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Sets of Continuous Functions on a Compact Space into a Linear Lattice

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Let E be a compact topological space, and $C(E)$ the set of all continuous real-valued functions on E. If A is a subset of $C(E)$, let $\Delta(A)$ denote the set of quadruples $(s, t; \lambda, \mu)$, where $s, t \in E$ and λ, μ are non-negative real numbers, such that $\lambda g(s) = \mu g(t)$ for all $g \in A$. A theorem of KAKUTANI [2] states that if A is a uniformly closed linear sublattice of $C(E)$, and if an element f of $C(E)$ is such that $\lambda f(s) = \mu f(t)$ for all $(s, t; \lambda, \mu) \in \Delta(A)$, then $f \in A$. BONSALL [1] extended this result to closed, convex sublattices of $C(E)$. In the present paper, it is shown how similar results can be stated and proved for functions mapping E into a linear lattice with a topology satisfying certain conditions.

We start by establishing some terminology and notation. Let X be a linear lattice, i.e. a real linear space with a lattice ordering such that

$$x \leqq y \Rightarrow x+z \leqq y+z \qquad (\text{all } z \in X),$$

$$x \geqq 0,\ \lambda \geqq 0 \Rightarrow \lambda x \geqq 0.$$

The positive cone, i.e. $\{x: x \geqq 0\}$, will be denoted by P. $y \geqq x$ is equivalent to $y \in x+P$. For $x \in X$, we have $x = x^+ - x^-$, where $x^+ = x \vee 0$, $x^- = (-x) \vee 0$. $x^+ + x^-$ is denoted by $|x|$. A subset A of X is *order-convex* (or *full*) if $a, b \in A$ and $a \leqq x \leqq b$ imply $x \in A$. Equivalently, $(A+P) \cap (A-P) = A$. A is *solid* if $a \in A$ and $|x| \leqq |a|$ imply $x \in A$.

If X is also a topological linear space, we say that X is *locally order-convex (locally solid)* if the order-convex (solid) neighbourhoods of 0 form a local base. Local order-convexity is equivalent to normality of P. It is shown in [4], p. 234 and [3], p. 40 that each of the following statements is equivalent to X being locally solid:

(i) X is locally order-convex, and the mapping $x \to x^+$ is continuous at 0,

(ii) the mapping $x \to x^+$ is uniformly continuous on X,

(iii) the mapping $(x, y) \to x \vee y$ is uniformly continuous on $X \times X$.

This is a very natural condition to impose on a linear lattice with a topology, and we shall suppose throughout that it is satisfied. For the results in this paper, we need also the far more special condition that there is a local base consisting of neighbourhoods U which are sublattices, i.e. if x and y are in U, then so are $x \vee y$ and $x \wedge y$. This is the natural generalisation to topological linear spaces

of the defining condition for AM-spaces, viz. $\|x \vee y\| = \|x\| \vee \|y\|$ for $x, y \geqq 0$. Locally solid spaces satisfying this condition will be called *topological M-spaces.* We notice that topological M-spaces are automatically locally convex. In fact, if U is a sublattice containing 0, then $u \in U$ implies $|u| = u^+ \vee u^- \in U$, and it is easily seen that

$$\{x\colon \exists\, u \in U \text{ such that } |x| \leqq |u|\}$$

is a solid, convex sublattice containing U.

A natural example of a topological M-space which is not normable is the set of all real-valued functions on an infinite set S, with the topology of pointwise convergence. By taking S to be the set of positive integers, we obtain the space of all real sequences, with its usual topology.

Suppose that X is a locally solid linear lattice, with local base $\mathcal{N}(X)$. Let E be a compact topological space, and let $C(E, X)$ be the set of all continuous functions mapping E into X. With the natural definitions of addition and scalar multiplication, this is a linear space. It becomes a topological linear space if it is given the topology of uniform convergence on E, in which the basic neighbourhoods of 0 are $\tilde{U}$ $(U \in \mathcal{N}(X))$, where

$$\tilde{U} = \{f\colon f(s) \in U \text{ for all } s \in E\}.$$

$\bar{A}$ will denote the closure of the set A with respect to this topology.

Take $f, g \in C(E, X)$. Since the mapping $(x, y) \to x \vee y$ is continuous, the function h defined by

$$h(s) = f(s) \vee g(s) \qquad (s \in E)$$

is continuous on E. Writing $h = f \vee g$, and defining $f \wedge g$ similarly, we see that $C(E, X)$ is a linear lattice. It is clear that if U is solid or a sublattice, then so is $\tilde{U}$.

A subset A of $C(E, X)$ will be said to *admit* $\vee$ *if*

$$f, g \in A \Rightarrow f \vee g \in A.$$

Without using the M-space condition or scalar multiplication, we have:

Theorem 1. *Suppose that X is locally solid, and that U, V are symmetric neighbourhoods of* 0 *in X such that $V + V + V \subseteq U$. Let f be a member of $C(E, X)$.*

(i) *If A is a subset of $C(E, X)$ admitting $\vee$, and for each $s \in E$, $\exists\, g_s \in A$ such that $g_s(s) - f(s) \in V + P$, then $\exists\, g \in A$ such that $g(s) - f(s) \in U + P$ for all $s \in E$.*

(ii) *If A is a subset of $C(E, X)$ admitting $\wedge$, and for each $s \in E$, $\exists\, g_s \in A$ such that $g_s(s) - f(s) \in V - P$, then $\exists\, g \in A$ such that $g(s) - f(s) \in U - P$ for all $s \in E$.*

Proof. We give the proof of (i); (ii) is entirely similar.

With g_s as stated, there is a neighbourhood $N(s)$ of s such that, for $t \in N(s)$, $g_s(t) - g_s(s)$ and $f(t) - f(s)$ are in V. For such t, $g_s(t) - f(t) \in U + P$.

Since E is compact, there is a finite set of points $s_1, \ldots, s_n$ such that

$$E = \bigcup_{i=1}^{n} N(s_i).$$

Let $g=g_{s_1}\vee\cdots\vee g_{s_n}$. Then $g\in A$, and for any $t\in E$, $t\in N(s_i)$ for some i, so

$$g(t)\in g_{s_i}(t)+P\subseteq f(t)+U+P+P=f(t)+U+P.$$

Theorem 2. *Let X be a topological M-space, and let A be a sublattice of $C(E, X)$. Suppose that f is an element of $C(E, X)$ such that, given $s, t\in E$ and a neighbourhood W of 0 in X, $\exists\, g\in A$ such that*

$$g(s)-f(s)\in W+P,$$
$$g(t)-f(t)\in W-P.$$

Then $f\in\bar{A}$.

Proof. Take an order-convex neighbourhood U of 0 in X. There exist symmetric neighbourhoods V, W which are sublattices and satisfy $V+V+V\subseteq U$, $W+W+W\subseteq V$.

Fix $t\in E$. Let $A_t=\{g\in A: g(t)-f(t)\in W-P\}$. By hypothesis, for each $s\in E$, $\exists\, g_s\in A_t$ such that $g_s(s)-f(s)\in W+P$. If $g_1, g_2\in A_t$, then $\exists\, w_i\in W$ such that $g_i(t)\leqq f(t)+w_i$ $(i=1, 2)$. Then

$$(g_1\vee g_2)(t)\leqq f(t)+w_1\vee w_2.$$

Since W is a sublattice, this shows that $g_1\vee g_2\in A_t$. Hence, by theorem 1(i), $\exists\, g\in A_t$ with $g(s)-f(s)\in V+P$ for all $s\in E$.

Let $A'=\{g\in A: g(s)-f(s)\in V+P$ for all $s\in E\}$. We have just shown that, for each $t\in E$, $\exists\, g_t\in A'$ such that $g_t(t)-f(t)\in W-P\subseteq V-P$. The fact that V is a sublattice implies, as above, that A' admits $\wedge$. Hence, by theorem 1(ii), $\exists\, g\in A'$ such that $g(s)-f(s)\in U-P$ for all $s\in E$. Thus, for all $s\in E$, we have

$$g(s)-f(s)\in(U+P)\cap(U-P)=U.$$

It follows that $f\in\bar{A}$.

The following corollaries of theorem 2 are of interest, though no use is made of them below:

Corollary 1. *If X is a topological M-space, and A is a sublattice of $C(E, X)$ such that if $s, t\in E$, $x, y\in X$ and a neighbourhood W of 0 are given, there is an element g of A such that*

$$g(s)-x\in W+P,$$
$$g(t)-y\in W-P,$$

then $\bar{A}=X$.

Proof. Immediate.

Corollary 2. *Let X be a topological M-space, and let A be a sublattice of $C(E, X)$ which admits addition. Suppose that $f\in C(E, X)$ and that, given $s, t\in E$ and a neighbourhood W of 0, $\exists\, g, h\in A$ such that*

$$g(s)-f(s)\in W+P,\quad g(t)\in W-P,$$
$$h(t)-f(t)\in W-P,\quad h(s)\in W+P.$$

Then $f\in\bar{A}$.

It is sufficient, in particular, if, given $s, t \in E$ and a neighbourhood W of 0, $\exists\, g \in A$ such that

$$g(s)-f(s) \in W, \quad g(t) \in W.$$

Proof. Given a neighbourhood V of 0, there is a neighbourhood W such that $W+W \subseteq V$. If g and h correspond to W as stated, then $g+h \in A$, and

$$(g+h)(s)-f(s) \in V+P,$$
$$(g+h)(t)-f(t) \in V-P.$$

We now give our generalisation of KAKUTANI's theorem. By a *positive* linear functional on X, we mean one which takes non-negative values on the positive cone P. We shall be concerned with the space $X \times X$, i.e. the direct sum of X with itself. If K and L are subsets of X, then $K \times L$ will denote the subset $\{(k, l): k \in K, l \in L\}$ of $X \times X$.

Theorem 3. *Suppose that X is a topological M-space, and that A is a convex sublattice of $C(E, X)$. Let $\Gamma(A)$ denote the set of quintuples $(s, t; \varphi, \psi; \beta)$, where $s, t \in E$, φ, ψ are positive continuous linear functionals on X, and β is real, such that*

$$\varphi(g(s)) \leqq \psi(g(t)) + \beta$$

for all $g \in A$. If f is an element of $C(E, X)$ such that

$$\varphi(f(s)) \leqq \psi(f(t)) + \beta$$

for all $(s, t; \varphi, \psi; \beta) \in \Gamma(A)$, then $f \in \bar{A}$.

Proof. If $f \notin \bar{A}$, then, by theorem 2, there exist s and t in E and a neighbourhood W of 0 in X such that, for each $g \in A$,

$$(g(s), g(t)) - (f(s), f(t)) \notin (W+P) \times (W-P). \tag{1}$$

Write

$$\{(g(s), g(t)): g \in A\} = H,$$
$$P \times (-P) = Q,$$
$$W \times W = \hat{W}.$$

Then H and Q are convex and $\hat{W}$ is a neighbourhood of 0 in $X \times X$. (1) can be written

$$\{H - (f(s), f(t))\} \cap (\hat{W} + Q) = \emptyset,$$

or

$$\{(f(s), f(t)) + \hat{W}\} \cap (H - Q) = \emptyset.$$

Hence there is a continuous linear functional χ on $X \times X$ and a real number β such that $\chi(f(s), f(t)) > \beta$, while $\chi(z) \leqq \beta$ for $z \in H - Q$. Writing $\varphi(x) = \chi(x, 0)$, $\psi(y) = -\chi(0, y)$, we have $\chi(x, y) = \varphi(x) - \psi(y)$. The fact that $\chi(x, y) \leqq \beta$ for $(x, y) \in H$ implies that $(s, t; \varphi, \psi; \beta) \in \Gamma(A)$, while $\chi(f(s), f(t)) > \beta$ gives

$$\varphi(f(s)) > \psi(f(t)) + \beta.$$

Finally, since χ is bounded above on $H-Q$, it is bounded below on Q. Since $\lambda Q=Q$ for $\lambda>0$, χ is non-negative on Q. It follows that φ and ψ are non-negative on P.

Various simpler forms of theorem 3 are obtained when further assumptions are made about A. First, suppose that A is a wedge, i.e. a set admitting addition and multiplication by non-negative scalars. If a linear functional is bounded above on a wedge, then it is bounded above by 0 there, and hence (more naturally from the proof of theorem 3 than from its statement) we obtain:

Corollary 1. *Suppose that X is a topological M-space, and that A is a wedge sublattice of $C(E, X)$. Let $\Delta(A)$ denote the set of quadruples $(s, t; \varphi, \psi)$, where $s, t \in E$ and φ, ψ are positive continuous linear functionals on X, such that*

$$\varphi(g(s)) \leqq \psi(g(t))$$

for all $g \in A$. If f is an element of $C(E, X)$ such that

$$\varphi(f(s)) \leqq \psi(f(t))$$

for all $(s, t; \varphi, \psi) \in \Delta(A)$, then $f \in \bar{A}$.

If H is a linear subspace, then a linear functional which is bounded above on H is zero on H. Hence when A is a linear sublattice of $C(E, X)$, $\Delta(A)$ becomes the set of $(s, t; \varphi, \psi)$ for which

$$\varphi(g(s)) = \psi(g(t))$$

for all $g \in A$.

Corresponding to each $x \in X$, we have the *constant* function in $C(E, X)$ defined by $f(s)=x$ for all $s \in E$. If A is a linear subspace containing all constant functions, then $(s, t; \varphi, \psi) \in \Delta(A)$ implies that $\varphi=\psi$. Hence we have:

Corollary 2. *Suppose that X is a topological M-space, and that A is a linear sublattice of $C(E, X)$ which contains all constant functions. Suppose that $f \in C(E, X)$ and that whenever $s, t \in E$ and the positive continuous linear functional φ are such that*

$$\varphi(g(s)) = \varphi(g(t))$$

for all $g \in A$, we have

$$\varphi(f(s)) \leqq \varphi(f(t)).$$

Then $f \in \bar{A}$.

Upper Semi-Equicontinuous Sets of Functions

A subset A of $C(E, X)$ is said to be *upper semi-equicontinuous* if, given $s \in E$ and a neighbourhood U of 0 in X, there is a neighbourhood $N(s)$ of s such that for $t \in N(s)$, $g(t)-g(s) \in U-P$ for all $g \in A$. By an argument exactly analogous to that given for real-valued functions in [*1*], p. 126, we obtain:

Theorem 4. *Let X be a topological M-space, and let A be an upper semi-equicontinuous subset of $C(E, X)$ which admits $\vee$. Suppose that $f \in C(E, X)$, and that, given a finite subset $s_0, s_1, \ldots, s_n$ of E and a neighbourhood W of 0 in X,*

there is an element g of A such that

$$g(s_0)-f(s_0)\in W+P,$$
$$g(s_i)-f(s_i)\in W-P \qquad (i=1,2,\dots,n).$$

Then $f\in\bar{A}$.

From this, we deduce:

Theorem 5. *Let X be a topological M-space, and let A be a convex, upper semi-equicontinuous subset of $C(E, X)$ which admits $\vee$. Suppose that $f\in C(E, X)$, and that whenever $\beta\in\mathbf{R}$, $s_0, s_1, \dots, s_n\in E$ and the positive continous linear functionals $\varphi_0, \varphi_1, \dots, \varphi_n$ are such that*

$$\varphi_0(g(s_0))\leqq\sum_{i=1}^{n}\varphi_i(g(s_i))+\beta$$

for all $g\in A$, we have

$$\varphi_0(f(s_0))\leqq\sum_{i=1}^{n}\varphi_i(f(s_i))+\beta.$$

Then $f\in\bar{A}$.

Proof. If $f\notin\bar{A}$, then there exist $s_0, s_1, \dots, s_n$ in E and a neighbourhood W of 0 in X such that, for each $g\in A$, the condition of theorem 4 is not satisfied.

In the space X^{n+1}, write

$$(f(s_0), f(s_1), \dots, f(s_n))=f(s),$$
$$\{g(s)\colon g\in A\}=H,$$
$$P\times(-P)\times\cdots\times(-P)=Q,$$
$$W\times W\times\cdots\times W=\hat{W}.$$

Then

$$(H-f(s))\cap(\hat{W}+Q)=\emptyset,$$

or

$$(f(s)+\hat{W})\cap(H-Q)=\emptyset.$$

Hence there is a continuous linear functional χ on X^{n+1} and a real number β such that $\chi(f(s))>\beta$, while $\chi(z)\leqq\beta$ for $z\in H-Q$. Define

$$\varphi_0(x_0)=\chi(x_0, 0, \dots, 0),$$
$$\varphi_i(x_i)=-\chi(0, \dots, x_i, \dots, 0) \qquad (i=1, \dots, n),$$

so that

$$\chi(x_0, x_1, \dots, x_n)=\varphi_0(x_0)-\sum_{i=1}^{n}\varphi_i(x_i).$$

Then the first inequality in the statement of the theorem is satisfied, while the second is not. χ is bounded below on Q, so non-negative there, from which it follows that each φ_i is non-negative on P.

An example given in [*1*] shows that this result can fail in the absence of upper semi-equicontinuity.

References

[*1*] Bonsall, F. F.: Semi-algebras of continuous functions. Proc. London Math. Soc. (3) **10**, 122—140 (1960).

[*2*] Kakutani, S.: Concrete representations of abstract (M)-spaces. Ann. of Math. (2) **42**, 994—1024 (1941).

[*3*] Namioka, I.: Partially ordered linear topological spaces. Mem. Amer. Math. Soc. No. 24, (1957).

[*4*] Schaefer, H. H.: Topological vector spaces. New York: Macmillan 1966.

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