Werk

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A convergence theorem concerning real-valued interval functions

By
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1. Introduction

In a previous paper the author [1] has shown the following theorem:

Theorem A. If each of h and m is a real-valued nondecreasing function on the number interval $[a, b]$, then the following two statements are equivalent:

1. h is absolutely continuous with respect to m, and

2. the integral (section 2)

$$\int_{[a,b]} \left| dh - \int_I (dh)^p (dm)^{1-p} \right| \to 0 \quad \text{as} \quad p \to 1$$

for $0<p<1$.

In this paper we show (theorem 2) that under the hypothesis of theorem A, independently of whether or not h is absolutely continuous with respect to m, there is a real-valued function g on $[a, b]$ such that

$$\int_{[a,b]} \left| dg - \int_I (dh)^p (dm)^{1-p} \right| \to 0 \quad \text{as} \quad p \to 1$$

for $0<p<1$.

2. Preliminary definitions and theorems

Throughout this paper all integrals discussed are HELLINGER [3] type limits of the appropriate sums. Thus, if H is a real-valued function of subintervals of the number interval $[a, b]$, then

$$\int_{[a,b]} H(I)$$

exists if and only if $\int_I H(J)$ exists for each subinterval I of $[a, b]$, in which case, letting

$$\int_{[c,c]} H(I) = 0 \qquad \text{for } a \leqq c \leqq b,$$

we see that for

$$a \leqq w \leqq x \leqq y \leqq b, \qquad \int_{[w,x]} H(I) + \int_{[x,y]} H(I) = \int_{[w,y]} H(I).$$

We state without proof a theorem of KOLMOGOROFF [4].

Theorem K. If H is a real-valued function of subintervals of $[a, b]$ such that

$$\int_{[a,b]}^{*} H(I)$$

exists, then

$$\int_{[a,b]} \Big| |H(I)| - \Big| \int_I H(J) \Big| \Big| = \int_{[a,b]} \Big| H(I) - \int_I H(J) \Big| = 0,$$

so that if $[x, y]$ is a subinterval of $[a, b]$, then

$$\int_{[x,y]} |H(I)|$$

exists if and only if

$$\int_{[x,y]} \Big| \int_I H(J) \Big|$$

exists, in which case equality holds.

We also state without proof a theorem of the author [2] which is a consequence of theorem K.

Theorem B. If each of U and V is a real-nonnegative-valued function of subintervals of $[a, b]$, such that each of

$$\int_{[a,b]} U(I) \quad \text{and} \quad \int_{[a,b]} V(I)$$

exists, then for all p in (0, 1)

$$\int_{[a,b]} [U(I)]^p [V(I)]^{1-p} = \int_{[a,b]} \Big[\int_I U(J) \Big]^p \Big[\int_I V(J) \Big]^{1-p}.$$

We see that from theorems K and B it follows that if each of U and V is a real-valued function of subintervals of $[a, b]$ such that each of the integrals

$$\int_{[a,b]} U(I), \quad \int_{[a,b]} V(I), \quad \int_{[a,b]} \Big| \int_I U(J) \Big| \quad \text{and} \quad \int_{[a,b]} \Big| \int_I V(J) \Big|$$

exists, then

$$\int_{[a,b]} |U(I)|^p |V(I)|^{1-p} = \int_{[a,b]} \Big(\int_I |U(J)| \Big)^p \Big(\int_I |V(J)| \Big)^{1-p}$$

for all p in (0, 1).

3. The integral $\int_{[a,b]} \min\{ds, dt\}$

If each of w, x, y, and z is a number, then $\min\{w, x\} + \min\{y, z\} \leqq \min\{w+y, x+z\}$. This implies that if each of s and t is a real-valued nondecreasing function on the number interval $[a, b]$, and E is a refinement of the subdivision D of $[a, b]$, then

$$0 \leqq \sum_E \min\{\Delta s, \Delta t\} \leqq \sum_D \min\{\Delta s, \Delta t\},$$

so that

$$\int_{[a,b]} \min\{ds, dt\}$$

exists.

We state a lemma whose proof follows by conventional methods.

Lemma C. *If each of s and t is a real-valued nondecreasing function on $[a, b]$, and for $a \leqq x \leqq b$,*

$$u(x) = \text{l.u.b.} \int_{[a,x]} \min\{ds, K\,dt\}$$

for $0 < K$, then

$$\int_{[a,b]} \left| du - \int_I \min\{ds, K\,dt\} \right| \to 0 \quad \text{as} \quad K \to \infty,$$

and u is absolutely continuous with respect to t.

Theorem 1. *If each of s and t is a real-valued nondecreasing function on $[a, b]$ and p is in $(0, 1)$, then*

$$\int_{[a,b]} (ds - \min\{ds, M\,dt\})^p (dt)^{1-p} \to 0 \quad \text{as} \quad M \to \infty.$$

Proof. Suppose c is a positive number. There is a positive number K' such that if $K' \leqq K$, then $[s(b) - s(a)]/[K^{1-p}] < c$.

If D is a subdivision of $[a, b]$ and $K' \leqq K$, then

$$\sum_D (\Delta s - \min\{\Delta s, K\,\Delta t\})^p (\Delta t)^{1-p} = \sum_{D^*} (\Delta s - K\,\Delta t)^p (\Delta t)^{1-p},$$

where D^* is the set (if any) of all I in D such that $K\,\Delta t \leqq \Delta s$. We see that

$$\sum_{D^*} \Delta t \leqq \left[\sum_{D^*} \Delta s\right]/K \leqq [s(b) - s(a)]/K.$$

Therefore

$$\sum_{D^*} (\Delta s - K\,\Delta t)^p (\Delta t)^{1-p} \leqq \left(\sum_{D^*} \Delta s\right)^p \left(\sum_{D^*} \Delta t\right)^{1-p}$$

$$\leqq [s(b) - s(a)]^p \left([s(b) - s(a)]/K\right)^{1-p} < c.$$

Therefore

$$\int_{[a,b]} (ds - \min\{ds, K\,dt\})^p (dt)^{1-p} \leqq c.$$

Therefore

$$\int_{[a,b]} (ds - \min\{ds, M\,dt\})^p (dt)^{1-p} \to 0 \quad \text{as} \quad M \to \infty.$$

4. The convergence theorem

We now prove the second theorem mentioned in the introduction.

Theorem 2. *If each of h and m is a real-valued nondecreasing function on the number interval $[a, b]$, then there is a real-valued function g on $[a, b]$ such that*

$$\int_{[a,b]} \left| dg - \int_I (dh)^p (dm)^{1-p} \right| \to 0 \quad \text{as} \quad p \to 1$$

for p in $(0, 1)$.

Proof. For $a \leqq x \leqq b$ let

$$g(x) = \text{l.u.b.} \int_{[a,x]} \min\{dh, M\,dm\}$$

for $0 < M$.

By lemma C,

$$\int_{[a,b]} \left| dg - \int_I \min\{dh, M\,dm\} \right| \to 0 \quad \text{as} \quad M \to \infty$$

and g is absolutely continuous with respect to m.

By well known inequalities and theorems K and B, for p in (0, 1) and $0 < M$,

$$\begin{aligned}
&\int_{[a,b]} \left| dg - \int_I (dh)^p (dm)^{1-p} \right| \\
&\quad \leqq \int_{[a,b]} \left| dg - \int_I (dg)^p (dm)^{1-p} \right| + \\
&\qquad + \int_{[a,b]} \left| \int_I (dg)^p (dm)^{1-p} - \int_I (\min\{dh, M\,dm\})^p (dm)^{1-p} \right| + \\
&\qquad + \int_{[a,b]} \left| \int_I (\min\{dh, M\,dm\})^p (dm)^{1-p} - \int_I (dh)^p (dm)^{1-p} \right| \\
&\quad = \int_{[a,b]} \left| dg - \int_I (dg)^p (dm)^{1-p} \right| + \\
&\qquad + \int_{[a,b]} |(dg)^p - (\min\{dh, M\,dm\})^p| \, (dm)^{1-p} + \\
&\qquad + \int_{[a,b]} |(dh)^p - (\min\{dh, M\,dm\})^p| \, (dm)^{1-p} \\
&\quad \leqq \int_{[a,b]} \left| dg - \int_I (dg)^p (dm)^{1-p} \right| + \\
&\qquad + \int_{[a,b]} |dg - \min\{dh, M\,dm\}|^p (dm)^{1-p} + \\
&\qquad + \int_{[a,b]} (dh - \min\{dh, M\,dm\})^p (dm)^{1-p} \\
&\quad \leqq \int_{[a,b]} \left| dg - \int_I (dg)^p (dm)^{1-p} \right| + \\
&\qquad + \left(\int_{[a,b]} |dg - \min\{dh, M\,dm\}| \right)^p [m(b) - m(a)]^{1-p} + \\
&\qquad + \int_{[a,b]} (dh - \min\{dh, M\,dm\})^p (dm)^{1-p}.
\end{aligned}$$

Suppose c is a positive number. By theorem A there is a number q in (0, 1) such that if $q \leqq p < 1$, then

$$\int_{[a,b]} \left| dg - \int_I (dg)^p (dm)^{1-p} \right| < c/3.$$

Suppose $q \leqq p < 1$.

There is a positive number M' such that if $M' \leqq M$, then

$$\begin{aligned}
\int_{[a,b]} |dg - \min\{dh, M\,dm\}| &= \int_{[a,b]} \left| dg - \int_I \min\{dh, M\,dm\} \right| < \\
&< ([c/3]/[1 + \{m(b) - m(a)\}^{1-p}])^{1/p}.
\end{aligned}$$

By theorem 1 there is a positive number M'' such that if $M'' \leqq M$, then

$$\int_{[a,b]} (dh - \min\{dh, M\,dm\})^p (dm)^{1-p} < c/3 .$$

Letting $N = \max\{M', M''\}$, we see that

$$\int_{[a,b]} \left| dg - \int_I (dh)^p (dm)^{1-p} \right| \leqq \int_{[a,b]} \left| dg - \int_I (dg)^p (dm)^{1-p} \right| +$$

$$+ \Big(\int_{[a,b]} |dg - \min\{dh, N\,dm\}| \Big)^p [m(b) - m(a)]^{1-p} +$$

$$+ \int_{[a,b]} (dh - \min\{dh, N\,dm\})^p (dm)^{1-p} < c/3 +$$

$$+ \big([c/3] / [1 + \{m(b) - m(a)\}^{1-p}] \big) [m(b) - m(a)]^{1-p} + c/3 \leqq c .$$

Therefore

$$\int_{[a,b]} \left| dg - \int_I (dh)^p (dm)^{1-p} \right| \to 0$$

as $p \to 1$ for p in (0, 1).

References

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