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On free metabelian products

By
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1. Introduction

CHEN [1] has provided us with a representation of the free metabelian group of rank two in a graded ring. The object of this note is to find a similar representation of the free metabelian product of two torsion-free abelian groups. An immediate consequence of this embedding is a simple new proof of the following theorem of REE [2] (cf. also BAUMSLAG [3]): *The free metabelian product of two torsion-free abelian groups is residually torsion-free nilpotent.*

2. A special semigroup product and an allied ring

Suppose that S and T are semigroups such that $S \cap T = \emptyset$. If $X \times Y$ denotes the set of all pairs (x, y) $(x \in X, y \in Y)$, for a given pair of sets X, Y, put

$$Z^* = S \cup T \cup S \times T \cup T \times S.$$

We required an enlarged version of Z^* viz.

$$Z = \{z\} \cup Z^*,$$

where z lies outside Z^*.

We define now isomorphisms ϑ, φ of S and T respectively into the semigroup Q of all mappings of Z into Z as follows: if $s, s' \in S$, $t, t' \in T$, then

$$(2.1)\quad \begin{cases} \text{(i)} & s'(s\vartheta) = s's, & s'(t\varphi) = (s', t) \\ \text{(ii)} & t'(s\vartheta) = (t', s), & t'(t\varphi) = t't \\ \text{(iii)} & (s', t')(s\vartheta) = (s's, t'), & (s', t')(t\varphi) = (s', t't) \\ \text{(iv)} & (t', s')(s\vartheta) = (t', s's), & (t', s')(t\varphi) = (t't, s') \\ \text{(v)} & z(s\vartheta) = s, & z(t\varphi) = t. \end{cases}$$

Now we put $P = sg(S\vartheta \cup T\varphi)$, the subsemigroup of Q generated by $S\vartheta \cup T\varphi$ and term P the semi-permutable product of S and T. As usual we identify the copies $S\vartheta$ and $T\varphi$ of S and T respectively with S and T in the obvious way. It follows immediately then from the definition of P that whenever $s_1, s_2 \in S$, $t_1, t_2 \in T$ we have

(a) $s_1 t_1 s_2 = s_1 s_2 t_1$

(b) $t_1 s_1 t_2 = t_1 t_2 s_1$

(c) $s_1 t_1 = s_2 t_2$ if and only if $s_1 = s_2$, $t_1 = t_2$.

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In addition by making use of (a) and (b) it follows that

(d) every element u of P can be uniquely written in one of the forms s, t, st, $t\,s$ ($s\in S$, $t\in T$).

Suppose now that S and T are graded semigroups in the sense that there exists a mapping δ from $S\cup T$ into the positive integers such that if $s_1, s_2\in S$ and $t_1, t_2\in T$, then

$$\delta(s_1\,s_2)=\delta(s_1)+\delta(s_2),\quad \delta(t_1\,t_2)=\delta(t_1)+\delta(t_2). \tag{2.2}$$

It is clear that if we put

$$\delta(s\,t)=\delta(s)+\delta(t),\quad \delta(t\,s)=\delta(t)+\delta(s),$$

then this turns P into a graded semigroup.

Next, suppose that K is a graded semigroup and that $K^+=K\cup\{e\}$ denotes K with an identity e adjoined. Denote by

$$\Pi=R\,[[K^+]]$$

the power series ring with coefficients in the ring R over K^+. Thus, $\Pi=\{f: K^+\to R \mid$ for each integer $n(>0)$ there are only a finite number of elements k in K with $\delta(k)=n$ and $f(k)\neq 0\}$; addition is coordinatewise and multiplication triangular:

$$(f+g)(k)=f(k)+g(k),$$

$$(f\,g)(k)=\sum_{k_1\,k_2=k} f(k_1)\,g(k_2).$$

As usual, we identify the elements f of Π with infinite "sums"

$$\sum_{k\in K} f(k)\,k.$$

We denote the element which is 0 everywhere excepting at e, where it takes the value r, simply by r, and follow the convention of omitting parts of an infinite sum of the form $0\cdot k$ whenever convenient.

If R has a unit element 1, then the elements of Π of the form

$$1+f\,(f\in\Pi),$$

where $f(e)=0$, form a subgroup $A=A(\Pi)$ of the multiplicative semigroup of Π, the so-called *adjoint group* of Π. If R is a field containing the additive group $\boldsymbol{Q}$ of rationals, then it follows easily that A is a D-group, i.e., a group in which extraction of n-th roots is always uniquely possible ($n\neq 0$).

3. Free metabelian products

Let S and T be graded semigroups and let P be the semipermutable product of S and T. Furthermore, let

$$\Pi=\boldsymbol{Q}\,[[P^+]].$$

Then

$$\boldsymbol{Q}[[S^+]] \quad \text{and} \quad \boldsymbol{Q}[[T^+]]$$

may be considered as subrings of Π in the natural way. Let A_1 and A_2 be subgroups of the adjoint groups of $\boldsymbol{Q}[[S^+]]$ and $\boldsymbol{Q}[[T^+]]$ respectively. If $[A_1, A_2]$ is the subgroup of $A(\Pi)$ generated by the commutators $[a_1, a_2] = a_1^{-1} a_2^{-1} a_1 a_2 (a_1 \in A_1, a_2 \in A_2)$ then

Lemma 1. *If S and T are commutative so is $[A_1, A_2]$.*

Proof. We observe that if $f, g, h \in \boldsymbol{Q}[[P^+]]$ and $h(e)=0$, then

$$h(f, g)=0, \tag{3.1}$$

where (f, g) denotes the ring commutator $fg-gf$. To see this, write h, f, g in their conventional forms:

$$h=\sum h(p)\, p, \quad g=\sum g(p)\, p, \quad f=\sum f(p)\, p.$$

Then,

$$fg=\sum_p \Big(\sum_{p_1 p_2 = p} f(p_1)\, g(p_2) \Big) p, \quad gf=\sum_p \Big(\sum_{p_1 p_2 = p} g(p_1)\, f(p_2) \Big) p.$$

Thus,

$$h(fg)=\sum_p \Big\{ \sum_{p_1 p_2 = p} h(p_1) \Big(\sum_{p_3 p_4 = p_2} f(p_3)\, g(p_4) \Big) \Big\} p, \tag{3.2}$$

and

$$h(gf)=\sum_p \Big\{ \sum_{p_1 p_2 = p} h(p_1) \Big(\sum_{p_4 p_3 = p_2} g(p_4)\, f(p_3) \Big) \Big\} p. \tag{3.3}$$

From (2.1) it follows that $p_1 p_3 p_4 = p_1 p_4 p_3$ if $p_1 \neq e$, so that, since $h(e)=0$, (3.2) and (3.3) are identical, i.e., $h(f, g)=0$.

Finally, if $1+f \in A_1$, $1+g \in A_2$, then

$$[1+f, 1+g]=1+(f, g)+\text{terms of the form } h(f, g),$$

where $h(e)=0$ – this may be verified by direct computation. Hence,

$$[1+f, 1+g]=1+(f, g). \tag{3.4}$$

Thus, in particular,

$$\begin{aligned}[[1+f_1, 1+g_1], [1+f_2, 1+g_2]] &= [1+(f_1, g_1), 1+(f_2, g_2)] \\ &= 1+((f_1, g_1), (f_2, g_2)) = 1+(f_1, g_1)(f_2, g_2)-(f_2, g_2)(f_1, g_1)=1.\end{aligned}$$

This completes the proof of the lemma.

Next, let U and V be torsion-free abelian groups and choose an index set I whose cardinality is at least that of U and V. We choose S and T to be free commutative semigroups freely generated by

$$X=\{x_i;\ i \in I\} \quad \text{and} \quad Y=\{y_i;\ i \in I\},$$

respectively. If we set $\delta(x_i)=\delta(y_i)=1$ for all $i \in I$ and extend δ to S and T by means of (2.2), then it is clear that S and T are graded semigroups.

Put

$$P = S \operatorname{sp} T \quad \text{and} \quad \Pi = \boldsymbol{Q}\,[[P^+]],$$

where $\boldsymbol{Q}$ is again the field of rationals.

As has been remarked, the adjoint groups of $\boldsymbol{Q}\,[[S^+]]$ and $\boldsymbol{Q}\,[[T^+]]$ are D-groups. Since they are obviously commutative, they are direct products of copies of the additive group of $\boldsymbol{Q}$ (cf., e.g., KAPLANSKY [4], p. 10). However, every torsion-free abelian group can be embedded in an abelian D-group of the same cardinality. Indeed, let $\{u_i;\ i \in J \subseteq I\}$ be a maximal independent subset of a torsion-free abelian group U whose elements are indexed by a subset J of I. Then the mapping

$$u_i \to 1 + x_i$$

determines an isomorphism of U into $A(\boldsymbol{Q}\,[[S^+]])$ (cf., KAPLANSKY [4], p. 10). Henceforth, we identify U with this copy of it in $A(\boldsymbol{Q}\,[[S^+]])$.

Similarly, we can identify the torsion-free abelian group V with a correspondingly chosen subgroup of $A(\boldsymbol{Q}\,[[T^+]])$.

Our aim is to show that U and V generate, qua subgroups of $A(\Pi)$, their free metabelian product M. It is sufficient, by Lemma 1, therefore, to show that the cross commutators

$$[u, v] (u \in U, v \in V, u \neq 1, v \neq 1)$$

are independent (cf., e.g., GRUENBERG [5]).

Thus, let

$$w = [u_1, v_1]^{n_1} \ldots [u_m, v_m]^{n_m} (u_i \in U, v_j \in V)$$

where the n_i are non-zero integers. It is clear that we may assume, without loss of generality, that

(3.5) $\quad u_i = u_j,\ i \neq j,$ implies $v_i \neq v_j$; $v_i = v_j,\ i \neq j,$ implies $u_i \neq u_j$.

Each $u_i = 1 + f_i \in A(\boldsymbol{Q}\,[[S^+]])$, $v_j = 1 + g_j \in A(\boldsymbol{Q}\,[[T^+]])$, so that

$$w = 1 + n_1(f_1, g_1) + \cdots + n_m(f_m, g_m),$$

where $1 + f_i$ is a product of terms $(1 + x_j)^\alpha$ with $\alpha \in \boldsymbol{Q}$, and $1 + g_i$ is a product of terms $(1 + y_j)^\beta$ with $\beta \in \boldsymbol{Q}$.

Let us assume that $w = 1$; then

$$n_1(f_1 g_1 - g_1 f_1) + \cdots + n_m(f_m g_m - g_m f_m) = 0. \tag{3.6}$$

The equation (3.5) may be rewritten in the form

$$(n_1 f_1 g_1 + \cdots + n_m f_m g_m) + (n_1 g_1 f_1 + \cdots + n_m g_m f_m) = 0;$$

here the first parenthesis consists of elements of P which begin with elements of S while the second consists of elements of P which begin with elements of T. It follows therefore that

$$n_1 f_1 g_1 + \cdots + n_m f_m g_m = 0. \tag{3.7}$$

We shall show that the special nature of the elements f_i, g_i that appear in (3.7) contradicts (3.7).

We remind the reader that $1+f_i$ and $1+g_i$ are respectively products of terms of the form $(1+x_j)^\alpha$, $(1+y_k)^\beta$ where α, β are non-zero rationals. Suppose that the exponents α, β that occur are integers; clearly there is no loss of generality in assuming that they are actually positive, since a relation of the form (3.7) arising from negative exponents gives rise to a relation of the form (3.7) where the corresponding exponents are all positive. But then by inspecting the terms in f_i and g_i involving respectively the most x's and y's it follows readily that (3.7) is untenable.

There is left then the case where some of the exponents α, β involved in the f_i, g_i are actually not integers. Choose an integer N so that αN, βN are integral. Now the mapping defined by

$$x_i \to (1+x_i)^N - 1, \qquad y_i \to (1+y_i)^N - 1$$

determines a homomorphism φ of P into P since the image elements satisfy all the relations between the originals. But then

$$0 = (n_1 f_1 g_1 + \cdots + n_m f_m g_m)\,\varphi = n_1 f_1 \varphi\, g_1 \varphi + \cdots + n_m f_m \varphi\, g_m \varphi$$

and the $1+f_i\varphi$, $1+g_i\varphi$ are products of terms of the form $(1+x_j)^{\alpha N}$, $(1+y_k)^{\beta N}$ and the exponents are all integers. So we are back to the situation already considered and again we have a contradiction. So $w \neq 1$ and the proof that M is the free metabelian product of U and V is complete.

The proof now of REE's theorem is the exact analogue of MAGNUS' proof that free groups are residually torsion-free nilpotent; one uses simply the analogue of the dimension subgroups *without* attempting to identify them with the terms of the lower central series.

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