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Universal Isols*

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Research in recursive equivalence types frequently starts with a study of cancellation laws. As prototypes of such laws consider the assertions

$$nA=nB\rightarrow A=B, \tag{1}$$

$$nA\leqq nB\rightarrow A\leqq B \tag{2}$$

where n is a positive integer and A and B are recursive equivalence types. Both (1) and (2) are theorems, the former due to FRIEDBERG (c.f. [3]), and the latter due to NERODE (from a personal communication with MYHILL). A natural generalization of these laws is to replace the linear function appearing in their antecedants by the canonical extension F of an arbitrary non-constant recursive combinatorial function f. Thus, we inquire whether either of

$$F(A)=F(B)\rightarrow A=B, \tag{3}$$

$$F(A)\leqq F(B)\rightarrow A\leqq B \tag{4}$$

holds, where A and B are recursive equivalence types. Unfortunately, if we take $A=RX$ and $B=R+X$ where R is the recursive equivalence type of an infinite recursively enumerable set and X is the recursive equivalence type of a retraceable immune set, then $A\not\leqq B$ but $F(A)=F(B)$ for every non-linear recursive combinatorial function F (c.f. [2]). This A and B serve as a uniform counterexample to (3) and (4) showing that the linear cancellation laws (1) and (2) are sharp, i.e., not capable of generalization along the proposed lines. The situation for (3) is considerably better if we restrict A and B to be isols. In this case MYHILL has shown (c.f. [5]) that if F does not reduce to a constant, then there is an integer k (depending on F) such that (3) holds for any isols A and B which are greater than k. The purpose of this paper is to show that no such generalization is possible for (4), i.e., that the linear cancellation law (2) is sharp for inequalities among the isols.

The diagonal counterexample method, first developed by MYHILL and NERODE, has been used with considerable force by both authors. In a series of

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papers (c.f. [6 – 8]) NERODE has systematically used this method to characterize all of the universal and some of the AE properties of isols, i.e., properties which correspond to sentences in prenex normal form in which existential quantifiers do not occur, etc. We shall pursue these methods here and show that there is a class of isols, the universal isols, each of which can uniformly serve as a counterexample in the theory of universal properties of isols. Let ε denote the non-negative integers and Λ the isols. A subset of ε is called *co-finite* if its relative complement in ε is finite. If we denote a combinatorial function by a lower case letter then we shall denote its canonical extension to Λ by an uppercase version of the same letter. Our principal definition is: X is a *universal isol* if X is an isol and for every pair of recursive combinatorial functions f and g, $F(X)=G(X)$ implies that $\{x: f(x)=g(x)\}$ is co-finite. Thus in a certain sense universal isols are generic for co-finite sets, i.e., as in the „Nullstellensatz". The existence of universal isols will be given by a diagonal argument, which, following the method of MYHILL and NERODE, is readily formulated in terms of the Baire Category theorem for topological spaces. Let $\mathfrak{X}$ be the set of all subsets of ε and Q the set of all finite subsets of ε. Denote the elements of $\mathfrak{X}$ by lower case Greek letters, and the relative complement of a set α with respect to ε by $\bar{\alpha}$. For $\alpha, \beta \in Q$ let

$$N(\alpha,\beta)=\{\xi: \alpha\subseteq\xi\subseteq\bar{\beta}\} \quad \text{and} \quad \mathcal{N}=\{N(\alpha,\beta): \alpha,\beta\in Q\}.$$

$\mathfrak{X}$ topologized by $\mathcal{N}$ as a neighborhood system is readily seen to be homeomorphic with the Cantor discontinuum and consequently is a compact Hausdorff space without isolated points. A set $A\subseteq\mathfrak{X}$ is called *nowhere dense* if every non-empty neighborhood contains a non-empty subneighborhood which is disjoint from A. A set $A\subseteq\mathfrak{X}$ is called a *Category I* set if it can be expressed as a countable union of nowhere dense sets. A set $A\subseteq\mathfrak{X}$ is called a *residual set* if it's relative complement with respect to $\mathfrak{X}$ is a Category I set. By a well-known version of the Baire Category theorem (c.f. [4]) every residual set in a compact Hausdorff space without isolated points has a cardinality $\geqq 2^{\aleph_0}$, and consequently by the preceding remarks, this theorem applies to the space $\mathfrak{X}$. Moreover, since the cardinality of $\mathfrak{X}$ itself is $2^{\aleph_0}$, every residual set in $\mathfrak{X}$ will contain exactly $2^{\aleph_0}$ elements.

For any infinite set $\omega\subseteq\varepsilon$ let $W(\omega)=\{\xi: \omega\subseteq\xi\subseteq\varepsilon\}$.

Lemma 1. *$W(\omega)$ is a nowhere dense set.*

Proof. Let $N(\alpha, \beta)$ be a non-empty neighborhood. Since ω is infinite and α is finite, choose an element $x\in\omega-\alpha$ and put $\beta'=\beta\cup\{x\}$. Then clearly $N(\alpha, \beta')$ is a non-empty subneighborhood of $N(\alpha, \beta)$ which is disjoint from $W(\omega)$. Q.e.d.

For any function f, denote its range and its domain by ρf and δf respectively. If $A\subseteq\delta f$ let $f[A]=\{f(x): x\in A\}$. A one one function p whose range and domain are contained in ε is called a *partial isomorphism*. If, moreover, p is partial recursive, then p is called a *partial recursive isomorphism*. If $\alpha, \beta\subseteq\varepsilon$ and there exists a partial recursive isomorphism p such that $\alpha\subseteq\delta p$ and $p[\alpha]=\beta$

then we say that α and β are *recursively equivalent* (in symbols $\alpha \simeq \beta$) and put Req $(\alpha) = \{\xi : \xi \simeq \alpha\}$. We presuppose that the reader is familiar with the notions of combinatorial function and normal combinatorial operator. A straightforward and short development of these concepts can be found in [5].

If Φ and Ψ are combinatorial operators and p is a partial isomorphism let

$$H(\Phi, \Psi, p) = \{\xi : \xi \subseteq \varepsilon \wedge \Phi(\xi) \subseteq \delta p \wedge p[\Phi(\xi)] = \Psi(\xi)\}.$$

Lemma 2. *Let f and g be a pair of combinatorial functions induced by the combinatorial operators Φ and Ψ respectively, and p a partial isomorphism. If $\{x : f(x) \neq g(x)\}$ is infinite then $H(\Phi, \Psi, p)$ is a nowhere dense set.*

Proof. Let $N(\alpha, \beta)$ be a non-empty neighborhood. By our hypothesis there is an integer n greater than the number of elements in α such that $f(n) \neq g(n)$. Choose a set γ containing exactly n elements such that $\alpha \subseteq \gamma \subseteq \bar{\beta}$. If $\Phi(\gamma) \nsubseteq \delta p$ then $\Phi(\xi) \nsubseteq \delta p$ for every $\gamma \subseteq \xi$ (since combinatorial operators are monotone) and consequently $N(\gamma, \beta)$ is a non-empty subneighborhood of $N(\alpha, \beta)$ which is disjoint from $H(\Phi, \Psi, p)$. Otherwise $\Phi(\gamma) \subseteq \delta p$ but $p[\Phi(\gamma)] \neq \Psi(\gamma)$ since $\Phi(\gamma)$ contains $f(n)$ elements, $\Psi(\gamma)$ contains $g(n)$ elements and $f(n) \neq g(n)$. If for some x, $x \in \Phi(\gamma)$ but $p(x) \notin \Psi(\gamma)$ then $\Phi^{-1}(x) \subseteq \gamma$ but $\Psi^{-1}(p(x)) \nsubseteq \gamma$. Choose $y \in \Psi^{-1}(p(x)) - \gamma$ and put $\beta' = \beta \cup \{y\}$. $N(\gamma, \beta')$ is a non-empty subneighborhood of $N(\alpha, \beta)$ and for any $\xi \in N(\gamma, \beta')$, $x \in \Phi(\xi)$ but $p(x) \notin \Psi(\xi)$ (since otherwise we would have $y \in \xi$). Hence $p[\Phi(\xi)] \neq \Psi(\xi)$ and consequently $N(\gamma, \beta')$ is disjoint from $H(\Phi, \Psi, p)$. If for some x, $x \notin \Phi(\gamma)$ but $p(x) \in \Psi(\gamma)$, then we use exactly the same argument to obtain a non-empty subneighborhood $N(\gamma, \beta')$ of $N(\alpha, \beta)$ which is disjoint from $H(\Phi, \Psi, p)$. But the three cases we have considered are exhaustive. Q.e.d.

Theorem 1. *There are exactly $2^{\aleph_0}$ universal isols.*

Proof. Let ω_n be an enumeration of all infinite recursively enumerable sets, $\langle f_n, g_n \rangle$ an enumeration of all pairs of recursive combinatorial functions $\langle f, g \rangle$ such that $\{x : f(x) \neq g(x)\}$ is infinite, and p_n an enumeration of all partial recursive isomorphisms. For each integer n let Φ_n and Ψ_n be the normal recursive combinatorial operator inducing f_n and g_n respectively and let j be a one one function mapping ε^2 onto ε. Define $W_n = W(\omega_n)$ and $H_{j(m,n)} = H(\Phi_m, \Psi_m, p_n)$ for integers m, n and let $C = \bigcup \{W_n \cup H_n : n \in \varepsilon\}$. By the preceding lemmas C is a Category I set and consequently its complement $R = \mathscr{X} - C$ is a residual set. Let $\Theta = \{\text{Req } (\xi) : \xi \in R\}$. Since each recursive equivalence type is countable, and R has cardinality $2^{\aleph_0}$ (by the Baire theorem), Θ has cardinality $2^{\aleph_0}$ as well. It only remains to show that Θ consists entirely of universal isols. Let $X \in \Theta$ and choose a set $\xi \in X \cap R$. Since $\xi \notin W_n$ for each integer n, ξ has no infinite recursively enumerable subset. Hence $X \in \Lambda$. Since $\xi \notin H_{j(m,n)}$ for every integer n, there is no partial recursive isomorphism p_n such that $\Phi_m(\xi) \subseteq \delta p_n$ and $p_n[\Phi_m(\xi)] = \Psi_m(\xi)$. Hence $\Phi_m(\xi)$ and $\Psi_m(\xi)$ are not recursively equivalent and consequently $F_m(X) \neq G_m(X)$. Hence if $F(X) = G(X)$ for recursive combinatorial functions f and g, the pair $\langle f, g \rangle$ will not appear in our enumeration

$\langle f_n, g_n\rangle$ and therefore $\{x: f(x) \neq g(x)\}$ is finite. But then X is a universal isol. Q.e.d.

Now that universal isols are available, we will show how they easily yield Myhill-Nerode type counterexamples. Before doing so, however, it is necessary to make the following remarks. NERODE does not initially deal with combinatorial functions. Instead, he gives a method (c.f. [6], p. 365) whereby any relation $R \subseteq \varepsilon^k$ is extended to a relation $R_\Lambda \subseteq \Lambda^k$. He then extends k-ary functions by considering them to be $k+1$-ary relations and shows (c.f. [6], Corollary 8.2) that for recursive combinatorial functions his extension procedure agrees with MYHILL's. More pertinently, he also shows (c.f. [6], Corollary 9.4), that if $R \subseteq \varepsilon^k$ is a recursive relation and f and g are a pair of k-ary recursive combinatorial functions such that $R=\{x \in \varepsilon^k: f(x)=g(x)\}$ then $R_\Lambda=\{X \in \Lambda^k: F(X)=G(X)\}$. Since for recursive R there will always be such a pair of recursive combinatorial functions f and g, we shall use the statement of NERODE's Corollary 9.4 as our definition for the extension of a recursive relation. By doing so we include a rather large fragment of the theory of relations in a pure combinatorial function theory without added expense. In these terms our principal definition becomes: X is a *universal isol* if X is an isol and for every recursive relation $R \subseteq \varepsilon$, $X \in R_\Lambda$ implies that R is co-finite.

Let $R \subseteq \varepsilon^k$ be a k-ary relation, I a proper subset of $\{1, \ldots, k\}$, and $h: I \to \varepsilon$. If $J=\{1, \ldots, k\}-I$ contains t elements let $j(i)$, $1 \leqq i \leqq t$, be a strictly increasing enumeration of J. For $x=\langle x_1, \ldots, x_t\rangle \in \Lambda^t$ let $h^*(x)$ be that element $y=\langle y_1, \ldots, y_k\rangle \in \Lambda^k$ such that for $i \in I$, $y_i=h(i)$ and $y_{j(i)}=x_i$ for $1 \leqq i \leqq t$. We call $S_h R=\{x \in \varepsilon^t: h^*(x) \in R\}$ a *k-specification* of R. Less formally, $S_h R$ is the relation which is obtained from R by only considering those elements of R whose i-th component, for $i \in I$, is $h(i)$ and then striking out all such components. A relation $R \subseteq \varepsilon^k$ is called *totally unbounded* if for every $n \in \varepsilon$ there is an $\langle x_1, \ldots, x_k\rangle \in R$ such that $x_i \geqq n$ for $1 \leqq i \leqq k$.

Let

$$f(x)=\sum c(i)\binom{x}{i}$$

be a unary combinatorial function. It is not difficult to show that the first n functional values $f(0), \ldots, f(n-1)$ depend only on the first n Stirling coefficients $c(0), \ldots, c(n-1)$ and conversely. Thus the only requirement on the value of $f(n)$ is that it be large enough to make

$$c(n)=f(n)-\sum c(i)\binom{n}{i} \geqq 0$$

(where the summation is taken for $i<n$). Consequently, if $R \subseteq \varepsilon$ is an infinite set, it is possible to construct a combinatorial function whose range is contained in R simply by successively choosing as functional values elements of R which are sufficiently distant.

Theorem 2 (Theorem 4.5 of [6], necessity part). *Let $n>1$ and let R, $R^{(1)}, \ldots, R^{(n)}$ be recursive subsets of ε^k. If for some proper subset I of $\{1, \ldots, k\}$ and function h: $I \to \varepsilon$ each k-specification $S_h(R-R^{(1)}), \ldots, S_h(R-R^{(n)})$ is totally unbounded then $R_\Lambda - (R_\Lambda^{(1)} \cup \cdots \cup R_\Lambda^{(n)})$ has cardinality $2^{\aleph_0}$.*

Proof. Define $J, t, j(i)$, and h^* as before. Clearly each $S_h(R-R^{(l)})$ is a recursive totally unbounded subset of ε^t. Construct t unary recursive combinatorial functions f_i, $1 \leqq i \leqq t$, by using a primitive recursive schema which requires that for each integer x, if x is congruent to l modulo n, $1 \leqq l \leqq n$, then $\langle f_1(x), \ldots, f_t(x)\rangle$ is the first element, with respect to some recursive enumeration of ε^t, in $S_h(R-R^{(l)})$ such that for each $1 \leqq i \leqq t$ the sequence of Stirling coefficients associated with $f_i(0), \ldots, f_i(x)$ are all positive. Since the $S_h(R-R^{(l)})$, $1 \leqq l \leqq n$, are totally unbounded, such a choice always exists. Consequently, the f_i, $1 \leqq i \leqq t$, are all combinatorial functions. Moreover, the recursiveness of the $S_h(R-R^{(l)})$ guarantees that the f_i are recursive as well. For every $x \in \varepsilon$ write $f(x) = \langle f_1(x), \ldots, f_t(x)\rangle$ and for every $X \in \Lambda$ write $F(X) = \langle F_1(X), \ldots, F_t(X)\rangle$ where F_i is the canonical extension of f_i, $1 \leqq i \leqq t$. For x congruent to l modulo n, $1 \leqq l \leqq n$, $h^*(f(x)) \in R - R^{(l)}$. Hence for all $x \in \varepsilon$, $h^*(f(x)) \in R$. Since identities between recursive combinatorial functions extend from ε to Λ, for all $X \in \Lambda$, $h^*(F(X)) \in R_\Lambda$. However, if X is a universal isol $h^*(F(X)) \notin R_\Lambda^{(l)}$ since otherwise $\{x: h^*(f(x)) \in R^{(l)}\}$ would be a co-finite set and this clearly contradicts our construction. Hence $h^*(F(X)) \in R_\Lambda - (R_\Lambda^{(1)} \cup \cdots \cup R_\Lambda^{(n)})$ for any universal isol X. Now by our construction each of the t functions f_i has an unbounded range. Hence by (3), for infinite isols X and Y, $F_i(X) = F_i(Y) \to X = Y$. Therefore as X ranges over the $2^{\aleph_0}$ universal isols, $h^*(F(X))$ ranges over $2^{\aleph_0}$ distinct members of $R_\Lambda - (R_\Lambda^{(1)} \cup \cdots \cup R_\Lambda^{(n)})$. Q.e.d.

Universal isols can also be used to make some of the properties of the isolic integers extremely transparent. Let ε^* denote the ring of rational integers and let Λ^* denote the ring of isolic integers. Every unary recursive function f: $\varepsilon \to \varepsilon^*$ can be expressed as a difference $f(x) = f^+(x) - f^-(x)$ of two unary recursive combinatorial functions f^+ and f^-. Using known results about Horn sentences in Λ, we can show that for $X \in \Lambda$, $F^+(X) - F^-(X)$ depends only on f and not on the particular pair of recursive combinatorial functions f^+ and f^- whose difference represents f. Consequently we may define the *canonical extension* of f to a function F: $\Lambda \to \Lambda^*$ by requiring that $F(X) = F^+(X) - F^-(X)$ for $X \in \Lambda$, where f^+ and f^- are any pair of recursive combinatorial functions whose difference function is f. In these terms our principal definition becomes: X is a *universal isol* if X is an isol and for any recursive function f: $\varepsilon \to \varepsilon^*$, $F(X) = 0$ implies that $\{x: f(x) = 0\}$ is co-finite.

Now define a pair of unary recursive functions f and g from ε into ε^* by: $f(x) = 0$ for x even, $f(x) = 1$ for x odd, and $g(x) = 1$ for x even and $g(x) = 0$ for x odd. Then for every $x \in \varepsilon$, $f(x) \cdot g(x) = 0$. Using the extension procedure of the preceding paragraph $F(X) \cdot G(X) = 0$ for all $X \in \Lambda$. However if X is a universal isol then $F(X) \neq 0$ and $G(X) \neq 0$ since otherwise at least one of $\{x: f(x) = 0\}$ or $\{x: g(x) = 0\}$ would be co-finite and this contradicts our

construction. Thus we have obtained the Dekker-Myhill result (c.f. [1], Theorem 95) that Λ^* contains divisors of zero. Moreover, notice that for all $x\in\varepsilon$, $(f(x))^2=f(x)$. Hence $(F(X))^2=F(X)$ for all $X\in\Lambda$. However, if X is a universal isol the above argument shows that $F(X)\neq 1$. Thus we have the Dekker-Myhill result (c.f. [1], Theorem 95) that Λ^* contains non-trivial idempotent elements. Incidentally this procedure also shows why integers p which are prime in ε^* have non-trivial factorizations in Λ^*. Construct a pair of functions f and g by requiring that: $f(x)=1$ for x even, $f(x)=p$ for x odd and $g(x)=p$ for x even, $g(x)=1$ for x odd. Then exactly as before $F(X)\cdot G(X)=p$ for $X\in\Lambda$ but $F(X)\neq 1$, $F(X)\neq p$, $G(X)\neq 1$, $G(X)\neq p$ if X is a universal isol.

These remarks suggest that there is some connection between the arithmetic of Λ^* and the arithmetic of function rings. Indeed, NERODE has shown (c.f. [8]) that the elementary theory of the arithmetic isolic integers $\Lambda^*(A)$ is elementarily equivalent to an appropriately defined function ring. However, the situation is considerably different for the ordinary isolic integers Λ^*. Let $\mathfrak{F}$ be the set of all recursive functions f: $\varepsilon\to\varepsilon^*$ and let $\mathfrak{A}$ be the set of all $f\in\mathfrak{F}$ such that $\{x: f(x)=0\}$ is co-finite. If we extend recursive functions f: $\varepsilon^{*k}\to\varepsilon^*$, pointwise (as in the direct product) to functions F: $\mathfrak{F}^k\to\mathfrak{F}$, then $\mathfrak{A}$ determines a congruence with respect to the *Fs*. Consequently we can extend recursive functions f: $\varepsilon^{*k}\to\varepsilon^*$ to functions F: $(\mathfrak{F}/\mathfrak{A})^k\to\mathfrak{F}/\mathfrak{A}$ where $\mathfrak{F}/\mathfrak{A}$ is the quotient of $\mathfrak{F}$ modulo the congruence relation determined by $\mathfrak{A}$. The principal results of [7] are that Λ^* and $\mathfrak{F}/\mathfrak{A}$ are not elementarily equivalent, but that they are universally elementarily equivalent, i.e., they share the same universal sentences. A direct and particularly simple proof of the latter may be obtained by using universal isols as noted above.

In order to show that cancellation laws for inequalities fail to extend from ε to Λ we extensively use Theorem 11.1 of [6], and in the sequel simply refer to this theorem as $*$. If we intend a quantifier to range over a set A we indicate this by appending the symbol "A" as a subscript to the quantifier bracket. A function f: $\varepsilon\to\varepsilon$ is called a *linear function* if it can be expressed in the form $f(x)=ax+b$ for some $a, b\in\varepsilon$. Now let

$$f(x)=\sum c(i)\binom{x}{i}\,.$$

be a recursive combinatorial function. If f does not reduce to a constant, then for some integer $k>0$ we have $c(k)>0$. Then we easily see that for values of x larger than $k-1$, f is a strictly increasing function. Consequently

$$(\forall x, y)_\varepsilon\big(f(x+k)=f(y+k)\to x+k=y+k\big), \tag{5}$$

$$(\forall x, y)_\varepsilon\big(f(x+k)\leqq f(y+k)\to x+k\leqq y+k\big). \tag{6}$$

Now by $*$, (5) extends from ε to Λ, and in the special case where f is a non-constant linear function, (6) extends from ε to Λ (c.f. [1], Theorem 40). However in general we have

Theorem 3. *If f is a unary non-linear recursive combinatorial function then there exist X, $Y \in \Lambda - \varepsilon$ such that $F(X) \leqq F(Y)$, but $X \not\leqq Y$.*

Proof. Since f is non-linear there will exist an integer k such that the difference function $f(x+1)-f(x)$ will be strictly increasing for values of x larger than k. Let ρ be the set of even integers. We construct two strictly increasing unary recursive combinatorial functions g_0 and g_1 such that $g_1(x)=1+g_0(x)$ for $x \in \rho$, $g_1(x)=2+g_0(x)$ for $x \in \varepsilon - \rho$, and such that the difference function $h(x)=f(g_1(x))-f(g_0(x))$ is a strictly increasing recursive combinatorial function. Clearly there are combinatorial functions satisfying these conditions, and the reader should have no difficulty in showing that they may be chosen to be recursive functions. Now by $*$ and the corresponding sentence which is true in ε we have

$$(\forall X)_{\Lambda}(F(G_0(X))+H(X)=F(G_1(X))), \tag{7}$$

$$(\forall X, Y)_{\Lambda}(G_1(X)=Y+G_0(X) \rightarrow (Y=1 \vee Y=2)). \tag{8}$$

If U is a universal isol then

$$G_1(U) \neq 1+G_0(U) \tag{9}$$

since otherwise $\{x: g_1(x)=1+g_0(x)\}$ would be co-finite. By a similar argument

$$G_1(U) \neq 2+G_0(U). \tag{10}$$

Take $X=G_0(U)$ and $Y=G_1(U)$. Clearly $U \in \Lambda - \varepsilon$ and since g_0 and g_1 are strictly increasing X, $Y \in \Lambda - \varepsilon$ as well. Now by (7) $F(X) \leqq F(Y)$. If $X \leqq Y$ then for some $Z \in \Lambda$ we have $X+Z=Y$. Then by (8), $Z=1$ or $Z=2$. But this contradicts (9) or (10) respectively. Hence $X \not\leqq Y$. Q.e.d.

Corollary. *If k is an integer, $k \geqq 2$, then*

a) *there exist X, $Y \in \Lambda - \varepsilon$ such that $X^k \leqq Y^k$ but $X \not\leqq Y$,*

b) *there exist X, $Y \in \Lambda - \varepsilon$ such that $k^X \leqq k^Y$, but $X \not\leqq Y$.*

This corollary answers those questions left open in [*1*], p. 187.

There are other cancellation laws which can only be expressed in terms of functions of two variables. Let

$$f(x, y)=\sum c(i, j) \binom{x}{i} \binom{y}{j}$$

be a binary recursive combinatorial function. If for some pair of integers $k>0$ and l, $c(k, l)>0$, then for each fixed $y \geqq l$, $f(x, y)$ as a function of x is strictly increasing for values of $x \geqq k$. Consequently

$$(\forall x, y, z)_{\varepsilon}(f(x+k, z+l)=f(y+k, z+l) \rightarrow x+k=y+k), \tag{11}$$

$$(\forall x, y, z)_{\varepsilon}(f(x+k, z+l) \leqq f(y+k, z+l) \rightarrow x+k \leqq y+k). \tag{12}$$

Now by $*$, (11) extends from ε to Λ. However for (12) we have

Theorem 4. *If f is a binary recursive combinatorial function such that for some pair of integers $k>0$ and $l>0$, $c(k, l)>0$, then there exist $X, Y, Z\in\Lambda-\varepsilon$ such that $F(X, Z)\leqq F(Y, Z)$, but $X\not\leqq Y$.*

Proof. Since $c(k, l)>0$ we can always make $f(x+1, y)-f(x, y)$ as large as we please by simply choosing sufficiently large values for x and y. Let ρ be the set of even integers. We construct three strictly increasing unary recursive combinatorial functions g_0, g_1, and g_2 such that $g_1(x)=1+g_0(x)$ for $x\in\rho$, $g_1(x)=2+g_0(x)$ for $x\in\varepsilon-\rho$ and such that the difference function $h(x)=f(g_1(x), g_2(x))-f(g_0(x), g_2(x))$ is a strictly increasing recursive combinatorial function. The existence of these functions readily follows from our hypotheses. For any universal isol U, take $X=G_0(U)$, $Y=G_1(U)$ and $Z=G_2(U)$. Then by an argument entirely analogous to the one in the proof of Theorem 3 we can show that $F(X, Z)\leqq F(Y, Z)$, but $X\not\leqq Y$. Q.e.d.

Although none of the obvious corollaries to Theorem 4 are new, it should be remarked that now they can be obtained by a uniform method.

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