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Titel: On Representation of Partially Ordered Sets.

Autor: Abian, Alexander; Deever, David

Jahr: 1966

PURL: https://resolver.sub.uni-goettingen.de/purl?266833020_0092|log57

Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

ABIAN, A., and D. DEEVER
Math. Zeitschr. 92, 353–355 (1966)

On Representation of Partially Ordered Sets

ALEXANDER ABIAN and DAVID DEEVER

Received November 5, 1965

In this paper we obtain conditions for representation of partially ordered sets by means of sequences made up of 0 and 1 and ordered by the principle of strong first differences.

Representation of a partially ordered set P by sequences of 0 and 1 of type $\overline{\overline{P}}$ is a consequence of Theorem 2.12 of [1] and the well-ordering principle. However, in this paper the type of sequences representing a partially ordered set P is determined by the cardinality of a weakly dense subset of P which may be less than $\overline{\overline{P}}$.

The ordering of sequences by the principle of strong first differences (in contrast to the ordering by first differences, i.e., lexicographically) introduced below is motivated by the notion of set-theoretical inclusion.

The results below generalize the theorem of [2] for simply ordered sets.

We first introduce some definitions. Let (s_i) and (t_i) be two sequences of finite or transfinite type. We say that (s_i) is less than or equal to (t_i) according to the principle of *strong first differences*, and we write

$$(s_i) \underline{\underline{\preceq}}^* (t_i) \tag{1}$$

if $s_i = 1$ implies $t_i = 1$ for every index i. As usual, if $(s_i) \underline{\underline{\preceq}}^* (t_i)$ and $(s_i) \neq (t_i)$ then we write $(s_i) \prec^* (t_i)$.

Clearly $\underline{\underline{\preceq}}^*$ partially orders any set of sequences of 0 and 1 of similar type.

Let D be a subset of a partially ordered set $(P, \leqq)$ and let p and q be any two elements of P with $q \not\leqq p$. Then we say that D is *weakly dense* in P if there exists a d_1 in D such that

$$d_1 \leqq q \quad \text{and} \quad d_1 \not\leqq p. \tag{2}$$

Furthermore, we say that D is *very weakly dense* in P if there exists a d_2 in D such that

$$d_2 \leqq q \quad \text{and} \quad d_2 \not< p. \tag{3}$$

Lemma 1. *Let $(P, \leqq)$ be a partially ordered set with a very weakly dense subset D. Then for every $d \in D$ there exists at most one immediate successor p of d with the property that $e \in D$ and $e \leqq p$ implies $e \leqq d$.*

Proof. Suppose, by way of contradiction, that there exists in P two distinct immediate successors, p and q, of d with the above mentioned property. Clearly $q \not\leqq p$, and thus since D is very weakly dense in P there exists an $e \in D$ such that

$e \leqq q$ and $e \not< p$. But $d < p$ and hence we cannot have $e \leqq d$. Thus, we have found an $e \in D$ such that $e \leqq q$ but $e \not\leqq d$, contrary to the assumption that q had the property mentioned in the statement of the lemma. This proves the lemma.

Theorem 1. *For every ordinal number λ, a partially ordered set $(P, \leqq)$ has a very weakly dense subset of power $\aleph_\lambda$ if and only if it has a weakly dense subset of power $\aleph_\lambda$.*

Proof. Let D be a very weakly dense subset of P with $\overline{\overline{D}} = \aleph_\lambda$. For each $d \in D$ add to D that one immediate successor of d which has the property mentioned in Lemma 1 and call the resulting set E. Clearly $\overline{\overline{E}} = \aleph_\lambda$.

We show that E is weakly dense in P.

Let p and q be elements of P with $q \not\leqq p$. Then by (3) there exists a $d \in D \subset E$ such that $d \leqq q$ and $d \not< p$. If $d \neq p$ then (2) is satisfied. On the other hand, if $d = p$ then we have $p < q$ with $p \in D$.

Now, in this case, if q is not the immediate successor of p then there exists an $r \in P$ such that $p < r < q$. Hence, since $q \not\leqq r$, by (3) there exists an $e \in D \subset E$ with $e \leqq q$ and $e \not< r$ and consequently we have $e \leqq q$ and $e \not\leqq p$, satisfying (2).

Finally, if q is the immediate successor of p, then since $p \in D$ and in view of the construction of E either there exists an $e \in D \subset E$ with $e \leqq q$ and $e \not\leqq p$ or else we must have $q \in E$, in which case $q \leqq q$ and $q \not\leqq p$, and in either case (2) is satisfied. This shows that E is weakly dense in P.

Since a weakly dense subset is clearly very weakly dense as well the theorem is proved.

Lemma 2. *Let D be a weakly dense subset of a partially ordered set $(P, \leqq)$ and let p and q be elements of P. Then $p \leqq q$ if and only if for every d in D we have $d \leqq p$ implies $d \leqq q$.*

Proof. Suppose that for every d in D we have $d \leqq p$ implies $d \leqq q$. If $p \not\leqq q$ then since D is weakly dense in P there exists a d_0 in D such that $d_0 \leqq p$ and $d_0 \not\leqq q$, contrary to hypothesis. Hence, $p \leqq q$ as desired.

The converse is obvious.

Theorem 2. *Let $(P, \leqq)$ be a partially ordered set with a very weakly dense subset of power less than or equal to $\aleph_\lambda$. Then P is isomorphic to a set of sequences of 0 and 1 of type ω_λ partially ordered by the principle of strong first differences.*

Proof. By Theorem 1 P has a weakly dense subset D with $\overline{\overline{D}} \leqq \aleph_\lambda$. Consider the mapping f from P onto a subset $\mathfrak{B}$ of the power set of D where for every $p \in P$ we have

$$f(p) = \{d \mid d \in D \text{ and } d \leqq p\}. \tag{4}$$

We show that if $\mathfrak{B}$ is partially ordered by inclusion then f is an isomorphism.

In view of (4) and Lemma 2 it is clear that $p \leqq q$ if and only if $f(p) \subset f(q)$. Furthermore, if $f(p) = f(q)$ then for every $d \in D$ we have $d \in f(p)$ if and only if $d \in f(q)$ and in view of (4) and Lemma 1 this implies that $p = q$. This shows that f is an isomorphism.

Now let $(d_i)_{i<v}$ be a well ordering of D with $v \leqq \omega_\lambda$. Let g be a mapping from P onto a set S of sequences of 0 and 1 of type ω_λ such that $g(p)=(p_i)_{i<\omega_\lambda}$ for every $p \in P$, where

$$p_i = \begin{cases} 1 & \text{if } d_i \in f(p) \\ 0 & \text{if } d_i \notin f(p). \end{cases} \tag{5}$$

In view of (5) and the fact that f is an isomorphism from $(P, \leqq)$ onto $(\mathfrak{B}, \subset)$ it follows immediately that g is an isomorphism from $(P, \leqq)$ onto $(S, \precneqq^*)$. This proves the theorem.

Theorem 3. *Let $(S_{\omega_\lambda}, \precneqq^*)$ be the set of all sequences of 0 and 1 of type ω_λ, partially ordered by the principle of strong first differences. Then S_{ω_λ} has a weakly dense subset of power $\aleph_\lambda$.*

Proof. For each $j<\omega_\lambda$ let $d_j=(\delta_i^j)_{i<\omega_\lambda}$ where δ_i^j is the familiar Kronecker δ. Let D be the set of all such d_j. Clearly $\overline{\overline{D}}=\aleph_\lambda$ and $D \subset S_{\omega_\lambda}$. We show that D is a weakly dense subset of S_{ω_λ}.

Let $s=(s_i)_{i<\omega_\lambda}$ and $t=(t_i)_{i<\omega_\lambda}$ be two elements of S_{ω_λ} with $s \not\precneqq^* t$. Then in view of (1) there exists an index j such that $s_j=1$ and $t_j=0$. Consider $d_j=(\delta_i^j)_{i<\omega_\lambda}$. Clearly $d_j \precneqq^* s$ and $d_j \not\precneqq^* t$. Hence, in view of (3), D is weakly dense in S_{ω_λ} as desired.

Corollary. *Let S be any set of sequences of 0 and 1 of type ω_λ, partially ordered by the principle of strong first differences. Then there exists a set S' of sequences of 0 and 1 of type ω_λ with $S \subset S'$ such that $\overline{\overline{S}}=\overline{\overline{S'}}$ and S' has a weakly dense subset of power less than or equal to $\aleph_\lambda$.*

Proof. If $\overline{\overline{S}} \leqq \aleph_\lambda$ let $S'=S$ and since S is weakly dense in itself we are done. If $\overline{\overline{S}} > \aleph_\lambda$ let $S'=S \cup D$ where D is the weakly dense subset of S_{ω_λ} in Theorem 3. Since $\overline{\overline{D}}=\aleph_\lambda$ we have $\overline{\overline{S'}}=\overline{\overline{S \cup D}}=\overline{\overline{S}}$ and D is clearly weakly dense in S'.

References

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Department of Mathematics, The Ohio State University, 231 W. 18th Avenue, Columbus, Ohio 43210, USA