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The n-Arc Property for Functions Meromorphic in the Disk

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1. Introduction

This paper deals with the n-arc property which is a generalization of ambiguous point. In 1961 YOUNG [*10*] proved that if f is bounded and analytic in the open unit disk and if a point p on the unit circle has the two-arc property, then p has the n-arc property for every n. We establish this result for meromorphic functions and give some related applications and examples.

Let f be a function from the open unit disk D (in the complex plane) into the extended complex plane S, and let p be a point on the unit circle C. The *cluster set* $C(f, p)$ of f at p is defined as the set of all points w in S such that there exists a sequence $\{z_j\}$ of points from D with $z_j \to p$ and $f(z_j) \to w$. The *boundary cluster set* $C_B(f, p)$ of f at p is the set $\bigcap \overline{\bigcup C(f, q)}$ where the intersection is taken over all neighborhoods N of p, the union over all q on $N \cap C$ with $q \neq p$, and the bar denotes closure. The *left boundary cluster set* $C_{BL}(f, p)$ of f at p is the set $\bigcap \overline{\bigcup C(f, q)}$ where the intersection is taken over all neighborhoods N of p, and the union over all q on $N \cap C$ with $\arg(p) < \arg(q) < \arg(p) + \pi/4$. The *right boundary cluster set* $C_{BR}(f, p)$ of f at p is defined in the same way except that the last condition is replaced by $\arg(p) - \pi/4 < \arg(q) < \arg(p)$. Observe that $C_B(f, p) = C_{BL}(f, p) \cup C_{BR}(f, p)$. If E is a subset of C we define $C_{B-E}(f, p) = \bigcap \overline{\bigcup C(f, q)}$ where the intersection is taken over all neighborhoods N of p, and the union over all q on $N \cap (C - E)$ with $q \neq p$.

If λ is an arc ending at p (by this we mean a simple arc lying in D except for one endpoint at p), the *arc cluster set* $C_\lambda(f, p)$ is defined to be the set of all points w in S for which there exists a sequence $\{z_j\}$ of points on $\lambda \cap D$ with $z_j \to p$ and $f(z_j) \to w$. The point p is called an *ambiguous point* for f if there exist two arcs λ and η ending at p such that $C_\lambda(f, p)$ and $C_\eta(f, p)$ are disjoint.

In 1955 BAGEMIHL [*1*] proved that an arbitrary function from D into S has at most countably many ambiguous points. LOHWATER has suggested the following generalization of ambiguous point.

Definition. The point p has the *n-arc property* (with respect to f) if there exist n arcs $\lambda_1, \lambda_2, \ldots, \lambda_n$ each ending at p, such that the intersection of all n of the sets $C_{\lambda_1}(f, p), C_{\lambda_2}(f, p), \ldots, C_{\lambda_n}(f. p)$ is void, and the intersection of any $n-1$ of them is non-void.

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Note that the two-arc property is the same as ambiguous point.

BAGEMIHL, PIRANIAN, and YOUNG [2] have given an example of a function for which each point of C has the three-arc property. Thus BAGEMIHL's ambiguous point theorem cannot be extended to the n-arc property for n greater than two.

In 1961 YOUNG [10] proved the following

Theorem. *Let f be meromorphic in the open unit disk D and let p be a point on the unit circle C. If $C_B(f, p)$ is not connected, then p has the two-arc property. If in addition f is bounded, then p has the n-arc property for every n.*

Corollary. *Let f be bounded analytic in D, and let p be a point on C. If p has the two-arc property, then p has the n-arc property for every n.*

We shall prove that if f is *meromorphic* in D and if p has the two-arc property, then p has the n-arc property for every n. Then this will be combined with a result by GROSS [5] to show that in the above theorem by YOUNG the condition *meromorphic* can be replaced by *continuous*, and *bounded analytic* replaced by *meromorphic*.

2. Some Preliminary Results

Theorem 1. *Let f be meromorphic in the open unit disk D and let p be a point on the unit circle C. Let J be a closed Jordan domain lying in $C(f, p) - C_B(f, p)$ and not containing the point ∞. Suppose further that the boundary of J misses the points $f(z)$ such that $f'(z)=0$. If J contains an arc cluster set of f at p in its interior, then there exists an arc η ending at p such that $C_\eta(f, p)$ is contained in J and contains the boundary of J.*

Proof. Let λ be an arc ending at p such that $C_\lambda(f, p)$ is contained in the interior J^0 of J. Let N be a circular neighborhood of p such that $C(f, q)$ is disjoint from J for each q on $N \cap C$ with $q \neq p$, and let K denote the component of $f^{-1}(J^0) \cap N$ that contains a subarc γ of λ that ends at p. We remark that $\overline{K} \cap C = \{p\}$, K need not be simply connected, and the boundary of K may fail to be a simple closed curve.

If K is not simply connected, let $H_1, H_2, \ldots$ denote the components of $D - K$ that lie inside simple closed curves lying in K, and let $L = K \cup H_1 \cup H_2 \cup \cdots$. If K is simply connected let $L = K$. Then L is open, connected, and simply connected. We now prove that the boundary of L is locally connected. Suppose the boundary of L is not locally connected. Then ([9], p. 18, Theorem 12.1) there exists in the boundary of L a non-degenerate subarc α such that the boundary of L fails to be locally connected at each point on α. Let z_0 be any point on $\alpha \cap D$. Then $f'(z_0) \neq 0$, since $f(z_0)$ lies on the boundary of J, and therefore f is locally a homeomorphism at z_0. But f maps $\alpha \cap D$ into the boundary of J which is locally connected, and we have a contradiction. Thus the boundary of L is locally connected.

This implies that each point in the boundary of L is arcwise accessible from all sides from L ([9], p. 112, Theorem 4.2). A well known result in prime end theory [3] is that each prime end of a simply connected plane domain contains precisely one point that is arcwise accessible with respect to any sequence of crosscuts that determines the prime end. Therefore each prime end of L contains exactly one point.

Let φ map the open unit disk D' in the ζ-plane in a one-to-one conformal manner onto L. The prime ends of L correspond in a one-to-one fashion, under φ, to the points on the unit circle C' in the ζ-plane [3]. Since each prime end of L is degenerate, we can extend φ continuously to C' by defining $\varphi(\zeta)$ to be the point in $C(\varphi, \zeta)$ for each point ζ on C'.

Let E denote the set of all points ζ on C' such that $\varphi(\zeta)=p$. Then E is a closed and therefore measurable subset of C'. Moreover φ has radial limit p at each point in E and φ is non-constant, so according to a theorem by F. and M. RIESZ [8] the measure (in C') of E is zero.

Let γ' denote the arc $\varphi^{-1}(\gamma)$ and let p' denote the point on C' at which γ' ends.

Suppose p' is not a limit of poles of the composite function $f\varphi$. Then some neighborhood of p' is disjoint from $\varphi^{-1}(H_j)$ $(j=1, 2, \ldots)$, and it follows that $C(f\varphi, p')$ must be contained entirely in J. Moreover, since the measure of E is zero, the boundary of $C(f\varphi, p')$ is contained in $C_{B-E}(f\varphi, p')$ [7] which is a subset of the boundary of J, and since $C(f\varphi, p')$ contains $C_\lambda(f, p)$ which lies in the interior of J, it follows that $C(f\varphi, p')$ must equal J. Now let η' be an arc ending at p' such that $C_{\eta'}(f\varphi, p')=C(f\varphi, p')$ [5]. Then $\eta=\varphi(\eta')$ is an arc ending at p and $C_\eta(f, p)=J$. Thus we have, in this case, a strong form of the conclusion of Theorem 1.

If p' is a limit of poles of $f\varphi$, then there exists a subsequence $\{H_k'\}$ of $\{\varphi^{-1}(H_j)\}$ such that $\{H_k'\} \to p$ (see [9] for convergence of sets). Each H_k' contains a pole of $f\varphi$ in its interior, and the boundary of H_k' is a simple closed curve since it is a component of the preimage of the boundary of J under f and f is a local homeomorphism on this set. Also f maps the boundary of H_k' onto the boundary of J.

Let a and b be distinct points in the boundary of J and let A and B denote the components of the complement of $\{a, b\}$ in the boundary of J. Let a_k and b_k be points in the boundary of H_k' such that $f\varphi(a_k)=a$ and $f\varphi(b_k)=b$, and let A_k and B_k denote the components of the complement of $\{a_k, b_k\}$ in the boundary of H_k' such that A is contained in $f\varphi(A_k)$ and B is contained in $f\varphi(B_k)$ $(k=1, 2, \ldots)$. Let H denote the complement in D' of $\varphi^{-1}(H_1) \cup \varphi^{-1}(H_2) \cup \cdots$. We construct a sequence of mutually disjoint arcs $\eta_1, \eta_2, \ldots$ such that η_k lies in H except for one endpoint that is a_k and the other that is b_{k+1} $(k=1, 2, \ldots)$, and such that $\{\eta_k\} \to \{p'\}$. This construction is possible since the sets $\varphi^{-1}(H_j)$ together with the remaining points in $D' \cup C'$ form an upper semi-continuous decomposition of $D' \cup C'$ for which the decomposition space is homeomorphic to $D' \cup C'$ ([9], p. 173, Corollary 3.1). So constructing the above

arcs $\eta_1, \eta_2, \ldots$ in H is topologically equivalent to constructing them in D' and in addition dodging a countable and isolated set of points, which offers no difficulty.

Let η' denote the arc $\eta_1 \cup A_2 \cup \eta_2 \cup B_3 \cup \eta_3 \cup A_4 \cup \eta_4 \cup B_5 \cup \cdots$. Then η' is an arc ending at p', and as ζ approaches p' along η' its image $w = f\varphi(\zeta)$ lies in J and repeatedly traverses the subarcs A and B in the boundary of J. Thus $C_{\eta'}(f\varphi, p')$ is contained in J and contains the boundary of J, and therefore $\eta = \varphi(\eta')$ is the desired arc ending at p. This completes the proof of Theorem 1.

Theorem 2. *Let a, b, $w_1, w_2, \ldots, w_n$ be distinct points in the (finite) complex plane E. Then there exist in $E-\{a, b\}$ closed Jordan domains $J_1, J_2, \ldots, J_n$ such that w_k lies in the interior of J_k $(k=1, 2, \ldots, n)$, $J_1 \cap J_2 \cap \cdots \cap J_n$ is void, and the intersection of any $n-1$ of the sets $bdry(J_1), bdry(J_2), \ldots, bdry(J_n)$ is non-void.*

Proof. We first construct Jordan domains about the points $1, 2, \ldots, n$ which intersect in the desired fashion.

Let

$$L_j = \{z \mid \tfrac{3}{8} \leqq |z-(j+\tfrac{1}{2})| \leqq \tfrac{5}{8} \text{ and } \mathscr{I}(z) \leqq 0\}$$
$$\cup \{z \mid |z-j| \leqq \tfrac{1}{16}\} \cup \{z \mid |z-(j+1)| \leqq \tfrac{1}{16}\} \qquad (j=1, 2, \ldots, n-1).$$

Let

$$U_j = \{z \mid \tfrac{7}{8} \leqq |z-j| \leqq \tfrac{9}{8} \text{ and } \mathscr{I}(z) \geqq 0\}$$
$$\cup \{z \mid |z-(j-1)| \leqq \tfrac{1}{16}\} \cup \{z \mid |z-(j+1)| \leqq \tfrac{1}{16}\} \quad (j=2, 3, \ldots, n-1).$$

Let $J_k = U_k \cup (\bigcup L_j)$ *where the second union is taken over all j between* 1 *and* $n-1$ *with* $j \neq k, k+1$ $(k=1, 2, \ldots, n-2)$. Let $J_{n-1} = L_1 \cup L_2 \cup \cdots \cup L_{n-2}$ and let $J_n = L_2 \cup L_3 \cup \cdots \cup L_{n-1}$. Then the closed Jordan domains $J_1, J_2, \ldots, J_n$ and their boundaries intersect in the desired fashion, and k lies in the interior of J_k $(k=1, 2, \ldots, n)$. Now let Γ be an arc with endpoints a and ∞ such that $b \in \Gamma$ and $w_j \notin \Gamma$ $(j=1, 2, \ldots, n)$. Let φ be a homeomorphism of the slit plane $E-\{z \mid \mathscr{I}(z)=0 \text{ and } \mathscr{R}(z) \leqq 0\}$ onto $E-\Gamma$ which maps the integers $1, 2, \ldots, n$ onto the points $w_1, w_2, \ldots, w_n$. Then $\varphi(J_1), \varphi(J_2), \ldots, \varphi(J_n)$ are the desired closed Jordan domains.

3. The Main Result and Some Applications

Theorem 3. *Let f be meromorphic in the open unit disk D and let p be a point on the unit circle C. If p has the two-arc property, then p has the n-arc property for every n.*

Proof. Let $A(f, p)$ denote the set of all asymptotic values of f at p (that is $w \in A(f, p)$ if and only if there exists an arc λ ending at p such that $C_\lambda(f, p) = \{w\}$). If $A(f, p)$ is finite, then YOUNG's proof [*10*], with trivial modifications, establishes the desired conclusion. If $A(f, p)$ is infinite, let a and b be distinct asymptotic values of f at p and let α and β be arcs ending at p such that

$C_\alpha(f,p)=\{a\}$ and $C_\beta(f,p)=\{b\}$. We can assume that α and β are disjoint, except for their common endpoint p. Let Γ be an arc in D joining a point on $\alpha\cap D$ to a point on $\beta\cap D$ such that the region Ω bounded by Γ, α and β is a Jordan domain containing p in its closure. Let g denote f restricted to Ω. Let φ map the open unit disk D' in the ζ-plane one-to-one and conformally onto Ω such that when extended to the boundary of D' φ takes 1 onto p. Then $C_B(g\varphi,1)=\{a,b\}$ which is not connected. If the composite function $g\varphi$ has only finitely many asymptotic values at 1 then, as above, YOUNG's proof [*10*] shows that 1 has the n-arc property with respect to $g\varphi$ for every n, which in turn implies that p has the n-arc property with respect to f for every n. So suppose that $g\varphi$ has infinitely many asymptotic values at 1. Then $C(g\varphi,1)$ is the entire extended plane S; for if $C(g\varphi,1)$ is a proper subset of S then $g\varphi$ omits at least three values at 1 and therefore has at most one asymptotic value there [*6*].

Let $w_1, w_2, \ldots, w_n$ be distinct finite asymptotic values of $g\varphi$ at 1 such that $w_k\neq a, b$ $(k=1,2,\ldots,n)$. By Theorem 2 there exist in $S-\{a,b,\infty\}$ closed Jordan domains $J_1, J_2, \ldots, J_n$ such that w_k lies in the interior of J_k $(k=1,2,\ldots,n)$, $J_1\cap J_2\cap\cdots\cap J_n$ is void, and the intersection of any $n-1$ of the sets $bdry(J_1)$, $bdry(J_2)$, ..., $bdry(J_n)$ is non-void. Each J_k is contained in $C(g\varphi,1)-C_B(g\varphi,1)=S-\{a,b\}$, and each contains an arc cluster set of $g\varphi$ at 1 in its interior. Since the set of points $g\varphi(z)$ such that $(g\varphi)'(z)=0$ is at most countable, there is no loss in assuming that $bdry(J_k)$ is disjoint from this set $(k=1,2,\ldots,n)$. Then according to Theorem 1 there exists in D' an arc λ_k ending at 1 such that $bdry(J_k)\subset C_{\lambda_k}(g\varphi,1)\subset J_k$ $(k=1,2,\ldots,n)$. Let $\eta_k=\varphi(\lambda_k)$ $(k=1,2,\ldots,n)$. Then $\eta_1,\eta_2,\ldots,\eta_n$ are arcs ending at p, and since $C_{\eta_k}(f,p)=C_{\lambda_k}(g\varphi,1)$ $(k=1,2,\ldots,n)$ the sets $C_{\eta_1}(f,p)$, $C_{\eta_2}(f,p)$, ..., $C_{\eta_n}(f,p)$ intersect in the desired fashion. Since n was arbitrary, this completes the proof.

We now state and prove the following improved version of the theorem by YOUNG mentioned in the introduction.

Theorem 4. *Let f be a continuous function from the open unit disk D into the extended complex plane S, and let p be a point on the unit circle C. If $C_B(f,p)$ is not connected, then p has the two-arc property. If in addition f is meromorphic, then p has the n-arc property for every n.*

Proof. Suppose f is continuous and $C_B(f,p)$ is not connected. Then since $C_B(f,p)=C_{BL}(f,p)\cup C_{BR}(f,p)$, and since $C_{BL}(f,p)$ and $C_{BR}(f,p)$ are continua, they must be disjoint. A little known result by GROSS [*5*] asserts that (for arbitrary f) there exist arcs λ and η ending at p such that $C_\lambda(f,p)=C_{BL}(f,p)$ and $C_\eta(f,p)=C_{BR}(f,p)$. Thus p has the two-arc property. If in addition f is meromorphic, then by Theorem 2 p has the n-arc property for every n. This completes the proof.

Theorem 5. *Let f be meromorphic in the open unit disk D and let p be a point on the unit circle C. If $C(f,p)-C_B(f,p)$ contains an omitted value of f at p, then p has the n-arc property for every n.*

Proof. Let w be an omitted value of f at p such that w lies in $C(f,p)-C_B(f,p)$. Then by the Gross-Iversen theorem (see, for example, [4], Theorem 1'), w is an asymptotic value of f at p. By a previously mentioned result by GROSS $C_{BL}(f,p)$ is an arc cluster set of f at p. Thus p has the two-arc property, and by Theorem 3 the n-arc property for every n.

Theorem 6. *Let f be meromorphic in the open unit disk D and suppose there exists an open arc A on the unit circle C such that the modulus of $f(re^{i\vartheta})$ has radial limit one for almost all points $e^{i\vartheta}$ on A. Let p be a point on A and suppose that p is not the limit of zeros or poles of f. If $C(f,p)-C_B(f,p)$ is non-void, then p has the n-arc property for every n.*

Proof. In [10] YOUNG proves that p has the two-arc property. Theorem 3 then shows that p has the n-arc property for every n.

4. Two Examples

A number of deep results from complex function theory were used in proving Theorem 3. The following example, of a *differentiable homeomorphism* such that there is a point having the two-arc property but failing to have the n-arc property for every n greater than two, shows that some complex differentiability seems necessary in Theorem 3.

Example 1. Let

$$f(x+iy)=x+i\frac{y}{1-x}, \qquad 0<x<1 \text{ and } 0<y<1-x.$$

Then f is a differentiable homeomorphism from the open triangle $\Delta=\{x+iy \mid 0<x<1 \text{ and } 0<y<1-x\}$ onto the open unit square $T=\{x+iy \mid 0<x<1 \text{ and } 0<y<1\}$, and the cluster set of f at 1 is the segment $L=\{x+iy \mid x=1 \text{ and } 0\leqq y\leqq 1\}$. Since each point on L is an asymptotic value of f at 1, 1 has the two-arc property. But each arc cluster set of f at 1 is a subcontinuum of L, and since it is impossible for three subcontinua of L to intersect pairwise but have no point in common, we see that 1 cannot have the three-arc property. Similarly 1 cannot have the n-arc property for any n greater than two. We transplant f to the open unit disk by means of the Riemann mapping theorem, and the example is complete.

Example 2. We define a function f from the open unit disk D into the extended plane as follows. Let $|r|$ be less than one. If ϑ is rational let $f(re^{i\vartheta})=0$. If ϑ is irrational and $0<\vartheta<\pi$ let $f(re^{i\vartheta})=1$. If ϑ is irrational and $\pi\leqq\vartheta<2\pi$ let $f(re^{i\vartheta})=2$. Set $f(0)=0$. Then $C_B(f,1)=\{0,1,2\}$ which is not connected. Moreover 1 does not have the two arc property since 0 belongs to each arc cluster set of f at 1.

Example 2 shows that in Theorem 4 the continuity assumption cannot be omitted, and Example 1 shows that meromorphic cannot be replaced by differentiable homeomorphism.

5. A Conjecture

We close by remarking that Theorem 3 is the case $n=2$ of the following

Conjecture. *If f is meromorphic in the open unit disk and if a point p on the unit circle has the n-arc property, then p has the k-arc property for every $k>n$.*

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