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Tauberian constants for general triangular matrices and certain special types of Hausdorff means

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1. Introduction. We consider the transformation

(1.1) $$t_n = \sum_{k=0}^{\infty} c_{n,k} s_k \qquad (n \geqq 0),$$

where

$$s_k = a_0 + a_1 + \cdots + a_k.$$

In various special cases, it has been found that theorems of the following general type hold. We suppose that p, n are related in an appropriate way (usually the assumption is that $p/n \to \alpha$ as $n \to \infty$ where $\alpha > 0$ is a constant). Suppose that

(1.2) $$n a_n = O(1).$$

Then there is a constant A such that

(1.3) $$\limsup_{n \to \infty} |s_p - t_n| \leqq A \limsup_{n \to \infty} |n a_n|.$$

There are also analogous results in which (1.1) is replaced by a sequence-to-function transformation

(1.4) $$\Phi(u) = \sum_{k=0}^{\infty} c_k(u) s_k.$$

It is, of course, desirable that the best possible value of the constant A should be determined.

Theorems of this type were first considered by HADWIGER [6], and have since been investigated by various authors; see for example AGNEW ([3], [4]) and TENENBAUM [11]. In particular, the case in which (1.1) is a Hausdorff transformation has been considered by JAKIMOVSKI [8].

Some similar theorems have been obtained with (1.2) replaced by the weaker condition

(1.5) $$\gamma_n = O(1),$$

where we write

$$\gamma_n = \frac{1}{n+1} \sum_{\nu=1}^{n} \nu a_\nu$$

see, for example DELANGE [5], RAJAGOPAL [9] and SHERIF [10].

In § 2 of the present paper, we consider the case in which (1.1) is any regular *triangular* transformation satisfying certain quite weak restrictions. Suppose that $\alpha>0$, and that p is an integer valued function of n such that $p/n\to\alpha$ as $n\to\infty$. We show that if (1.5) holds, then

$$\limsup|s_p-t_n|\leqq D(\alpha)\limsup|\gamma_n| \tag{1.6}$$

where $D(\alpha)$ is a Tauberian constant which will be investigated.

In § 3, we shall then show that, in certain special cases, the results obtained can be simplified; and in particular, we shall consider $D(\alpha)$ for certain classes of Hausdorff transformations.

A new feature which arises is that in the case $\alpha=1$ we must distinguish between the case in which $p=n$ and that in which $p/n\to1$ but $p\neq n$ for all sufficiently large n. That such a distinction is sometimes necessary is shown by the trivial example in which (1.1) reduces to $t_n=s_n$. If $p=n$, then s_p-t_n is identically zero; but if $p=n-1$, then s_p-t_n need to tend to zero under the hypothesis (1.5) (which differs in this respect from (1.2)).

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2. Throughout this paragraph, we will write

$$A_{n,p}=\begin{cases}\displaystyle\sum_{\nu=1}^{p-1}\left|\frac{1}{\nu}\sum_{k=0}^{\nu-1}c_{n,k}-c_{n,\nu}\right|+\left|1+\frac{1}{p}\sum_{k=0}^{p-1}c_{n,k}-c_{n,k}\right|+\\ \displaystyle\qquad+\sum_{\nu=p+1}^{n}\left|\frac{1}{\nu}\sum_{k=0}^{n}c_{n,k}+c_{n,\nu}\right| \quad \text{for} \quad (p\leqq n),\\ \displaystyle\sum_{\nu=1}^{n}\left|\frac{1}{\nu}\sum_{k=0}^{\nu-1}c_{n,k}-c_{n,\nu}\right|+\sum_{\nu=n+1}^{p}\frac{1}{\nu}+1+o(1) \quad \text{for} \quad (p>n).\end{cases} \tag{2.1}$$

Theorem 2.1. *Suppose that*

$$\sum_{k=0}^{n}c_{n,k}=1, \tag{2.2}$$

$$\sum_{k=0}^{n}|c_{n,k}| \tag{2.3}$$

bounded,

$$\max_{0\leqq k\leqq n-1}|c_{n,k}|\to0 \quad as \quad n\to\infty. \tag{2.4}$$

Suppose that (1.5) *holds. Let* $\alpha>0$, *and let* p *be an integer valued function of the integer* n *such that* $p/n\to\alpha$ *as* $n\to\infty$; *in the case* $\alpha=1$, *we suppose further that* $p\neq n$ *for sufficiently large* n. *Then*

(i) (1.6) *holds with*

$$D(\alpha)=\limsup_{n\to\infty}A_{n,p}; \tag{2.5}$$

this limit depends only on α, and not on the particular function p chosen. This result is the best possible in the sense that equality can occur in (1.6).

(ii) *If* $\alpha>1$, *then*

$$D(\alpha)=D(1)+\log\alpha. \tag{2.6}$$

(iii) *If* $p=n$, *then a similar result holds, but with* $D(1)$ *replaced by* $\bar{D}(1)$, *where*

$$\bar{D}(1)=\limsup_{n\to\infty} A_{n,n}=\limsup_{n\to\infty}\{A_{n,n-1}+|1-c_{n,n}|-|c_{n,n}|-1\}. \tag{2.7}$$

For the proof of Theorem 2.1, we require the following lemma.

Lemma 2.1. *Suppose that the transformation* (1.1) *has the properties*

(i) $c_{n,k}\to 0$ *as* $n\to\infty$ *for any fixed* k,

(ii) $\sum_{k=0}^{\infty}|c_{n,k}|$ *is bounded.*

Let

$$A_n=\sum_{k=0}^{\infty}|c_{n,k}|$$

and let

$$A=\limsup_{n\to\infty} A_n.$$

Then for any bounded sequence $\{s_n\}$

$$\limsup_{n\to\infty}|t_n|\leqq A\limsup_{n\to\infty}|s_n|. \tag{2.8}$$

This result is the best possible; that is to say $\{s_n\}$ *can be chosen so that there is equality in* (2.8).

This result is essentially due to AGNEW ([*2*], Lemma 3.1). AGNEW gives the analogous result for sequence-to-function transforms but only obvious modifications of AGNEW's argument are required.

Proof of Theorem 2.1. We have

$$a_n=(1+1/n)\gamma_n-\gamma_{n-1} \qquad (n\geqq 1).$$

Hence

$$s_n=a_0+\sum_{\nu=1}^{n}\gamma_\nu/\nu+\gamma_n \qquad (n\geqq 1). \tag{2.9}$$

Thus, by (2.2)

$$\begin{cases} t_n=a_0+\sum_{\nu=1}^{n}c_{n,\nu}\gamma_\nu+\sum_{k=1}^{n}c_{n,k}\sum_{\nu=1}^{k}\frac{\gamma_\nu}{\nu} \\ \quad =a_0+\sum_{\nu=0}^{n}\gamma_\nu\left(c_{n,\nu}+\frac{1}{\nu}\sum_{k=\nu}^{n}c_{n,k}\right). \end{cases} \tag{2.10}$$

From (2.9) and (2.10) (using (2.2) to obtain a slight modification) we have an expression for $s_p - t_n$,

$$(2.11)\quad s_p - t_n = \begin{cases} \sum_{\nu=1}^{p-1} \gamma_\nu \left(\frac{1}{\nu} \sum_{k=0}^{\nu-1} c_{n,k} - c_{n,\nu} \right) + \gamma_p \left\{ 1 + \frac{1}{p} \sum_{k=0}^{p-1} c_{n,k} - c_{n,p} \right\} - \\ \qquad - \sum_{\nu=p+1}^{n} \gamma_\nu \left(\frac{1}{\nu} \sum_{k=\nu}^{n} c_{n,k} + c_{n,\nu} \right) \quad \text{for} \quad (p \leqq n), \\ \sum_{\nu=1}^{n} \gamma_\nu \left(\frac{1}{\nu} \sum_{k=0}^{\nu-1} c_{n,k} - c_{n,\nu} \right) + \sum_{\nu=n+1}^{p-1} \frac{\gamma_\nu}{\nu} + \gamma_p \left(1 + \frac{1}{p} \right) \quad \text{for} \ (p > n). \end{cases}$$

Part (i) of the theorem will clearly follow at once with the aid of Lemma 2.1 once we have proved that the limit (2.5) depends only on α. If p, q are two integer valued functions of n with $p/n \to \alpha$, $q/n \to \alpha$, then $p - q = o(n)$. It is thus enough to show that if $p - q = o(n)$, $q \geqq c\,n$ (where c is a positive constant), $p \neq n$, $q \neq n$, then

$$(2.12)\qquad A_{n,p} - A_{n,q} \to 0$$

as $n \to \infty$.

There is clearly no loss of generality in taking $p > q$. If $p > q > n$, then (2.12) follows at once from the second equality of (2.1), next suppose that $n > p > q$. We deduce from (2.4) that

$$\sum_{\nu=q}^{p} \frac{1}{\nu} \left| \sum_{k=0}^{\nu-1} c_{n,k} \right| \to 0$$

as $n \to \infty$. It therefore follows from the first equality of (2.1) that

$$A_{n,p} - A_{n,q} = |1 - c_{n,p}| + |c_{n,q}| - |1 - c_{n,q}| - |c_{n,p}| + o(1),$$

and (2.12) now follows with the aid of (2.4). In order to dispose of the case in which $p > n > q$, it is enough, in view of the cases already considered, to take $p = n+1$, $q = n-1$. By an argument similar to that given above, we deduce from (2.1), with the aid of (2.3) and (2.4), that

$$A_{n,n+1} - A_{n,n-1} \to 0$$

as $n \to \infty$. The proof of (i) is now completed.

Part (ii) of the theorem now follows at once from the second equality of (2.1).

Part (iii), apart from the second half of (2.7), follows at once from (2.11) and Lemma 2.1. Further, again arguing as above, we find that

$$(2.13)\qquad A_{n,n} - A_{n,n-1} = |1 - c_{n,n}| - |c_{n,n}| - 1 + o(1),$$

and this completes the proof of the theorem.

Theorem 2.2. *Suppose that the conditions of Theorem 2.1 hold.*

(i) *If, further*

$$(2.14)\qquad 0 \leqq c_{n,0} \leqq c_{n,1} \leqq c_{n,2} \leqq \cdots \leqq c_{n,n},$$

then

(2.15) $$D(\alpha)=D(1)+\log(1/\alpha) \qquad (\alpha\leqq 1),$$

and

$$D(1)=\limsup_{n\to\infty} B_n,$$

where

(2.16) $$B_n=\sum_{\nu=1}^{n}\left\{c_{n,\nu}-\frac{1}{\nu}\sum_{k=0}^{\nu-1}c_{n,k}\right\}+1.$$

(ii) *If, further*

(2.17) $$c_{n,0}\geqq c_{n,1}\geqq c_{n,2}\geqq\cdots\geqq c_{n,n}\geqq 0,$$

then for $p\leqq n$, *the quantity* $A_{n,p}$ *defined by the first equality of* (2.1) *can be written*

(2.18) $$A_{n,p}=\begin{cases}2\sum\limits_{\nu=p+1}^{n}c_{n,\nu}+\sum\limits_{\nu=1}^{n}\frac{1}{\nu}\sum\limits_{k=0}^{\nu-1}c_{n,k}-\\ \qquad -2\sum\limits_{\nu=p+1}^{n}\frac{1}{\nu}\sum\limits_{k=0}^{\nu-1}c_{n,k}+\log\left(\frac{1}{\alpha}\right)+o(1).\end{cases}$$

Thus for $\alpha\leqq 1$, $D(\alpha)$ *is given by* (2.5), *where* $A_{n,p}$ *is defined by* (2.18).

Proof of Theorem 2.2. Under the hypothesis (2.14), the term inside the modulus in the last sum in the first equality of (2.1) can be written

$$c_{n,\nu}+\frac{1}{\nu}-\frac{1}{\nu}\sum_{k=0}^{\nu-1}c_{n,k}.$$

So

(2.19) $$\begin{cases}A_{n,p}=\sum\limits_{\nu=1}^{n}\left\{c_{n,\nu}-\frac{1}{\nu}\sum\limits_{k=0}^{\nu-1}c_{n,k}\right\}+\sum\limits_{\nu=p+1}^{n}\frac{1}{\nu}+1+o(1);\\ \qquad =B_n+\log(1/\alpha)+o(1),\end{cases}$$

where B_n is defined by (2.16). On letting $n\to\infty$ (2.15) follows from (2.5) and (2.19). Under the hypothesis (2.17), the term inside the modulus of the first sum in the first equality of (2.1) is positive. So,

$$A_{n,p}=\sum_{\nu=1}^{p}\left(\frac{1}{\nu}\sum_{k=0}^{\nu-1}c_{n,k}-c_{n,\nu}\right)+1+o(1)+\sum_{\nu=p+1}^{n}\left(c_{n,\nu}+\frac{1}{\nu}-\frac{1}{\nu}\sum_{k=0}^{\nu-1}c_{n,k}\right).$$

(2.20) $$\begin{cases}A_{n,p}=2\sum\limits_{\nu=p+1}^{n}c_{n,\nu}+\sum\limits_{\nu=1}^{n}\frac{1}{\nu}\sum\limits_{k=0}^{\nu-1}c_{n,k}-\\ \qquad -2\sum\limits_{\nu=p+1}^{n}\frac{1}{\nu}\sum\limits_{k=0}^{\nu-1}c_{n,k}+\sum\limits_{\nu=p+1}^{n}\frac{1}{\nu}+o(1).\end{cases}$$

Hence, on letting $n\to\infty$, (2.18) follows from (2.20).

3. Definition. Let $\{\mu_n\}$ $(n\geqq 0)$ be a fixed sequence of real or complex numbers. The Hausdorff transform $\{t_n\}$ of a sequence $\{s_n\}$ by means of the fixed sequence $\{\mu_n\}$ (or, in short, the $[H, \mu_n]$ transform) is given by

$$t_n=\sum_{\nu=0}^{n}\binom{n}{\nu}(\Delta^{n-\nu}\mu_\nu)\, s_\nu \quad \text{for} \quad (n\geqq 0), \tag{3.1}$$

where, for $p, q>0$,

$$\Delta^p \mu_q=\sum_{r=0}^{p}(-1)^r\binom{p}{r}\mu_{q+r}. \tag{3.2}$$

Necessary and sufficient condition for a $[H, \mu_n]$ transformation to be regular, that is, that it will transform each convergent sequence to a convergent transform having the same limit, were given by HAUSDORFF (see HARDY [7], Chapter XI).

These conditions are the following:

$$\beta(t) \text{ is of bounded variation in } 0\leqq t\leqq 1, \tag{3.3}$$

$$\beta(0)=\beta(0+)=0, \quad \beta(1)=1, \tag{3.4}$$

$$\mu_n=\int_0^1 t^n\, d\beta(t) \quad \text{for} \quad n=1,2,\dots. \tag{3.5}$$

Further properties of the Hausdorff methods have been investigated by AGNEW [*1*].

Theorem 3.1. *Suppose that* (1.1) *is the regular Hausdorff transformation given by* (3.1)$-$(3.5); *suppose also that either* (2.14) *or* (2.17) *holds; the conditions of Theorem 2.1 are then necessarily satisfied. In the case in which* (2.17) *is assumed, we make the further assumption that* $\beta(t)/t$ *is integrable in* (0, 1). *Then under the condition* (2.14)

$$D(\alpha)=2-\int_0^1 \frac{\beta(t)}{t}\,dt+\log\left(\frac{1}{\alpha}\right) \quad \textit{for} \quad (\alpha\leqq 1), \tag{3.6}$$

and

$$\bar{D}(1)=2\beta(-1)-\int_0^1 \frac{\beta(t)}{t}\,dt. \tag{3.7}$$

Under the condition (2.17),

$$D(\alpha)=2(1-\beta(\alpha))+\int_0^\alpha \frac{\beta(t)}{t}\,dt+\int_\alpha^1 \frac{1-\beta(t)}{t}\,dt \quad \textit{for} \quad (\alpha\leqq 1), \tag{3.8}$$

and

$$\bar{D}(1)=D(1). \tag{3.9}$$

For the proof of Theorem 3.1, we require the following lemmas:

Lemma 3.1. *For any regular Hausdorff method given by* (3.1)$-$(3.5), *if* (2.14) *holds, then* $\beta(t)/t$ *is integrable in* (0, 1).

Proof. It is clear that (2.14) includes the assumption that $c_{n,\nu}\geqq 0$ for all ν. Hence, by HARDY ([7], Theorem 207), $\beta(t)$ is non decreasing. JAKIMOVSKI [8] has shown that

$$0\leqq\sum_{\nu=1}^{n}\binom{n}{\nu}u^{\nu}(1-u)^{n-\nu}\leqq\sum_{\nu=0}^{n}\binom{n}{\nu}u^{\nu}(1-u)^{n-\nu}=1, \tag{3.10}$$

$$\lim_{\substack{n\to\infty\\ p/n\to\alpha}}\sum_{\nu=p+1}^{n}\binom{n}{\nu}u^{\nu}(1-u)^{n-\nu}=\begin{cases}0 & \text{for } 0\leqq u<\alpha\\ 1 & \text{for } \alpha<u\leqq 1.\end{cases} \tag{3.11}$$

Let δ be any number with $0<\delta<1$ at which $\beta(t)$ has not got a discontinuity. Let p be a function of n with $p/n\to\delta$. Then

$$\sum_{\nu=0}^{p}c_{n,\nu}=1-\sum_{\nu=p+1}^{n}c_{n,\nu}=1-\int_0^1\left\{\sum_{\nu=p+1}^{n}\binom{n}{\nu}u^{\nu}(1-u)^{n-\nu}\right\}d\beta(u)\to$$
$$\to 1-\int_\delta^1 d\beta(u)=1-[\beta(u)-\beta(\delta)]$$

as $n\to\infty$ by dominated convergence (using (3.10) and (3.11)). Since $\beta(1)=1$, this gives

$$\sum_{\nu=0}^{p}c_{n,\nu}\to\beta(\delta). \tag{3.12}$$

But, by (2.14)

$$\sum_{\nu=0}^{p}c_{n,\nu}\leqq(p+1)\,c_{n,p}\leqq\frac{p+1}{n-p}\sum_{\nu=p+1}^{n}c_{n,\nu},$$

i.e.

$$(n-p)\sum_{\nu=0}^{p}c_{n,\nu}\leqq(p+1)\sum_{\nu=p+1}^{n}c_{n,\nu}=(p+1)\left(1-\sum_{\nu=0}^{p}c_{n,\nu}\right)$$

so that

$$(n+1)\sum_{\nu=0}^{p}c_{n,\nu}\leqq p+1.$$

Thus

$$\sum_{\nu=0}^{p}c_{n,\nu}\leqq\frac{p+1}{n+1}\to\delta \quad\text{as}\quad n\to\infty. \tag{3.13}$$

Comparing (3.12) and (3.13) we have

$$\beta(\delta)\leqq\delta. \tag{3.14}$$

This has been proved for all δ at which $\beta(u)$ has a discontinuity; but; since $\beta(u)$ has at most an enumerable set of discontinuities, and since $\beta(\delta)$ is non-decreasing we see that (3.14) holds for all δ. This clearly implies the integrability of $\beta(u)/u$.

Lemma 3.2. *For any regular Hausdorff method given by* (3.1)–(3.5), *if* (2.17) *holds then* $\beta(t)$ *is continuous in* (0, 1).

Proof. We note that (2.17) includes the assertion that $c_{n,\nu}\geqq 0$ for all ν. Hence, by HARDY ([7], Theorem 207), $\beta(t)$ is non decreasing.

It is easy to see that (2.17) implies that $\beta(u)$ is continuous at 1; and for any regular Hausdorff method it is continuous at 0 (cf. (2.14)). So we need consider only $0<u<1$. If the result is false, then there is some u_0 $(0<u_0<1)$ at which $\beta(u)$ is continuous; suppose that the jump of $\beta(u)$ at u_0 is λ. (Thus $\lambda>0$). Choose α, λ with

$$0<\alpha<u_0<\gamma<1 \tag{3.15}$$

such that $\beta(u)$ is continuous at the points α, γ and such that

$$\frac{\alpha\lambda}{\gamma-\alpha}>1. \tag{3.16}$$

This is possible, since (3.16) is satisfied whenever α, λ satisfy (3.15) and are sufficiently near to u_0; and since $\beta(u)$ has only an enumerable set of discontinuities, and hence has some points of continuity in any assigned interval.

Let p, q be functions of n with $p/n\to\alpha$, $q/n\to\gamma$. Consider

$$\sum_{\nu=p+1}^{q} c_{n,\nu}=\int_0^1\left\{\sum_{\nu=p+1}^{n}\binom{n}{\nu}u^\nu(1-u)^{n-\nu}\right\}d\beta(u)-$$
$$-\int_0^1\left\{\sum_{\nu=q+1}^{n}\binom{n}{\nu}u^\nu(1-u)^{n-\nu}\right\}d\beta(u)\to\int_\alpha^1 d\beta(u)-\int_\gamma^1 d\beta(u).$$

But the expression on the right

$$=\int_\alpha^\gamma d\beta(u)$$
$$\geqq\lambda,$$

(since $\beta(u)$ is non-decreasing). So

$$\sum_{\nu=p+1}^{q} c_{n,\nu}\geqq\lambda+o(1), \qquad (\text{as } n\to\infty).$$

But

$$\sum_{\nu=0}^{p} c_{n,\nu}\geqq(p+1)\,c_{n,p}\geqq\frac{(p+1)}{(q-p)}\sum_{\nu=p+1}^{q} c_{n,\nu}\geqq\frac{(p+1)}{(q-p)}(\lambda+o(1))\to\frac{\alpha\lambda}{\gamma-\alpha} \quad \text{as} \quad n\to\infty.$$

But

$$\sum_{\nu=0}^{p} c_{n,\nu}\leqq\sum_{\nu=0}^{n} c_{n,\nu}=1.$$

Thus in view of (3.16) we have a contradiction. This gives the result.

We are now in position to prove Theorem 3.1. If (2.14) holds, it follows that

$$\max_{0\leqq k\leqq n-1}|c_{n,k}|=c_{n,n-1}.$$

Similarly, if (2.17) holds, we have

$$\max_{0\leqq k\leqq n-1}|c_{n,k}|=c_{n,0}.$$

Now $c_{n,0}\to 0$ by the conditions for regularity. Also it follows from the argument given in HARDY [7] that, for a regular Hausdorff method, $c_{n,n-1}\to 0$. Thus (2.4) follows. Using (3.5) we have

$$\text{(3.17)}\qquad \begin{cases} c_{n,\nu}=\binom{n}{\nu}\Delta^{n-\nu}\mu_\nu \\ \quad =\binom{n}{\nu}\int\limits_0^1 t^\nu(1-t)^{n-\nu}\,d\beta(t). \end{cases}$$

We suppose first that (2.14) holds. Hence, it follows from (3.17) that B_n defined by (2.16) is

$$\text{(3.18)}\qquad \begin{cases} B_n=2-\sum\limits_{\nu=1}^{n}\frac{1}{\nu}\sum\limits_{k=0}^{\nu-1}\binom{n}{k}\Delta^{n-k}\mu_k \\ \quad =2-\int\limits_0^1\sum\limits_{\nu=1}^{n}\frac{1}{\nu}\sum\limits_{k=0}^{\nu=1}\binom{n}{k}t^k(1-t)^{n-k}\,d\beta(t)+o(1). \end{cases}$$

Now

$$\text{(3.19)}\qquad \begin{cases} \frac{d}{dt}\left\{\sum\limits_{k=0}^{\nu-1}\binom{n}{k}t^k(1-t)^{n-k}\right\} \\ \quad =\sum\limits_{k=1}^{\nu-1}k\binom{n}{k}t^{k-1}(1-t)^{n-k}-\sum\limits_{k=0}^{\nu-1}(n-k)\binom{n}{k}t^k(1-t)^{n-1-k}. \end{cases}$$

Since

$$(n-k)\binom{n}{k}=(k+1)\binom{n}{k+1};$$

we see on replacing k by $k+1$ in the first sum that the expression (3.19) reduces to

$$-\nu\binom{n}{\nu}t^{\nu-1}(1-t)^{n-\nu}.$$

Also, for $\nu\leqq n$ the expression inside the curly brackets on the left of (3.19) vanishes when $t=1$. Hence

$$\sum_{\nu=1}^{n}\frac{1}{\nu}\sum_{k=0}^{\nu-1}\binom{n}{k}t^k(1-t)^{n-k}=\int\limits_t^1\frac{1}{u}\sum_{\nu=1}^{n}\binom{n}{\nu}u^\nu(1-u)^{n-\nu}\,du.$$

Thus, it follows from (3.18) that

$$\text{(3.20)}\qquad \begin{cases} \sum\limits_{\nu=1}^{n}\frac{1}{\nu}\sum\limits_{k=0}^{\nu-1}\binom{n}{k}(\Delta^{n-k}\mu_k)=\int\limits_0^1 d\beta(t)\int\limits_t^1\frac{1}{u}\sum\limits_{\nu=1}^{n}\binom{n}{\nu}u^\nu(1-u)^{n-\nu}\,du \\ \quad =\int\limits_0^1\frac{\beta(u)}{u}\sum\limits_{\nu=1}^{n}\binom{n}{\nu}u^\nu(1-u)^{n-\nu}\,du. \end{cases}$$

We have

$$\sum_{\nu=1}^{n}\binom{n}{\nu}u^\nu(1-u)^{n-\nu}=\sum_{\nu=1}^{n}\binom{n}{\nu}u^\nu(1-u)^{n-\nu}-(1-u)^n=1-(1-u)^n.$$

Thus the expression (3.20) equals

$$\int_0^1 \frac{\beta(u)}{u}\,du - \int_0^u (1-u)^n \frac{\beta(u)}{u}\,du. \tag{3.21}$$

For any fixed n with $0<u<1$, $(1-u)^n \to 0$ hence the second term in (3.21) tends to 0 by dominated convergence and by using Lemma 3.1.

Combining this with (2.19) and (3.18), we have

$$\limsup_{n\to\infty} A_{n,p} = 2 - \int_0^1 \frac{\beta(t)}{t}\,dt + \log\left(\frac{1}{\alpha}\right). \tag{3.22}$$

Thus (3.6) follows from (2.5) and (3.22).

We now consider that (2.17) holds with the further assumption that $\beta(t)/t$ is integrable in (0, 1). It follows from (3.17) that the first term of the R.H.S. of (2.20) is equal to

$$2\int_0^1 \sum_{\nu=p+1}^{n} \binom{n}{\nu} t^\nu (1-t)^{n-\nu}\, d\beta(t). \tag{3.23}$$

We have

$$0 \leqq \sum_{\nu=p+1}^{n} \binom{n}{\nu} t^\nu (1-t)^{n-\nu} \leqq \sum_{\nu=0}^{n} \binom{n}{\nu} t^\nu (1-t)^{n-\nu},$$

thus it follows from (3.11) and Lemma 3.2, that the expression (3.23) tends to

$$\begin{cases} 2\int_\alpha^1 d\beta(t) \quad \text{as} \quad n\to\infty \qquad \text{(by dominated convergence)}, \\ = 2(\beta(1)-\beta(\alpha)). \end{cases} \tag{3.24}$$

Using (3.4), the expression (3.24) is equal to

$$2(1-\beta(\alpha)). \tag{3.25}$$

It follows from (3.17) that the third term of (2.20) is equal to

$$-2 \sum_{\nu=p+1}^{n} \frac{1}{\nu} \sum_{k=0}^{\nu-1} \binom{n}{k} \Delta^{n-k} \mu_k. \tag{3.26}$$

But

$$\sum_{k=0}^{\nu-1} \binom{n}{k} \Delta^{n-k} \mu_k = 1 - \sum_{k=\nu}^{n} \binom{n}{k} \Delta^{n-k} \mu_k.$$

Thus, by a similar argument of obtaining (3.20), (3.26) is equal to

$$\begin{cases} -2 \sum_{\nu=p+1}^{n} \frac{1}{\nu} + 2 \sum_{\nu=p+1}^{n} \frac{1}{\nu} \sum_{k=\nu}^{n} \Delta^{n-k} \mu_k \\ = -2\log\left(\frac{1}{\alpha}\right) + 2\int_0^1 \frac{1-\beta(u)}{u} \sum_{\nu=p+1}^{n} \binom{n}{k} u^\nu (1-u)^{n-\nu}\, du + o(1). \end{cases} \tag{3.27}$$

Since

$$0 \leqq \sum_{\nu=p+1}^{n} \binom{n}{\nu} u^{\nu}(1-u)^{n-\nu} \leqq \sum_{\nu=0}^{n} \binom{n}{\nu} u^{\nu}(1-u)^{n-\nu} = 1,$$

it follows from (3.11) and Lemma 3.2 that (3.27) tends to

(3.28) $-2\log\left(\frac{1}{\alpha}\right) + \int_{\alpha}^{1} \frac{1-\beta(t)}{t}\,dt$ as $n \to \infty$ (by dominated convergence).

Using the previous argument of obtaining (3.21), we find that as $n \to \infty$, $p/n \to \alpha$ the second term of the R.H.S. of (2.20) in view of (3.17) tends to

(3.29) $$\int_{0}^{1} \frac{\beta(t)}{t}\,dt.$$

Collecting (3.25), (3.28) and (3.29), it follows from (2.20) and (2.5) that, as $n \to \infty$, $p/n \to \alpha$,

(3.30) $$D(\alpha) = 2(1-\beta(\alpha)) - \log\left(\frac{1}{\alpha}\right) + \int_{0}^{1} \frac{\beta(t)}{t}\,dt + 2\int_{\alpha}^{1} \frac{1-\beta(t)}{t}\,dt.$$

But

(3.31) $$2\int_{\alpha}^{1} \frac{1-\beta(t)}{t}\,dt = \int_{\alpha}^{1} \frac{1-\beta(t)}{t}\,dt + \int_{\alpha}^{1} \frac{dt}{t} - \int_{\alpha}^{1} \frac{\beta(t)}{t}\,dt.$$

Using (3.31) in (3.30), we find that (3.6) holds.

We now prove (3.7) and (3.9). For any regular Hausdorff method,

$$c_{n,n} = \mu_n \to \beta(1) - \beta(1-) = \mu \quad \text{(say)},$$

as $n \to \infty$. Thus, under the conditions of either part of Theorem 3.1

(3.32) $$\bar{D}(1) = D(1) + |1-\mu| - |\mu| - 1.$$

But since

$$\sum_{k=0}^{n} c_{n,k} = 1,$$

and since, under either of the assumptions (2.14), (2.17),

$$c_{n,k} \geqq 0 \quad \text{(for all } n, k),$$

we must have

$$0 \leqq c_{n,k} \leqq 1 \quad \text{(for all } n);$$

and hence

$$0 \leqq \mu \leqq 1.$$

Thus (3.32) gives

(3.33) $$\begin{cases} \bar{D}(1) = D(1) - 2\mu, \\ \quad\;\; = D(1) - 2\beta(1) + 2\beta(1-); \end{cases}$$

and since $\beta(1) = 1$, (3.6) and (3.33) give (3.7). However (2.4) and (2.17) show that $c_{n,n} \to 0$, thus under the hypothesis (2.17) we must have $\mu = 0$ which gives (3.9).

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