_n) *with index,* λ, or summable $|N, p_n|_\lambda$, if

$$\sum_{n=1}^{\infty} n^{\lambda-1} |t_n - t_{n-1}|^\lambda < \infty . \tag{2.21}$$

When $\lambda=1$, this definition reduces to the customary definition of absolute Nörlund summability, as given by MEARS [5] for example. See also [1].

We recall now the standard definition of strong Cesàro summability $[C, \alpha+1]_\lambda$ and show that is it equivalent to our definition. For $\lambda>0$, $\alpha>-1$, the series

$$\sum_{r=0}^{\infty} a_r$$

is said to be summable $[C, \alpha+1]_\lambda$ to s, if

$$\frac{1}{n+1} \sum_{r=0}^{n} |s_r^\alpha - s|^\lambda = o(1)$$

where

$$s_n^\alpha=\frac{1}{\varepsilon_n^\alpha}\sum_{r=0}^{n}\varepsilon_{n-r}^{\alpha-1}s_r.$$

Since the Cesàro method of summability $(C,\alpha+1)$ is the method (N,p_n) with $p_n=\varepsilon_n^\alpha$, in order to show that our definition of strong Cesàro summability as strong Nörlund summability is equivalent to the standard definition of strong Cesàro summability it suffices to observe that $(\overline{N},1)\Leftrightarrow(\overline{N},\varepsilon_n^\alpha)$ for $\alpha>-1$. This follows from Proposition 3, because $\{\varepsilon_n^\alpha\}$ is non-increasing when $-1<\alpha\leqq 0$, non-decreasing when $\alpha>0$, and

$$\varepsilon_n^{\alpha+1}/\varepsilon_n^\alpha\sim n/(\alpha+1)\qquad\text{as } n\to\infty.$$

3. Inclusion Theorems

In this section we shall prove certain theorems giving sufficient conditions for one strong Nörlund method of summability to include another. Before doing so however, we make the following simplifying remark.

Remark. If (N,p_n) is a Nörlund method with $p_n\neq 0$ for all values of n, and $\{s_n\}$ is a sequence, then $t_n^\Delta-s$ is the n-th $(N,\Delta p_n)$ transform of the sequence $\{s_n-s\}$. Thus we have,

$$\sum_{r=0}^{\infty}a_r=s[N,p_n]_\lambda$$

if and only if $s_n-s\to 0[N,p_n]_\lambda$. Hence in order to prove that $[N,p_n]_\lambda\Rightarrow[N,q_n]_\lambda$, it is sufficient to prove that $s_n\to 0[N,p_n]_\lambda$ implies $s_n\to 0[N,q_n]_\lambda$. For the remainder of this paper, we shall assume that if (N,p_n) is a Nörlund method, then $p_n\neq 0$ for all values of n, unless mention is made to the contrary.

Theorem 1. *If* $(N,p_n)\Rightarrow(N,q_n)$ *then* $[N,p_n]_1\Rightarrow[N,q_n]_1$.

Proof. Referring to (2.4) and (2.5), we have, for a given sequence $\{s_n\}$,

$$q_n u_n^\Delta=\sum_{r=0}^{n}k_{n-r}p_r t_r^\Delta.$$

Thus

$$|q_r||u_r^\Delta|\leqq\sum_{\nu=0}^{r}|k_{r-\nu}||p_\nu||t_\nu^\Delta|$$

and hence

$$\sum_{r=0}^{n}|q_r||u_r^\Delta|\leqq\sum_{r=0}^{n}\sum_{\nu=0}^{r}|k_\nu||p_{r-\nu}||t_{r-\nu}^\Delta|$$

$$=\sum_{\nu=0}^{n}|k_\nu|\sum_{r=\nu}^{n}|p_{r-\nu}||t_{r-\nu}^\Delta|.$$

Setting

$$\varphi_n=\sum_{\nu=0}^{n}|p_\nu||t_\nu^\Delta|,$$

we see that

$$\text{(a)} \qquad \sum_{r=0}^{n} |q_r|\,|u_r^{\Delta}| \leqq \sum_{\nu=0}^{n} |k_\nu|\,\varphi_{n-\nu} = \sum_{\nu=0}^{n} |k_{n-\nu}|\,|P_\nu|\,\frac{\varphi_\nu}{|P_\nu|}.$$

Supposing now, that $s_n \to 0\,[N, p_n]_1$, we have $\varphi_n/|P_n| = o(1)$. Thus

$$\text{(b)} \qquad \frac{1}{|Q_n|}\sum_{\nu=0}^{n} |k_{n-\nu}|\,|P_\nu|\,\frac{\varphi_\nu}{|P_\nu|} = o(1),$$

provided

$$|k_0|\,|P_n| + \cdots + |k_n|\,|P_0| = O(|Q_n|)$$

and

$$|k_{n-\nu}| = o(|Q_n|) \qquad \text{for each } \nu.$$

But, by Proposition 1 this is equivalent to our hypothesis $(N, p_n) \Rightarrow (N, q_n)$. It follows from (a) and (b) that $s_n \to 0\,[N, q_n]_1$.

Thus

$$[N, p_n]_1 \Rightarrow [N, q_n]_1,$$

and the proof is complete.

Corollary 1. *If* $(N, p_n) \Leftrightarrow (N, q_n)$, *then* $[N, p_n]_1 \Leftrightarrow [N, q_n]_1$.

Theorem 2. *If* $(N, p_n) \Rightarrow (N, q_n)$ *and*

$$\text{(3.1)} \qquad \sum_{r=0}^{n} |k_{n-r}|\,|p_r| = O(|q_n|),$$

then

$$[N, p_n]_\lambda \Rightarrow [N, q_n]_\lambda \qquad \textit{for } \lambda > 1.$$

Proof.

$$|q_r|^\lambda\,|u_r^{\Delta}|^\lambda \leqq \left(\sum_{\nu=0}^{r} |k_{r-\nu}|\,|p_\nu|\,|t_\nu^{\Delta}|\right)^\lambda.$$

Using HÖLDER's inequality, we obtain

$$|q_r|^\lambda\,|u_r^{\Delta}|^\lambda \leqq \left(\sum_{\nu=0}^{r} |k_{r-\nu}|\,|p_\nu|\,|t_\nu^{\Delta}|^\lambda\right)\left(\sum_{\nu=0}^{r} |k_{r-\nu}|\,|p_\nu|\right)^{\lambda-1}$$

$$\leqq H\,|q_r|^{\lambda-1}\sum_{\nu=0}^{r} |k_{r-\nu}|\,|p_\nu|\,|t_\nu^{\Delta}|^\lambda.$$

Thus

$$\sum_{r=0}^{n} |q_r|\,|u_r^{\Delta}|^\lambda \leqq H\sum_{r=0}^{n}\sum_{\nu=0}^{r} |k_{r-\nu}|\,|p_\nu|\,|t_\nu^{\Delta}|^\lambda.$$

To complete the proof we now set

$$\varphi_n = \sum_{r=0}^{n} |p_r|\,|t_r^{\Delta}|^\lambda,$$

and proceed as in the proof of Theorem 1.

Corollary 2. *If* $Q_n^* = O(|Q_n|)$,

$$k_{n-r}/Q_n \to 0 \qquad \text{for each } r \tag{3.2}$$

and (3.1) *holds, then*

$$[N, p_n]_\lambda \Rightarrow [N, q_n]_\lambda \qquad \text{for } \lambda > 1.$$

Proof. By summing both sides of (3.1) and using the fact that $Q_n^* = O(|Q_n|)$ we find that,

$$\sum_{r=0}^{n} |k_{n-r}|\,|P_r| = O(|Q_n|),$$

which together with (3.2) implies that $(N, p_n) \Rightarrow (N, q_n)$. The result now follows from Theorem 2.

Corollary 3. *If* (N, p_n) *and* (N, q_n) *are regular Nörlund methods with* $\{|p_n|\}$ *and* $\{|q_n|\}$ *non-decreasing, and if* $(N, p_n) \Leftrightarrow (N, q_n)$, *then*

$$[N, p_n]_\lambda \Leftrightarrow [N, q_n]_\lambda \qquad \text{for } \lambda > 1.$$

Proof. By Proposition 2, we know that

$$\sum_{n=0}^{\infty} |k_n| < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} |l_n| < \infty.$$

Thus

$$\sum_{r=0}^{n} |k_{n-r}|\,|p_r| \leqq H\,|p_n|$$

and

$$|p_n| \leqq \sum_{r=0}^{n} |l_{n-r}|\,|q_r| \leqq H_1\,|q_n|.$$

Hence

$$\sum_{r=0}^{n} |k_{n-r}|\,|p_r| = O(|q_n|).$$

Similarly

$$\sum_{r=0}^{n} |l_{n-r}|\,|q_r| = O(|p_n|).$$

The conclusion now follows from Theorem 2.

As (3.1) is in general a difficult condition to check, we proceed now to obtain some theorems which guarantee that (3.1) is satisfied. We shall suppose now that the sequences $\{p_n\}$ and $\{q_n\}$ are sequences of positive terms. We begin by quoting part of a result proved by HARDY [3, Theorem 22].

If (N, p_n) is a regular Nörlund method and

$$p_0 = 1, \quad p_n > 0, \quad p_{n+1}/p_n \geqq p_n/p_{n-1} \qquad (n > 0) \tag{3.3}$$

then $\gamma_0 = 1$ and $\gamma_n \leqq 0$ for all $n > 0$, where the γ_n are defined by (2.12) and (2.13).

Theorem 3. *If (N, q_n) is a regular Nörlund method, $\{p_n\}$ satisfies* (3.3), $q_n>0$, $q_n/q_{n-1}\leqq p_n/p_{n-1}$ *for* $n>0$, *and* $p_n=O(q_n)$, *then* (N, p_n) *is regular,* (3.1) *holds, and* $(N, p_n)\Rightarrow(N, q_n)$.

Proof.

$$\frac{1}{q_0}Q_n\leqq P_n$$

and

$$p_n\leqq H q_n \quad \text{so that} \quad \frac{p_n}{P_n}\leqq q_0\frac{H q_n}{Q_n} \quad \text{and hence} \quad \frac{p_n}{P_n}\to 0.$$

The condition $P_n^*=O(|P_n|)$ is satisfied because $p_n>0$. Thus (N, p_n) is regular.

We write $\gamma_n=-c_n$ for $n>0$, so that, in view of the above mentioned result of HARDY, $c_n\geqq 0$ for $n>0$.

Now

$$p_n-c_1 p_{n-1}-\cdots-c_n p_0=0 \qquad \text{for } n>0$$

and by (2.10) and (2.11)

$$q_n-c_1 q_{n-1}-\cdots-c_n q_0=k_n \qquad \text{for } n\geqq 0.$$

Hence

$$\frac{k_n}{q_n}=1-c_1\frac{q_{n-1}}{q_n}-\cdots-c_n\frac{q_0}{q_n}$$

$$\leqq 1-c_1\frac{p_{n-1}}{p_n}-\cdots-c_n\frac{p_0}{p_n}=0 \qquad \text{for } n>0$$

and thus

$$k_n\leqq 0 \qquad \text{for } n>0.$$

So we have

$$\begin{aligned}|k_0|\,p_n+\cdots+|k_n|\,p_0&=k_0 p_n-k_1 p_{n-1}-\cdots-k_n p_0\\&=2k_0 p_n-k_0 p_n-\cdots-k_n p_0\\&=2k_0 p_n-q_n\leqq 2k_0 p_n+q_n\leqq H_1 q_n.\end{aligned}$$

Since $p_n=O(q_n)$. This proves (3.1). It follows by summing both sides of (3.1) that

$$\sum_{r=0}^{n}|k_{n-r}|\,P_r=O(Q_n).$$

Also we have

$$|k_n|\,p_0\leqq|k_0|\,p_n+\cdots+|k_n|\,p_0=O(q_n)=o(Q_n)$$

since (N, q_n) is regular. Further $0<Q_n\leqq Q_{n+r}$ for $r>0$ and so

$$k_{n-r}/Q_n\to 0 \qquad \text{as } n\to\infty.$$

It follows now from Proposition 1 that $(N, p_n)\Rightarrow(N, q_n)$.

Corollary 4. *If* (N, p_n) *and* (N, q_n) *satisfy the hypotheses of Theorem 3, then*

$$[N, p_n]_\lambda \Rightarrow [N, q_n]_\lambda \quad \text{for } \lambda > 1.$$

Proof. This result is an immediate consequence of Theorems 2 and 3.

Theorem 4. *If* (N, p_n) *and* (N, q_n) *are regular Nörlund methods,* $\{p_n\}$ *satisfies* (3.3), *and* $q_n > 0$, $p_n/p_{n-1} \leqq q_n/q_{n-1}$ *for* $n > n_0$, *then*

$$[N, p_n]_\lambda \Rightarrow [N, q_n]_\lambda \quad \text{for } \lambda > 1.$$

Proof. In the case $n_0 = 0$ the result is an immediate consequence of Theorem 2 and HARDY's Theorem 23 in [3], which yields $(N, p_n) \Rightarrow (N, q_n)$ and $k_n \geqq 0$ for all n, so that

$$|k_0|\, p_n + \cdots + |k_n|\, p_0 = k_0\, p_n + \cdots + k_n\, p_0 = q_n .$$

For the general case we modify the second part of HARDY's proof of Theorem 23 in [3].

We have

$$p_n/p_{n-1} \leqq q_n/q_{n-1} \quad \text{for } n = n_0+1,\ n_0+2,\ \ldots .$$

Let

$$r_n = p_n \quad \text{for } n = n_0,\ n_0+1,\ \ldots$$

and define r_n recursively for $n = n_0-1, n_0-2, \ldots, 0$, so that $r_n > 0$ and

$$r_{n+1}/r_n \leqq \min(r_{n+2}/r_{n+1},\ q_{n+1}/q_n,\ p_{n+1}/p_n) .$$

Let $\eta_n = r_n/r_0$, then

$$\eta_0 = 1, \quad \eta_n > 0, \quad \eta_{n+1}/\eta_n \geqq \eta_n/\eta_{n-1}, \quad \eta_n/\eta_{n-1} \leqq q_n/q_{n-1},$$

and

$$\eta_n/\eta_{n-1} \leqq p_n/p_{n-1}$$

for $n > 0$. Also we have $p_n = O(\eta_n)$.

Thus we have $[N, p_n]_\lambda \Rightarrow [N, \eta_n]_\lambda$ by Corollary 4 and $[N, \eta_n]_\lambda \Rightarrow [N, q_n]_\lambda$ by the case $n_0 = 0$. Thus $[N, p_n]_\lambda \Rightarrow [N, q_n]_\lambda$ for $\lambda > 1$, as required.

Theorem 5. *If* (N, p_n) *and* (N, q_n) *are regular Nörlund methods,*

$$p_0 = q_0 = 1, \quad p_n > 0, \quad q_n > 0, \quad p_{n+1}/p_n \geqq p_n/p_{n-1}, \quad q_{n+1}/q_n \geqq q_n/q_{n-1} \quad \text{for } n > 0$$

$$p_n/p_{n-1} \leqq q_n/q_{n-1} \quad \text{for } n > n_0$$

and $q_n = O(p_n)$, *then*

$$(N, p_n) \Leftrightarrow (N, q_n)$$

and

$$[N, p_n]_\lambda \Leftrightarrow [N, q_n]_\lambda \quad \text{for } \lambda \geqq 1.$$

Proof. That $(N, p_n) \Rightarrow (N, q_n)$ follows directly from HARDY's Theorem 23 in [3].

To prove that $(N, q_n) \Rightarrow (N, p_n)$ let $\{\eta_n\}$ be defined as in the proof of Theorem 4.

Now, $(N, q_n) \Rightarrow (N, \eta_n)$ by Theorem 3, $(N, \eta_n) \Rightarrow (N, p_n)$ by HARDY's Theorem 23 in [3], and so $(N, q_n) \Rightarrow (N, p_n)$.

Thus $(N, p_n) \Leftrightarrow (N, q_n)$ and consequently, by Corollary 1, $[N, p_n]_1 \Leftrightarrow [N, q_n]_1$.

To show that $[N, p_n]_\lambda \Leftrightarrow [N, q_n]_\lambda$ for $\lambda > 1$, we observe that $[N, p_n]_\lambda \Rightarrow [N, q_n]_\lambda$ by Theorem 4, $[N, \eta_n]_\lambda \Rightarrow [N, p_n]_\lambda$ by the case $n_0 = 0$ of Theorem 4, and $[N, q_n] \Rightarrow [N, \eta_n]_\lambda$ by Corollary 4 and hence $[N, q_n]_\lambda \Rightarrow [N, p_n]_\lambda$. This completes the proof.

4. Relations Between Strong Nörlund, Absolute Nörlund, and Nörlund Summability Methods

Theorem 6. $[N, p_n]_1 \Rightarrow (N, p_n)$.

Proof. Suppose $s_n \to s\,[N, p_n]_1$.

Now

$$\frac{1}{|P_n|}\left|\sum_{r=0}^{n} p_r (t_r^\Delta - s)\right| \leqq \frac{1}{|P_n|}\sum_{r=0}^{n} |p_r|\,|t_r^\Delta - s|\,.$$

Thus

$$\frac{1}{P_n}\sum_{r=0}^{n} p_r\, t_r^\Delta \to s\,.$$

But

$$\frac{1}{P_n}\sum_{r=0}^{n} p_r\, t_r^\Delta = \frac{1}{P_n}\sum_{r=0}^{n} p_r\, s_{n-r}\,, \tag{4.1}$$

and so

$$s_n \to s(N, p_n)\,.$$

Theorem 7. *If* $P_n^* = O(|P_n|)$ *then* $[N, p_n]_\lambda \Rightarrow (N, p_n)$ *for* $\lambda > 1$.

Proof. Using the fact that $P_n^* = O(|P_n|)$ in conjunction with Theorem 1 in [1], we find that $[N, p_n]_\lambda \Rightarrow [N, p_n]_1$ for $\lambda > 1$. The result now follows from Theorem 6.

Theorem 8. *If* $(\overline{N}, p_n)$ *is regular, and* $\lambda \geqq 1$ *then,* $s_n \to s\,[N, p_n]_\lambda$ *if and only if,*

$$s_n \to s(N, p_n)$$

and

$$\frac{1}{|P_n|}\sum_{r=0}^{n} |p_r|\,|t_r^\Delta - t_r|^\lambda = o(1)\,.$$

The proof of this theorem follows closely the proof of BORWEIN's Theorem 7 in [1].

Proof. We have to prove that

$$\frac{1}{|P_n|}\sum_{r=0}^{n} |p_r|\,|t_r^\Delta - s|^\lambda = o(1) \tag{4.2}$$

if and only if

$$t_n \to s \tag{4.3}$$

and

$$\frac{1}{|P_n|}\sum_{r=0}^{n}|p_r||t_r^{\Delta}-t_r|^{\lambda}=o(1). \tag{4.4}$$

(i) Suppose that (4.2) holds. Then, by Theorem 7, (4.3) holds, and so

$$\frac{1}{|P_n|}\sum_{r=0}^{n}|p_r||t_r-s|^{\lambda}=o(1) \tag{4.5}$$

since $(\overline{N}, p_n)$ is regular. Hence, by MINKOWSKI's inequality and (4.2),

$$\begin{aligned}\left\{\frac{1}{|P_n|}\sum_{r=0}^{n}|p_r||t_r^{\Delta}-t_r|^{\lambda}\right\}^{1/\lambda} &\leqq \left\{\frac{1}{|P_n|}\sum_{r=0}^{n}|p_r||t_r^{\Delta}-s|^{\lambda}\right\}^{1/\lambda}\\ &\quad+\left\{\frac{1}{|P_n|}\sum_{r=0}^{n}|p_r||t_r-s|^{\lambda}\right\}^{1/\lambda}\\ &=o(1)\end{aligned}$$

and (4.4) follows.

(ii) Suppose that (4.3) and (4.4) hold. Again, (4.5) holds. Hence, by MINKOWSKI's inequality and (4.4),

$$\begin{aligned}\left\{\frac{1}{|P_n|}\sum_{r=0}^{n}|p_r||t_r^{\Delta}-s|^{\lambda}\right\}^{1/\lambda} &\leqq \left\{\frac{1}{|P_n|}\sum_{r=0}^{n}|p_r||t_r^{\Delta}-t_r|^{\lambda}\right\}^{1/\lambda}\\ &\quad+\left\{\frac{1}{|P_n|}\sum_{r=0}^{n}|p_r||t_r-s|^{\lambda}\right\}^{1/\lambda}\\ &=o(1),\end{aligned}$$

so that (4.2) holds. The proof is thus complete.

Remark.

$$\text{If } \sum_{r=0}^{\infty}a_n \text{ is summable } |N, p_n|_1, \quad \text{then} \quad \sum_{n=0}^{\infty}a_n=s(N, p_n)$$

where

$$s=\sum_{n=1}^{\infty}(t_n-t_{n-1})+t_0. \tag{4.6}$$

Theorem 9. *If* $(\overline{N}, p_n)$ *is regular and*

$$\sum_{n=0}^{\infty}a_n \quad \textit{is summable } |N, p_n|_1,$$

then

$$\sum_{n=0}^{\infty}a_n=s[N, p_n]_1,$$

where s is given by (4.6).

Proof. By Theorem 8 it suffices to prove that,

$$\frac{1}{P_n^*}\sum_{r=0}^{n}|p_r|\,|t_r^{\Delta}-t_r|=o(1). \tag{4.7}$$

Now, for $r>0$,

$$\begin{aligned}t_r^{\Delta}-t_r&=\frac{1}{p_r}\sum_{v=0}^{r}\Delta p_v s_{r-v}-\frac{1}{P_r}\sum_{v=0}^{r}p_v s_{r-v}\\&=\frac{P_r\sum_{v=0}^{r}p_v s_{r-v}-P_r\sum_{v=0}^{r-1}p_v s_{r-v-1}-p_r\sum_{v=0}^{r}p_v s_{r-v}}{p_r P_r}\\&=\frac{P_{r-1}\sum_{v=0}^{r}p_v s_{r-v}-P_r\sum_{v=0}^{r-1}p_v s_{r-v-1}}{p_r P_r}\\&=\frac{P_{r-1}t_r-P_{r-1}t_{r-1}}{p_r},\end{aligned}$$

so that

$$p_r(t_r^{\Delta}-t_r)=P_{r-1}(t_r-t_{r-1}), \tag{4.8}$$

and hence

$$|t_r^{\Delta}-t_r|\leqq\frac{P_{r-1}^*}{|p_r|}|t_r-t_{r-1}|.$$

Consequently, since $t_0^{\Delta}=t_0$,

$$\frac{1}{P_n^*}\sum_{r=0}^{n}|p_r|\,|t_r^{\Delta}-t_r|\leqq\frac{1}{P_n^*}\sum_{r=1}^{n}P_{r-1}^*|t_r-t_{r-1}|.$$

Let

$$b_r=|t_r-t_{r-1}|,\qquad B_n=\sum_{r=1}^{n}b_r;$$

then

$$\frac{1}{P_n^*}\sum_{r=1}^{n}P_{r-1}^*b_r=B_n-\frac{1}{P_n^*}\sum_{r=1}^{n}B_r|p_r|=o(1)$$

by the regularity of $(\overline{N},p_n)$. The required conclusion follows.

Theorem 10. *If $\lambda>1$, $(\overline{N},p_n)$ is regular and*

$$P_{n-1}^*=O(n|p_n|); \tag{4.9}$$

and if the series

$$\sum_{n=0}^{\infty}a_n$$

is summable (N,p_n) to s and is summable $|N,p_n|_\lambda$ then the series is summable $[N,p_n]_\lambda$ to s.

Proof. Using (4.8), we find that

$$\frac{1}{P_n^*}\sum_{r=0}^{n}|p_r|\,|t_r^{\Delta}-t_r|^{\lambda}\leqq\frac{1}{P_n^*}\sum_{r=1}^{n}\frac{P_{r-1}^*}{|p_r|^{\lambda-1}}\,|t_r-t_{r-1}|^{\lambda}$$

$$\leqq\frac{1}{P_n^*}\sum_{r=1}^{n}P_{r-1}^*\,r^{\lambda-1}\,|t_r-t_{r-1}|^{\lambda}.$$

Using the same technique as in the proof of Theorem 9 we find that the final term is $o(1)$, so that

$$\frac{1}{P_n^*}\sum_{r=0}^{n}|p_r|\,|t_r^{\Delta}-t_r|^{\lambda}=o(1).$$

Finally, using Theorem 8, we obtain the desired conclusion.

Remark. Condition (4.9) is satisfied when $\{|p_n|\}$ is non-decreasing, and also if (N, p_n) is the (C, α) method of summability with $\alpha>-1$.

5. Construction of a Scale of Nörlund Methods

We shall restrict ourselves now to Nörlund methods (N, p_n) for which $p_0>0$ and $p_n\geqq 0$.

Given any sequence $\{v_n\}$ we use the notation

$$v_n^{\alpha}=\sum_{r=0}^{n}\varepsilon_r^{\alpha-1}\,v_{n-r}, \tag{5.1}$$

so that

$$\Delta v_n=v_n^{-1}. \tag{5.2}$$

The following identities are immediate:

$$\sum_{r=0}^{n}\varepsilon_r^{\beta-1}\,v_{n-r}^{\alpha}=v_n^{\alpha+\beta}, \tag{5.3}$$

$$P_n^{\alpha}=p_n^{\alpha+1}=\sum_{r=0}^{n}p_r^{\alpha}. \tag{5.4}$$

We are going to consider the family of Nörlund methods (N, p_n^{α}) for $\alpha>-1$, and, when $p_n\neq 0$ for all values of n, we shall alow $\alpha=-1$. In the special case $p_0=1$, $p_n=0$ for $n>0$ we have $p_n=\varepsilon_n^{\alpha-1}$, so that (N, p_n^{α}) is the Cesàro method (C, α).

Theorem 11. *If either* (i) $\beta>\alpha>-1$ *for* (ii) $\beta>\alpha=-1, p_n>0$ *and* $P_n\to\infty$, *then* $(N, p_n^{\alpha})\Rightarrow(N, p_n^{\beta})$.

Proof. We use Proposition 1 with p_n^{α} and p_n^{β} in place of p_n and q_n. Let $\beta-\alpha=\delta>0$.

Now the appropriate $k_n=\varepsilon_n^{\delta-1}>0$ and, by (5.3),

$$\varepsilon_n^{\delta-1}P_0^{\alpha}+\cdots+\varepsilon_0^{\delta-1}P_n^{\alpha}=P_n^{\beta}.$$

So the appropriate form of (2.14) in Proposition 1 holds.

Since in this case $k_{n-r}\sim k_n$, in order to verify the appropriate form of (2.15) it suffices to show that

$$\varepsilon_n^{\delta-1}/P_n^{\beta}=o(1). \tag{5.5}$$

Now, for $\alpha>-1$,

$$\varepsilon_n^{\delta-1}/P_n^{\alpha+\delta}=\varepsilon_n^{\delta-1}\Big/\sum_{r=0}^{n}\varepsilon_r^{\alpha+\delta}p_{n-r}\leqq\varepsilon_n^{\delta-1}/\varepsilon_n^{\alpha+\delta}p_0=O(1/n^{1+\alpha})=o(1).$$

Finally when $\alpha=-1$ and $P_n\to\infty$, we may suppose without loss in generality that $0<\delta\leqq 1$ and then obtain

$$\varepsilon_n^{\delta-1}/P_n^{\alpha+\delta}\leqq\varepsilon_n^{\delta-1}/\varepsilon_n^{\delta-1}P_n=1/P_n=o(1).$$

This completes the proof of the theorem.

Corollary 5. *If (N,p_n) is a regular Nörlund method, then so also is (N,p_n^{α}) for $\alpha>0$.*

Theorem 12. *If either* (i) $\alpha>0$ *or* (ii) $\alpha=0$, $p_n>0$ *and* $P_n\to\infty$, *then* $(N,p_n^{\alpha-1})\Rightarrow[N,p_n^{\alpha}]_{\lambda}$ *for* $\lambda>0$.

Proof. Let $s_n\to s(N,p_n^{\alpha-1})$, i.e. let

$$w_n=\frac{1}{P_n^{\alpha-1}}\sum_{r=0}^{n}p_{n-r}^{\alpha-1}s_r\to s.$$

Then

$$\frac{1}{P_n^{\alpha}}\sum_{r=0}^{n}p_r^{\alpha}|w_r-s|^{\lambda}=o(1) \tag{5.6}$$

if $(\overline{N},p_n^{\alpha})$ is regular, which is the case when either (i) or (ii) is satisfied. Since (5.6) is equivalent to $s_n\to s\,[N,p_n^{\alpha}]_{\lambda}$, this completes the proof.

Theorem 13. *For $\alpha\geqq 0$ and $\lambda\geqq 1$,*

$$[N,p_n^{\alpha}]_{\lambda}\Rightarrow(N,p_n^{\alpha})$$

provided $p_n>0$ when $\alpha=0$.

Proof. This result follows immediately from Theorems 6 and 7, and p_n^{α} in place of p_n.

Theorem 14. *If $(N,p_n)\Rightarrow(N,q_n)$, then $(N,p_n^{\alpha})\Rightarrow(N,q_n^{\alpha})$ for $\alpha>0$.*

Proof. We have, by hypothesis and Proposition 1,

$$|k_0|P_n+\cdots+|k_n|P_0\leqq HQ_n.$$

Now

$$\begin{aligned}|k_0|P_n^\alpha+\cdots+|k_n|P_0^\alpha &= \sum_{r=0}^{n}|k_{n-r}|\sum_{v=0}^{r}\varepsilon_v^{\alpha-1}P_{r-v}\\ &= \sum_{v=0}^{n}\varepsilon_v^{\alpha-1}\sum_{r=v}^{n}|k_{n-r}|P_{r-v}\\ &\leqq H\sum_{v=0}^{n}\varepsilon_v^{\alpha-1}Q_{n-v}\\ &= HQ_n^\alpha.\end{aligned}$$

Also $|k_{n-r}|/Q_n^\alpha \leqq |k_{n-r}|/Q_n = o(1)$ by hypothesis and Proposition 1. The conclusion now follows from Proposition 1.

Corollary 6. *If (N, p_n) is regular, then, for $\alpha>0$.*

$$(C,\alpha)\Rightarrow(N,p_n^\alpha) \quad \textit{and} \quad [C,\alpha]_\lambda\Rightarrow[Np_n^\alpha]_\lambda \quad \textit{for } \lambda\geqq 1.$$

Theorem 15. *If $\beta>\alpha\geqq 0$ and $\lambda\geqq 1$ then $[N,p_n^\alpha]_\lambda\Rightarrow[N,p_n^\beta]_\lambda$ provided $p_n>0$ when $\alpha=0$.*

Proof. This follows immediately from Theorems 1 and 11 in the case $\lambda=1$, and from Theorems 2 and 11 in the case $\lambda>1$, because

$$\sum_{r=0}^{n}\varepsilon_r^{\beta-\alpha-1}p_{n-r}^\alpha=p_n^\beta \quad \text{and} \quad \varepsilon_r^{\beta-\alpha-1}>0.$$

6. An Application

The method (C^*, μ) is defined by BORWEIN [2] as follows: Let $\mu=m+\delta$, where m is a non-negative integer, and $0\leqq\delta<1$, and let

$$\pi_\mu(x)=m!\,\varepsilon_m^x(x+m+1)^\delta=(x+1)\ldots(x+m)(x+m+1)^\delta.$$

A series

$$\sum_{r=0}^{\infty}a_r$$

is said to be summable (C^*, μ) to s if,

$$\sigma_n=\frac{1}{\pi_\mu(n)}\sum_{r=0}^{n}\pi_\mu(n-r)\,a_r\to s.$$

The method (C^*, μ) is the Nörlund method (N, p_n) with

$$p_n=\pi_\mu(n)-\pi_\mu(n-1). \tag{6.1}$$

BORWEIN [2] has proved that

$$(C^*,\mu)\Leftrightarrow(C,\mu) \quad \text{for } \mu\geqq 0. \tag{6.2}$$

We now define the strong method $[C^*, \mu]_\lambda$ to be the method $[N, p_n]_\lambda$ with p_n given by (6.1), and prove the following theorem.

Theorem 16. *For* $\mu>0$, $\lambda\geqq 1$, $[C^*, \mu]_\lambda \Leftrightarrow [C, \mu]_\lambda$.

Proof. The case $\lambda=1$ follows immediately from (6.2) and Theorem 1. Suppose therefore that $\lambda>1$, and let $q_n=\varepsilon_n^{\mu-1}$. We consider two cases, (i) $\mu\geqq 1$ and (ii) $\mu<1$.

Case (i) $\mu>1$. Now $\pi_\mu(n)\sim\Gamma(\mu+1)\,\varepsilon_n^\mu$.

Also

$$\frac{\pi_\mu(n)-\pi_\mu(n-1)}{\pi_{\mu-1}(n)}=(n+m)\left[(1+1/(n+m))^\delta-1\right]+m$$
$$\to \delta+m=\mu \quad \text{as} \quad n\to\infty.$$

Thus

$$p_n=\pi_\mu(n)-\pi_\mu(n-1)\sim\mu\,\pi_{\mu-1}(n)\sim\Gamma(\mu+1)\,\varepsilon_n^{\mu-1}=\Gamma(\mu+1)\,q_n.$$

Now $q_n=\varepsilon_n^{\mu-1}\leqq q_{n+1}$ since $\mu\geqq 1$, and, by (6.2) and Proposition 2,

$$\sum_{n=0}^{\infty}|l_n|<\infty \quad \text{and} \quad \sum_{n=0}^{\infty}|k_n|<\infty.$$

Hence

$$|k_0|\,p_n+\cdots+|k_n|\,p_0=O(|k_0|\,q_n+\cdots+|k_n|\,q_0)=O(q_n)$$

and

$$|l_0|\,q_n+\cdots+|l_n|\,q_0=O(q_n)=O(p_n).$$

The desired result now follows from Theorem 2.

Case (ii) $\mu<1$ i.e. $\mu=\delta$ with $0<\delta<1$.

Now

(6.3) $$q_{n+1}/q_n\geqq q_n/q_{n-1} \qquad \text{for } n>0,$$

because

$$\frac{q_{n+1}\,q_{n-1}}{q_n^2}=\frac{n^2+n\delta}{n^2+n\delta+\delta-1}\geqq 1 \qquad \text{for } n>0.$$

Also

(6.4) $$p_{n+1}/p_n\geqq p_n/p_{n-1} \qquad \text{for } n>0,$$

because

$$p_{n+1}/p_n=\left(\frac{n+1+\theta_n}{n+\theta_n}\right)^{\delta-1} \qquad (0<\theta_n<1)$$
$$=\left(1+\frac{1}{n+\theta_n}\right)^{\delta-1}$$
$$\geqq\left(1+\frac{1}{n-1+\theta_{n-1}}\right)^{\delta-1}$$
$$=p_n/p_{n-1}.$$

We show next that there is an integer n_0 such that

$$p_{n+1}/p_n \geqq q_{n+1}/q_n \qquad \text{for } n > n_0 . \tag{6.5}$$

Let

$$f(x) = \sum_{n=0}^{\infty} f_n x^n$$

$$= \tfrac{1}{2}(1+2x)^{1+\delta} + \tfrac{1}{2}(1+2x)^{\delta} + 1 + \delta x - (1+\delta)(1+x)^{1+\delta} - (1-\delta)(1+x)^{\delta}.$$

An elementary computation shows that

$$f_0 = f_1 = f_2 = 0 \quad \text{and} \quad 2f_3 = \delta(1-\delta)^2 > 0 .$$

Hence, for n sufficiently large,

$$n^{1+\delta} f(1/n) = (n+1)\{(n+2)^{\delta} - (n+1)^{\delta}\} - (n+\delta)\{(n+1)^{\delta} - n^{\delta}\} > 0$$

and (6.5) follows, because

$$\frac{p_{n+1}\, q_n}{p_n\, q_{n+1}} = \frac{n+1}{n+\delta} \cdot \frac{(n+2)^{\delta} - (n+1)^{\delta}}{(n+1)^{\delta} - n^{\delta}} .$$

Since $p_0 = q_0 = 1$, $p_n = O(q_n)$ and (6.3), (6.4) and (6.5) hold, we obtain the desired conclusion in Case (ii) by appealing to Theorem 5.

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Math. Zeitschr. 103, 112–116 (1968)

Terminality of the Maximal Unipotent Subgroups of Chevalley Groups

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Received June 7, 1967

In connection with the theory of group extensions, EVENS [2] has made the following definition: A nilpotent group G of class n will be said to be terminal if whenever $\tilde{G}$ is a nilpotent group such that $\tilde{G}/\Gamma_{n+1}(\tilde{G}) \cong G$, then $\Gamma_{n+1}(\tilde{G}) = 1$.

Due to the fact that the r-Sylow subgroups of the finite classical groups, for odd r different from the field characteristic, are direct sums of wreath products of cyclic groups, these r-Sylow subgroups are found to be terminal if they are non-abelian [2].

The question then arises concerning terminality of the p-Sylow subgroups of these groups for p the field characteristic. Here we consider the question, in fact, for the maximal unipotent subgroup U of a Chevalley group. The answer to the question for U translates by means of REE's identifications to the answer for most of the classical groups [5], but in addition we have the result for all the exceptional cases except G_2. Finiteness of field is not required in what is to follow, nor are there any restrictions on field characteristic other than those stated explicitly.

First we deal with certain preliminaries concerning roots, Chevalley constants, and a central series defined by CHEVALLEY. Greek letters $\alpha, \beta, \ldots$ will denote roots. Relative to an ordering of the additive group generated by the roots, $\alpha_1, \alpha_2, \ldots, \alpha_n$ will denote the fundamental roots in the standard way [1, p. 18]. Height of a root is defined as usual, and we may assume the ordering to be chosen so that $ht\beta > ht\alpha$ implies $\beta > \alpha$. We will always denote the root of greatest height by θ.

Let U be the maximal unipotent subgroup generated by the one-parameter groups X_α, where α is any positive root. Let $\alpha < \beta$ be positive roots, and let $x_\alpha(s) \in X_\alpha$, $x_\beta(t) \in X_\beta$. CHEVALLEY has established the following commutator relations within U:

$$\begin{aligned} x_\beta(t)^{-1} x_\alpha(s) x_\beta(t) x_\alpha(s)^{-1} &= x_\beta(t)^{-1} x_\alpha(-s)^{-1} x_\beta(t) x_\alpha(-s) \\ &= \prod_{\substack{i,j \in Z^+ \\ i\alpha + j\beta \in \Delta^+}} x_{i\alpha + j\beta}(C_{ij,\alpha\beta} s^i t^j) \end{aligned} \tag{1}$$

where Z^+ denotes the positive integers and Δ^+ denotes the set of positive roots [1, p. 33], and $C_{ij,\alpha\beta}$ is an integer.

Let U_m be the subgroup of U generated by the X_α where α is a positive root of height $\geqq m$. It follows from the above formula that the U_m form a central series for U, and in particular that U is nilpotent of class at most h, where h is the height of θ [1, p. 39]. We shall prove that with certain exceptions in case the field characteristic is 2 or 3, the U_m form the lower central series for U.

Consider the Lie ring [4] associated with the central series given by the U_m. In this ring, the commutator relations (1) become

$$(2) \qquad [x_\alpha(s), x_\beta(t)] = [x_\beta(t), x_\alpha(-s)] = \begin{cases} 0 & \text{if } \alpha+\beta \notin \Delta^+ \\ x_{\alpha+\beta}(C_{11,\alpha\beta}\, s\, t) & \text{if } \alpha+\beta \in \Delta^+. \end{cases}$$

For any results which can be deduced from the structure of this Lie ring, we need only evaluate the simplest of the Chevalley constants $C_{11,\alpha\beta}$, where $\alpha+\beta$ is a root.

Lemma 1. $|C_{11,\alpha\beta}| \leqq k$, *where k is the greatest multiplicity of any bond occurring in the Dynkin diagram of the Lie algebra giving rise to the Chevalley group.*

Proof. CHEVALLEY [1, pp. 24–27] establishes that $C_{11,\alpha\beta} = \pm(p+1)$, where p is the greatest non-negative integer such that $\beta - p\alpha$ is a root. It is well-known that the α-string through β contains at most four roots [3, p. 117], so $p \leqq 2$, so $|C_{11,\alpha\beta}| \leqq 3$.

The only circumstance under which we can have $p=2$ is if the Lie algebra is G_2 [1, p. 20]. It remains to show that if the Dynkin diagram contains only single bonds, then $p=0$. This is well-known, but no one appears to have set down a proof, so we give one here. We choose a new ordering [1, p. 20] of the root group in which β is a fundamental root, and $\alpha = c_1\beta + c_2\gamma$, with γ another fundamental root. Since the Dynkin diagram contains no multiple bonds, we can only have $\alpha = \beta+\gamma$ or γ. The first of these cannot occur, since then $\alpha+\beta = 2\beta+\gamma$ could not be a root. In the second case, $\beta-\alpha = \beta-\gamma$ is the difference of two fundamental roots and so cannot be a root.

The following result is needed in the proof of terminality, but is also of some interest in its own right.

Theorem 1. *Let U be a maximal unipotent subgroup of a Chevalley group, and let the characteristic of the field be greater than k, where k is the highest multiplicity of a bond in the Dynkin diagram of the Lie algebra. Then:*

i) *U is generated by the X_{α_i}, for $\{\alpha_i\}$ a fundamental system of roots.*

ii) *The U_m form the lower central series of U.*

Proof. It suffices to show that in the Lie ring associated with the central series U_m, every one of the graded summands U_m/U_{m+1} is generated by m-fold simple commutators of the elements $x_{\alpha_i}(s)$. To show this, it suffices to show that every $x_\alpha(s)$, as an element of the Lie ring, is such an m-fold commutator for $m = h\,t\,\alpha$.

It is known that every positive root α can be written as a sum

$$\alpha=\alpha_{i_1}+\alpha_{i_2}+\cdots+\alpha_{i_m}$$

where every partial sum $\alpha_{i_1}+\alpha_{i_2}+\cdots+\alpha_{i_l}$ is a root [1, p.20] and where we choose $\alpha_{i_1}>\alpha_{i_2}$.

Then by repeated application of (2) we have:

$$x_\alpha(s)=[x_{\alpha_{i_m}}(1),[\ldots[x_{\alpha_{i_2}}(1),x_{\alpha_{i_1}}(s/c)]\ldots]],$$

where

$$c=C_{11,\alpha_{i_2}\alpha_{i_1}}C_{11,\alpha_{i_3}\alpha_{i_1}+\alpha_{i_2}}\cdots C_{11,\alpha_{i_m}\alpha_{i_1}+\cdots\alpha_{i_{m-1}}}.$$

We have $c\neq 0$, since we are assuming the field characteristic is greater than $k\geqq|C_{11,\alpha\beta}|$.

Theorem 2. *Let U be a maximal unipotent subgroup of a Chevalley group of any of the following types:*

A_l,	$l\geqq 4$;	
B_l,	$l\geqq 4$,	*field characteristic* $\neq 2$;
C_l,	$l\geqq 4$,	*field characteristic* $\neq 2$;
D_l,	$l\geqq 4$,	*field characteristic* $\neq 2$;
F_4,		*field characteristic* $\neq 2$;
E_6, E_7, E_8.		

Then U is terminal.

Proof. If there exists a nilpotent $\tilde{U}$ such that $\tilde{U}/\Gamma_{h+1}(\tilde{U})\cong U$, where h is the class of U, then also $(\tilde{U}/\Gamma_{h+2}(\tilde{U}))/\Gamma_{h+1}(\tilde{U}/\Gamma_{h+2}(\tilde{U}))\cong U$, so in proving terminality by contradiction, we may assume that $\Gamma_{h+2}(\tilde{U})=1$.

We form the Lie rings associated with the lower central series of $\tilde{U}$ and U. The homomorphism of U onto $\tilde{U}$ induces a homomorphism of the Lie ring of $\tilde{U}$ onto the Lie ring of U with kernel $\Gamma_{h+1}(\tilde{U})$. In this homomorphism, let $y_\alpha(s)$ be the element of $\Gamma_m(\tilde{U})/\Gamma_{m+1}(\tilde{U})$ mapping onto the element $x_\alpha(s)$ in the Lie ring of U, m being the height of α. Since the $y_{\alpha_i}(s)$ generate the first summand $\tilde{U}/\Gamma_2(\tilde{U})$ of the Lie ring of $\tilde{U}$, if we show that each $y_{\alpha_i}(s)$ commutes with $y_\theta(t)$, an arbitrary generator of $\Gamma_h(\tilde{U})/\Gamma_{h+1}(\tilde{U})$, it will follow that $\Gamma_{h+1}(\tilde{U})=0$, a contradiction.

Now suppose we can find β a positive root, such that $\theta-\beta$ is also a positive root, $\beta<\theta-\beta$, and such that neither of $\alpha_i+\beta$, $\alpha_i+\theta-\beta$ is a root.

On account of the $y_\alpha(s)$ being uniquely determined by the $x_\alpha(s)$, the commutator formula (2) holds with x replaced by y and "$=0$" replaced by "$\in\Gamma_{h+1}(\tilde{U})$". Then by the Jacobi identity,

$$\begin{aligned}[y_{\alpha_i}(s),y_\theta(t)]&=[y_{\alpha_i}(s),[y_\beta(1/C_{11,\beta\theta-\beta}),y_{\theta-\beta}(t)]]\\&=-[y_\beta(1/C_{11,\beta\theta-\beta}),[y_{\theta-\beta}(t),y_{\alpha_i}(s)]]\\&\quad-[y_{\theta-\beta}(t),[y_{\alpha_i}(s),y_\beta(1/C_{11,\beta\theta-\beta})]].\end{aligned}$$

Division by $C_{11,\beta\theta-\beta}$ is possible by our assumptions on the field characteristic. By our assumptions on β, each of the interior commutators in the last two lines belongs to $\Gamma_{h+1}(\tilde{U})$, the last summand of the Lie ring, and so each of the last two lines is 0, so $y_{\alpha_i}(s)$ commutes with $y_\theta(t)$.

It therefore suffices to give examples of β for each of the types of Lie algebras.

For A_l, $l\geqq 4$, the highest root $\theta=\alpha_1+\alpha_2+\cdots+\alpha_l$. Suppose $i\neq 1, 2$. Then if $\beta=\alpha_1$, neither $\alpha_i+\beta$ nor $\alpha_i+\theta-\beta$ is a root. Suppose $i=1$ or 2. Then if $\beta=\alpha_l$, neither $\alpha_i+\beta$ nor $\alpha_i+\theta-\beta$ is a root, since $l>3$.

For C_l, $l\geqq 4$, the highest root $\theta=2\alpha_1+2\alpha_2+\cdots+2\alpha_{l-1}$. If $i\neq 1, 2$, let $\beta=\alpha_1$. Suppose $i=1$ or 2. Then if $\beta=\alpha_1+\alpha_2+\cdots+\alpha_{l-1}$, neither $\alpha_i+\beta$ nor $\alpha_i+\theta-\beta$ is a root, since $l>3$.

For F_4, the highest root $\theta=2\alpha_1+4\alpha_2+3\alpha_3+2\alpha_4$. If $i=1$ or 2, let $\beta=\alpha_4$, while if $i=3$ or 4, let $\beta=\alpha_1+\alpha_2+\alpha_3+\alpha_4$.

For E_6, $\theta=\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+\alpha_5+2\alpha_6$. If $i\neq 3, 6$, let $\beta=\alpha_6$, while if $i=3$ or 6, let $\beta=\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6$.

For E_7, $\theta=\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+2\alpha_7$. If $i\neq 5, 6$, let $\beta=\alpha_6$, while if $i=5$ or 6, let $\beta=\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7$.

For E_8, $\theta=2\alpha_1+3\alpha_2+4\alpha_3+5\alpha_4+6\alpha_5+4\alpha_6+2\alpha_7+3\alpha_8$. If $i\neq 1, 2$, let $\beta=\alpha_1$, while if $i=1$ or 2, let $\beta=\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8$.

The two remaining cases are B_l and D_l. Here we will be able to use the preceding method of calculating roots to show $y_{\alpha_i}(s)$ commutes with $y_\theta(t)$ for every value of i except $i=1$, and in the case of D_4, $i=1, 3, 4$. This remaining case happily can be taken care of by a very interesting computation which was first discovered in the case D_4 by EVENS.

For B_l, $l\geqq 4$, the highest root $\theta=\alpha_1+2\alpha_2+2\alpha_3+\cdots+2\alpha_{l-1}+2\alpha_l$. If $i\neq 1, 2, 3$, let $\beta=\alpha_2$, while if $i=2$ or 3, let $\beta=\alpha_2+\alpha_3+\cdots+\alpha_l$.

For D_l, $l\geqq 5$, $\theta=\alpha_1+2\alpha_2+2\alpha_3+\cdots+2\alpha_{l-2}+\alpha_{l-1}+\alpha_l$. If $i\neq 1, 2, 3$, let $\beta=\alpha_2$, while if $i=2$ or 3, let $\beta=\alpha_2+\cdots+\alpha_{l-1}$. In particular, $\alpha_3+\theta-\beta$ is not a root since $3<l-1$.

For D_4, $\theta=\alpha_1+2\alpha_2+\alpha_3+\alpha_4$. For $i=2$, let $\beta=\alpha_2+\alpha_3$.

In D_4 the roots α_1, α_3, and α_4 play symmetrical roles. Hence, by reordering we may assume that all we need to prove here is that $y_{\alpha_1}(s)$ commutes with $y_\theta(t)$.

To show that $y_{\alpha_1}(s)$ commutes with $y_\theta(t)$, we show that

$$[y_{\alpha_1}(s), y_\theta(t)]=-[y_{\alpha_1}(s), y_\theta(t)]. \tag{3}$$

This implies that

$$[y_{\alpha_1}(s), y_\theta(2t)]=0,$$

and, since we are assuming here that the field characteristic is different from 2, this shows that $y_{\alpha_1}(s)$ commutes with an arbitrary $y_\theta(t)$. The computation to prove (3) is the same for both B_l and D_l, since in both cases $\theta-\alpha_2$ and

$\theta-\alpha_2-\alpha_1$ are roots:

$$\begin{aligned}[y_{\alpha_1}(s), y_\theta(t)] &= [y_{\alpha_1}(s), [y_{\alpha_2}(1), y_{\theta-\alpha_2}(t/C_{11,\alpha_2\theta-\alpha_2})]] \\ &= -[y_{\alpha_2}(1), [y_{\theta-\alpha_2}(t/C_{11,\alpha_2\theta-\alpha_2}), y_{\alpha_1}(s)]] \\ &\quad -[y_{\theta-\alpha_2}(t/C_{11,\alpha_2\theta-\alpha_2}), [y_{\alpha_1}(s), y_{\alpha_2}(1)]].\end{aligned}$$

The first of the last two terms is 0, since $\theta-\alpha_2+\alpha_1$ is not a root. We proceed with the second term:

$$\begin{aligned}&[y_{\alpha_1}(s), y_\theta(t)] \\ &\quad = [[y_{\alpha_1}(s), y_{\alpha_2}(1)], [y_{\alpha_1}(s), y_{\theta-\alpha_2-\alpha_1}(t/\{s\, C_{11,\alpha_2\theta-\alpha_2} C_{11,\alpha_1\theta-\alpha_2-\alpha_1}\})]] \\ &\quad = -[y_{\alpha_1}(s), [y_{\theta-\alpha_2-\alpha_1}(t/\{s\, C_{11,\alpha_2\theta-\alpha_2} C_{11,\alpha_1\theta-\alpha_2-\alpha_1}\}), [y_{\alpha_1}(s), y_{\alpha_2}(1)]]] \\ &\qquad -[y_{\theta-\alpha_2-\alpha_1}(t/\{s\, C_{11,\alpha_2\theta-\alpha_2} C_{11,\alpha_1\theta-\alpha_2-\alpha_1}\}), [[y_{\alpha_1}(s), y_{\alpha_2}(1)], y_{\alpha_1}(s)]].\end{aligned}$$

Here the last term is 0, since $2\alpha_1+\alpha_2$ is not a root. Now

$$\begin{aligned}&[y_{\theta-\alpha_2-\alpha_1}(t/\{s\, C_{11,\alpha_2\theta-\alpha_2} C_{11,\alpha_1\theta-\alpha_2-\alpha_1}\}), [y_{\alpha_1}(s), y_{\alpha_2}(1)]] \\ &\quad = -[y_{\alpha_1}(s), [y_{\alpha_2}(1), y_{\theta-\alpha_2-\alpha_1}(t/\{s\, C_{11,\alpha_2\theta-\alpha_2} C_{11,\alpha_1\theta-\alpha_2-\alpha_1}\})]] \\ &\qquad -[y_{\alpha_2}(1), [y_{\theta-\alpha_2-\alpha_1}(t/\{s\, C_{11,\alpha_2\theta-\alpha_2} C_{11,\alpha_1\theta-\alpha_2-\alpha_1}\}), y_{\alpha_1}(s)]],\end{aligned}$$

of which the first term is 0, since $\theta-\alpha_1$ is not a root. The second term is:

$$\begin{aligned}&[y_{\alpha_2}(1), [y_{\alpha_1}(s), y_{\theta-\alpha_2-\alpha_1}(t/\{s\, C_{11,\alpha_2\theta-\alpha_2} C_{11,\alpha_1\theta-\alpha_2-\alpha_1}\})]] \\ &\quad = [y_{\alpha_2}(1), y_{\theta-\alpha_2}(t/C_{11,\alpha_2\theta-\alpha_2})] = y_\theta(t).\end{aligned}$$

Thus $[y_{\alpha_1}(s), y_\theta(t)] = -[y_{\alpha_1}(s), y_\theta(t)]$, which completes the proof.

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Math. Zeitschr. 103, 117—121 (1968)

Zur Fortsetzung der Integrationshalbgruppe

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Eingegangen am 4. August 1967

Das bekannte Riemann-Liouville-Integral ([4], 23.16)

$$(J(\lambda)x)(t)=\frac{1}{\Gamma(\lambda)}\int_0^t (t-u)^{\lambda-1}x(u)\,du$$

bettet die Operation n-facher Integration einer komplexwertigen Funktion x aus $L^1[0,\infty)$ in eine für alle komplexen λ mit Realteil $\lambda>0$ definierte Halbgruppe $J(\lambda)$ ein. Es gilt

$$(J(\lambda+1)x)(t)=\int_0^t J(\lambda)x(u)\,du$$

sowie

$$\frac{d}{dt}(J(\lambda+1)x)(t)=(J(\lambda)x)(t).$$

Im folgenden wird gezeigt, daß man die Integrationshalbgruppe $J(\lambda)$ so fortsetzen kann, daß ihr Definitionsbereich aus Teilmengen einer Banachalgebra bzw. vollständigen p-normierten Algebra (kurz p-Algebra) besteht. Außerdem ist der Operator $J(\lambda)$ auch auf Funktionen mit Werten in einem p-normierten Raum ([2, 3, 6]) anwendbar.

Mit $L^1[0,a]$, $0<a\leqq\infty$, bezeichnen wir den Unterraum von $L^1(-\infty,+\infty)$ der bezüglich des Lebesgueschen Maßes integrierbaren Funktionen mit $x(t)=0$, $t<0$ und $t>a$ falls $a<\infty$,

$$\|x\|=\int_{-\infty}^{+\infty}|x(t)|\,dt.$$

Die Einschränkung der Faltung auf $L^1[0,a]$ hat dann die Gestalt ([1], S. 134)

$$x*y=\int_0^t x(u)\,y(t-u)\,du.$$

In der kommutativen Banachalgebra $L^1[0,a]$ $(0<a<\infty)$ hat $(J(\lambda)x)(t)$ die Form

$$\left(\frac{1}{\Gamma(\lambda)}t^{\lambda-1}\right)*x.$$

$$\frac{1}{\Gamma(\lambda)}t^{\lambda-1}\text{ ist Element von }L^1[0,a]\qquad(0<a<\infty).$$

Mit $R[0,a]$ bezeichnen wir die durch Adjunktion des Einselementes e zu $L^1[0,a]$ entstehende kommutative Banachalgebra.

Satz 1. *Die Integrationshalbgruppe* $J(\lambda)$, $\operatorname{Re}\lambda>0$, *läßt sich in die Teilmenge*

$$R_+[0,a]=\{x\in R[0,a]\colon x=\lambda e+f,\ \operatorname{Re}\lambda>0,\ f\in L^1[0,a]\}$$

vortsetzen. Für $x, y\in R_+[0,a]$ *gilt*

$$J(x)*J(y)=J(x+y).$$

$J(x)$ *ist eine auf* $R_+[0,a]$ *analytische Funktion im Sinne der auf Teilmengen von Banachalgebren definierten Funktionen* ([4], 26.3.9, [3], 9.1).

Zum Beweis beachten wir zunächst, daß $J(\lambda)=\Gamma(\lambda)^{-1}t^{\lambda-1}\in L^1[0,a]$ in der rechten offenen Halbebene eine in λ holomorphe Funktion ([4], S. 664) mit Werten in $L^1[0,a]$ ist. Wie für komplexwertige holomorphe Funktionen $f(\lambda)$ ([4], 5.2.5) gilt auch für Funktionen mit Werten in einer kommutativen Banachalgebra X für die Abbildung φ des Raumes $\mathscr{H}(G,X)$ der auf einem Gebiet G definierten holomorphen Funktionen mit Werten in X in die Algebra X die Multiplikativität,

$$f(x):=\varphi(f):=\frac{1}{2\pi i}\int_C f(\lambda)(\lambda e-x)^{-1}\,d\lambda,$$

und zwar für alle x mit $\sigma(x)\subset G$. Als Funktion von x aufgefaßt ist $f(x)$ analytisch:

$$f(x)=\sum_{k=0}^{\infty}(x-x_0)^k\frac{1}{2\pi i}\int_C f(\lambda)(\lambda e-x)^{-k-1}\,d\lambda.$$

Auf diese Weise erhalten wir nun

$$J(x):=\int_C J(\lambda)(\lambda e-x)^{-1}\,d\lambda,\qquad x\in R_+[0,a],$$

daß $\sigma(x)$ genau aus einem Element der rechten offenen Halbebene besteht, da die Banachalgebra $L^1[0,a]$ mit ihrem Radikal identisch ist. Dies sieht man leicht folgendermaßen: Für die Funktion 1, die auf $[0,a]$ identisch 1 ist, gilt

$$1*1=t,\qquad 1^n=\frac{t^{n-1}}{(n-1)!};$$

demnach liegt der in $L^1[0,a]$ dichte Teilraum der Polynome im Radikal, das wegen seiner Abgeschlossenheit mit $L^1[0,a]$ übereinstimmt. Bei geeigneter Wahl der Integrationswege C_1 und C_2 gilt wegen $J(\lambda)*J(\mu)=J(\lambda+\mu)$

$$\begin{aligned}J(x)*J(y)&=\frac{1}{(2\pi i)^2}\int_{C_2}\int_{C_1}J(\lambda+\mu)(\lambda e-x)^{-1}(\mu e-y)^{-1}\,d\lambda\,d\mu\\(\zeta=\lambda+\mu)\qquad&=\frac{1}{(2\pi i)^2}\int_{C_3}J(\zeta)\int_{C_1}(\lambda e-x)^{-1}\big(\zeta e-(y+\lambda)\big)^{-1}\,d\lambda\,d\zeta\\&=\frac{1}{2\pi i}\int_{C_3}J(\zeta)\big(\zeta e-(x+y)\big)^{-1}\,d\zeta=J(x+y),\end{aligned}$$

wobei C_3 ein das Spektrum von $x+y$ umschließender Weg ist.

Auch auf Teilmengen von $R[0,\infty)$ läßt sich die Integrationshalbgruppe fortsetzen, indem man die Funktionen

$$\frac{1}{\Gamma(\lambda)}t^{\lambda-1}$$

außerhalb eines endlichen Intervalls 0 setzt; dies läuft aber auf die in Satz 1 gegebene Erweiterung hinaus.

Die von VOGT [6] aufgestellte Integrationstheorie für Funktionen mit Werten in einem vollständigen p-normierten Raum X ermöglicht die Anwendung von $J(\lambda)$ auf den Funktionenraum

$$X \underset{p}{\hat{\otimes}} L^1[0,\infty).$$

Für einen Banachraum X ist es ein bekanntes Resultat von GROTHENDIECK, daß speziell $L^1([0,\infty)\,X)$ mit dem projektiven Tensorprodukt $X\hat{\otimes}L^1[0,\infty)$ übereinstimmt.

Bemerkung. Sei

$$F(u)\in X \underset{p}{\hat{\otimes}} L^1[0,\infty),$$

dann gilt

$$\big(J(\lambda)*J(\mu)*F\big)(u)=\big(J(\lambda+\mu)*F\big)(u)$$

für $\mathrm{Re}(\lambda)>0$ *bzw.* $\mathrm{Re}\,(\mu)>0$.

Beweis. Wir können eine spezielle Entwicklung

$$F(t)=\sum_k a_k\otimes f_k(t)$$

wählen mit

$$\sum_k \|a_k\|_p\,\|f_k\|^p<\infty\,,\qquad a_k\in X,\ f_k\in L^1[0,\infty).$$

Es gilt dann

$$J(\lambda)*J(\mu)*F=\sum_k a_k\otimes J(\lambda+\mu)*f_k=J(\lambda+\mu)*F,$$

wie sich aus dem Analogon für den Satz von FUBINI in [6], Satz 8, ergibt.

Natürlich brauchen die Funktionen $(J(\lambda)F)(t)$ entgegen dem Fall der Beschränkung auf endliche Intervalle nicht mehr in

$$X \underset{p}{\hat{\otimes}} L^1(0,\infty)$$

zu liegen.

Im folgenden setzen wir $J(\lambda)$ bzw. $J(\lambda)-J(\lambda_0)$ auf Teilmengen von topologischen Algebren fort, die sich in der Form

$$X \underset{p}{\hat{\otimes}} R[0,a] \quad \text{bzw.} \quad X \underset{p}{\hat{\otimes}} L^1[0,a]$$

darstellen lassen, wobei X eine p-Algebra mit Einselement ist.

Für kommutative p-Algebren X und Y definieren wir auf

$$X \hat{\underset{p}{\otimes}} Y$$

in üblicher Weise eine Multiplikation: seien

$$u, v \in X \hat{\underset{p}{\otimes}} Y, \qquad u = \sum_i a_i \otimes f_i \text{ und } v = \sum_k b_k \otimes g_k,$$

so setzen wir

$$u\,v = \sum_{i,k} a_i\, b_k \otimes f_i\, g_k.$$

Zunächst gilt

$$\|u\,v\|\hat{} \leqq \sum_{i,k} \|a_i\|_p \|b_k\|_p \|f_i\|^p \|g_k\|^p \leqq \Big(\sum_i \|a_i\|_p \|f_i\|^p\Big)\Big(\sum_k \|b_k\|_p \|g_k\|^p\Big).$$

Außerdem ist die Multiplikation darstellungsunabhängig, wie man durch Fortsetzung der bilinearen Abbildung

$$(X \times Y) \times (X \times Y) \to X \hat{\underset{p}{\otimes}} Y,$$

$$(a, f) \times (b, g) \to a\, b \otimes f\, g$$

auf

$$(X \hat{\underset{p}{\otimes}} Y) \times (X \hat{\underset{p}{\otimes}} Y)$$

sieht.

Für den Raum $\mathfrak{M}$ der maximalen bzw. regulären maximalen Ideale gilt $\mathfrak{M} = \mathfrak{M}_X \times \mathfrak{M}_Y$ (vgl. [5]).

Satz 2. *Sei X eine kommutative p-Algebra mit Einselement e_X. Dann läßt sich die Integrationshalbgruppe auf die Teilmenge T der p-Algebra*

$$X \hat{\underset{p}{\otimes}} R[0, a], \qquad 0 < a < \infty,$$

fortsetzen, für deren Elemente u $\sigma(u) \subset \{\lambda\colon \mathrm{Re}(\lambda) > 0\}$ erfüllt ist.

Beweis. In der Form $J^*(\lambda) = e_X \otimes J(\lambda)$ kann man $J(\lambda)$ in

$$X \hat{\underset{p}{\otimes}} R[0, a]$$

einbetten. Die für skalare holomorphe Funktionen angegebene Fortsetzung in Teilmengen einer p-Algebra ([3], 9.1) behält auch für vektorwertige Funktionen mit Werten in einer kommutativen p-Algebra ihre Gültigkeit. Die Halbgruppeneigenschaft ergibt sich hier aus dem Funktionalkalkül für zwei Veränderlichen ([3]).

Mit dem Funktionalkalkül für Algebren ohne Einselement ([4], 24.5) gewinnt man auch die Fortsetzung der Operatoren $J(\lambda) - J(\lambda_0)$, $\mathrm{Re}\,\lambda_0 > 0$, auf Teilmengen der p-Algebra

$$X \hat{\underset{p}{\otimes}} L^1[0, a].$$

Sei $d(y)$ die Dissolventenmenge des Elementes y der kommutativen p-Algebra Y (wir setzen $\alpha(y) = $ Komplement $d(y)$) ([4], 24.5) und $D(\lambda, y)$ die Dissol-

vente von y. Dann ordnet man vermöge der Integrationsmethode von [2] jeder für $\lambda=0$ verschwindenden holomorphen Funktion f mit Werten in Y das Element

$$f(y)=-\int_C f(\lambda)\,D(\lambda,y)\,\frac{d\lambda}{\lambda}$$

zu, falls f auf einer Umgebung von $(\alpha(y)\cup\{0\})$ holomorph ist und $C\,\alpha(y)$ umschließt. Wir erweitern nun ein Resultat aus [3], 9.4, so daß wir es auf den Integrationsoperator anwenden können.

Sei G eine offene, den Nullpunkt enthaltende Menge in $\mathbb{C}$, dann bezeichnen wir mit $\mathscr{H}_0(G,Y)$ die Algebra der auf G lokalholomorphen Funktionen f, $f(0)=0$, mit Werten in der kommutativen p-Algebra Y, ferner sei $Y(G)$ die Teilmenge von Y, für deren Elemente y $\alpha(y)\subset G$ gilt. $\mathscr{H}(Y(G),Y)$ sei die Algebra der auf $Y(G)$ holomorphen Funktionen ([3], 9.1). Es gilt

Satz 3. *Durch die Formel*

$$f(y)=-\int_C f(\lambda)\,D(\lambda,y)\,\frac{d\lambda}{\lambda}$$

wird ein multiplikativer Homomorphismus von $\mathscr{H}_0(G,Y)$ in $\mathscr{H}(Y(G),Y)$ gegeben.

Beachtet man, daß auch das Produkt zweier holomorpher Funktionen mit Werten in einer p-Algebra wieder holomorph und daher auch integrierbar ist ([2]), so ergibt sich der Beweis mit kleinen Abänderungen wie in [3], § 9 unter Verwendung von 24.5 in [4]. Wir verzichten daher auf die Durchführung.

Analog zu Satz 2 erhalten wir demnach

Satz 4. *Der Operator $J(\lambda)-J(\lambda_0)$, $\mathrm{Re}(\lambda_0)>0$, kann auf die Teilmenge der*

$$u\in X\mathbin{\hat{\otimes}_p} L^1[0,a],\qquad 0<a<\infty,$$

holomorph fortgesetzt werden, für die $\alpha(u)\subset\{\lambda\colon \mathrm{Re}\,\lambda>-\mathrm{Re}\,\lambda_0\}$ erfüllt ist.

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Math. Zeitschr. 103, 122–128 (1968)

Spectral Interpolation in L_p Spaces

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Received February 3, 1967

Introduction

Let (X, Σ, μ) be a finite measure space. We shall write L_p for $L_p(X, \Sigma, \mu)$. Let $1 \leqq r \leqq p \leqq s \leqq \infty$. Let T be a spectral operator on L_r which leaves L_s invariant. Let T_s ($\equiv T$ restricted to L_s) be a spectral operator on L_s. We show that if $E(\cdot)$ is the spectral measure of T then $E(\cdot)/L_s$ is the spectral measure of T_s and that T_p is also a spectral operator on L_p with $E(\cdot)/L_p$ as the corresponding spectral measure. Also, we obtain a necessary and sufficient condition for T_p to be a spectral operator whenever T is a spectral operator and vice-versa. The only properties of the L_p spaces used are that for $r<p$, L_p is a dense subset of L_r, that the topology of L_p is stronger than that of L_r and the Riesz-Thorin Theorem; so that the results are applicable to other interpolation spaces satisfying these conditions.

1. Preliminaries

Most of the following definitions are taken from [1]. Let Γ be the class of the Borel subsets of the complex plane C. Let B be a complex Banach space and $L(B)$, the space of all continuous linear operators on B. A set function $E(\cdot)$ defined on Γ with values in $L(B)$ is called a *spectral measure* on B if

(a) $E(\emptyset)=0$ and $E(C)=I$ ($\emptyset$ is the null set and I the identy operator on B);

(b) $E(\sigma \cap \delta)=E(\sigma)\,E(\delta)$ for all $\sigma, \delta \in \Gamma$;

(c) $E(\cdot)$ is strongly countably additive, i.e., the vector valued function $E(\cdot)x$ is countably additive for each $x \in B$; and

(d) $E(\cdot)$ is uniformly bounded so that there exists a constant K such that

$$\| E(\sigma) \| \leqq K \qquad \text{for all } \sigma \in \Gamma.$$

An operator $S \in L(B)$ is called a *scalar operator* on B if there exists a (unique) spectral measure $E(\cdot)$ on B such that

$$S = \int \lambda \, dE(\lambda)$$

where the integral exists as a RIEMANN-STIELTJE'S integral in the norm topology of $L(B)$. $T \in L(B)$ is called a *spectral operator* on B if there exists a (unique) spectral measure $E(\cdot)$ on B such that

(i) T commutes with $E(\cdot)$, i.e., $TE(\sigma)=E(\sigma)T$ for all $\sigma \in \Gamma$; and

(ii) $\sigma(T | E(\sigma)\,B) \subset \bar{\sigma}$ for all $\sigma \in \Gamma$. ($\sigma(T)$ is the spectrum of T.)

$N \in L(B)$ is called quasi-nilpotent if

$$\lim_{n\to\infty} \| N^n \|^{1/n} = 0$$

or equivalently, if

$$\lim_{n\to\infty} |\langle N^n x, x' \rangle|^{1/n} = 0$$

for each $x \in B$ and $x' \in B'$, the dual of B.

A spectral operator T has a unique canonical decomposition $T = S + N$ where S is a scalar operator having the same spectral measure as T and N is a quasi-nilpotent operator commuting with S.

Let $R(\lambda, T)\ (\equiv(\lambda I - T)^{-1})$ denote the resolvent of T. Then for each $x \in B$, $R(\lambda, T)x$ has single valued extension property ([1]; p. 225) and there exists a maximal open set (called the resolvent of x) $\rho_T(x)$ containing $\rho(T)$ which serves as the domain of definition of a (unique) vector valued analytic function $\tilde{x}(\lambda)$ which satisfies the equation

$$(\lambda I - T)\,\tilde{x}(\lambda) = x \qquad \text{for all } \lambda \in \rho_T(x).$$

The complement of $\rho_T(x)$ in C is called the spectrum of x and is denoted by $\sigma_T(x)$. The following result is due to DUNFORD.

Lemma 1. *Let T be a spectral operator on B with spectral measure $E(\cdot)$. Let σ be any closed set in C. Then*

$$E(\sigma)B = \{x \in B : \sigma_T(x) \subset \sigma\}.$$

For the details of these results the reader is referred to [1].

In what follows, the statement "a particular subset is invariant under $E(\cdot)$" would mean that it is invariant under each $E(\sigma)$ $(\sigma \in \Gamma)$. Also, whenever we write T_q it is understood that the operator acts on L_q and that all the considerations are with respect to the topology of L_q.

2. Spectral Interpolation

Throughout this section we assume that $1 \leqq r \leqq p \leqq s \leqq \infty$. T will denote a continuous linear operator on L_r which leaves L_s invariant and we denote the restriction of T to L_s by T_s. We assume that T_s is continuous on L_s.

Lemma 2. *T leaves L_p invariant and T_p is continuous on L_p.*

Proof. If f is a simple function then by Riesz-Thorin theorem ([5]; Chapter XII, Theorem 1.11)

$$\| Tf \|_p \leqq \| T \|_r^{1-t} \| T_s \|_s^t \| f \|_p;$$

where t is given by

$$\frac{1}{p} = (1-t)\frac{1}{r} + t\frac{1}{s}.$$

Since, the simple functions are dense in L_p (we may obviously assume that $p \neq \infty$) we have for each $f \in L_p$,

$$\|Tf\|_p \leqq \|T\|_r^{1-t} \|T_s\|_s^t \|f\|_p;$$

so that $Tf \in L_p$ and T_p is continuous on L_p.

Now, let T and T_s be spectral operators on L_r and L_s, respectively and let $E(\cdot)$ be the spectral measure of T.

Proposition 1. *L_s is invariant under $E(\cdot)$ and $E(\cdot)/L_s$ is the spectral measure of T_s.*

Proof. Let $F(\cdot)$ be the spectral measure of T_s and let $f \in L_s \subset L_r$ be arbitrary. Let $\lambda \to \tilde{f}(\lambda)$ be the analytic function (with values in L_s) defined on $\rho_{T_s}(f)$ such that

$$(\lambda I - T)\tilde{f}(\lambda) = f; \qquad \lambda \in \rho_{T_s}(f).$$

Since, the topology of L_s is stronger than the topology of L_r, $\lambda \to \tilde{f}(\lambda)$ is also analytic in L_r. This shows that $\rho_{T_s}(f) \subset \rho_T(f)$ and hence $\sigma_T(f) \subset \sigma_{T_s}(f)$.

Let σ be any closed set in C. By Lemma 1,

$$\sigma_{T_s}(F(\sigma) f) \subset \sigma.$$

Hence, $\sigma_T(F(\sigma)f) \subset \sigma$; so that, again by Lemma 1, $F(\sigma)f \in E(\sigma)L_r$ and hence

$$E(\sigma)\, F(\sigma)\, f = F(\sigma)\, f.$$

If δ is any closed set disjoint from σ then by operating on $E(\delta)\, F(\delta) f = F(\delta) f$ with $E(\sigma)$ we have,

$$E(\sigma)\, F(\delta)\, f = 0.$$

Since, $F(\cdot)f$ is countably additive in L_s, it is also countably additive in L_r. Therefore, if σ' denotes the complement of σ in C then

$$E(\sigma)\, F(\sigma')\, f = 0.$$

But,

$$E(\sigma)\big(F(\sigma) + F(\sigma')\big) f = E(\sigma) f,$$

so that,

$$E(\sigma) f = E(\sigma)\, F(\sigma) f$$

$$= F(\sigma) f.$$

Since, both $E(\cdot)f$ and $F(\cdot)f$ are countably additive in L_r we have

$$E(\sigma) f = F(\sigma) f;$$

for every Borel set σ. This proves the proposition.

Corollary 1. *$E(\cdot)$ leaves L_s invariant and $\sigma(T) = \sigma(T_s)$.*

We now proceed to prove that T_p is a spectral operator on L_p.

Lemma 3. *L_p is invariant under $E(\cdot)$ and there exists a constant K such that*

$$\| E(\sigma)/L_p \|_p \leqq K; \qquad \text{for all } \sigma \in \Gamma.$$

Proof. The lemma follows immediately from Riesz-Thorin theorem and the fact that there exist constants K_1 and K_2 such that for each $\sigma \in \Gamma$

$$\| E(\sigma) \|_r \leqq K_1 \quad \text{and} \quad \| E(\sigma)/L_s \|_s \leqq K_2 .$$

Proposition 2. *$E(\cdot)/L_p$ is a spectral measure on L_p.*

Proof. By Lemma 3, $E(\cdot)/L_p$ is uniformly bounded.

It is immediate that it satisfies (a) and (b). We show that it also satisfies (c).

Suppose (c) is not true so that there exists an $f \in L_p$, a sequence $\sigma_i \downarrow \emptyset$ of Borel sets and an $\varepsilon > 0$ such that for each positive integer n there exists an integer $m \geqq n$ such that

$$\| E(\sigma_m) f \|_p \geqq \varepsilon .$$

Let (f_k) be a sequence in L_s such that $f_k \to f$ in L_p. From the uniform boundedness of $E(\cdot)/L_p$ there exists an integer N such that

$$\| E(\sigma)(f - f_N) \|_p < \varepsilon/2; \qquad \text{for all } \sigma \in \Gamma.$$

Now,

$$\begin{aligned} \| E(\sigma_m) f_N \|_s &\geqq \| E(\sigma_m) f_N \|_p \\ &\geqq \| E(\sigma_m) f \|_p - \| E(\sigma_m)(f - f_N) \|_p \\ &> \varepsilon/2 . \end{aligned}$$

This contradicts the fact that $E(\sigma_i) f_N \to 0$ in L_s. Hence $E(\cdot)/L_p$ must also satisfy (c).

Theorem 1. *T_p is a spectral operator on L_p with $E(\cdot)/L_p$ as the corresponding spectral measure.*

Proof. T_p, of course, commutes with $E(\cdot)/L_p$.

We show that $\sigma(T_p/E(\sigma)L_p) \subset \overline{\sigma}$; for each $\sigma \in \Gamma$.

It is easy to verify that $E(\sigma)L_s$ is dense in $E(\sigma)L_p$. Let $\lambda \notin \overline{\sigma}$ so that $\lambda \in \rho(T/E(\sigma)L_r)$ and also $\lambda \in \rho(T_s/E(\sigma)L_s)$. Since $(\lambda I - T)$ is bijective on $E(\sigma)L_r$ as also on $E(\sigma)L_s$, it follows that $(\lambda I - T_p)$ is injective on $E(\sigma)L_p$ and it has dense range in the same. Thus, λ is not in the point spectrum or the residual spectrum of $T_p/E(\sigma)L_p$. Since, $((\lambda I - T)/E(\sigma)L_r)^{-1}$ and $((\lambda I - T_s)/E(\sigma)L_s)^{-1}$ are continuous, Riesz-Thorin theorem shows that $((\lambda I - T_p)/E(\sigma)L_p)^{-1}$ is continuous on its domain of definition so that λ is not in the continuous spectrum of $T_p/E(\sigma)L_p$. Thus, $\lambda \in \rho(T_p/E(\sigma)L_p)$. This proves the theorem.

3. Uniform Boundedness of the Spectral Measure

FIXMAN [3] proved that if the measure space X is not a finite union of atoms then there exist spectral operators on L_2 which are not spectral on any L_p $(p \neq 2)$. DUNFORD ([1]; p. 251) remarked that for some special type of operators it is the uniform boundedness of the spectral measure which ensures that the operator is a spectral operator. We now show that this, in fact, is the necessary and sufficient condition for an operator to be a spectral operator on $L_p (1 \leqq p \leqq \infty)$ whenever it is a spectral operator on L_r $(1 \leqq r \leqq \infty)$. As in the preceeding section, we assume that $1 \leqq r \leqq p \leqq \infty$ and that T is a continuous linear operator on L_r which leaves L_p invariant and that T_p is continuous on L_p.

Proposition 3. (a) *Let $E(\cdot)$ be a spectral measure on L_r which leaves L_p invariant. Let there exist a constant K such that for each $\sigma \in \Gamma$, $\| E(\sigma)/L_p \|_p \leqq K$. Then $E(\cdot)/L_p$ is a spectral measure on L_p.*

(b) *Let $E(\cdot)$ be a spectral measure on L_p and let for each $\sigma \in \Gamma$, $E(\sigma)$ be continuous in the topology of L_r so that it may be extended to all of L_r by continuity. (We denote the extension also by $E(\sigma)$.) Let $\| E(\sigma) \|_r \leqq K$ for some constant K and for all $\sigma \in \Gamma$. Then $E(\cdot)$ is a spectral measure on L_r.*

Proof. In both the cases we need to prove only the strong countable additivity of $E(\cdot)$ or $E(\cdot)/L_p$. (b) may be proved by repeating the steps in the proof of Proposition 2.

To prove (a), by the Orlicz-Pettis theorem it is sufficient to prove weak countable additivity of $E(\cdot)/L_p$. Let p' and r' be such that

$$\frac{1}{p}+\frac{1}{p'}=1 \quad \text{and} \quad \frac{1}{r}+\frac{1}{r'}=1.$$

We have $L_{r'} \subset L_{p'}$ as a dense subset.

Let $f \in L_p$ be arbitrary. Let $\sigma_i \downarrow \emptyset$ be any sequence of Borel sets. Since $E(\cdot)f$ is countably additive in L_r, for each $g \in L_{r'}$ we have

$$\lim_i \langle E(\sigma_i) f, g \rangle = 0.$$

Consider the operators T_i: $L_{r'} \to C$ defined by

$$T_i g = \langle E(\sigma_i) f, g \rangle.$$

Now,

$$\lim_i T_i g$$

exists for each g in a dense subset (namely $L_{r'}$) of $L_{p'}$. Also, $(T_i g) = (\langle E(\sigma_i) f, g \rangle)$ is bounded for each $g \in L_{p'}$. In fact,

$$|\langle E(\sigma_i) f, g \rangle| \leqq K \| f \|_p \| g \|_{p'}.$$

Hence, by the principle of uniform boundedness ([2]; Theorem II. 1.18)

$$\lim_i T_i g$$

exists for each $g \in L_{p'}$ and this limit is zero. Thus

$$\lim_i \langle E(\sigma_i) f, g\rangle = 0; \qquad \text{for each } g \in L_{p'}.$$

This, proves the weak, and hence the strong, countable additivity of $E(\cdot)/L_p$.

For the proof of our main result in this section we shall need the following lemma which may be established in a manner analogous to the proof of the above proposition.

Lemma 4. (a) *Let N be a quasi-nilpotent operator on L_r which leaves L_p invariant. Let N_p be continuous on L_p. Then N_p is quasi-nilpotent on L_p.*

(b) *Let N be a continuous linear operator on L_r which leaves L_p invariant. Let N_p be quasi-nilpotent on L_p. Then N is quasi-nilpotent on L_r.*

Theorem 2. (a) *Let T be a spectral operator on L_r with $E(\cdot)$ as the corresponding spectral measure. Let T leave L_p invariant and let T_p be continuous on L_p. Then a necessary and sufficient condition for T_p to be a spectral operator on L_p is that $E(\cdot)$ leaves L_p invariant and that there exists a constant K such that for each $\sigma \in \Gamma$, $\| E(\cdot)/L_p \|_p \leqq K$.*

(b) *Let T be a continuous linear operator on L_r which leaves L_p invariant. Let T_p be a spectral operator on L_p with $E(\cdot)$ as its spectral measure. Then a necessary and sufficient condition for T to be a spectral operator is that for all $\sigma \in \Gamma$, $E(\sigma)$ is continuous in the topology of L_r (so that it may be extended by continuity to all of L_r) and that there exists a constant K such that $\| E(\sigma) \|_r \leqq K$.*

Proof. We shall prove only (a). (b) will follow similarly. Necessity of the condition is provided by Proposition 1.

To prove the sufficiency, let $T = S + N$ be the canonical decomposition of T into its scalar part and the radical. Consider the operator S' on L_p given by

$$S' = \int \lambda \, d(E(\lambda)/L_p).$$

Since, $E(\cdot)/L_p$ is a spectral measure on L_p [Proposition 3], S' exists and is a scalar operator. From the corresponding expression for S and the fact that

$$Sf = \int \lambda \, d(E(\lambda) f); \qquad f \in L_r,$$

it follows that S leaves L_p invariant and that $S' = S_p$. S_p is continuous on L_p and hence N_p is also continuous on L_p. By Lemma 4, N_p is quasi-nilpotent and since $T_p = S_p + N_p$, T_p is a spectral operator. (It is immediate that N_p commutes with S_p.) This proves the theorem.

Remark. If in the above theorem T is a scalar operator then T_p is also scalar. In fact, if T is a spectral operator of type n then so is T_p. Also, in this case as a consequence of ([4]; Theorem 1) the point spectra of T and T_p are same and so are the continuous spectra and the residual spectra are void.

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Math. Zeitschr. 103, 129–138 (1968)

Summability in Topological Groups. II*

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Received April 20, 1967

Introduction

This paper contains further developments of the results of PRULLAGE [2]. The notation and definitions are the same as those in the above cited paper.

Specifically, this paper will include: (1) a proof that for every regular triangular limitation method, $\{f(m)\}$, on the circle, there is a method consisting of continuous functions that is equivalent to and consistent with $\{f(m)\}$, (2) an example of a group, G, and a sequence, $\{x(n)\}$, in $S(G)$ such that no regular triangular method on G will transform $\{x(n)\}$ into a Cauchy sequence, (3) a generalization of a theorem of BUCK [1], and (4) a proof of a restricted Toeplitz theorem for non-triangular limitation methods.

1. The Circle

I have not succeeded in proving that for any group, G, and any regular triangular method, $\{f(m)\}$, on G, there is a method consisting of continuous functions which is equivalent to and consistent with $\{f(m)\}$. There are, however, reasons for proving this theorem for the case where G is the circle group. It is well-known that the only continuous homomorphisms from the circle, G, to itself are those which send an element into an integral power of itself. These mappings are continuous homomorphisms on any topological group, so it seems that proving the theorem in the case of the circle should indicate that the theorem is probably true in general. One possible approach to a proof of the general theorem might be through representation theory, and so this is one more reason to present the following theorem.

Theorem 1. *If $\{f(m)\}$ is a regular triangular limitation method on G, then there exists a method, $\{g(m)\}$, with $g(m)$ a continuous homomorphism from G^m into G, for which $\{f(m)\}$ and $\{g(m)\}$ are equivalent and consistent.*

Proof. Since $\{f(m)\}$ is regular, the Toeplitz theorem (Theorem 1 of [2], p. 261) implies the existence of an increasing sequence of positive integers, $N(k)$, and a decreasing sequence of positive numbers, $\delta(k)$, such that if $m > N(k)+1$ and if $d(1, \exp(i\,x(n))) < \delta(k)$ for $n = N(k)+1,\ N(k)+2, \ldots, m$ then

$$d\big(1, f(m: 1, 1, \ldots, 1, \exp(i\,x(N(k)+1)), \exp(i\,x(N(k)+2)), \ldots, \exp(i(x(m))))\big) < 2^{-k}.$$

* Part of the research for this paper was done while the author held a University of Kentucky Faculty Research Fellowship.

Since each $\{f(m)\}$ is a homomorphism from G^m into G, it follows that for any integers, r, m, and n there is an integer s such that

$$f(m: 1, 1, \ldots, 1, \exp(i\pi/r), 1, 1, \ldots, 1) = \exp(i s \pi/r)$$

where $\exp(i\pi/r)$ appears in the n-th position. So now I shall define a triangular matrix, $\|a(m, n)\|$, whose entries are integers. Let $a(m, n) = 0$ for all m and for $n = 1, 2, \ldots, N(1)$. There is an increasing sequence of positive integers, $r(k)$, such that $\pi/2^{r(k)} < \delta(k)$ for each $k = 1, 2, \ldots$. Then for the strip $N(k) < m \leqq N(k+1)$ and $n = N(1)+1, N(1)+2, \ldots, m$, define $a(m, n)$ to be an integer of least absolute value for which

$$\exp(i\, a(m, n)\, \pi/2^{r(k)}) = f(m: 1, 1, \ldots, 1, \exp(i\pi/2^{r(k)}), 1, 1, \ldots, 1)$$

where $\exp(i\pi/2^{r(k)})$ appears in the n-th position.

Now I define $g(m)$ by setting

$$g(m: \exp(i x(1)), \exp(i x(2)), \ldots, \exp(i x(m))) = \exp\left(i \sum_{n=1}^{m} a(m, n)\, x(n)\right).$$

It is clear that each $g(m)$ is a continuous homomorphism from G^m to G.

Let n be any integer, $n > N(1)$. Then for any $m \geqq n$, there is a k such that $N(k) < m \leqq N(k+1)$. So the choice of $a(m, n)$ implies that

$$d(1, f(m: 1, 1, \ldots, 1, \exp(i\pi/2^{r(k)}), 1, 1, \ldots, 1)) = \min_j |a(m, n) \cdot \pi/2^{r(k)} + 2\pi j| = |a(m, n)| \cdot \pi/2^{r(k)} < 1$$

since $\pi/2^{r(k)} < \pi/2^{r(1)} < \delta(1)$. It follows that $|a(m, n)|\, \pi/2^{r(k)-1} < 2$ so that

$$d(1, \exp(i\, |a(m, n)|\, \pi/2^{r(k)-1})) = |a(m, n)|\, \pi/2^{r(k)-1}.$$

Since $\pi/2^{r(k)-1} < \delta(1)$, $|a(m, n)|\, \pi/2^{r(k)-1} < 1$. This process may be continued, so it follows that $|a(m, n)|\, \pi/2^{r(1)} < 1$ or $|a(m, n)| < 2^{r(1)}/\pi$ for all $m \geqq n > N(1)$.

Let t be a positive integer greater than $r(1)$. The first condition of Theorem 1 of [2] implies that

$$\lim_{m \to \infty} f(m: 1, 1, \ldots, 1, \exp(i\pi/2^t), 1, 1, \ldots, 1) = 1$$

for $\exp(i\pi/2^t)$ in the n-th position with $n > N(1)$. There is a k such that $2^{r(k)} > 2^t$. For any $m > N(k)$ it follows that

$$f(m: 1, 1, \ldots, 1, \exp(i\pi/2^t), 1, 1, \ldots, 1) = \exp(i\, a(m, n)\, \pi/2^t)$$

so that

$$\lim_{m \to \infty} \exp(i\, a(m, n)\, \pi/2^t) = 1.$$

Thus $a(m, n)\, \pi/2^t$ is an integral multiple of 2π for all sufficiently large m. But

$$|a(m, n)| \cdot \pi/2^t < (2^{r(1)}/\pi) \cdot (\pi/2^t) < 1.$$

Hence for each $n>N(1)$ there is an integer M such that if $m>M$ then $a(m, n)=0$. Since each $a(m, n)$ is zero if $n\leqq N(1)$ it follows that each column of $\|a(m, n)\|$ is eventually zero.

Let $\varepsilon>0$ be given and let $\{\exp(i\,x(n))\}$ be a sequence in $S(G)$. Without loss of generality we may assume that $|x(n)|\leqq\pi$ for all $n=1, 2, \ldots$. Let k be chosen so that $2^{-k}<\varepsilon/3$. We may choose an integer $M(1)$ such that if $m>M(1)$ then

$$d\big(f(m\colon \exp(i\,x(1)), \exp(i\,x(2)), \ldots, \exp(i\,x(N(k))), 1, 1, \ldots, 1), 1\big)<\varepsilon/3\,.$$

There is also an integer $M(2)$ such that $m>M(2)$ implies that

$$a(m,1)=a(m,2)=\cdots=a\big(m, N(k)\big)=0\,.$$

Let m be any integer, $m>\max\,[M(1), M(2), N(k)+1]$. Choose elements $y(n)=p(n)\,\pi/2^{r(k)}$ with $p(n)$ an integer of least absolute value which will minimize $|x(n)-y(n)|$ for $n=N(k)+1, N(k)+2, \ldots, m$. Clearly

$$d\big(\exp(i\,x(n)), \exp(i\,y(n))\big)\leqq\pi/2^{r(k)+1}<\delta(k)$$

and

$$f(m\colon 1, 1, \ldots, 1, \exp(i\,y(n)), 1, 1, \ldots, 1)=g\big(m\colon 1, 1, \ldots, 1, \exp(i\,y(n)), 1, 1, \ldots, 1\big)$$

for $n=N(k)+1,\ N(k)+2, \ldots, m$. Thus

$$\begin{aligned}
&d\big(f(m\colon \exp(i\,x(1)), \exp(i\,x(2)), \ldots, \exp(i\,x(m))),\\
&\qquad g(m\colon \exp(i\,x(1)), \exp(i\,x(2)), \ldots, \exp(i\,x(m)))\big)\\
&\leqq d\big(f(m\colon \exp(i\,x(1)), \exp(i\,x(2)), \ldots, \exp(i\,x(N(k))), 1, 1, \ldots, 1), 1\big)\\
&\quad + d\big(g(m\colon \exp(i\,x(1)), \exp(i\,x(2)), \ldots, \exp(i\,x(N(k))), 1, 1, \ldots, 1), 1\big)\\
&\quad + d\big(f(m\colon 1, 1, \ldots, 1, \exp(i[x(N(k)+1)-y(N(k)+1)]),\\
&\qquad \exp(i[x(N(k)+2)-y(N(k)+2)]), \ldots, \exp(i[x(m)-y(m)])), 1\big)\\
&\quad + d\big(g(m\colon 1, 1, \ldots, 1, \exp(i[x(N(k)+1)-y(N(k)+1)]),\\
&\qquad \exp(i[x(N(k)+2)-y(N(k)+2)]), \ldots, \exp(i[x(m)-y(m)])), 1\big)\\
&<\varepsilon/3+0+2^{-k}+\min_j\left|\sum_{n=N(k)+1}^{m} a(m, n)[x(n)-y(n)]+2\pi j\right|\\
&\leqq 2\varepsilon/3+\sum_{n=N(k)+1}^{m}|a(m, n)|\cdot|x(n)-y(n)|\\
&<2\varepsilon/3+\sum_{n=N(k)+1}^{m} a(m, n)\cdot\operatorname{sgn} a(m, n)\cdot\pi/2^{r(k)}.
\end{aligned}$$

Now

$$\begin{aligned}
&d\big(1, f(m\colon 1, 1, \ldots, 1, \exp(i\operatorname{sgn} a(m, n)\,\pi/2^{r(k)}), 1, 1, \ldots, 1)\big)\\
&\qquad=\min_j\big||a(m, n)|\,\pi/2^{r(k)}+2\pi j\big|=|a(m, n)|\,\pi/2^{r(k)}<1,
\end{aligned}$$

for each $n=N(k)+1, N(k)+2, \ldots, m$. Then by an argument similar to the one used above to prove that $|a(m, n)|<2^{r(1)}/\pi$, it follows that

$$\sum_{n=N(k)+1}^{m} a(m, n) \cdot \operatorname{sgn} a(m, n) \cdot \pi/2^{r(k)}<1$$

so that

$$\sum_{n=N(k)+1}^{m} a(m, n) \cdot \operatorname{sgn} a(m, n) \cdot \pi/2^{r(k)}$$
$$=d\big(1, f(m: 1, 1, \ldots, 1, \exp(i \operatorname{sgn} a(m, N(k)+1)\pi/2^{r(k)}),$$
$$\exp(i \operatorname{sgn} a(m, N(k)+2)\pi/2^{r(k)}), \ldots, \exp(i \operatorname{sgn} a(m, m)\pi/2^{r(k)}))\big)<2^{-k}.$$

Hence it follows that

$$\lim_{m\to\infty} d\big(f(m: \exp(i x(1)), \exp(i x(2)), \ldots, \exp(i x(m))),$$
$$g(m: \exp(i x(1)), \exp(i x(2)), \ldots, \exp(i x(m)))\big)=0.$$

It is now quite easy to show that $\{f(m)\}$ and $\{g(m)\}$ are equivalent and consistent.

I shall now briefly consider the case of the topological group product of j copies of G where j is some positive integer. So let $\{f(m)\}$ be a regular triangular limitation method on G^j. I shall define a triangular matrix, $\|A(m, n)\|$, whose entries are j by j matrices of integers. Let $\{N(k)\}$, $\{\delta(k)\}$, and $\{r(k)\}$ be chosen in a manner similar to the way they were chosen in Theorem 1. Let $A(m, n)$ be the zero matrix for all m and for all $n=1, 2, \ldots, N(1)$. The rest of the matrix $\|A(m, n)\|$ will be defined in strips as in Theorem 1.

So let m and n be integers, $N(k)<m\leqq N(k+1)$ and $N(1)<n\leqq m$. Denote the (s, t)-th entry in $A(m, n)$ by $a(m, n: s, t)$. Let

$$X(n)=\big(1, 1, \ldots, 1, \exp(i\pi/2^{r(k)}), 1, 1, \ldots, 1\big),$$

with $\exp(i\pi/2^{r(k)})$ in the t-th position, and let $X(l)=(1, 1, \ldots, 1)$ for all integers l, $1\leqq l\leqq m$ and $l\neq n$. Then define $a(m, n: s, t)$ to be an integer of least absolute value for which the s-th coordinate of

$$f\big(m: X(1), X(2), \ldots, X(m)\big)=\exp\big(i\, a(m, n: s, t)\pi/2^{r(k)}\big).$$

Denote $(\exp(i\, y(n: 1)), \exp(i\, y(n: 2)), \ldots, \exp(i\, y(n: j)))$ in G^j by $Y(n)$. Then set

$$g\big(m: Y(1), Y(2), \ldots, Y(m)\big)=\Big(\exp\Big(i\sum_{n=1}^{m}\sum_{t=1}^{j} a(m, n: 1, t)\, y(n: t)\Big),$$
$$\exp\Big(i\sum_{n=1}^{m}\sum_{t=1}^{j} a(m, n: 2, t)\, y(n: t)\Big), \ldots, \exp\Big(i\sum_{n=1}^{m}\sum_{t=1}^{j} a(m, n: j, t)\, y(n: t)\Big)\Big)$$

for all $\{Y(n)\}$ in $S(G^j)$.

The following two theorems may now be proved using techniques similar to those used in proving Theorem 1 above and Theorems 8 and 12 of [2]. Since the proofs are also quite tedious they will be omitted.

Theorem 2. *The method $\{g(m)\}$, which consists of continuous homomorphisms, is equivalent to and consistent with $\{f(m)\}$.*

If we start with a matrix $\|A(m, n)\|$ of the same form as above and define a method $\{g(m)\}$ as before then the following Toeplitz-type theorem holds.

Theorem 3. *The method $\{g(m)\}$, induced by the matrix $\|A(m, n)\|$, is regular if and only if the following three conditions are satisfied.*

(1) *For each n, s, and t, the sequence*

$$a(1, n: s, t),\ a(2, n: s, t),\ \ldots$$

is eventually zero.

(2) *There is a positive integer M such that if $m > M$ then*

$$\sum_{n=1}^{m} a(m, n: s, t) = \delta(s, t),$$

the Kronecker delta.

(3) *There exists a number N such that for all m and s,*

$$\sum_{n=1}^{m} \sum_{t=1}^{j} |a(m, n: s, t)| < N.$$

2. A Non-limitable Sequence

Example. Let G be the additive group of integers with the metric

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

Let $\{f(m)\}$ be any regular triangular limitation method on G. If we define $a(m, n)$ to be the integer $f(m: 0, 0, \ldots, 0, 1, 0, 0, \ldots, 0)$, where 1 is in the n-th position, it is easy to see that

$$f(m: x(1), x(2), \ldots, x(m)) = \sum_{n=1}^{m} a(m, n)\, x(n)$$

for each integer m and each $(x(1), x(2), \ldots, x(m))$ in G^m. Thus we need consider only methods given by triangular matrices.

Since G is discrete, condition (3) of the Toeplitz-type Theorem 1 of [2] is automatically satisfied. Thus $\{f(m)\}$ is regular if and only if each column $a(1, n)$, $a(2, n)$, ... is eventually zero and the sums

$$\sum_{n=1}^{m} a(m, n)$$

are eventually one. It is also clear that a sequence in G is Cauchy if and only if it is eventually constant.

Let $\{x(n)\}$ be the sequence $3, 3^2, \ldots, 3^n, \ldots$ and let $\|a(m, n)\|$ be as above. If M is any non-zero integer choose a positive integer j such that 3^j does not divide M. Since each column is eventually zero, there is a positive integer N such that if $m>N$ then $a(m, 1), a(m, 2), \ldots, a(m, j-1)$ are all zero. Then

$$f(m: x(1), x(2), \ldots, x(m))=\sum_{n=j}^{m} a(m, n) \cdot 3^n.$$

Since 3^j divides $f(m:x(1), x(2), \ldots, x(m))$ and 3^j does not divide M, it follows that $\{f(m:x(1), x(2), \ldots, x(m))\}$ does not converge to M.

Also $\{f(m:x(1), x(2), \ldots, x(m))\}$ does not converge to zero for if it did then for all sufficiently large m,

$$\sum_{n=1}^{m} a(m, n)=1 \quad \text{and} \quad \sum_{n=1}^{m} 3^n \cdot a(m, n)=0.$$

But then

$$\sum_{n=1}^{m} (1-3^n)\, a(m, n)=1.$$

However the left member of this equality is even and the right member is odd. Thus $\{x(n)\}$ is not limitable by any regular triangular method.

3. A Generalization of a Theorem of Buck

In [1], BUCK proved the following theorem.

If there exists a regular matrix which sums every subsequence of a sequence $\{x(n)\}$ *then* $\{x(n)\}$ *is convergent.*

There is little hope of proving such a strong theorem in the topological group setting. The fact that a subset of the reals is totally bounded if and only if it is bounded is crucial in proving the above theorem. Of course, this property is not satisfied by all topological Hausdorff groups, G, which satisfy the first axiom of countability. The above example indicates that even some very basic summability theorems do not generalize to this setting. The following theorem is the best I can prove at this time.

Theorem 4. *Let* $\{x(n)\}$ *be a totally bounded sequence in* $S(G)$.

(a) *There is a regular method which transforms* $\{x(n)\}$ *into a Cauchy sequence.*

(b) *If there is a regular triangular method which transforms every subsequence of* $\{x(n)\}$ *into a Cauchy sequence then* $\{x(n)\}$ *is Cauchy.*

Proof. (a) It is easy to prove that if $\{x(n)\}$ is totally bounded then there is a subsequence, $\{x(n(j))\}$, of $\{x(n)\}$ which is Cauchy. Define

$$f(m: x(1), x(2), \ldots, x(m))=\begin{cases} 0 & \text{for} \quad 1 \leqq m<n(1) \\ x(n(j)) & \text{for } n(j) \leqq m<n(j+1) \end{cases}$$

for all $j=1, 2, \ldots$. It is clear that $\{f(m)\}$ is a regular triangular limitation method and that $\{f(m:x(1), x(2), \ldots, x(m))\}$ is Cauchy.

(b) Assume that $\{x(n)\}$ is not Cauchy. Then there are at least two subsequences $\{x(i(n))\}$ and $\{x(j(n))\}$ which are both Cauchy but

$$\lim_{n\to\infty}(x(i(n))-x(j(n)))\neq 0.$$

Let $\{f(m)\}$ be any regular triangular method on G and let $\{V(n)\}$ be a sequence of neighborhoods of the identity for which

$$\bigcap_{n=1}^{\infty} V(n)=\{0\}.$$

Since $\{f(m)\}$ is regular there is a positive integer $N(1)$ such that if $m\geqq N(1)$ then

$$x(i(m))-f(N(1): x(i(1)), x(i(2)), \ldots, x(i(N(1))))\in V(1).$$

Let us delete some of the terms in the sequence $\{x(j(n))\}$, if necessary, in order that $j(N(1))>i(N(1))$. Clearly, the sequence

$$x(i(1)), x(i(2)), \ldots, x(i(N(1))), x(j(N(1)+1)), x(j(N(1)+2)), \ldots$$

is Cauchy. So we can choose an integer $N(2)>N(1)$ so that if $m\geqq N(2)$, then

$$x(j(m))-f(N(2): x(i(1)), x(i(2)), \ldots, x(i(N(1))), x(j(N(1)+1)),$$
$$x(j(N(1)+2)), \ldots, x(j(N(2))))\in V(2).$$

It is clear that we can continue to choose integers $N(2)<N(3)<N(4)<\cdots$ and blocks of length $N(k)-N(k-1)$ from the sequences $\{x(i(n))\}$ and $\{x(j(n))\}$ alternately so that if $m\geqq N(k)$ then

$$x(j(m))-f(N(k): x(i(1)), x(i(2)), \ldots, x(i(N(k-1))), x(j(N(k-1)+1)),$$
$$x(j(N(k-1)+2)), \ldots, x(j(N(k))))$$

or

$$x(i(m))-f(N(k): x(i(1)), x(i(2)), \ldots, x(j(N(k-1))), x(i(N(k-1)+1)),$$
$$x(i(N(k-1)+2)), \ldots, x(i(N(k))))$$

belongs to $V(k)$.

Now we may choose a neighborhood U of 0 and a positive integer M so that $x(i(n))-x(j(n))$ is not in U for $n>M$. Let W be a neighborhood of 0 for which if $x, y, z\in W$ then $x+y-z\in U$. Also let $M(1)$ be any positive integer and let k be an odd integer for which $N(k)>\max(M, M(1))$. Then if $q>N(k+1)$ it follows that

$$\begin{aligned} x(i(q))-x(j(q))=&[x(i(q))-f(i(N(k)): x(i(1)), x(i(2)), \ldots, x(i(N(k))))]\\ &+[f(i(N(k)): x(i(1)), x(i(2)), \ldots, x(i(N(k))))\\ &-f(j(N(k+1)): x(i(1)), x(i(2)), \ldots, x(j(N(k+1))))]\\ &-[x(j(q))-f(j(N(k+1)): x(i(1)), x(i(2)), \ldots, x(j(N(k+1))))]\notin U. \end{aligned}$$

Hence the middle bracketed term does not belong to W. Thus $\{f(m)\}$ does not transform the constructed subsequence of $\{x(n)\}$ to a Cauchy sequence.

I wish to point out here that part (a) of the above theorem proves that every totally bounded sequence is transformed by some regular method into a Cauchy sequence, while the example in section 2 shows that not every bounded sequence is so transformable.

4. Non-triangular Methods

In this section I will prove a Toeplitz-type theorem for non-triangular limitation methods on groups which is a considerable improvement on Lemma 1 of [2]. Condition (c) of that lemma was quite unsatisfactory. The following theorem contains an extra hypothesis. This hypothesis is automatically fulfilled in the usual cases and so is rather natural.

Theorem 5. *Let $\{f(m)\}$ be a limitation method which satisfies the following condition: for each positive integer m and each neighborhood U of 0 there is an integer M such that if $x(n)$ is in U for $n=M, M+1, \ldots,$ then*

$$f(m: 0, 0, \ldots, 0, x(M), x(M+1), \ldots) \in U .$$

Then $\{f(m)\}$ is regular if and only if these three conditions are satisfied.

(1) $\lim_{m\to\infty} f(m: 0, 0, \ldots, 0, x, 0, 0, \ldots)=0$ *for all x in G in any fixed position.*

(2) $\lim_{m\to\infty} f(m: x, x, \ldots)=x$ *for all x in G.*

(3) *For each neighborhood U of 0 there exists a neighborhood V of 0 and an integer N such that if j, k and m are integers satisfying $j>k>N$ and $m>N$ and if $x(n)$ is in V for $n=k, k+1, \ldots, j.$ then*

$$f(m: 0, 0, \ldots, 0, x(k), x(k+1), \ldots, x(j), 0, 0, \ldots) \in U .$$

Proof of Sufficiency. Let $\{x(n)\}$ be a Cauchy sequence and let W be a neighborhood of 0. Choose a neighborhood U of 0 such that if s, t, u, v and w belong to U then $s+t-u-v-w$ is in W. Let V and N be chosen as in condition (3). Let $V(1)$ be a neighborhood of 0 which is contained in $V \cap U$. Since $\{x(n)\}$ is Cauchy there is an integer P greater than N such that if $q, r \geqq P$ then $x(q)-x(r)$ is in $V(1)$. Condition (1) implies the existence of an integer $N(1)$ such that if $m>N(1)$ then

$$f(m: x(1)-x(P), x(2)-x(P), \ldots, x(P-1)-x(P), 0, 0, \ldots) \in U .$$

From condition (2) there is an integer $N(2)$ such that if $m>N(2)$ then

$$x(P)-f(m: x(P), x(P), \ldots) \in U .$$

Now choose an integer m so that

$$m>\max[N, N(1), N(2)] .$$

The extra condition implies that there is an integer $N(3)$ such that if $y(n)$ is in U for $n=N(3), N(3)+1, \ldots$, then

$$f(m: 0, 0, \ldots, y(N(3)), y(N(3)+1), \ldots) \in U .$$

It is clear that we may choose $N(3)>P+1$. Then if $n>P$ it follows that

$$\begin{aligned} x(n)-f(m: x(1), x(2), \ldots)=&[x(n)-x(P)]+[x(P)-f(m: x(P), x(P), \ldots)] \\ &-f(m: x(1)-x(P), x(2)-x(P), \ldots, x(P-1)-x(P), 0, 0, \ldots) \\ &-f(m: 0, 0, \ldots, 0, x(P+1)-x(P), x(P+2)-x(P), \ldots, \\ &\quad x(N(3)-1)-x(P), 0, 0, \ldots) \\ &-f(m: 0, 0, \ldots, x(N(3))-x(P), x(N(3)+1)-x(P), \ldots) \in W. \end{aligned}$$

Thus $\{f(m)\}$ is regular.

Proof of Necessity. Conditions (1) and (2) are clearly necessary.

To prove that (3) is necessary assume that it does not hold. Then let U be a neighborhood of 0 for which (3) fails. Choose a neighborhood W of 0 such that if u, v, and w are in W then $u-v-w$ is in U. Let $\{V(i)\}$ be a sequence of neighborhood of 0 for which

$$\bigcap_{i=1}^{\infty} V(i)=\{0\}, \quad V(1) \subset W, \quad \text{and} \quad V(i+1) \subset V(i) \quad \text{for } i=1, 2, \ldots .$$

Choose a positive integer $N(1)$ and elements $x(n)$ in $V(1)$ for $n=1, 2, \ldots, N(1)$ arbitrarily. Now suppose that integers $N(i)$ have been constructed for which $N(i)<N(i+1)$ for $i=1, 2, \ldots, r-1$ and that elements $x(1), x(2), \ldots, x(N(r))$ have been chosen. By the assumption that (3) does not hold there are integers $j(r+1)$, $k(r+1)$, and $m(r+1)$ satisfying $j(r+1)>k(r+1)>N(r)$ and $m(r+1)>N(r)$ and there are elements $x(n)$ in $V(r+1)$ for $n=k(r+1), k(r+1)+1, \ldots, j(r+1)$ for which

$$f(m(r+1): 0, 0, \ldots, 0, x(k(r+1)), x(k(r+1)+1), \ldots, x(j(r+1)), 0, 0, \ldots) \notin U .$$

Condition (1) implies that there is an integer $M(r+1)>j(r+1)$ so that if $m>M(r+1)$ then

$$\begin{aligned} f(m: x(1), x(2), \ldots, x(N(r)), 0, 0, \ldots, 0, x(k(r+1)), x(k(r+1)+1), \ldots, \\ x(j(r+1)), 0, 0, \ldots) \in W. \end{aligned}$$

Also there is an integer $N(r+1)>M(r+1)$ so that if $y(n)$ is in W for $n=N(r+1)+1, N(r+1)+2, \ldots$ then

$$f(m(r+1): 0, 0, \ldots, 0, y(N(r+1)+1), y(N(r+1)+2), \ldots) \in W.$$

Finally choose $x(n)=0$ for $n=N(r)+1, N(r)+2, \ldots, k(r+1)-1$ and for $n=j(r+1)+1, j(r+1)+2, \ldots, N(r+1)$.

It is clear that $\{x(n)\}$ converges to 0. Let N be any positive integer. Choose a positive integer r such that $m(r+1)>N$. Then

$$f(m(r+1): x(1), x(2), \ldots) - f(m(r+1): x(1), x(2), \ldots, x(N(r)), 0, 0, \ldots)$$

$$-f(m(r+1): 0, 0, \ldots, 0, x(N(r+1)+1), x(N(r+1)+2), \ldots)$$

$$= f(m(r+1): 0, 0, \ldots, 0, x(k(r+1)), x(k(r+1)+1), \ldots, x(j(r+1)), 0, 0, \ldots) \notin U .$$

Hence $f(m(r+1): x(1), x(2), \ldots) \notin W$. Thus $\{f(m)\}$ would not be regular. This contradiction establishes the necessity of (3) and completes the proof of Theorem 4.

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Math. Zeitschr. 103, 139–150 (1968)

Topological M-spaces

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Received June 2, 1967

1. Introduction

Topological M-spaces were introduced in [3], where it was shown that KAKUTANI's theorem [4] characterising a closed linear lattice of continuous real-valued functions can be generalised to functions with values in a topological M-space. Nothing was said, however, about the properties of topological M-spaces themselves, and in the present paper an attempt is made to fill that gap. Topological M-spaces are a natural generalisation of normed M-spaces, and it is shown here how they admit analogous of the well-known theorems (also dating from [4]) which apply to the latter. Most of the theory applies to a weaker structure, i.e. locally convex linear lattices in which there is a local base consisting of sublattices. The cardinal feature of such spaces (which are called here *pseudo-M-spaces*) is the existence of plenty of real lattice homomorphisms. The main result (theorem 5) is a theorem on separation by lattice homomorphisms; it does not appear to have been stated previously, even in the normed case. Among other things, it enables us to represent topological M-spaces as lattives of continuous functions (theorem 6).

An example given in § 4 shows that, in a linear lattice with a topology, continuity of the mapping $x \to x^+$ does not imply uniform continuity.

2. Terminology and Elementary Properties

Let X be a linear space over the real field $\boldsymbol{R}$, and suppose that a partial ordering $\leqq$ (not necessarily antisymmetric) is defined on X such that

$$x \leqq y \quad \text{implies} \quad x+z \leqq y+z \qquad \text{for all } z \in X,$$

$$x \geqq 0, \quad \lambda \geqq 0 \quad \text{implies} \quad \lambda x \geqq 0.$$

The positive cone, i.e. $\{x: x \geqq 0\}$, will be denoted by P, so that $x \leqq y$ is equivalent to $y \in x+P$. A subset A of X is *order-convex* if $a, b \in A$ and $a \leqq x \leqq b$ implies that $x \in A$. Equivalently, $(A+P) \cap (A-P) = A$. (NAMIOKA uses the term *full*, SCHAEFER the term *P-saturated*.) X' will denote the space of all linear functionals on X. We say that $f \in X'$ is *positive* if it takes non-negative values on P, and use the same notation $\leqq$ for the partial ordering thus introduced into X'.

If X is a topological linear space, let $\mathcal{N}(X)$ denote the family of all neighbourhoods of 0 in X, and X^* the space of all continuous linear functionals on X. By a *local base* we mean a base of neighbourhoods of 0. X is locally order-convex if and only if the positive cone P is *normal*, i.e. given $U\in\mathcal{N}(X)$, there exists $V\in\mathcal{N}(X)$ such that if $x, y\in P$ and $x+y\in V$, then $x, y\in U$. (For a slightly stronger result, see lemma 2, § 6.) P is said to give an *open decomposition* of X if, for each $U\in\mathcal{N}(X)$, $(P\cap U-P\cap U)\in\mathcal{N}(X)$. This implies the continuity of any $g\in X'$ which is such that $0\leqq g\leqq f$ for some $f\in X^*$.

If $\leqq$ is a lattice ordering, we write $x^+=x\vee 0$, $x^-=(-x)\vee 0$, $|x|=x\vee(-x)$. The following elementary relations hold:

$$(1)\qquad (x\vee y)+z=(x+z)\vee(y+z),$$

$$(2)\qquad x\vee y+x\wedge y=x+y,$$

$$(3)\qquad |x|=x^++x^-=x^+\vee x^-,$$

$$(4)\qquad |x\vee y|\leqq|x|\vee|y|,\quad |x\wedge y|\leqq|x|\vee|y|,$$

$$(5)\qquad |x+y|\leqq|x|+|y|.$$

Also, the lattice ordering is necessarily distributive, so the sublattice generated by a subset A is the set of infima of finite subsets of $S(A)$, where $S(A)$ is the set of suprema of finite subsets of A. We say that a subset A *admits* $\vee$ if $a, b\in A$ implies $a\vee b\in A$, and that A is *solid* if $a\in A$ and $|x|\leqq|a|$ implies that $x\in A$. A linear mapping T of one linear lattice into another is said to be a *lattice homomorphism* if $T(x\vee y)=(Tx)\vee(Ty)$ for all x, y. In this paper, the term "lattice homomorphism" is used with the meaning "lattice homomorphism into $\boldsymbol{R}$". Lattice homomorphisms are positive linear functionals, and either of the following conditions is necessary and sufficient for a positive linear functional f to be a lattice homomorphism:

(i) f is *indecomposable*, i.e. if g, h are positive linear functionals such that $g+h=f$, then g and h are scalar multiples of f;

(ii) the kernel of f is solid.

Indecomposable positive functionals can, of course, be defined on partially ordered linear spaces that are not lattices; in fact, this opens up the possibility of generalising results about lattice homomorphisms to such spaces.

(For these and other elementary facts about linear lattices, see [5, 8].)

In a linear lattice with a topology, it is immediate that continuity of the mapping $x\to x^+$ at 0 implies that P gives an open decomposition of X, and that, if X is Hausdorff, continuity of $x\to x^+$ for all x implies that P is closed. (We notice that if P is closed, then X must be Hausdorff, since if $x\neq 0$, then $x\notin P$ or $x\notin -P$.) The basic result connecting the lattice ordering with the topology is the following. It can be obtained by combining results published

in [8] (p. 234), [7] (p. 40) and [6] (p. 89), but no simple proof of the equivalence of all five statements seems to have appeared yet, so we give one:

Theorem 1. *Let X be a topological linear space with a lattice ordering, the positive cone being denoted by P. Then the following conditions are equivalent:*

(i) *the mapping $(x, y)\to x\vee y$ is uniformly continuous on $X\times X$,*

(ii) *the mapping $x\to x^+$ is uniformly continuous on X,*

(iii) *P is normal and gives an open decomposition of X,*

(iv) *P is normal and $x\to x^+$ is continuous at 0,*

(v) *X is locally solid.*

Proof. (i)$\Rightarrow$(ii). *A priori.*

(ii)$\Rightarrow$(iii). Open decomposition follows from continuity of $x\to x^+$, as mentioned above. To show that P is normal, take $U\in\mathcal{N}(X)$. Then $V\in\mathcal{N}(X)$ exists such that if $x-y\in V$, then $x^+-y^+\in U$. If $a, b\in P$ and $a+b\in V$, then $a=a^+-(-b)^+\in U$.

(iii)$\Rightarrow$(iv). Given order-convex $U\in\mathcal{N}(X)$, $V=P\cap U-P\cap U\in\mathcal{N}(X)$. Take $x\in V$, so that $x=u_1-u_2$, where $u_i\in P\cap U$ $(i=1, 2)$. Then $0\leqq x^+\leqq u_1$, so $x^+\in U$.

(iv)$\Rightarrow$(v). Take order-convex $U\in\mathcal{N}(X)$. $x\to|x|$ is continuous at 0, so $V\in\mathcal{N}(X)$ exists such that $x\in V$ implies $\pm|x|\in U$. If $x\in V$ and $|y|\leqq|x|$, then $y\in U$. Hence $\{y\colon \exists x\in V$ such that $|y|\leqq|x|\}$ is a solid neighbourhood of 0 contained in U.

(v)$\Rightarrow$(i). Take solid $U\in\mathcal{N}(X)$. Solid $V\in\mathcal{N}(X)$ exists such that $V+V\subseteq U$. Take $v_1, v_2\in V$ and $x, y\in X$. Let $z=(x+v_1)\vee(y+v_2)-x\vee y$. Then $v_1\wedge v_2\leqq z\leqq v_1\vee v_2$, so $|z|\leqq|v_1|+|v_2|$, and $z\in U$.

We notice that scalar multiplication makes no appearance in theorem 1; in fact, the theorem applies to commutative topological groups. An example given in § 4 shows that $x\to x^+$ can be continuous on X without being uniformly continuous (the contrary is stated in [8], p. 234).

3. M-spaces and Pseudo-M-spaces

Let X be a topological linear space with a lattice ordering. We say that X satisfies *condition* (M) if there is a local base consisting of sublattices. If U is a sublattice, then $U\cap(-U)$ is a symmetric sublattice, so the condition implies that there is a local base consisting of symmetric sublattices. Further, it is sufficient if the neighbourhoods which admit $\vee$ form a local base, for then those which admit $\wedge$ also do so, and if U admits $\vee$, then the set of infima of finite subsets of U is a sublattice.

It is clear that condition (M) implies the continuity of $x\to x^+$ at 0. However, it does not imply that this mapping is continuous at other points, as is shown by the lexicographic ordering of $\boldsymbol{R}^2$ (since the ordering here is total, every subset is a sublattice).

The above definition makes no use of scalar multiplication, and can be made in a commutative topological group. From now on, however, we shall only be considering locally convex topological linear spaces. A locally convex space will be called a (topological) *pseudo-M-space* if it satisfies condition (M), and an *M-space* if it is also locally solid. Theorem 1 shows that a pseudo-M-space is an M-space if and only if the positive cone is normal. In fact, local convexity is then implied by the other assumptions, as the following theorem shows:

Theorem 2. *Let X be a linear lattice with a topology. If condition (M) is satisfied and the positive cone is normal, then there is a local base consisting of solid, convex sublattices. The topology can be given by seminorms p satisfying*

$$|x| \geqq |y| \Rightarrow p(x) \geqq p(y),$$

$$p(x \vee y) = p(x) \vee p(y) \qquad \text{for } x, y \geqq 0.$$

(A seminorm satisfying these two conditions will be called an *M-seminorm.*)

Proof. Any neighbourhood of 0 contains an order-convex neighbourhood U. U contains a symmetric sublattice $V \in \mathcal{N}(X)$. If $v \in V$, then $|v| = v \vee (-v) \in V$. Let $[V]$ be the order-convex cover of V. Then $[V] \subseteq U$. We show that $x \in [V]$ if and only if $v \in V$ exists such that $|x| \leqq v$, from which it will follow that $[V]$ is solid. If $v \in V$ and $|x| \leqq v$, then $-v \leqq x \leqq v$, so $x \in [V]$. Conversely, if $x \in [V]$, then there exist $v_1, v_2 \in V$ such that $-v_1 \leqq x \leqq v_2$. Then $\pm x \leqq v_1 \vee v_2$, so $|x| \leqq v_1 \vee v_2 \in V$, as required.

Take $x, y \in [V]$. Then there exist $u, v \in V$ such that $|x| \leqq u$, $|y| \leqq v$. Writing $w = u \vee v$, we have $w \in V$. Now $|x \vee y| \leqq |x| \vee |y| \leqq w$, so $x \vee y \in [V]$. Similarly, $x \wedge y \in [V]$. If $0 < \lambda < 1$, then

$$\begin{aligned} |\lambda x + (1-\lambda) y| &\leqq \lambda |x| + (1-\lambda) |y| \\ &\leqq \lambda w + (1-\lambda) w = w, \end{aligned}$$

so $\lambda x + (1-\lambda) y \in [V]$. Hence $[V]$ is a convex sublattice.

$[V]$ is symmetric, so its Minkowski functional p is a seminorm. Suppose that $|x| \geqq |y|$. If $\lambda > p(x)$, then $x/\lambda \in [V]$, so $y/\lambda \in [V]$, and $\lambda \geqq p(y)$. Hence $p(x) \geqq p(y)$. Now suppose that $x, y \geqq 0$. By the result just proved, $p(x \vee y) \geqq p(x) \vee p(y)$. Conversely, if $\lambda > p(x) \vee p(y)$, then x/λ, $y/\lambda \in [V]$, so $(x \vee y)/\lambda \in [V]$, and $\lambda \geqq p(x \vee y)$. Hence $p(\chi \vee \gamma) = p(x) \vee p(y)$, as stated.

Thus an M-space which is normable can have its topology given by an M-norm; this is the classical definition of an "abstract M-space". KAKUTANI [4] showed that such spaces can be embedded in spaces of continuous real functions on a compact, Hausdorff space.

A typical example of an M-space which is not normable is the set of all real-valued functions on an infinite set S, with the topology of pointwise convergence. More generally, we can consider the set of all continuous real functions on a topological space S, with the topology of uniform convergence

on a specified family of compact sets. Later (§ 6) we shall see that every Hausdorff M-space can be embedded in such a space.

(It is not included in the definition that an M-space should be Hausdorff; for instance, a space with a topology given by one M-seminorm is an M-space. Clearly, if the topology is Hausdorff, then the positive cone is closed.)

Corresponding to theorem 2 for pseudo-M-spaces, we have:

Theorem 3. *In a pseudo-M-space, there is a local base consisting of convex sublattices.*

Proof. It is sufficient to show that the sublattice generated by a convex set is convex. Let A be convex, and let $a=a_1\vee\cdots\vee a_m$, $b=b_1\vee\cdots\vee b_n$, where $a_i, b_j\in A$. Take $\lambda\in(0,1)$. Then

$$\lambda a+(1-\lambda)b=(\lambda a_1+(1-\lambda)b)\vee\cdots\vee(\lambda a_m+(1-\lambda)b)$$
$$=\sup_{i,j}(\lambda a_i+(1-\lambda)b_j).$$

This shows that the set of suprema of finite subsets of A is convex. A similar argument applied to the infima of this set completes the proof.

The Minkowski functional of a convex sublattice satisfies $p(x\vee y)\leqq p(x)\vee p(y)$ for all x, y, but equality does not necessarily hold for positive x, y.

A necessary and sufficient condition for a normed linear space with a lattice ordering to satisfy condition (M) is the existsnce of some $K>0$ such that, for any finite family of elements $x_1,\ldots,x_n$ with norm not greater than 1, $\|x_1\vee\cdots\vee x_n\|\leqq K$. For then a similar relation holds for infima, and the sublattice generated by the unit ball is contained in $\{x\colon\|x\|\leqq K^2\}$. In particular, it is sufficient if $\|x\vee y\|\leqq\|x\|\vee\|y\|$ for all x, y.

The following theorem is not used below, but is included to give an example of what can be done with M-seminorms. It is a straightforward generalisation of a result of KAKUTANI ([4], p. 1002).

Theorem 4. *Let p be an M-seminorm, and suppose that $x>0$, $p(x)>0$ and that f is a linear functional dominated by p such that $f(x)=p(x)$. Then $f\geqq 0$, and $x\wedge y=0$ implies $f(y)=0$.*

Proof. Take $y>0$. If $p(y)=0$, then $f(y)=0$. If $p(y)>0$, let $y'=(p(x)/p(y))y$, so that $p(y')=p(x)$. Now $x-y'$ and $y'-x\leqq x\vee y'$, so $|x-y'|\leqq x\vee y'$, and $p(x-y')\leqq p(x)$. Thus $f(x-y')\leqq p(x)=f(x)$, so $f(y')\geqq 0$.

Now $p(x\vee y')=p(x)$, so $f(x\vee y')=f(x)$. If $x\wedge y=0$, then $x\wedge y'=0$, so $x\vee y'=x+y'$. It follows that $f(y')=0$.

4. Cones of Sequences with Non-negative Partial Sums

In the usual way, let m denote the space of all bounded real sequences, and c_0 the subspace of m consisting of sequences convergent to 0. The norm in both m and c_0 is defined by $\|(\xi_n)\|=\sup|\xi_n|$.

In either m or c_0, let P_s denote the set of sequences having all partial sums non-negative. This is a closed cone, and (as a subset of m), it is the dual of the cone of decreasing positive sequences in l_1. It provides examples of several situations of interest in the theory of partially ordered linear spaces.

Firstly, P_s is not normal. Given n, two elements of P_s are

$$x_n=(1, \ldots, 1, -n), \quad y_n=(0, \ldots, 0, n),$$

the terms $-n$ and n occurring in place $n+1$ (we close the brackets to indicate that all further terms are 0). $\|x_n\|=\|y_n\|=n$, while $\|x_n+y_n\|=1$.

We show that P_s makes m and c_0 pseudo-M-spaces. Given $x=(\xi_n)$, $y=(\eta_n)$, let $X_n=\xi_1+\cdots+\xi_n$, $Y_n=\eta_1+\cdots+\eta_n$,

$$\zeta_n=(X_n \vee Y_n)-(X_{n-1} \vee Y_{n-1}) \qquad (n \geqq 2),$$
$$\zeta_1=X_1 \vee Y_1.$$

It is easily seen that $|\zeta_n| \leqq |\xi_n| \vee |\eta_n|$ for all n, so if $z=(\zeta_n)$, then $\|z\| \leqq \|x\| \vee \|y\|$. Also, $\zeta_1+\cdots+\zeta_n=X_n \vee Y_n$, so that, with respcet to the ordering given by P_s, $z=x \vee y$.

In particular, the mapping $x \to x^+$ is continuous at 0. In m, it is easy to verify that it is discontinuous at the point $(1, -1, 1, -1, \ldots)$. By contrast, in c_0 it is continuous for all x (though theorem 1 shows that it is not uniformly continuous). To show this, take $x=(\xi_n) \in c_0$ and $\varepsilon \geqq 0$. An integer N exists such that $|\xi_r| \leqq \varepsilon$ for $r>N$. Take $y=(\eta_n) \in c_0$ such that $\|y-x\| \leqq \varepsilon/N$. Now $x^+=(\alpha_n)$, $y^+=(\beta_n)$, where

$$\alpha_r=X_r^+-X_{r-1}^+ \quad (r \geqq 2), \qquad \alpha_1=X_1^+,$$
$$\beta_r=Y_r^+-Y_{r-1}^+ \quad (r \geqq 2), \qquad \beta_1=Y_1^+,$$

and X_r, Y_r are as above. We use the fact that $|\lambda^+-\mu^+| \leqq |\lambda-\mu|$ for all λ, μ. For $r \leqq N$,

$$|Y_r-X_r|=\left|\sum_{s=1}^{r}(\eta_s-\xi_s)\right| \leqq r\varepsilon/N \leqq \varepsilon,$$

so $|Y_r^+-X_r^+| \leqq \varepsilon$, and $|\beta_r-\alpha_r| \leqq 2\varepsilon$. For $r>N$,

$$|\alpha_r| \leqq |\xi_r| \leqq \varepsilon, \quad |\beta_r| \leqq |\eta_r| \leqq \varepsilon+\|y-x\| \leqq 2\varepsilon,$$

so $|\beta_r-\alpha_r| \leqq 3\varepsilon$. Hence $\|y^+-x^+\| \leqq 3\varepsilon$.

Roughly speaking, the continuity of $x \to x^+$ in c_0 is due to the fact that each element is close to a finite dimensional subspace, but the continuity is not uniform because the dimension of the subspace required depends on x.

P_s is too large to be normal, and large enough to give an open decomposition of c_0. By contrast, the cone of decreasing positive sequences is small enough

to be normal, and too small to give an open decomposition of c_0 (in fact, it only generates a dense subspace).

It is shown by WESTON [9] that if a partial ordering in a locally convex space is given by a normal cone, then every monotonic, weakly convergent sequence is convergent. P_s (in m) gives a simple example of this result failing when the cone is not normal. Let e_n denote the sequence having 1 in place n and 0 elsewhere. Then (e_n) is a decreasing sequence which converges to 0 in the weak topology but not in the norm topology.

5. Separation by Lattice Homomorphisms

We now turn to the separation theorem which is the main result of this paper.

If (X, Y) is a dual pair of linear spaces, then $\sigma(X)$ will denote the weak topology induced by X on Y and its subsets.

We use the following variant of the Krein-Milman theorem (see [5], p. 131): *if A is a compact, convex subset of a Hausdorff, locally convex space, then every closed support of A contains an extreme point of A.* Here, a *support* of A is a non-empty convex subset S of A such that if $a_1, a_2 \in A$ and $\lambda a_1 + (1-\lambda) a_2 \in S$ for some $\lambda \in (0, 1)$, then $a_1, a_2 \in S$.

Let p be a sublinear functional on a linear space X over $\boldsymbol{R}$, and let H be the set of linear functionals dominated by p, i.e. such that $f(x) \leqq p(x)$ for all x. Then H is $\sigma(X)$-compact. For any $x \in X$,

$$\{f \in H : f(x) = p(x)\}$$

is non-empty, by the Hahn-Banach theorem, and is clearly a $\sigma(X)$-closed support of H. Hence it contains an extreme point of H. An alternative proof of this result, using no topology, is given by BONSALL in [1].

The following lemma is the key to our separation theorem:

Lemma 1. *Let X be a linear lattice with a topology, and suppose that K is a convex neighbourhood of 0 which admits $\vee$ and satisfies $K - P = K$. Let p be the Minkowski functional of K, and let H be the set of linear functionals dominated by p. Then the extreme points of H are* (a) 0, *and* (b) *the indecomposable positive functionals f for which $\sup f(K) = 1$. For each $x \in X$, there is a lattice homomorphism f dominated by p such that $f(x) = p(x)$.*

Proof. We notice first that $f \in H$ if and only if $\sup f(K) \leqq 1$. Members of H are continuous, since $K \in \mathcal{N}(X)$, and positive, since they are bounded below on P.

(i) 0 is an extreme point of H, since $H \cap (-H) = \{0\}$. Suppose that f is indecomposable and $\sup f(K) = 1$. Suppose, further, that $f = \lambda g + (1-\lambda) h$, where $g, h \in H$ and $0 < \lambda < 1$. Then $\lambda g = \mu f$ for some $\mu > 0$. Since $\sup g(K) \leqq 1$, we have $\mu \leqq \lambda$. Also, $(1-\lambda) h = (1-\mu) f$, so $1 - \mu \leqq 1 - \lambda$. Hence $\mu = \lambda$ and $g = h = f$. Thus f is an extreme point of H.

(ii) Suppose that f is a non-zero extreme point of H. Let $\sup f(K)=\mu$. Then $\mu>0$, since $f\neq 0$. If $\mu<1$, then $f/\mu\in H$, contradicting the fact that f is an extreme point. Hence $\mu=1$.

Suppose that $f=g+h$, where g, h are non-zero positive functionals. If $k_1, k_2\in K$, then $k=k_1\vee k_2\in K$, and

$$g(k_1)+h(k_2)\leqq g(k)+h(k)=f(k)\leqq 1.$$

Hence if $\mu=\sup g(K)$, then $0<\mu<1$, and $1-\mu=\sup h(K)$. Let $g'=g/\mu$, $h'=h/(1-\mu)$. Then $g', h'\in H$ and $f=\mu g'+(1-\mu)h'$. Since f is extreme, it follows that $g=\mu f$.

The last statement in the lemma follows from (ii) by the variant of the Krein-Milman theorem quoted above and the fact that indecomposable positive functionals are lattice homomorphisms.

We remark that implication (i) does not require that K should admit $\vee$. Implication (ii), however, is the one which interests us here. The main theorem follows from lemma 1 in much the same way as the standard separation theorem follows from the Hahn-Banach theorem. $\bar{A}$ denotes the closure of A.

Theorem 5. *Let X be a pseudo-M-space with positive cone P. Let A be a subset of X, and L the set of continuous lattice homomorphisms on X. Then:*

(i) *If A admits $\vee$ and $x\notin\overline{A-P}$, then there exists $f\in L$ such that $f(x)>\sup f(A)$.*

(ii) *If A admits $\wedge$ and $x\notin\overline{A+P}$, then there exists $f\notin L$ such that $f(x)<\inf f(A)$.*

Proof. (ii) can be deduced from (i), so we prove (i).

We show first that $A-P$ is convex. Take $x, y\in A-P$ and $\lambda\in(0, 1)$. There exist $a, b\in A$ such that $x\leqq a$, $y\leqq b$. Then $\lambda x+(1-\lambda)y\leqq\lambda a+(1-\lambda)b\leqq a\vee b\in A$, so $\lambda x+(1-\lambda)y\in A-P$.

By theorem 3, there is a convex sublattice $U\in\mathcal{N}(X)$ such that $(x-U)\cap(A-P)=\emptyset$. Take $a_0\in A$ and let $K=U+(A-a_0)-P$. Then K is a convex neighbourhood of 0, and $x-a_0\notin K$. Also, K admits $\vee$, since if $k_1, k_2\in K$, then there exist $u_i\in U$ and $a_i\in A$ such that $k_i\leqq u_i+a_i-a_0$ $(i=1, 2)$, and then $k_1\vee k_2\leqq(u_1\vee u_2)+(a_1\vee a_2)-a_0$. Let p be the Minkowski functional of K. Then $p(x-a_0)\geqq 1$, so by lemma 1, there is a lattice homomorphism f dominated by p such that $f(x-a_0)\geqq 1$. Since f is non-zero, $u\in U$ exists such that $f(u)>0$, say $f(u)=\delta$. Take $a\in A$. Then

$$f(u+a-a_0)=\delta+f(a)-f(a_0)\leqq 1,$$

or $f(a_0)-f(a)\geqq\delta-1$. Thus $f(x)-f(a)>\delta$ $(a\in A)$, and the proof is complete.

Corollary 1. *If X is a pseudo-M-space with a closed positive cone, then:*

(i) *if $f(x)\geqq 0$ for all $f\in L$, then $x\geqq 0$,*

(ii) *L separates points of X.*

Proof. (ii) follows from (i). To prove (i), take $x \notin P$ and apply part (ii) of the theorem, with $A = \{0\}$.

Corollary 2. *If X is a pseudo-M-space with a closed positive cone, A is a compact, order-convex sublattice of X, and $x \notin A$, then there is a continuous lattice homomorphism separating x from A.*

Proof. Since A is order-convex, $A = (A+P) \cap (A-P)$, so if $x \notin A$, then $x \notin A+P$ or $x \notin A-P$. Since A is compact and P is closed, $A+P$ and $A-P$ are closed. The result follows.

Remarks. (1) Conversely, if $f \in L$ exists such that $f(x) > \sup f(A)$, then it is clear that $x \notin \overline{S(A)-P}$, where $S(A)$ denotes the set of suprema of finite subsets of A. As a simple example, consider $\boldsymbol{R}^2$ with the usual order. Let $A = \{(\xi, \eta) : \xi + \eta = 0\}$, $x = (1, 1)$. Then $x \notin \overline{A-P}$, but $S(A) = \boldsymbol{R}^2$. The only lattice homomorphisms are positive multiplies of e_1 and e_2, where $e_1(\xi, \eta) = \xi$, $e_2(\xi, \eta) = \eta$.

(2) Condition (M) is not dispensable. The space $L_1[0, 1]$, with the usual norm and order, is a locally solid linear lattice, but has no non-zero continuous lattice homomorphisms at all ([5], p. 246).

(3) The lattice ordering is not strictly necessary for theorem 5. Everything works if X has a local base consisting of convex neighbourhoods which are directed upwards, and A is directed upwards. The conclusion is then that there is an indecomposable positive functional which separates x from A.

6. Representation of M-spaces

Recall that M-spaces are pseudo-M-spaces in which the positive cone is normal. We shall need the following lemma:

Lemma 2. *If X is a topological linear space, P is a normal cone in X, and $U \in \mathcal{N}(X)$ is given, then $V \in \mathcal{N}(X)$ exists such that $\overline{V+P} \cap \overline{V-P} \subseteq U$.*

Proof. There exists $U_1 \in \mathcal{N}(X)$ such that $U_1 + U_1 \subseteq U$. Since P is normal, there exists $V_1 \in \mathcal{N}(X)$ such that $V_1 \subseteq U_1$ and if $p, q \in P$ and $p+q \in V_1$, then $p \in U_1$. Take symmetric $V \in \mathcal{N}(X)$ such that $V+V+V+V \subseteq V_1$.

Take $x \in \overline{V+P} \cap \overline{V-P}$. There exist $v_i, w_i \in V$ and $p_i \in P$ $(i = 1, 2)$ such that

$$x + w_1 = v_1 + p_1, \qquad x + w_2 = v_2 - p_2 .$$

Then $p_1 + p_2 = w_1 - w_2 - v_1 + v_2 \in V_1$, so $p_1 \in U_1$ and $x \in U$.

If A is a subset of X (or X^*), A^0 will denote its polar in X^* (or X). It is well-known that if X is locally convex and $U \in \mathcal{N}(X)$, then U^0 is $\sigma(X)$-compact. $C(S)$ will denote the space of all continuous real-valued functions on the topological space S.

Theorem 6. *Let X be a Hausdorff M-space, and let L be the set of its continuous lattice homomorphisms, endowed with the topology $\sigma(X)$. Then X,*

with all its structure, is embedded in the space $C(L)$, the latter having the topology of uniform convergence on the sets $\{U^0 \cap L : U \in \mathcal{N}(X)\}$. If X is barrelled, this coincides with the topology of uniform convergence on compact sets.

Proof. Corresponding to each $x \in X$, a continuous function $\hat{x}$ on L is defined by $\hat{x}(f) = f(x)$ $(f \in L)$. The correspondence $x \to \hat{x}$ is clearly linear, and corollary 1 to theorem 5 shows that it is one-to-one. Also, with respect to the lattice ordering of $C(L)$, it maps $x \vee y$ to $\hat{x} \vee \hat{y}$. For the rest of the proof, we identify $\hat{x}$ with x.

For $U \in \mathcal{N}(X)$, $U^0 \cap L$ is $\sigma(X)$-compact, so the topology (τ) (for $C(L)$) of uniform convergence on these sets is defined, and not larger than the topology of uniform convergence on compact sets. Restricted to X, a local base for τ is $\{(U^0 \cap L)^0 : U \in \mathcal{N}(X)\}$. Now $(U^0 \cap L)^0 \supseteq U^{00} \supseteq U$, so each such set is in $\mathcal{N}(X)$. Conversely, take $U \in \mathcal{N}(X)$. We must find $V \in \mathcal{N}(X)$ such that $(V^0 \cap L)^0 \subseteq U$. By lemma 2, there is a symmetric sublattice $V \in \mathcal{N}(X)$ such that $\overline{V+P} \cap \overline{V-P} \subseteq U$. Take $x \notin U$. Then $x \notin \overline{V+P}$ or $x \notin \overline{V-P}$. If $x \notin \overline{V-P}$, then, by theorem 5(i), $f \in L$ exists such that $\sup f(V) \leqq 1$, while $f(x) > 1$. Since V is symmetric, $f \in V^0$. Similarly, by theorem 5(ii), if $x \notin \overline{V+P}$, then there exists $f \in V^0 \cap L$ such that $f(x) < -1$. In either case, $x \notin (V^0 \cap L)^0$, showing that V is as required.

If X is barrelled and K is a $\sigma(X)$-compact subset of L, then $K^0 \in \mathcal{N}(X)$. Now $K \subseteq K^{00} \cap L$, and it follows that τ coincides with the topology (for $C(L)$) of uniform convergence on compact sets.

Of course, there is no question of identifying X with the whole of $C(L)$, because multiplication by positive scalars is possible in L, and for each $x \in X$, we must have

$$x(\lambda f) = \lambda x(f) \quad (f \in L, \lambda \geqq 0).$$

In contrast to the normed case, there is no obvious way of choosing one representative of each ray in L.

If X is not barrelled, then the topology of uniform convergence on compact sets may be strictly larger than τ. To show this, let X be the space of all finite real sequences with the supremum norm. For each n, a lattice homomorphism e_n is defined by $e_n(x) = \xi_n$, where $x = (\xi_j)$. Then $ne_n \to 0$ with respect to $\sigma(X)$, so $E = \{ne_n\} \cup \{0\}$ is $\sigma(X)$-compact. However, $E^0 \notin N(X)$.

The well-known theorem on isometric embedding which applies in the normed case follows without difficulty from lemma 1, without making use of order-units or the second conjugate. Let $\| \ \|$ be an M-norm on X, and denote the unit ball by U. Let L_1 be the set of lattice homomorphisms on X with unit norm. If p is the Minkowski functional of $U - P$, then it is easily seen that $\|x\| = p(x) \vee p(-x)$ for all x. Lemma 1 now shows that, given $x \in X$, there exists $f \in L_1$ such that $|f(x)| = \|x\|$.

We conclude with a theorem on separation by pairs of lattice homomorphisms. We make use of the following result, proved by BONSALL in [2] (lemma 2, p. 124) and generalised in [3] (theorem 2):

Lemma 3. *Suppose that E is compact and that A is a sublattice of $C(E)$. Suppose, also, that $\varphi \in C(E)$, and that, given $s, t \in E$, there exists $\psi \in A$ such that*

$$\varphi(s) < \psi(s) + 1,$$

$$\varphi(t) > \psi(t) - 1.$$

Then there exists $\psi \in A$ such that $|\varphi(s) - \psi(s)| < 1$ for all $s \in E$.

Theorem 7. *Let X be an M-space, and A a convex sublattice of X. If $x \notin \bar{A}$, then there exist continuous lattice homomorphisms f, g on X such that $f - g$ separates x from A.*

Proof. There exists $U \in \mathcal{N}(X)$ such that $(x+U) \cap A = \emptyset$. By lemma 2, $V \in \mathcal{N}(X)$ exists such that $\overline{V+P} \cap \overline{V-P} \subseteq U$. Let L_V be the set of lattice homomorphisms in V^0. Then L_V is $\sigma(X)$-compact. Given $a \in A$, $a - x \notin \overline{V+P}$ or $a - x \notin \overline{V-P}$, and as in the proof of theorem 6, $f \notin L_V$ exists such that $|f(a-x)| > 1$.

For each $x \in X$, a function $\hat{x} \in C(L_V)$ is defined by $\hat{x}(f) = f(x)$ $(f \in L_V)$. (This time, the mapping $x \to \hat{x}$ does not need to be one-to-one, but this does not affect our proof.) $\{\hat{a} : a \in A\}$ is a sublattice of $C(L_V)$, so by lemma 3, there exist $f', g' \in L_V$ such that, given $a \in A$, either

(1) $$f'(x) \geqq f'(a) + 1$$

or

(2) $$g'(x) \leqq g'(a) - 1.$$

In $\boldsymbol{R}^2$, write

$$\{(f'(a), -g'(a)) : a \in A\} = H,$$

$$\{(\xi, \eta) : \xi \geqq 0 \text{ and } \eta \geqq 0\} = Q.$$

Then H is convex, and relations (1) and (2) show that

$$(f'(x), -g'(x)) \notin \overline{H - Q}.$$

Hence there is a linear functional χ on $\boldsymbol{R}^2$ such that

$$\chi(f'(x), -g'(x)) > \sup \chi(H - Q).$$

χ is bounded below on Q, so is of the form

$$\chi(\xi, \eta) = \lambda \xi + \mu \eta,$$

where $\lambda, \mu \geqq 0$. Write $f = \lambda f'$, $g = \mu g'$. Then

$$f(x) - g(x) > \sup \{f(a) - g(a) : a \in A\}.$$

Applied to spaces $C(E)$ (E compact), theorem 7 says the same as theorem 1 of [2], since the lattice homomorphisms on $C(E)$ are the functionals of the form f_s $(s \in E)$, where $f_s(x) = x(s)$.

Theorem 7 shows that, in the special case when the compact set E consists of one point only, positive linear functionals can be replaced by lattice homomorphisms in theorem 3 of [3]. It remains an open question whether this is true in all cases.

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Math. Zeitschr. 103, 151–161 (1968)

A Fixpoint Theorem for Complete Categories

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Received March 1, 1967

Classical results on lattices provide a fruitful source of inspiration for discoveries about categories, in view of the fact that a partially ordered set may be regarded as a category in which there is at most one map between any two objects. Our aim here is to take a look at a fixpoint theorem for lattices, due to BIRKHOFF [1], Chapter IV, and TARSKI [6], and to extend it to categories. We begin by stating the result in its general form given by TARSKI.

Theorem. *Let A be a complete lattice, T a commutative set of increasing functions $t: A \to A$, F the set of all fixpoints common to all $t \in T$, then F is non-empty and a complete lattice. In particular*

$$\sup F = \sup \{a \in A \mid \forall_{t \in T}\, a \leqq t(a)\} \in F,$$

$$\inf F = \inf \{a \in A \mid \forall_{t \in T}\, t(a) \leqq a\} \in F.$$

Here t: $A \to A$ is said to be *increasing* if $a \leqq a' \Rightarrow t(a) \leqq t(a')$ for all $a, a' \in A$. A set T of such functions is said to be *commutative* if $tt' = t't$ for all $t, t' \in T$. An element $f \in A$ is said to be a *fixpoint* of t if $t(f) = f$.

1. Preliminaries

We find it convenient to have two different stylistic conventions for denoting categories, objects, and maps: $\mathscr{A}$, A, and a, or X, x, and ξ respectively.

Our main tool will be the concept of "supremum" ("infimum") of a functor Γ: $X \to \mathscr{A}$. This is also called "colimit" ("limit") or "right root" ("left root") in the literature.

To review this notion, let us write C_A: $X \to \mathscr{A}$ for the functor with constant value A in $\mathscr{A}$. Thus

$$C_A(x) = A, \quad C_A(\xi) = 1_A,$$

for each object x and map ξ of X. Then (A, u) is said to be a *supremum* of Γ provided u: $\Gamma \to C_A$ is a natural transformation such that, for every natural transformation u': $\Gamma \to C_{A'}$, there exists a unique map a: $A \to A'$ with $C_a u = u'$.

* These results were obtained at the Mathematical Research Institute of the E.T.H. in Zürich.

Here $C_a(x)=a$, of course, hence the last equation means

$$a\,u(x)=u'(x)$$

for all objects x of X. Actually the supremum is unique up to isomorphism, and we write $\sup \Gamma \cong (A, u)$.

We call $\mathscr{A}$ *sup-complete* if $\sup \Gamma$ exists for any small "index" category X. In particular, this implies that $\mathscr{A}$ has an *initial* object Λ: Let X be the empty category, Γ: $X \to \mathscr{A}$ the inclusion functor and Λ a supremum of Γ, then the above condition asserts that each object A' of $\mathscr{A}$ determines a unique map a: $\Lambda \to A'$.

The dual notions *infimum*, *inf-complete*, and *terminal* object are of course defined dually. We shall use the fact that a terminal object can be expressed not only as an infimum of a functor with empty index category, but also as the supremum of an identity functor.

The following result appears to be well-known, but I have not been able to find it in the literature.

Proposition 1.1. *Let V be a terminal object of $\mathscr{A}$, so that for each object A of $\mathscr{A}$ there exists a unique map $t(A)$: $A \to V$. Then t: $1_{\mathscr{A}} \to C_V$ is a natural transformation and* $\sup 1_{\mathscr{A}} \cong (V, t)$. *Conversely, if* $\sup 1_{\mathscr{A}} \cong (V, t)$ *exists, then V is a terminal object of $\mathscr{A}$.*

Proof. First, assume V is terminal and $t(A)$: $A \to V$. Then, for any a: $A \to A'$, $t(A')\,a=t(A)$ by uniqueness, and so t is natural. Now suppose t': $1_{\mathscr{A}} \to C_V$, is also natural, then $t'(V)\,t(A)=t'(A)$. Suppose there is any map x: $V \to V'$ such that $x\,t(A)=t'(A)$ for all objects A of $\mathscr{A}$. Then, in particular, $x\,t(V)=t'(V)$. But $t(V)=1$, being the unique map $V \to V$, hence $x=t'(V)$. Thus $\sup 1_{\mathscr{A}} \cong (V, t)$.

Conversely, assume that $\sup 1_{\mathscr{A}} \cong (V, t)$. Hence there exists a unique map v: $V \to V$ such that $v\,t(A)=t(A)$ for all objects A of $\mathscr{A}$. But naturality of t yields $t(V)\,t(A)=t(A)$, and so $t(V)=v=1$. Now suppose a: $A \to V$ for a fixed object A of $\mathscr{A}$. Then naturality of t yields $t(V)\,a=t(A)$, hence $a=t(A)$. Thus $t(A)$ is the unique map $A \to V$, and so V is terminal.

Unfortunately, the index category $\mathscr{A}$ of $1_{\mathscr{A}}$ is not in general small, hence we cannot infer the existence of a terminal object from sup-completeness alone. By "subcategory" we shall always mean "full subcategory".

Proposition 1.2. *A sup-complete category $\mathscr{A}$ has a terminal object V if and only if it has a small subcategory $\mathscr{S}$ such that every object A of $\mathscr{A}$ admits a map $A \to S$, where S is an object of $\mathscr{S}$.*

Proof. To see the necessity of the condition, we take $\mathscr{S}$ the subcategory whose sole object is V.

Conversely, assume the condition, and let the supremum of the inclusion functor $\mathscr{S} \to \mathscr{A}$ be (V, v). We try to define t: $1_{\mathscr{A}} \to V$ by $t(A)=v(S)\,a$, where a: $A \to S$. For this definition to make sense, we must show that it does not depend on the choice of S and a.

Suppose also $a'\colon A \to S'$ with S' in $\mathscr{S}$. We wish to show that $v(S)\,a = v(S')\,a'$. To this purpose, let $p\colon S \to P$ and $p'\colon S' \to P$ be maps such that $p\,a = p'\,a'$. This may be done by forming a pushout, a special kind of supremum. By assumption there exists a map $a^*\colon P \to S^*$ with S^* in $\mathscr{S}$. Using naturality of v, we then have

$$v(S)\,a = v(S^*)\,a^*\,p\,a = v(S^*)\,a^*\,p'\,a' = v(S')\,a',$$

as required.

It is easily verified that $t\colon 1_{\mathscr{A}} \to C_V$ is a natural transformation and that $t|\mathscr{S} = v$. Now suppose $t'\colon 1_{\mathscr{A}} \to C_{V'}$ is another natural transformation. Then $t'|\mathscr{S}$ is still natural, hence there exists a unique map $x\colon V \to V'$ such that $t'(S) = x\,t(S)$ for all S in $\mathscr{S}$. Take any A in $\mathscr{A}$ and pick $a\colon A \to S$ with S in $\mathscr{S}$. Then naturality of t' gives

$$t'(A) = t'(S)\,a = x\,t(S)\,a = x\,v(S)\,a = x\,t(A).$$

It is clear that x is unique with this property, and so $\sup 1_{\mathscr{A}} \cong (V, t)$. By Proposition 1.1, V is a terminal object of $\mathscr{A}$.

2. Fixpoint of a Single Functor

For expository reasons we shall first consider a single functor $T\colon \mathscr{A} \to \mathscr{A}$. With this functor we associate a category $(1, T)$ as follows: Its objects are all maps $a\colon A \to T(A)$, where A is any object of $\mathscr{A}$. Its maps are triples $\alpha = (a, a', f)$ such that $T(f)\,a = a'f$. In other words, α is a commutative diagram:

$$\begin{array}{ccc} A' & \xrightarrow{a'} & T(A') \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle T(f)} \\ A & \xrightarrow{a} & T(A). \end{array}$$

There is a canonical functor $\Delta\colon (1, T) \to \mathscr{A}$, where $\Delta(a) = A$ such that $a\colon A \to T(A)$, and $\Delta(\alpha) = f$. (The reason for the notation "$(1, T)$" is that this is a special case of a construction involving two functors. In fact, there is a dual category $(T, 1)$, which will play a rôle later.)

Lemma 2.1. *A functor $\Gamma\colon X \to (1, T)$ will have a sup if $\Delta\Gamma\colon X \to \mathscr{A}$ has a sup. In particular, $(1, T)$ is sup-complete if $\mathscr{A}$ is.*

Proof. Assume $\sup \Delta\Gamma \cong (A, u)$. The map $T(u(x))\,\Gamma(x)\colon \Delta(\Gamma(x)) \to T(A)$ gives rise to a natural transformation $\Delta\Gamma \to C_{T(A)}$. Therefore there exists a unique map $a\colon A \to T(A)$ such that $a\,u(x) = T(u(x))\,\Gamma(x)$. Let $\alpha(x) = (\Gamma(x), a, u(x))$, then it is easily verified that $\sup \Gamma \cong (a, \alpha)$.

Lemma 2.2. *If $(1, T)$ has a terminal object $f\colon F \to T(F)$, then f is an isomorphism. In fact, let $(a, f, u(a))$ be the unique map $a \to f$ in $(1, T)$, then $f^{-1} = u(T(f))$.*

Proof. Put $f'=u(T(f))$, then the top square of the following diagram commutes:

$$\begin{array}{ccc} F & \xrightarrow{f} & T(F) \\ {\scriptstyle f'}\uparrow & & \uparrow{\scriptstyle T(f')} \\ T(F) & \xrightarrow{T(f)} & T(T(F)) \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle T(f)} \\ F & \xrightarrow{f} & T(F). \end{array}$$

Since the bottom square commutes trivially, we read off that $f'f=u(f)=1$. Hence, from the top square, $ff'=T(f'f)=T(1)=1$.

Proposition 2.3. *Assume that $\mathscr{A}$ is sup-complete and that $T\colon \mathscr{A}\to\mathscr{A}$. There exists an object F in $\mathscr{A}$ such that $T(F)\cong F$ if the following condition holds: $\mathscr{A}$ has a small subcategory $\mathscr{S}$ and, for each map $a\colon A\to T(A)$ with A in $\mathscr{A}$, there exist maps $x\colon A\to S$ and $s\colon S\to T(S)$ with S in $\mathscr{S}$ such that $sx=T(x)\,a$.*

Proof. By Lemma 2.1, $(1, T)$ is sup-complete. Under the assumed condition, $(1, T)$ has a small subcategory whose objects are the maps $s\colon S\to T(S)$ with S in $\mathscr{S}$; moreover, each object a of $(1, T)$ admits a map $(a, s, x)\colon a\to s$, where s is in the small subcategory. By Proposition 1.2, $(1, T)$ has a terminal object $f\colon F\to T(F)$. By Lemma 2.2, f is an isomorphism.

That the sufficient condition of Proposition 2.3 is not necessary is seen by taking $\mathscr{A}$ to be the ordered class of all ordinals regarded as a category: when $m\leqq n$ there is exactly one map $m\to n$. $\mathscr{A}$ is sup-complete (but not inf-complete, the empty set having no infimum). Let $T(m)$ be the predecessor of m if there is one, otherwise put $T(m)=m$. Thus T has plenty of fixpoints, but the condition of Proposition 2.3 is not satisfied.

Corollary 2.4. *If $\mathscr{A}$ is a small sup-complete category and $T\colon \mathscr{A}\to\mathscr{A}$, there exists an object F of $\mathscr{A}$ such that $T(F)\cong F$.*

Of course, in this case $\mathscr{A}$ is essentially a complete lattice (see [3], p. 78), hence this is essentially contained in the Birkhoff-Tarski Theorem.

3. Commutative Sets of Functors

It turns out that we shall obtain a necessary and sufficient condition for the existence of a fixpoint of a single functor only after considering the corresponding problem for a commutative set of functors. What is a commutative set of functors?

Let $\mathscr{T}$ be a set of functors $T\colon \mathscr{A}\to\mathscr{A}$. One might wish to call $\mathscr{T}$ "strictly commutative" if $TT'=T'T$ for each $T, T'\in\mathscr{T}$. It is indeed possible to develop a fixpoint theory for strictly commutative sets of functors. However, this concept is not too satisfactory, as it fails to be preserved by natural isomorphisms of functors. One would really like to consider a set $\mathscr{T}$ for which there are natural isomorphisms $TT'\cong T'T$. This is not quite enough, as one soon discovers trying to work with this notion.

Definition 3.1. *A commutative set of functors* on $\mathscr{A}$ is a pair $(\mathscr{T}, g)$, where $\mathscr{T}$ is a set of functors $T\colon \mathscr{A}\to\mathscr{A}$ and g is a family of natural isomorphisms $g(T, T')\colon TT'\to T'T$ satisfying the following three conditions:

(1) $$g(T, T)=1_{TT},$$

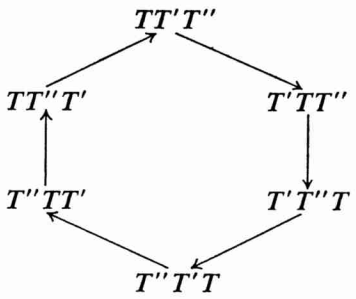
(2) $$g(T, T')\,g(T', T)=1_{TT},$$

(3) $$g(T'', T, T')\,g(T', T'', T)\,g(T, T', T'')=1_{TT'T''},$$

where

$$g(T, T', T'')=\big(T'\,g(T, T'')\big)\big(g(T, T')\,T''\big).$$

Condition (3) is illustrated by the following commutative hexagon:



Remark 1. Conditions (1) and (2) are not serious restrictions. Assume that $TT'\cong T'T$ for any $T, T'\in\mathscr{T}$, and suppose that $\mathscr{T}$ is simply ordered. When $T<T'$, let $g(T, T')$ be the given isomorphism $TT'\to T'T$. When $T=T'$, put $g(T, T')=1_{TT}$. When $T>T'$, put $g(T, T')=g(T', T)^{-1}$. Then (1) and (2) hold.

Remark 2. A simple calculation shows that (3) is automatically satisfied when $T'=T''$. Thus (3) holds when $\mathscr{T}$ has at most two elements.

Remark 3. Let $T_1, \ldots, T_n$ be members of $\mathscr{T}$, then there is a natural transformation

$$T_1 \ldots T_{i-1}\, g(T_i, T_{i+1})\, T_{i+2} \ldots T_n\colon T_1 \ldots T_i\, T_{i+1} \ldots T_n \to T_1 \ldots T_{i+1}\, T_i \ldots T_n.$$

Now suppose $t\colon T_1 \ldots T_n \to T_1 \ldots T_n$ is a product of transformations of this type, we may ask whether necessarily $t=1$. The answer due to MACLANE is yes (see [4]). This fact is shown by him to be a consequence of MOORE's presentation of the symmetric group in terms of generators and relations. (See [2], p. 63, in particular (6.22).)

Definition 3.2. Given a commutative set of functors $(\mathscr{T}, g)$ on $\mathscr{A}$, let $a=\{a_T \mid T\in\mathscr{T}\}$ be a family of maps $a_T\colon A\to T(A)$. We say that (A, a) is *compatible* with $(\mathscr{T}, g)$ if, for any $T, T'\in\mathscr{T}$,

$$T(a_{T'})\,a_T=g(T', T)(A)\,T'(a_T)\,a_{T'}.$$

If moreover the a_T are all isomorphisms, we call (A, a) a *fixpoint* of $(\mathscr{T}, g)$.

We shall find it convenient to abbreviate

$$g(T', T)(A)\, T'(a_T) = (T'\, a)_T .$$

The compatibility of (A, a) is then illustrated by the family of commutative squares:

$$\begin{array}{ccc} T'(A) & \xrightarrow{(T'\,a)_T} & TT'(A) \\ {\scriptstyle a_{T'}}\uparrow & & \uparrow{\scriptstyle T(a_{T'})} \\ A & \xrightarrow{a_T} & T(A). \end{array}$$

We hasten to point out the following auxiliary result, which depends heavily on Condition (3):

Lemma 3.3. *Let $(\mathscr{T}, g)$ be a commutative set of functors on $\mathscr{A}$. If (A, a) is compatible with $(\mathscr{T}, g)$, then so is $(T^*(A), T^* a)$, for any $T^* \in \mathscr{T}$.*

Proof.

$$\begin{aligned} & T\big((T^* a)_{T'}\big)(T^* a)_T \\ &= T\big(g(T^*, T')(A)\big)\, TT^*(a_{T'})\, g(T^*, T)(A)\, T^*(a_T) \\ &= X_1\, g(T^*, T)\big(T'(A)\big)\, T^*\, T(a_{T'})\, T^*(a_T), \end{aligned}$$

in view of the naturality of $g(T^*, T)$. Factoring out T^* from the last two factors and using the compatibility of (A, a), we see that this

$$\begin{aligned} &= X_1 X_2\, T^*\big(g(T', T)(A)\big)\, T^*\, T'(a_T)\, T^*(a_{T'}) \\ &= X_1 X_2 X_3\, T^*\, T'(a_T)\, g(T', T^*)(A)(T^* a)_{T'} . \end{aligned}$$

Using naturality again, we see that this

$$\begin{aligned} &= X_1 X_2 X_3\, g(T', T^*)\big(T(A)\big)\, T'\, T^*(a_T)(T^* a)_{T'} \\ &= X_1 X_2 X_3 X_4\, T'\big(g(T, T^*)(A)\big)\, T'\big((T^* a)_T\big)(T^* a)_{T'} \\ &= X_1 X_2 X_3 X_4 X_5\, g(T, T')\big(T^*(A)\big)(T'\, T^* a)_T (T^* a)_{T'} \\ &= (T'\, T^* a)_T (T^* a)_{T'} , \end{aligned}$$

since $X_1 X_2 X_3 X_4 X_5 X_6 = 1$, by Condition (3).

Let $(\mathscr{T}, g)$ be a commutative set of functors on $\mathscr{A}$. We construct the category $(1, (\mathscr{T}, g))$ as follows: Its objects are all families of maps $a = \{a_T \mid T \in \mathscr{T}\}$, a_T: $A \to T(A)$, such that (A, a) is compatible with $(\mathscr{T}, g)$. Its maps are triples (a, a', x): $a \to a'$, where (A, a) and (A', a') are compatible with $(\mathscr{T}, g)$ and x: $A \to A'$ is a map such that

$$a'_T\, x = T(x)\, a_T$$

for all $t \in \mathscr{T}$. Again we have a functor Δ: $(1, (\mathscr{T}, g)) \to \mathscr{A}$, where $\Delta(a) = A$ and $\Delta((a, a', x)) = x$.

The following generalizes Lemma 2.1.

Lemma 3.4. *Assume* $(\mathscr{T}, g)$ *is a commutative set of functors on* $\mathscr{A}$. *A functor* $\Gamma\colon X \to (1, (\mathscr{T}, g))$ *will have a supremum if* $\Delta\Gamma\colon X \to \mathscr{A}$ *has. In particular,* $(1, (\mathscr{T}, g))$ *is sup-complete if* $\mathscr{A}$ *is.*

Proof. Assume $\sup \Delta\Gamma \cong (A, u)$. As in Lemma 2.1, we obtain for each T a unique map $a_T\colon A \to T(A)$ such that

$$a_T\, u(x) = T\big(u(x)\big)\, \Gamma(x)_T .$$

Putting $a = \{a_T \mid T \in \mathscr{T}\}$, we shall verify that (A, a) is compatible with $(\mathscr{T}, g)$. Indeed

$$\begin{aligned} T(a_{T'})\, a_T\, u(x) &= T\big(a_{T'}\, u(x)\big)\, \Gamma(x)_T \\ &= T\, T'\big(u(x)\big)\, T\big(\Gamma(x)_{T'}\big)\, \Gamma(x)_T \\ &= T\, T'\big(u(x)\big)\, g(T', T)\big(\Delta\Gamma(x)\big)\, T'\big(\Gamma(x)_T\big)\, \Gamma(x)_{T'} , \end{aligned}$$

since $(\Delta\Gamma(x), \Gamma(x))$ is compatible with $(\mathscr{T}, g)$. Since $g(T', T)$ is a natural transformation, this

$$\begin{aligned} &= g(T', T)(A)\, T'\, T\big(u(x)\big)\, T'\big(\Gamma(x)_T\big)\, \Gamma(x)_{T'} \\ &= g(T', T)(A)\, T'\big(a_T\, u(x)\big)\, \Gamma(x)_{T'} \\ &= g(T', T)(A)\, T'(a_T)\, a_{T'}\, u(x) . \end{aligned}$$

Now this is true for all objects x of X, hence

$$T(a_{T'})\, a_T = g(T', T)(A)\, T'(a_T)\, a_{T'} ,$$

as required.

Finally, it is easily verified that $\sup \Gamma \cong (a, \alpha)$, where α is a suitable natural transformation, as in the proof of Lemma 2.1.

Next we generalize Lemma 2.2.

Lemma 3.5. *If* $(1, (\mathscr{T}, g))$ *has a terminal object* $f = \{f_T \mid T \in \mathscr{T}\}$, $f_T\colon F \to T(F)$, *then all* f_T *are isomorphisms, and so* (F, f) *is a fixpoint of* $(\mathscr{T}, g)$. *In fact, let* $(a, f, u(a))$ *be the unique map* $a \to f$ *in* $(1, (\mathscr{T}, g))$, *then* $f_T^{-1} = u(Tf)$.

Proof. Consider the following diagram:

$$\begin{array}{ccc} F & \xrightarrow{f_T} & T(F) \\ {\scriptstyle u(T'f)}\uparrow & & \uparrow{\scriptstyle T(u(T'f))} \\ T'(F) & \xrightarrow{(T'f)_T} & T\,T'(F) \\ {\scriptstyle f_{T'}}\uparrow & & \uparrow{\scriptstyle T(f_{T'})} \\ F & \xrightarrow{f_T} & T(F) . \end{array}$$

The top square commutes by definition of u. The bottom square commutes by compatibility of (F, f). Therefore

$$u(T'f)\,f_{T'}=u(f)=1.$$

Putting $T=T'$ in the top square, we then get

$$f_{T'}\,u(T'f)=T'\big(u(T'f)\,f_{T'}\big)=T'(1)=1.$$

(This argument may be simplified slightly by taking $T=T'$ in the first place.)

We refrain from generalizing Proposition 2.3. Instead we prove the following:

Proposition 3.6. *Assume that $\mathscr{A}$ is sup-complete and locally small, that $(\mathscr{T}, g)$ is a commutative set of functors on $\mathscr{A}$, and that $T^0 \in \mathscr{T}$ has small image. Then $(\mathscr{T}, g)$ has a fixpoint (F, f), the terminal object of $(1, (\mathscr{T}, g))$.*

A category $\mathscr{A}$ is called *locally small* if, for any two objects A, A' of $\mathscr{A}$, the maps $A \to A'$ form a set. By saying that T^0 has *small image* we mean that it factors thus: $\mathscr{A} \to \mathscr{S} \to \mathscr{A}$, where $\mathscr{S}$ is a small subcategory of $\mathscr{A}$, $\mathscr{S} \to \mathscr{A}$ being the inclusion functor.

Proof. By Lemma 3.4, $(1, (\mathscr{T}, g))$ is sup-complete. Let a be a given object of this category, then so is $T^0 a$, by Lemma 3.3. In fact, $T^0 a$ lies in the small subcategory of $(1, (\mathscr{T}, g))$ whose objects s have components s_T: $S \to T(S)$ with S in $\mathscr{S}$. Since

$$(T^0 a)_T\, a_{T^0}=T(a_{T^0})\, a_T$$

by compatibility, $(a, T^0 a, a_{T^0})$ is a map $a \to T^0 a$. It then follows from Proposition 1.2 that $(1, (\mathscr{T}, g))$ has a terminal object f with components f_T: $F \to T(F)$. By Lemma 3.5, the f_T are isomorphisms, and so (F, f) is a fixpoint.

4. Necessary and Sufficient Condition for the Existence of a Fixpoint

Lemma 4.1. *Let (F, f) be a fixpoint of the commutative set of functors $(\mathscr{T}, g)$ on $\mathscr{A}$. Then the latter may be extended to a commutative set $(\mathscr{T} \cup \{C_F\}, g')$ by putting*

$$g'(T, T')=g(T, T'), \qquad g'(C_F, C_F)=1,$$

$$g'(C_F, T)=f_T, \qquad g'(T, C_F)=f_T^{-1}.$$

Proof. It suffices to verify Condition (3) with T'' replaced by C_F. Indeed, this condition then becomes

$$T(f_{T'})\, f_T\, f_{T'}^{-1}\, T'(f_T^{-1})\, g(T, T')(F)=1,$$

which is equivalent to the compatibility of (F, f).

Theorem 4.2. *Assume that $\mathscr{A}$ is sup-complete and locally small, and that $(\mathscr{T}, g)$ is a commutative set of functors on $\mathscr{A}$. Then the latter has a fixpoint if and only if it can be extended to a commutative set $(\mathscr{T} \cup \{U\}, g')$ where U has small image.*

Proof. If $(\mathscr{T}, g)$ has a fixpoint (F, f), take $U = C_F$ and apply Lemma 4.1. Conversely, if the condition holds, apply Proposition 3.6 to the extended commutative set. Then this has a fixpoint, which is a fortiori a fixpoint of $(\mathscr{T}, g)$.

Corollary 4.3. *Assume that $\mathscr{A}$ is sup-complete and locally small and that $T\colon \mathscr{A} \to \mathscr{A}$. There exists an object F of $\mathscr{A}$ such that $T(F) \cong F$ if and only if there exists a functor $U\colon \mathscr{A} \to \mathscr{A}$ with small image such that $TU \cong UT$.*

It is instructive to compare this corollary with Proposition 2.3. The proof of the earlier result invoked a terminal object of the category $(1, T)$, while the proof of the present result depends on the category $(1, (\{T, U\}, g))$ instead, where $g(T, U)\colon TU \to UT$ is a natural isomorphism.

To illustrate the above corollary, let $\mathscr{A}$ be the category of sets, and let $T\colon \mathscr{A} \to \mathscr{A}$ be the covariant functor such that $T(A) = 2^A$ and $T(f)(X) = \{f(x) \mid x \in X\}$ for $f\colon A \to A'$ and $X \in 2^A$. Cantor showed that T has no fixpoint, hence we may conclude that there is no functor $U\colon \mathscr{A} \to \mathscr{A}$ with small image such that $TU \cong UT$. It follows that the condition of the corollary is not redundant.

For a positive application of the corollary, a functor U with small image should be given to begin with. Here are some examples of such functors:

Example 1. $U = C_B$, the functor with constant value B.

Example 2. Assume $\mathscr{A}$ has an initial object Λ and a fixed object $B \neq \Lambda$. Define $U(A) = \Lambda$ if there exists a map $A \to B$, $U(A) = B$ otherwise. Moreover, for any map $a\colon A \to A'$ in $\mathscr{A}$, put $U(a) = 1_\Lambda$ if there is a map $A' \to B$, $U(a) = 1_B$ if there is no map $A \to B$, and $U(a) =$ the unique map $\Lambda \to B$ in the remaining case. Then U is a functor with image $\{\Lambda, B\}$.

Example 3. Let $\mathscr{A}$ be the category of associative rings with unity element. For any ring A, let $U(A)$ be the characteristic subring of A, that is the subring generated by 1. Then $U(A) \cong Z/n\,Z$ where n is a non-negative integer.

This example may be generalized to any category $\mathscr{A}$ with initial object Λ, coequalizers, and pullbacks, provided Λ has a representative set of those quotient objects which are coequalizers. Indeed, let $U(A)$ be the codomain of the coequalizer of the pair of maps which form the pullback of the diagram $\Lambda \to A \leftarrow \Lambda$, then U will have small image.

Unfortunately the application of Corollary 4.3 to these three examples is not very interesting. For, if we assume that T commutes with U, we can easily show that T has a fixpoint without quoting Corollary 4.3. In the first example this is trivial, in the second easy, in the last a consequence of the Birkhoff-Tarski Theorem. In fact, assume that U sends $\mathscr{A}$ onto a subcategory $\mathscr{L}$ which is a complete lattice and that $U \mid \mathscr{L} = 1$, then $UT \mid \mathscr{L}$ has a fixpoint F by the classical result, hence

$$T(F) \cong T\,U(F) \cong U\,T(F) \cong F.$$

An interesting application of Theorem 4.2 or its corollary remains to be found.

5. Completeness of the Category of Fixpoints

Lemma 5.1. *Let $(\mathscr{T}, g)$ be a commutative set of functors on the locally small inf-complete category $\mathscr{A}$, and assume $T^0 \in \mathscr{T}$ has small image. Then the category $(1, (\mathscr{T}, g))$ has a reflective subcategory* Fix $(\mathscr{T}, g)$, *whose objects are all families of isomorphisms a_T: $A \to T(A)$ in this category.*

Proof. We saw in Section 2 how to construct a category $(1, T)$ with each functor T: $\mathscr{A} \to \mathscr{A}$. Dually one may of course construct a category $(T, 1)$ whose objects are all maps a: $T(A) \to A$ with A in $\mathscr{A}$. Similarly the category $((\mathscr{T}, g), 1)$ has as objects all families of maps a_T: $T(A) \to A$ satisfying a dual compatibility condition (see Definition 3.2). We refrain from spelling out the details.

Let a: $A \to T(A)$ be an object of $(1, (\mathscr{T}, g))$, which is fixed for the remainder of this proof. We consider the category $(C_A, 1)$ associated with the constant functor C_A. We observe that any member T of $\mathscr{T}$ induces a functor T_a: $(C_A, 1) \to (C_A, 1)$ as follows: For any object x: $A \to B$ of $(C_A, 1)$, put $T_a(x) = T(x)\, a_T$: $A \to T(B)$; for any map $\beta = (x, x', b)$ of $(C_A, 1)$, put $T_a(\beta) = (T_a(x), T_a(x'), T(b))$. We thus obtain a set $\mathscr{T}_a$ of functors on $\mathscr{A}$.

Next we introduce the natural transformation $g_a(T, T')$: $T_a T'_a \to T'_a T_a$ by defining

$$g_a(T, T')(x) = \big(T_a\, T'_a(x), T'_a\, T_a(x), g(T, T')(B)\big).$$

This is a map in $(C_A, 1)$ because

$$\begin{aligned} g(T, T')(B)\, T_a\, T'_a(x) &= g(T, T')(B)\, T\, T'(x)\, T(a_{T'})\, a_T \\ &= T'\, T(x)\, g(T, T')(A)\, T(a_{T'})\, a_T \\ &= T'\, T(x)\, T'(a_T)\, a_{T'} \\ &= T'_a\, T_a(x), \end{aligned}$$

by naturality of $g(T, T')$ and by compatibility of (A, a). A routine calculation shows that g_a satisfies Conditions (1) to (3) of Definition 3.1. Therefore $(\mathscr{T}_a, g_a)$ is a commutative set of functors on $(C_A, 1)$.

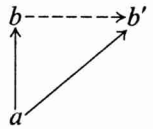
We observe that T_a^0 has small image. Indeed, we know that T^0: $\mathscr{A} \to \mathscr{S} \to \mathscr{A}$, where $\mathscr{S}$ is small and $\mathscr{S} \to \mathscr{A}$ is the inclusion functor. Hence T_a^0: $(C_A, 1) \to \mathscr{S}_a \to (C_A, 1)$, where the second arrow again represents inclusion and $\mathscr{S}_a$ is small, since

$$\begin{aligned} \mathscr{S}_a &= \{T_a^0(x) \mid \exists_{B \text{ in } \mathscr{A}}\, x: A \to B\} \\ &\subseteq \{y \mid \exists_{S \text{ in } \mathscr{S}}\, y: A \to S\}. \end{aligned}$$

Now, by the dual of Lemma 2.1, $(C_A, 1)$ is inf-complete. Hence, by the dual of Lemma 3.4, $((\mathscr{T}_a, g_a), 1)$ is inf-complete. Therefore, by the dual of Proposition 3.6, this category has an initial object φ. Here φ will be a dually compatible family of isomorphisms φ_{T_a}: $T_a(x) \to x$ in the category $(C_A, 1)$.

We shall write $\varphi_{T_a}=(T_a(x), x, b_T^{-1})$, where $T(x)\, a_T = b_T x$. It is easily verified that b is an object of $(1, (\mathscr{T}, g))$, in fact of Fix $(\mathscr{T}, g)$. Hence (a, b, x) is a map $a \to b$ in $(1, (\mathscr{T}, g))$.

We claim that $a \to b$ is a "best approximation" of a in Fix $(\mathscr{T}, g)$. Indeed, suppose (a, b', x'): $a \to b'$, where b' is an object of Fix $(\mathscr{T}, g)$. Put $\varphi'_{T_a} = (T_a(x'), x', b'^{-1}_T)$, then φ'_{T_a}: $T_a(x') \to x'$, and φ' is an object of $((\mathscr{T}_a, g_a), 1)$. However, φ was an initial object of this category, hence ther exists a unique map ξ: $x \to x'$ such that $\xi\, \varphi_{T_a} = \varphi'_{T_a} T_a(\xi)$. Write $\xi = (x, x', c)$, then $cx = x'$ and $c b_T^{-1} = b'^{-1}_T T(c)$. Therefore (b, b', c): $b \to b'$, and this is easily seen to be the only map which will make the following diagram commute:



It follows that Fix $(\mathscr{T}, g)$ is a reflective subcategory of $(1, (\mathscr{T}, g))$.

Proposition 5.2. *Let $(\mathscr{T}, g)$ be a commutative set of functors on a locally small sup- and inf-complete category $\mathscr{A}$, and assume that $T^0 \in \mathscr{T}$ has small image. Then the category of fixpoints of $(\mathscr{T}, g)$ is both sup- and inf-complete.*

Proof. By Lemma 5.1, Fix $(\mathscr{T}, g)$ is a reflective subcategory of $(1, (\mathscr{T}, g))$. Now the latter is sup-complete since $\mathscr{A}$ is, by Lemma 3.4. By a well-known (and easily proved) result, it follows that Fix $(\mathscr{T}, g)$ is also sup-complete. (See e.g. [5], p. 129, Proposition 5.1. "Reflective" is there called "co-reflective").

Finally we observe that Fix$(\mathscr{T}, g) \cong$ Fix$'(\mathscr{T}, g)$, the subcategory of $((\mathscr{T}, g), 1)$ consisting of isomorphisms. By duality, the latter is inf-complete.

Postscript. The referee observes that fixpoints of T may be viewed as algebras over a certain idempotent triple on LAWVERE's comma category $(\mathscr{A}, T)$. It is conceivable that various statements in this paper might then be reduced to statements about triples.

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Math. Zeitschr. 103, 162—172 (1968)

On Some Varieties of Soluble Groups. II

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Received June 3, 1967

Introduction

In the present note we continue the investigation begun in Part I (LEVIN [4]) of groups belonging to varieties associated with commutator laws of the form

$$[[x_1, \ldots, x_m], [x_{m+1}, \ldots, x_{m+n}]]=1, \tag{0.1}$$

which we will abbreviate by $[m, n]=1$. In particular, we wish to investigate possible extensions of the observation made in Part I that either of the laws $[2, 2]=1$ or $[x_1, x_2, x_3, x_4]=[x_1, x_2, x_4, x_3]$ may be used equivalently to define the variety of metabelian groups.

If a group G satisfies the law

$$[x_1, x_2, x_3, \ldots, x_k]=[x_1, x_2, x_{\varphi(3)}, \ldots, x_{\varphi(k)}], \tag{0.2}$$

where φ is a permutation of $\{3, \ldots, k\}$, we shall write that G satisfies $C(k, \varphi)$, while if G satisfies $C(k, \varphi)$ for all permutations φ of $\{3, \ldots, k\}$, we write that G satisfies $C(k)$.

A central result here (Lemma 4.1) states that G satisfies $C(n+2)$, $n \geqq 2$, if and only G satisfies the laws $[n-k, 2+k]=1$ for all $k=0, 1, \ldots, n-2$. From this it follows that $[3, 2]=1$ is equivalent with $C(5)$. Further, $[n, 2]=1$, $n \geqq 3$, implies $C(2n-1)$, and an example (Section 4) shows that this result is best possible. On the other hand, the results of Section 5 show that, in general, for $m, n \geqq 3$, the law $[m, n]=1$ will not imply any nontrivial law $C(k, \varphi)$.

1. Notation

The notation used here is that of Part I (cf., e.g., MAGNUS, KARRASS and SOLITAR [7], Chapt. 5). For $x_i \in G$, we define

$$[x_1, x_2]=x_1^{-1} x_2^{-1} x_1 x_2,$$

and, by induction

$$[x_1, \ldots, x_n]=[[x_1, \ldots, x_{n-1}], x_n]; \tag{1.1}$$

$$x^y=y^{-1} x y, \quad x^{-y}=y^{-1} x^{-1} y. \tag{1.2}$$

Thus, for $x, y, z \in G$,

$$[x, y, z]=[x^{-1} x^y, z]=x^{-y} x x^{-z} x^{yz}. \tag{1.3}$$

* The author gratefully acknowledges support by the National Science Foundation.

$Z(G)$ will denote the center of G, and G' will denote the commutator subgroup of G. In addition, the statement that G satisfies $[n, m]=1$ denotes that G satisfies the law

$$[[x_1, \ldots, x_n], [x_{n+1}, \ldots, x_{n+m}]]=1. \tag{1.4}$$

In particular, by (1.1), $[n, m]=1$ implies $[n+p, m+q]=1$ for any nonnegative numbers p and q.

The following standard identities will be required in the sequel (cf., [7], Section 5.2):

$$[x, y]=[y, x]^{-1}; \tag{1.5}$$

$$[xy, z]=[x, z]^y[y, z], \quad [x, yz]=[x, z][x, y]^z; \tag{1.6}$$

$$[x, y]^z=[x^z, y^z]; \tag{1.7}$$

$$[x, y, z^x][z, x, y^z][y, z, x^y]=1. \tag{1.8}$$

Further, we shall require the following lemma.

Lemma 1.1. *Let $a\in G$ be a fixed element of a group G. If $[a, x, a]=1$ for all $x\in G$, then $[a^y, a^z]=1$ for all $y, z\in G$.*

Proof. By (1.3), $[a, x, a]=[a^{-1}a^x, a]=[a^x, a]=1$, for all $x\in G$. Replacing x by yz^{-1} yields $[a^y, a^z]^{z^{-1}}=1$, whence, $[a^y, a^z]=1$.

Next, we note that if G satisfies $C(k, \varphi_1)$ and $C(k, \varphi_2)$ then G satisfies $C(k, \varphi_1\varphi_2^{-1})$. It follows that G satisfies $C(k, \varphi)$ for any φ in the group generated by φ_1 and φ_2.

This remark leads to the following lemma.

Lemma 1.2. *If G satisfies $C(k)$, then G satisfies $C(m)$ for all $m\geqq k$.*

Proof. (Induction.) The case $m=k$ is true by hypothesis. Thus, assume the Lemma for all $m\leqq p-1$ and let G satisfy $C(p-1)$. Then G satisfies $C(p, \varphi_1)$ for the cyclic permutation $\varphi_1=(3\,4\ldots p-1)$ and, by expressing $[x_1, \ldots, x_p]$ in the form $[[x_1, x_2], x_3, \ldots, x_p]$, $C(p, \varphi_2)$ for $\varphi_2=(p-1\,p)$ also. This completes the proof since φ_1 and φ_2 generate the group of all permutations of $\{3, \ldots, p\}$.

2. Some Lemmas

In this section some results for arbitrary $n\geqq 2$ will be listed.

Lemma 2.1 (cf., [4]). *The following statements concerning a group G and a fixed element $a\in G$ are equivalent:*

$$[a, [x, y]]=1, \qquad \textit{for all } x, y\in G; \tag{2.1}$$

$$[a, x, y]=[a, y, x], \qquad \textit{for all } x, y\in G. \tag{2.2}$$

Proof. First, we note that either (2.1) or (2.2) with the substitution $x=a$ implies that $[a, y, a]=1$ for all $y\in G$. Hence, by Lemma 1.1 and (1.3),

$$[a, x, y][a, y, x]^{-1}=a^{-x}a\,a^{-y}a^{xy}a^{y}a^{-1}a^{x}a^{-yx}=a^{xy}a^{-yx},$$

for all $x, y\in G$, and $a^{xy}=a^{yx}$ for all $x, y\in G$ if and only if $[a, [x, y]]=1$.

In this paper Lemma 2.1 will be applied exclusively to the case where $a\in G$ has the form $a=[x_1, \ldots, x_n]$.

Lemma 2.2. *Let G be a group satisfying $[n, 2]=1$, $n\geqq 2$. If $G/Z(G)$ satisfies $C(n+1)$, then G satisfies $C(n+2)$.*

Proof. By Lemma 2.1, G satisfies $C(n+2, \varphi_1)$ for $\varphi_1=(n+1\ n+2)$. Since $G/Z(G)$ satisfies $C(n+1)$ it satisfies, in particular, $C(n+1, \varphi_2)$ for $\varphi_2=(3\ 4\ldots n+1)$. Thus, G satisfies

$$[[x_1, \ldots, x_{n+1}][x_1, x_2, x_4, \ldots, x_{n+1}, x_3]^{-1}, x_{n+2}]=1. \tag{2.3}$$

Since $[n, 2]=1$, we have the following situation: G satisfies a law of the form $[ab^{-1}, x]=1$ for all $x\in G$, where $a, b\in Z(G')$. Thus, by (1.6), $[ab^{-1}, x]=[a, x]^{b^{-1}}[b^{-1}, x]=[a, x][b^{-1}, x]=1$, and, since $[b^{-1}b, x]=[b^{-1}, x]^{b}[b, x]=[b^{-1}, x][b, x]=1$, it follows that $[a, x]=[b, x]$. Thus, G satisfies $C(n+2, \varphi_2)$, and since G satisfies $C(n+2, \varphi_1)$, it follows that G satisfies $C(n+2)$.

Lemma 2.3. *If G satisfies $[n, m]=1$ and $[n-1, m+1]=1$, then $G/Z(G)$ satisfies $[n-1, m]=1$. In particular, if G satisfies $[n, n-1]=1$, then $G/Z(G)$ satisfies $[n-1, n-1]=1$.*

Proof. Let $c=[x_1, x_2, \ldots, x_{n-1}]$ and $d=[x_{n+1}, \ldots, x_{n+m}]$. By hypothesis, G satisfies $[c, x, d]=1$ and $[d, y, c]=1$ for any $x, y\in G$. By (1.8), we have

$$[c, x, d^{c}][d, c, x^{d}][x, d, c^{x}]=1, \tag{2.4}$$

so that $[d, c, x^{d}]=1$, and $[d, c, z]=1$ for any $z\in G$. Thus, $G/Z(G)$ satisfies $[d, c]=1$, that is, $[n-1, m]=1$. Finally, if G satisfies $[n, n-1]=1$, then, by (1.5), G satisfies $[n-1, n]=1$ so that, by the above result, $G/Z(G)$ satisfies $[n-1, n-1]$.

Lemma 2.4. *If G satisfies $[n, m]=1$ and $G/Z(G)$ satisfies $[n-1, m]=1$, then G satisfies $[m+1, n-1]$.*

Proof. By hypothesis and (2.4), $[x, d, c^{x}]=1$ for all $x, x_i\in G$.

3. The Variety $[3, 2]=1$

The following theorem follows directly from the preceding lemmas.

Theorem 3.1. *G satisfies $[3, 2]=1$ if and only if G satisfies $C(5)$.*

Proof. If G satisfies $[3, 2]=1$, then, by Lemma 2.3, $G/Z(G)$ satisfies $[2, 2]=1$, that is, $G/Z(G)$ is metabelian. Thus, $G/Z(G)$ satisfies $C(4)$, and, by Lemma 2.2, G satisfies $C(5)$. Conversely, if G satisfies $C(5)$, then, in

particular, G satisfies $C(5, \varphi_1)$ for $\varphi_1=(4\ 5)$ so that, by Lemma 2.1, G satisfies $[3, 2]=1$.

Corollary 3.2. *If G satisfies $[3, 2]=1$ or, equivalently, $C(5)$, then G is 2-metabelian.*

Proof. If G satisfies $[3, 2]=1$, then for any $x, y\in G$,

$$[[x, y, y^{-1}], [x, y]]=[[y, x][x, y]^{y^{-1}}, [x, y]]=[[x, y]^{y^{-1}}, [x, y]]=1,$$

or, $[[y^{-1}, x], [x, y]]=1$. By a result of HIGMAN [3], the latter identity implies that G is 2-metabelian.

Corollary 3.3. *G satisfies $C(n)$, for some $n\geqq 5$, if and only if G satisfies $C(n, \varphi)$ for $\varphi_1=(4\ 5 \ldots n-1)$ and $\varphi_2=(n-1\ n)$.*

Proof. If G satisfies $C(n)$ then, trivially, G satisfies $C(n, \varphi_i)$, $i=1, 2$. Conversely, let G satisfy $C(n, \varphi_i)$, $i=1, 2$. We proceed by induction. The case $n=5$ is just Theorem 3.1. Let $n>5$. By the remarks in Section 1, G satisfies $C(n, \varphi)$ for any φ in $gp\{\varphi_1, \varphi_2\}$, that is, for any φ such that $\varphi(3)=3$. Since G satisfies $C(n, \varphi_1)$, $G/Z(G)$ satisfies $C(n-1, \varphi_1)$, and, since G satisfies $C(n, (n-1\ n-2))$, $G/Z(G)$ satisfies $C(n-1, (n-1\ n-2))$. Hence, by induction, $G/Z(G)$ satisfies $C(n-1)$, and it follows from Lemma 2.2 that G satisfies $C(n)$.

Clearly, φ_1 and φ_2 in Corollary 3.3 may be replaced by any other set of generators of the group of permutations of $\{4, \ldots, n\}$.

Any nilpotent group of class $\leqq 4$ satisfies $C(5)$. The following example exhibits a group which satisfies $[3, 2]=1$ but which is neither 3-metabelian nor satisfies any finite engel condition.

Let $A(Z, 3)$ be the associative Z-algebra of formal power series in the non-commuting variables x, y, z with integer coefficients[1]. The elements $1+x$, $1+y$, $1+z$ generate a free multiplicative group $F(3)$. Let (r_1, r_2) denote $r_1 r_2-r_2 r_1$ for $r_i\in A(Z, 3)$. If the relations $x_1(x_2, x_3)=0$ are added to $F(3)$ for any monomials $x_i\in A(Z, 3)$, $x_i\notin Z$, whenever the degree (as a monomial in x, y, z) of $x_1 x_2 x_3$ is $\geqq 5$, the resulting group H satisfies $[3, 2]=1$. This fact is easily verified by noting that for any elements $1+x_i\in H$,

$$(x_1, x_2)=(1+x_1)(1+x_2)-(1+x_2)(1+x_1),$$

so that

$$\begin{aligned}(1+x_1)^{-1}(1+x_2)^{-1}(1+x_1)(1+x_2)&=1+(1+x_1)^{-1}(1+x_2)^{-1}(x_1, x_2)\\ &=1+\sum_{i=1}^{\infty} r_i(x_1, x_2),\end{aligned} \tag{3.1}$$

where $r_1=1$ and the degrees of r_i, $i\geqq 2$, are $\geqq 1$ (as monomials in x, y, z). It follows readily that $[1+x_1, 1+x_2, 1+x_3]=1+\sum r_i'$, where the degrees of

[1] For more details concerning $A(Z, r)$ and $F(r)$ see MAGNUS, KARRASS, SOLITAR [7], Chapt. 5.

r_i' are $\geqq 3$. Thus,

$$(3.2)\qquad \begin{aligned}[[1+x_1, 1+x_2], [1+x_3, 1+x_4, 1+x_5]] &= [1+\sum r_i(x_1, x_2), 1+\sum r_i''] \\ &= 1+(\sum r_i(x_1, x_2), \sum r_i'')+\sum r_i'''(\sum r_i(x_1, x_2), \sum r_i'')=1,\end{aligned}$$

by hypothesis, since the parentheses operation in $A(Z, 3)$ is distributive over addition and each term in (3.2) other than 1 has degree at least 5.

However, H is not 3-metabelian since, for instance, $[[1+x, 1+y], [1+x, 1+z]] \neq 1$ in H. Moreover, H satisfies no finite Engel condition[2].

Further examples of groups satisfying $[3, 2]=1$ have been given by MACDONALD [5], who shows that any 3-metabelian group satisfies $[3, 2]=1$ and that groups satisfying $[x, y]^2=1$ are 3-metabelian and, hence, satisfy $[3, 2]=1$.

4. The Varieties $[n, 2]=1$, $n \geqq 3$

In general, $[n, 2]=1$ does not imply $C(k)$ for any $k \leqq 2n-2$, as is illustrated by the following example.

The example to be constructed will again use $A(Z, 3)$, but for $n>3$ the group $F(2)$ generated by $1+x$ and $1+y$ will suffice. Thus, let $H=gp\{1+x, 1+y\}$ with the following relations for monomials r appearing in $A(Z, 3)$:

(i) For r of degree $n+k$, $2 \leqq k \leqq n-2$, with $r=x_1 \ldots x_{n+k}$, where each x_i has degree 1, the following relations hold:

$$x_1 \ldots x_{n+k} = x_{\varphi(1)} x_{\varphi(2)} \ldots x_{\varphi(k)} x_{k+1} \ldots x_n x_{\psi(n+1)} \ldots x_{\psi(n+k)},$$

for all permutations φ of $\{1, \ldots, k\}$ and all permutations ψ of $\{n+1, \ldots, n+k\}$;

(ii) For r of degree $\geqq 2n-1$ and for any factorization $r=r_1 r_2 r_3$ in terms of monomials r_i of degrees $\geqq 1$, the relation $r_1(r_2, r_3)=0$ holds.

H satisfies $[n, 2]=1$. To see this, we note first that for any $1+x_i \in H$,

$$(4.1)\qquad [1+x_1, 1+x_2, \ldots, 1+x_n]=1+(x_1, \ldots, x_n)+\sum r_i'(x_1, x_2) r_i'',$$

where $(x_1, \ldots, x_n)$ is defined inductively by $(x_1, \ldots, x_n)=((x_1, \ldots, x_{n-1}), x_n)$, the r_i' and r_i'' are monomials, each term in the summation has degree (as a monomial in x, y) at least $n+1$, and the degree of each r_i' is at least 1. This can be proved by a straightforward induction starting from (3.1). Thus, for any $1+x_i \in H$,

$$[[1+x_1, 1+x_2], [1+x_3, \ldots, 1+x_{n+2}]]=[1+\sum r_i(x_1, x_2), 1+\sum r_i'''],$$

where the r_i are as defined in (3.1) and the r_i''' are monomials of degrees $\geqq n$. As in (3.2), we have

$$(4.2)\qquad \begin{aligned}&[1+\sum r_i(x_1, x_2), 1+\sum r_i'''] \\ &\quad = 1+(\sum r_i(x_1, x_2), \sum r_i''')+r_i''''(\sum r_i(x_1, x_2), \sum r_i''').\end{aligned}$$

[2] The construction of H is based on one used in BAUMSLAG and LEVIN [1]. In particular, CHEN [2] shows that omitting the restriction on the degree of $x_1 x_2 x_3$ yields a free metabelian group.

(4.2) can be expressed as a sum

$$1+s_i(x_1, x_2)\,s_i', \tag{4.3}$$

where either s_i or s_i' has degree $\geqq n$. Let $s(x_1, x_2)\,s'$ be a term in (4.3). Since the law (i) is symmetrical in φ and ψ, we may assume without loss of generality that deg $s' \geqq n$. If deg $s(x_1, x_2) \leqq n-2$, then (i) or (ii) apply and $s(x_1, s_2)\,s'=0$. If deg $s(x_1, x_2) > n-2$, then (ii) applies and again $s(x_1, x_2)\,s'=0$. Thus, H satisfies $[n, 2]=1$.

H does not satisfy $[n-1, n-1]=1$. To see this, we note that, as can be shown by an easy induction,

$$[1+x, 1+y, \underbrace{1+w, \ldots, 1+w}_{n-3 \text{ times}}] = 1+\sum_{k=0}^{n-3}\binom{n-3}{k}(-1)^k w^k(x, y)\,w^{n-3-k}, \tag{4.4}$$

for either $w=x$ or $w=y$. Thus,

$$\begin{aligned}
&[[1+x, 1+y, \underbrace{1+x, \ldots, 1+x}_{n-3 \text{ times}}], [1+x, 1+y, \underbrace{1+y, \ldots, 1+y}_{n-3 \text{ times}}]] \\
&= \left[1+\sum_{k=0}^{n-3}\binom{n-3}{k}(-1)^k x^k(x, y)\,x^{n-3-k}, 1+\sum_{k=0}^{n-3}\binom{n-3}{k}(-1)^k y^k(x, y)\,y^{n-3-k}\right] \\
&= 1+\left(\sum_{k=0}^{n-3}\binom{n-3}{k}(-1)^k x^k(x, y)\,x^{n-3-k}, \sum_{k=0}^{n-3}\binom{n-3}{k}(-1)^k y^k(x, y)\,y^{n-3-k}\right) \\
&\quad + \text{terms of degree} \geqq 2n-1.
\end{aligned}$$

By (ii), the terms of degree $\geqq 2n-1$ vanish, and by (i), the remaining expression reduces to

$$1+x^{n-3}(x, y)(x, y)\,y^{n-3}-y^{n-3}(x, y)(x, y)\,x^{n-3}. \tag{4.5}$$

By (i), the central two factors in each term of (4.5) remain unchanged throughout all permissible permutations. For instance, the term in (4.5) with central factor yx is

$$x^{n-3}x \cdot yx \cdot y\,y^{n-3}-y^{n-3}x \cdot yx \cdot y\,x^{n-3},$$

and this expression clearly does not vanish by (i). Thus, H does not satisfy $[n-1, n-1]=1$. In Lemma 4.1 it will be shown that $C(2n-2)$ implies $[n-1, n-1]=1$; thus, H does not satisfy $C(2n-2)$. Also, H satisfies no finite Engel condition.

If in $F(2)$ we add a relation that $r=0$ for any r of degree $\geqq n$, the resulting group is nilpotent and satisfies $C(n)$. However, it is clear that this group does not satisfy $[n-1-k, 2+k]=1$ for any $0 \leqq k \leqq n-3$. Thus, a generalization of Corollary 3.2 will not hold in general for $n>3$.

Lemma 4.1. *G satisfies $C(n+2)$, $n \geqq 2$, if and only if G satisfies $[n-k, 2+k]=1$ for all $k=0, 1, \ldots, n-2$.*

Proof. Induction on n. The cases $n=2$ and $n=3$ have been dealt with previously. Thus, let $n>3$. If G satisfies $C(n+2)$, then, in particular, G satisfies $[n, 2]=1$, by Lemma 2.1. Further, G satisfies $C(n+2, \varphi)$ for any φ which fixes $n+2$, and it follows by (1.6) and $[n, 2]=1$ that $G/Z(G)$ satisfies $C(n+1)$. Hence, by the induction hypothesis, $G/Z(G)$ satisfies $[n-1-k, 2+k]=1$ for any nonnegative k such that $n-1-k\geqq 2$. Finally, assuming, by another induction, that G satisfies $[n-k, 2+k]=1$ it follows from Lemma 2.4 that G satisfies $[n-1-k, 2+k+1]=1$, which proves the first part of the Lemma.

Conversely, let G satisfy $[n-k, 2+k]=1$ for all nonnegative $k\leqq n-2$, and assume by induction that if G satisfies $[n-1-k, 2+k]=1$ for all $0\leqq k\leqq n-3$ then G satisfies $C(n+1)$. Since G satisfies $[n-k, 2+k]=[n-k-1, 2+k+1]=1$, it follows, by Lemma 2.3, that $G/Z(G)$ satisfies $[n-k-1, 2+k]=1$, so that, by hypothesis, $G/Z(G)$ satisfies $C(n+1)$. Thus, by Lemma 2.2, G satisfies $C(n+2)$.

Remark. By (1.5), G satisfies $[n, m]=1$ if and only if G satisfies $[m, n]=1$. Thus, the hypotheses of Lemma 4.1 may be replaced by: G satisfies $C(n+2)$ if and only if G satisfies

$$[n, 2]=[n-1, 3]=\cdots=[n-s, 2+s]=1, \tag{4.6}$$

where $2s=n-2$ if n is even, and $2s=n-3$ if n is odd.

Theorem 4.2. *If G satisfies*

$$[n, 2]=[n-1, 3]=\cdots=[n-k, 2+k]=1, \tag{4.7}$$

for some $k<s$, then G satisfies $C(2n-2k-1)$.

Proof. Let G satisfy (4.7). Then, since $[n-m, 2+m]=1$ implies $[n-m+p, 2+m+q]=1$ for any $p, q\geqq 0$, G satisfies

$$[2n-2k-3, 2]=[2n-2k-4, 3]=\cdots=[n-k, n-k-1]=1, \tag{4.8}$$

and it follows from Lemma 4.1 that G satisfies $C(2n-2k-1)$.

In particular, for $k=0$ we have the following result.

Corollary 4.3. *If G satisfies $[n, 2]=1$, $n\geqq 3$, then G satisfies $C(2n-1)$.*

Theorem 4.2 is best possible, for, if G satisfying (4.7) would imply $C(2n-2k-2)$ for some $k<s$, then $[n-k, 2]=1$ would imply $C(2n-2k-2)$, since $[n-k, 2]=1$ implies (4.7) in the same way that (4.7) implies (4.8).

Corollary 4.4. *If G satisfies $C(n, \varphi)$, $n\geqq 5$, for all which leave fixed $3, \ldots, m$, $m\leqq n-2$, or for any set of generators of the group of permutations of $\{m+1, \ldots, n\}$, $m\geqq 3$, then G satisfies $C(n+m-3)$.*

We omit the details and merely note that the proof is by induction on m and is almost identical with the proof of Corollary 3.3.

The next theorem again generalizes Lemma 4.1.

Theorem 4.5. *Let G satisfy*

$$[n, 2]=[n-k_1, 2+k_1]=\cdots=[n-k_m, 2+k_m]=[n-s, 2+s]=1, \tag{4.9}$$

where $0\leqq k_1\leqq k_2\leqq\cdots\leqq k_m\leqq s$, with s defined as in (4.6). Then G satisfies $C(n+1+t)$, where

$$t=\max(k_1, k_2-k_1, \ldots, k_m-k_{m-1}, s-k_m).$$

Proof. As (4.7) implies (4.8), so (4.9) implies that G satisfies

$$[n+t-1, 2]=[n+t-2, 3]=\cdots=[2, n+t-1]=1,$$

so that G satisfies $C(n+1+t)$, by Lemma 4.1.

Finally, we comment briefly on the situation for

$$[x_1, \ldots, x_n]=[x_1, x_{\varphi(2)}, \ldots, x_{\varphi(n)}]. \tag{4.10}$$

Theorem 4.6. *G satisfies* (4.10) *for all permutations φ of $\{2, \ldots, k\}$ if and only if G is nilpotent of class $\leqq n-1$.*

Remark. Theorem 4.6 may be derived from MACDONALD [6] who shows that G is nilpotent of class $\leqq n-1$, $n\geqq 4$, if (4.10) holds for $\varphi=(2\ n)$, but our hypothesis allow the following short direct proof.

Proof. (Induction.) For $n=3$, $[x_1, x_2, x_3]=[x_1, x_3, x_2]$ implies $[x_1, [x_2, x_3]]=1$, by Lemma 2.1, and conversely. If G satisfies the hypotheses for $n>3$, then G satisfies $[n-2, 2]=1$ by Lemma 2.1, and G satisfies (4.10) for any φ which fixes n. Thus, as in the proof of Lemma 2.2, $G/Z(G)$ satisfies (4.10) with n replaced by $n-1$ for any permutation φ of $\{2, \ldots, n-1\}$, so that, by induction, $G/Z(G)$ is nilpotent of class $\leqq n-2$, and G is nilpotent of class $\leqq n-1$. The proof of the converse is trivial.

5. The Varieties $[m, n]=1$, $m, n\geqq 3$

As noted above, the law $[n, 2]=1$, $n\geqq 3$, implies $C(2n-1)$. The law $[n, 1]=1$, that is, nilpotence class n, implies $C(n+1)$ trivially. In general, however, a law $[m, n]=1$, $m, n\geqq 3$, will not imply $C(k, \varphi)$ for any k and nontrivial φ; it suffices to show this for $[3, 3]=1$. First, we note that the group K constructed from $F(2)=gp(1+x, 1+y)$ by adding the relations $r(r_1, r_2)=0$, for any monomial $r\in A(Z, 3)$ of degree $\geqq 3$, satisfies $[3, 3]=1$. K does not satisfy $[n, 2]=1$, for any n; for instance,

$$[[1+x, 1+y, \underbrace{1+x, \ldots, 1+x}_{n-2 \text{ times}}], [1+x, 1+y]]\neq 1$$

in K, as can be verified by direct computation using (4.4). Together with the following lemma, this implies that if $[3, 3]=1$ implies $C(n, \varphi)$, then $\varphi(3)=3$.

Lemma 5.1. *If G satisfies $[3,3]=1$ and $C(n,\varphi)$, $n\geqq 4$, where $\varphi(m)=3$, $m\neq 3$, then the two-generator subgroups of G satisfy $C(n+1)$.*

Proof. For $n=4$, G satisfies $C(4)$ trivially. Next, let $n\geqq 5$, $m=n$, and let $H=gp(x,y)$, $x,y\in G$. The hypotheses $[3,3]=1$ and $C(n,\varphi)$ imply, respectively, that G satisfies, for any x_i, $z\in G$, $[x,y,x_3,\ldots,x_{n-1},[x,y,z]]=1$ and $[x,y,x_3,\ldots,x_{n-1},[x,y]]=1$. Thus, by (1.1), $[x,y,x_3,\ldots,x_{n-1},[x,y]^z]=1$ so that G satisfies $[x,y,x_3,\ldots,x_{n-1},w]=1$, for any $w\in H'$. Hence, G satisfies $[v,w]=1$, $w\in H'$, for any v in the normal closure of the commutators $[x,y,x_3,\ldots,x_{n-1}]$ and, similarly, for any v in the normal closure of the commutators $[y,x,x_3,\ldots,x_{n-1}]$. It follows that H satisfies $[v,w]=1$, $w\in H'$, $v=[x_1,\ldots,x_{n-1}]$, for any $x_i\in H$. Thus, H satisfies $[n-1,2]=1$, and since G satisfies $[n-2,3]=[n-3,4]=\cdots=[3,n-2]=1$, it follows that H satisfies $C(n+1)$, by Lemma 4.1. For $m<n$, the proof is by induction on $n-m$. Thus, suppose the Lemma proved for the case $\varphi(m+1)=3$, $m\neq 2$, and consider $\varphi(m)=3$, $m\neq 3$. By the induction hypothesis, $H/Z(H)$ satisfies $C(n)$ and, hence, $[n-2,2]=1$. Since H satisfies $[n-2,3]=1$, it follows from Lemma 2.4 that H satisfies $[n-1,2]=1$ so that, as above, $[3,3]=1$ now implies that H satisfies $C(n+1)$.

Next, suppose that $[3,3]=1$ implies $C(m+n+3,\varphi)$, where $\varphi(m+n+3)=m+3$, $n>0$. Let K_{m+n} denote the group obtained from K by adding the relations: $r=0$ for any monomial $r\in A(Z,3)$ of degree $>m+n+3$. Then, the expressions

$$1+f=[1+x,1+y,\underbrace{1+x,\ldots,1+x}_{m+n\text{ times}},1+y]$$

and

$$1+g=[1+x,1+y,\underbrace{1+x,\ldots,1+x}_{m\text{ times}},1+y,\underbrace{1+x,\ldots,1+x}_{n\text{ times}}]$$

are equal in K_{m+n}. Using (4.4), it follows that

$$1+f=1+\left(\sum_{k=0}^{m+n}\binom{m+n}{k}(-1)^k x^k(x,y)\,x^{m+n-k},y\right)$$

and

$$1+g=1+\sum_{p=0}^{n}\binom{n}{p}(-1)^p x^p\left(\sum_{k=0}^{m}\binom{m}{k}(-1)^k x^k(x,y)\,x^{m-k},y\right)x^{n-p},$$

and, using the relations for K_{m+n}, a straightforward calculation shows that $(1+f)-(1+g)=xh+nyx(x,y)\,x^{m+n-1}$, where h is a polynomial in x,y. If $f=g$ this can happen only if $nyx(x,y)\,x^{m+n-1}=0$, which is impossible since yx has degree 2 and $n>0$. To show that $C(m+n+3+k,\varphi)$, where $\varphi(m+n+3)=m+3>3$, $n>0$, does not hold in K_{m+n+k}, we compare

$$1+f_1=[1+f,\underbrace{1+x,\ldots,1+x}_{k\text{ times}}]\quad\text{and}\quad 1+g_1=[1+g,\underbrace{1+x,\ldots,1+x}_{k\text{ times}}].$$

It follows from the above argument that $f_1 = g_1$ will imply that

$$nyx(x, y)\, x^{m+n+k-1} = 0,$$

which again is true only if $n = 0$. Hence, we have proved the following theorem

Theorem 5.2. *The law* $[m, n] = 1$, $m, n \geqq 3$, *does not imply* $C(k, \varphi)$ *for any* $k \geqq 4$ *and any nontrivial* φ.

6. Groups of Type $[m \to n]$

A group G is said to be of type $[m \to n]$ if every m-generator subgroup of G is nilpotent of class n. NEWMAN and WIEGOLD (H. NEUMANN [9], p. 97) have shown that a metabelian group of type $[k \to k+1]$, for $k = 2$ and $k = 3$, is of type $[k \to k+1]$, for all k, and give an example of a metabelian non-nilpotent group satisfying these conditions. In this section we will generalize this result to groups satisfying $C(m)$, for some m.

Lemma 6.1. *Let* G *satisfy* $C(m+1)$ *and be of type* $[n \to n+k]$, *for some* $n \geqq 3$, $k \leqq n-2$. *Then* G *is of type* $[p \to q]$, *for all* $p \geqq n$ *and* $q = \max(m, p+k)$.

Proof. Let $H = gp(h_1, \ldots, h_p)$ be a p-generator sugroup of G. To show that H is nilpotent of class q, it suffices to show that all commutators $[x_1, \ldots, x_{q+1}]$, where the x_i are generators of H, are 1. Let $c = [x_1, \ldots, x_{q+1}]$ and $N(j)$ denote the number of repetitions of x_j in $\{x_3, \ldots, x_{q+1}\}$. Since G satisfies $C(m+1)$ we may assume, with no loss in generality, that $N(3) \geqq N(4) \geqq \cdots \geqq N(q+1)$. If $c \neq 1$, there must be at least $n+1$ distinct elements in the set $\{[x_1, x_2], x_3, \ldots, x_{n+k+2}\}$ and, hence, $\geqq n$ distinct h_i among the $n+k \leqq 2n-2$ elements in $\{x_3, \ldots, x_{n+k+2}\}$. Since $N(3) \geqq N(4) \geqq \cdots$, this is possible only if $N(n+k+1) = N(n+k+2) = 1$ so that $N(j) = 1$, $j \geqq n+k+1$. We consider two cases:

1) $N(1) = N(2) = 0$. Then $\{x_1, x_2, \ldots, x_{n+k+2}\}$ contains $\geqq n+2$ distince h_i while $\{x_{n+k+3}, \ldots, x_{q+1}\}$ contains $q+1-(n+k+2)$ further distinct h_i. Thus, $p \geqq n+2+q+1-(n+k+2) \geqq q-n+3$ or $q \leqq p+n-3$, which contradicts the hypothesis on q.

2) $N(1) \geqq 1$ or $N(2) \geqq 1$. First we note that if $N(t) = 1$ for some $t \geqq 3$, it may be assumed that all occurrences of x_1 and x_2 in c appear in $\{x_1, \ldots, x_{t+1}\}$. For, if $N(j) > 1$, $j = 1$ or 2, then all x_j appear in $\{x_1, \ldots, x_t\}$, by hypothesis, while if $N(j) = 1$, $j = 1$ or 2, then the x_i with $N(x_i) = 1$, $i \geqq 3$, may be permuted so that $x_j = x_t$ or $x_j = x_{t+1}$. Since $N(n+k+1) = 1$, it may be assumed that either x_1 or x_2 occurs in $\{x_3, \ldots, x_{n+k+1}\}$. If $c \neq 1$, then there are $\geqq n+1$ distinct h_i in $\{x_1, \ldots, x_{n+k+1}\}$ so that there are $\geqq n-1$ distinct h_i ($\neq x_1$ or x_2) among $n+k-2 \leqq 2n-4$ of the elements $\{x_3, \ldots, x_{n+k+1}\}$. This is possible only if $N(n+k) = 1$ so that we may assume that all occurrences of x_1 and x_2 appear in $\{x_1, \ldots, x_{n+k+1}\}$, and, hence, there are $\geqq n+1$ distinct h_i among $\{x_1, \ldots, x_{n+k+1}\}$. Thus, as in Case 1, $p \geqq n+1+q+1-(n+k+1) \geqq q-n+3$, which again contradicts the hypothesis on q.

Lemma 6.1 may be applied to most of the results of Section 4. An example is given by the following theorem, which generalizes the above mentioned result of NEWMAN and WIEGOLD.

Theorem 6.2. *Let G satisfy $[4, 2]=1$ and be of type $[n \to n+k]$, for some $n \geqq 4$, $k \leqq n-2$. Then G is of type $[p \to p+k]$, for all $p \geqq n$.*

Proof. For $n>4$ or $n=4$, $k \geqq 1$, the proof is an immediate consequence of Lemma 6.1 and $C(7)$. For $k=0$, $n=4$, G is nilpotent of class 4 (HEINEKEN [8]).

In the same way an application of Lemma 6.1 to Theorem 3.1 yields the following result.

Theorem 6.3. *Let G satisfy $[3, 2]=1$ and be of type $[n \to n+k]$, for some $n \geqq 3$, $k \leqq n-2$. Then G is of type $[p \to p+k]$, for all $p \geqq n$.*

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