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Titel: A Note on Compactifications.

Autor: Magill, Kenneth D. Jr.

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Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

MAGILL, K.D., jr.
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A Note on Compactifications

KENNETH D. MAGILL, jr.

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1. Introduction. Our terminology here will agree with that in [1]. In particular, a compact space X is a Hausdorff space with the property that from any open covering of X, one can extract a finite subcovering. By a compactification of a space X, we mean a pair $(\alpha X, h)$ where αX is compact and h is a homeomorphism from X onto a dense subset of αX. Therefore, in a certain sense, one compactifies a space X by adjoining a space K to a homeomorphic image of X in such a manner that certain prescribed conditions are satisfied. In this paper, we consider the problem of determining possibilities for the space K. More specifically, we consider the following question: "Given a space X and a space K, does there exist a compactification $(\alpha X, h)$ of X such that $\alpha X - h(X)$ is homeomorphic to K?" It is shown that the answer is affirmative if X is locally compact, normal and contains an infinite discrete closed subset and K is any Peano space (i.e., compact, connected, locally connected, metric space). Thus, in particular, if X is any locally compact, noncompact metric space and K is any Peano space, there exists a compactification $(\alpha X, h)$ of X such that $\alpha X - h(X)$ is homeomorphic to K.

In concluding the introduction, we take the opportunity to express our appreciation to the referee for his valuable suggestions.

2. As is customary, we will use the symbol βX to denote the Stone-Čech compactification of a completely regular space X. We will assume X is actually a subspace of βX. Let K be any Hausdorff space. It follows from Theorem 6.12, p. 92 of [2] that if there exists a compactification $(\alpha X, h)$ of X such that $\alpha X - h(X)$ is homeomorphic to K, then K is a continuous image of $\beta X - X$. The converse, however, is not true. To see this, let X be any completely regular space which is not locally compact and let K be the space consisting of one point. Then K is a continuous image of $\beta X - X$ but there exists no compactification $(\alpha X, h)$ of X such that $\alpha X - h(X)$ is homeomorphic to K since X does not have a one-point compactification. For locally compact spaces, however, the converse does hold and we have the following:

Theorem (2.1). *Suppose X is locally compact and K is Hausdorff. Then there exists a compactification $(\alpha X, h)$ of X such that $\alpha X - h(X)$ is homeomorphic to K if and only if K is a continuous image of $\beta X - X$.*

Proof. As we observed previously, the necessity of the condition is a consequence of theorem 6.12, p. 92 of [2] so we will prove only that the condition is sufficient. Suppose, then, there exists a continuous function f mapping $\beta X - X$ onto K. There will be no loss in generality if we assume X and K are disjoint. Define the mapping g from βX onto $X \cup K$ by

$$g(x) = x \text{ if } x \in X$$

and

$$g(x) = f(x) \text{ if } x \in \beta X - X.$$

Let R be the equivalence relation on βX which is induced by g (i.e., $(x, y) \in R$ if and only if $g(x) = g(y)$) and let φ denote the canonical mapping from βX onto the quotient space $\beta X / R$ which we will hereafter denote by αX. For a subset H of βX, we will let R_H denote the equivalence relation which R induces on H. Furthermore, if a subset H of βX has the property that $x \in H$ and $(x, y) \in R$ implies $y \in H$, then H will be referred to as an R-saturated subset.

Since X is locally compact, X is an open subset of βX and since X is R-saturated, it follows from Corollary 1 [*1*, p. 46] that the restriction φ^* of φ to X which maps X onto $\varphi(X)$ is a homeomorphism. Similarly, since $A = \beta X - X$ is a closed subset of βX and is also R-saturated, the same corollary implies A/R_A is homeomorphic to $\varphi(A)$. Furthermore, since the function f mapping A onto K satisfies c) of Proposition 8 [*1*, p. 44], it follows from a) of that same proposition that A/R_A is homeomorphic to K. Thus $\varphi(A) = \alpha X - \varphi^*(X)$ is homeomorphic to K.

If follows easily that $\varphi^*(X)$ is a dense subset of αX. Now let us show that αX is compact. According to Proposition 8 [*1*, p. 121] it is sufficient to show R is closed and Proposition 10 [*1*, p. 62] assures us that R is closed if each equivalence class M of βX has a basis of R-saturated neighborhoods. Since X is open in βX, it follows immediately that a class of the form $g^{-1}(g(x))$, x in X has such a basis. Now suppose $x \in A$ and let $F = g^{-1}(g(x))$. Then F is a closed subset of A and hence also a closed subset of βX. Let U be any neighborhood of F in βX. Then $U \cap A$ is a neighborhood of F in A and it follows that $\complement_A(g^{-1}(g(\complement_A(U \cap A))))$ is an R_A-saturated neighborhood of F in A which is contained in $U \cap A$. There exists an open subset U' of βX such that $U' \cap A = \complement_A(g^{-1}(g(\complement_A(U \cap A))))$. Let $V = U \cap U'$. Then $F \subset V \subset U$ and since $V \cap A = U' \cap A$ is R_A-saturated, it follows that V is R-saturated. Thus, a class of the form $g^{-1}(g(x))$, $x \in A$ also has a basis of R-saturated neighborhoods. This proves that αX is compact and hence that $(\alpha X, \varphi^*)$ is a compactification of X with the property that $\alpha X - \varphi^*(X)$ is homeomorphic to K.

Before stating the next result, we recall once again that a compact, connected, locally connected, metric space is referred to as a Peano space.

Theorem (2.2). *Suppose X is a locally compact, normal space which contains an infinite, discrete, closed subset. Then for any Peano space K, there exists a compactification $(\alpha X, h)$ of X such that $\alpha X - h(X)$ is homeomorphic to K.*

Proof. Since X contains an infinite, discrete, closed subset, we can extract from this a countably infinite subset N which must necessarily also be discrete and closed. Topologically, N may be regarded as the space of natural numbers with the discrete topology. Let f be any bounded, real-valued function on N. Since N is a closed subset of X and X is normal, it follows from the Tietze Extension Theorem that there exists a continuous extension f' of f which is also bounded and whose domain is all of X. Then f' can be continuously extended to a function f'' whose domain is βX. Note that the restriction of f'' to $\mathrm{cl}_{\beta X} N$ is a continuous extension of f. Therefore, $\mathrm{cl}_{\beta X} N$ is a compactification of N with the property that every bounded real-valued function on N can be continuously extended to a function on $\mathrm{cl}_{\beta X} N$. But these properties characterize βN so we conclude $\mathrm{cl}_{\beta X} N = \beta N$. Since no point of $X-N$ belongs to $\mathrm{cl}_{\beta X} N$, it follows that

$$\beta N - N = [\mathrm{cl}_{\beta X} N] - N \subset \beta X - X.$$

From 6.10 (a) p. 91 of [2], we see that $\beta N - N$ contains a copy of βN. Thus, there exists a countably infinite, discrete space Y such that $\beta Y \subset \beta X - X$. Let g be any function mapping Y onto the rationals in $[0, 1]$. Then g is continuous on Y and hence has a continuous extension g' which maps βY into $[0, 1]$. Since βY is compact, $g'[\beta Y]$ is compact and must therefore be all of $[0, 1]$. Once again we appeal to the Tietze Extension Theorem and conclude that g' can be continuously extended to a function g'' whose domain is $\beta X - X$. Now let K be any Peano space. By the well known Hahn-Mazurkiewicz Theorem, there exists a continuous function k mapping $[0, 1]$ onto K. Therefore, the composition kg'' of k and g'' is a continuous mapping from $\beta X - X$ onto K and the desired result now follows from the previous theorem.

Since, in a metric space, countable compactness is equivalent to compactness, any noncompact metric space must contain an infinite, discrete, closed subset. Since, in addition, every metric space is normal, the following corollary is a consequence of Theorem (2.2).

Corollary (2.3). *Let X be any locally compact, noncompact, metric space and let Y be any Peano space. Then there exists a compactification $(\alpha X, h)$ of X such that $\alpha X - h(X)$ is homeomorphic to K.*

In closing, we make a few remarks about Theorem (2.2). There are three conditions placed on the space X in the statement of Theorem (2.2):

(i) X is locally compact,

(ii) X is normal,

(iii) X contains an infinite, discrete, closed subset.

If any one of these conditions is deleted from the hypothesis of Theorem (2.2), the resulting statement is not true. For example, the space of all ordinals less than the first uncountable ordinal [2, p. 72–76] satisfies conditions (i) and (ii) but not (iii). Its only compactification is the one-point compactification and thus the conclusion of Theorem (2.2) does not hold for this space.

The Tychonoff Plank [2, p. 123–125] is an example of a space which satisfies conditions (i) and (iii) (the "right edge" is an infinite, discrete, closed subset) but not (ii). As in the previous example, its only compactification is the one-point compactification and the conclusion of Theorem (2.2) does not hold for this space either.

Finally, suppose X is not locally compact. Then for any compactification $(\alpha X, h)$ of X, $h(X)$ is not an open subset of αX. Therefore, $\alpha X - h(X)$ is not compact and thus cannot be homeomorphic to any Peano space.

References

[1] Bourbaki, N.: Éléments de mathématique, premiere partie, livre 111, Topologie général, Kap. I; troisieme édition. Paris: Hermann 1961.

[2] Gillmann, L., and M. Jerison: Rings of continuous functions. New York: D. van Nostrand 1960.

Department of Mathematics, State University of New York at Buffalo
Buffalo, N.Y. 14214, USA