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# A note on the global dimension of some algebras

By
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**1.** Let $\mathfrak{g}$ be a Lie algebra over a commutative ring $K$ with unit element. We assume that $\mathfrak{g}$ is free of rank $n$ as a $K$-module. Let $f$ be a 2-cocycle on the standard complex of $\mathfrak{g}$ with values in $K$, i.e. $f$ is a bilinear map of $\mathfrak{g}\times\mathfrak{g}$ into $K$ such that $f(x, x)=0$ and $f\big(x, [y, z]\big)+f\big(y, [z, x]\big)+f\big(z, [x, y]\big)=0$ for every $x, y, z\in\mathfrak{g}$. Then the associative algebra $T(\mathfrak{g})/U_f(\mathfrak{g})$, where $T(\mathfrak{g})$ is the tensor algebra of $\mathfrak{g}$ and $U_f(\mathfrak{g})$ is the two-sided ideal generated by elements of the type $x\otimes y-y\otimes x-[x, y]-f(x, y)$, $x, y\in\mathfrak{g}$, is denoted by $\mathfrak{g}_f$.

M. PAVAMAN MURTHY and R. SRIDHARAN [4] have recently proved that $\mathfrak{g}_f$ has left global dimension $=n+$ global dimension of $K$ if $\mathfrak{g}$ is a nilpotent Lie algebra over a ring of prime characteristic. We show in this note that this is valid for *any* Lie algebra ($K$ being of prime characteristic). When $\mathfrak{g}$ is an abelian Lie algebra over a field of characteristic $>0$, this result follows also from the results of RINEHART [5], who moreover gives an example to show that the theorem is not true if the characteristic is zero.

**2.** We assume in what follows that $K$ is a commutative ring with unit, of characteristic $p$, a prime, and $\mathfrak{g}$ a Lie algebra which is a free $K$-module of rank $n$. If $f$ is a 2-cocycle on $\mathfrak{g}$ with values in $K$, we shall denote the associated algebra by $\mathfrak{g}_f$ (see §1). We will require the following facts about $\mathfrak{g}_f$: The canonical map $\mathfrak{g}\to\mathfrak{g}_f$ is a monomorphism [6, cor. 2.8]. $\mathfrak{g}_f$ possesses a natural filtration induced from that of $T(\mathfrak{g})$, the tensor algebra of $\mathfrak{g}$, and the associated graded algebra $E^0(\mathfrak{g}_f)$ is isomorphic to the polynomial algebra $K[X_1, \ldots, X_n]$ in $n$ variables over $K$ [6, Th. 2.5]. In fact, if $g_1, \ldots, g_n$ form a base for $\mathfrak{g}$, *the linear map* $\varphi$ of $K[X_1, \ldots, X_n]$ into $\mathfrak{g}_f$ got by mapping the monomials $X_1^{i_1}\ldots X_n^{i_n}$ onto the corresponding elements $g_1^{i_1}\ldots g_n^{i_n}$ induces an isomorphism

$$E^0(\varphi): E^0\big(K[X_1, \ldots, X_n]\big) = K[X_1, \ldots, X_n]\to E^0(\mathfrak{g}_f)$$

of graded *algebras*.

The main step in our computation of the global dimension of $\mathfrak{g}_f$ is the following

**Proposition 1.** *$\mathfrak{g}_f$ contains an algebra $\Gamma$ which is isomorphic to the algebra of polynomials in $n$ variables over $K$ such that $\mathfrak{g}_f$ is free as a $\Gamma$-module.*

Proposition 1 follows from Lemmas 1 and 2 proved below.

**Lemma 1.** *For every $g\in\mathfrak{g}$, there exists a monic polynomial $F\in K[X]$ such that $F(g)$ is in the centre of $\mathfrak{g}_f$ ($F(g)$ having the obvious meaning as an element of $\mathfrak{g}_f$).*

*Proof.* For $g \in \mathfrak{g}$, we shall denote by $L_g$ the endomorphism $x \to gx - xg$ of $\mathfrak{g}_f$. The submodule $\mathfrak{g}+K$ of $\mathfrak{g}_f$ is stable under $L_g$ and we denote by $A_g$ the restriction of $L_g$ to $\mathfrak{g}+K$. Then there exists a monic polynomial $f \in K[X]$ such that $f(A_g)=0$. (For instance, we could take for $f$ the characteristic polynomial of $A_g$ [3, p. 101].) Now we shall show that there exists a *monic* polynomial $F \in K[X]$ *of the form* $\sum a_i X^{p^i}$, $a_i \in K$ such that $F(A_g)=0$. The characteristic of $K$ being $p$, $K$ contains the prime field of order $p$. Let $K'$ be the subring of $K$ generated by the coefficients of $f$ over the prime field. $K'$ is obviously Noetherian. Let $A = K'[X]/(f)$, and $\eta : K'[X] \to A$ the canonical map, where $(f)$ is the principal ideal generated by $f$. Then $A$ is generated as a $K'$-module by $1, \eta(X), \ldots, \eta(X^{m-1})$ where $m$ is the degree of $f$, since $f$ is monic. The $K'$-submodule of $A$ generated by $\eta(X^{p^i})$, $i=0, 1, \ldots$ is therefore finitely generated, say, by

$$e_l = \sum_{0 \leqq i \leqq r_l} \lambda_{i,l}\, \eta(X^{p^i}), \qquad l = 0, 1, \ldots, k.$$

If $s = \sup\limits_{0 \leqq l \leqq k} r_l + 1$, we have $\eta(X^{p^s}) = \sum\limits_{0 \leqq l \leqq k} \mu_l\, e_l = \sum\limits_{i=0}^{s-1} k_i\, \eta(X^{p^i})$ for some $k_i \in K'$. This means that the polynomial $F = X^{p^s} - \sum k_i X^{p^i} \in (f)$ or again that $F(A_g)=0$.

On the other hand, one has the formula

$$L_{g^n}(h) = \sum_{j=1}^{n} \binom{n}{j} (L_g)^j(h)\, g^{n-j} \qquad \text{(easily proved by induction)}$$

for $g \in \mathfrak{g}$, $h \in \mathfrak{g}_f$. In particular, we have $L_{g^{p^i}}(h) = (L_g)^{p^i}(h)$ since $\binom{p^i}{m} \equiv 0 \pmod{p}$ if $0 < m < p^i$. Therefore $L_{F(g)} = F(L_g)$. But $L_{F(g)}$ is a derivation of the algebra $\mathfrak{g}_f$ and $F(L_g)$ resticted to $\mathfrak{g}+K$ is $F(A_g)=0$. Since $\mathfrak{g}+K$ generates $\mathfrak{g}_f$, we must have $L_{F(g)}=0$, or $F(g)$ is in the centre of $\mathfrak{g}_f$. This proves Lemma 1.

Now let $g_1, \ldots, g_n$ be a base of $\mathfrak{g}$ and $F_1, \ldots, F_n$ monic polynomials of degrees $m_1, \ldots, m_n$ respectively such that $F_i(g_i)$ is in the centre of $\mathfrak{g}_f$ for every $i$. We denote by $\Gamma$ the subalgebra of $\mathfrak{g}_f$ generated by $F_i(g_i) = h_i$.

**Lemma 2.** *$\Gamma$ is isomorphic to the algebra of polynomials in $n$ variables over $K$ and $\mathfrak{g}_f$ is free as a $\Gamma$-module.*

*Proof.* Consider the filtration on $\Gamma$ induced by the canonical filtration on $\mathfrak{g}_f$. Let us denote by $\zeta_q$ the map $\mathrm{F}_q(\mathfrak{g}_f) \to E_q^0(\mathfrak{g}_f)$ (as well as the map $\mathrm{F}_q(\Gamma) \to E_q^0(\Gamma)$). Then we claim that $E^0(\Gamma)$ is the subalgebra of $E^0(\mathfrak{g}_f)$ generated by $\zeta_{m_i}(h_i)$, $i=1, \ldots, n$. It is clear that $\zeta_{m_i}(h_i) \in E^0(\Gamma)$. On the other hand, for any $x \in E_q^0(\Gamma)$ there exists $y \in \mathrm{F}_q\Gamma$ expressible in the form $\sum \lambda_{j_1, \ldots, j_n} h_1^{j_1} \ldots h_n^{j_n}$ such that $\zeta_q(y) = x$. Now if $r = \sum m_i j_i$, we have

$$\begin{aligned} \zeta_r(h_1^{j_1} \ldots h_n^{j_n}) &= (\zeta_{m_1} h_1)^{j_1} \ldots (\zeta_{m_n} h_n)^{j_n} \\ &= \left(E^0(\varphi)\, X_1^{m_1}\right)^{j_1} \ldots \left(E^0(\varphi)\, X_n^{m_n}\right)^{j_n} \\ &= E^0(\varphi)\left(X_1^{m_1 j_1} \ldots X_n^{m_n j_n}\right). \end{aligned}$$

From this it is clear that all the monomials $h_1^{j_1} \ldots h_n^{j_n}$ occurring in the expression of $y$ are in $\mathrm{F}_q\,\mathfrak{g}_f$. Moreover $x = \zeta_q(y) = \sum \lambda_{j_1, \ldots, j_n} \zeta_q(h_1^{j_1} \ldots h_n^{j_n})$ where the

summation runs through $(j_1, \ldots, j_n)$ such that $\sum m_i j_i = q$. But

$$\sum \lambda_{j_1, \ldots, j_n} (\zeta_{m_1}(h_1))^{j_1} \ldots (\zeta_{m_n}(h_n))^{j_n} = \sum \lambda_{j_1 \ldots, j_n} \zeta_q (h_1^{j_1} \ldots h_n^{j_n}) .$$

Hence $E^0(\Gamma)$ is generated by $\zeta_{m_i}(h_i) = E^0(\varphi)(X_i^{m_i})$. Thus $E^0(\Gamma)$ is isomorphic to $K[X_1^{m_1}, \ldots, X_n^{m_n}]$ and from this it is easy to see that $\Gamma$ is itself isomorphic to the algebra of polynomials in $n$ variables. Finally, $K[X_1, \ldots, X_n]$ is a free module over $K[X_1^{m_1}, \ldots, X_n^{m_n}]$ with the set of monomials of the type $X_1^{t_1} \ldots X_n^{t_n}$ with $0 \leqq t_i < m_i$ as base. Since $\zeta_{\sum m_i t_i}(h_1^{t_1} \ldots h_n^{t_n}) = E^0(\varphi)(X_1^{t_1} \ldots X_n^{t_n})$ it follows [*6*, Prop. 2, Appendix] that the set of elements $h_1^{t_1} \ldots h_n^{t_n}$ with $0 \leqq t_i < m_i$ form a base for $\mathfrak{g}_f$ as a $\Gamma$-module. This completes the proof of Lemma 2.

**Lemma 3.** *If $\Lambda$ is an algebra over a commutative ring $\Gamma$ and $\Lambda$ is free as a $\Gamma$-module, then left global dimension of $\Lambda \geqq$ global dimension of $\Gamma$.*

A proof of this can be found in [*2*, Prop. 2, p. 74] or [*4*, Lemma 1].

**Theorem.** *Let $K$ be a commutative ring with unit of prime characteristic. Let $\mathfrak{g}$ be a Lie algebra over $K$ which is free of rank $n$ as a $K$-module and $f$ a 2-cocycle on the standard complex of $\mathfrak{g}$ with values in $K$. Then*

*Left global dimension of $\mathfrak{g}_f = n +$ global dimension of $K$.*

*Proof.* Let $\Gamma$ be the subalgebra of $\mathfrak{g}_f$ constructed in proposition 1. We then have gl.dim.$\Gamma = n +$ gl.dim.$K$ [*2*, Th. 6]. Hence it follows from Lemma 3 that l.gl.dim.$\mathfrak{g}_f \geqq n +$ gl.dim.$K$. On the other hand l.gl.dim.$\mathfrak{g}_f \leqq$ alg.dim.$\mathfrak{g}_f +$ gl.dim.$K$ [*2*, p. 75]. But one has alg.dim.$\mathfrak{g}_f = n$ [*6*, Th. 5.4]. Hence we have l.gl.dim.$\mathfrak{g}_f = n +$ gl.dim.$K$.

*Remarks.* 1. As in [*4*, prop. 1] one can prove that for *any* Lie algebra $\mathfrak{g}$ over $\boldsymbol{Z}$ (the ring of rational integers) which is $\boldsymbol{Z}$-free of rank $n$, left global dimension of $\mathfrak{g}_f = n + 1$.

2. Let $\mathfrak{g}$ be abelian of rank 2 over a field of characteristic zero and $f$ a nondegenerate skew-symmetric form on $\mathfrak{g}$. RINEHART [*5*] has proved that gl.dim.$\mathfrak{g}_f = 1$ which shows that our theorem is not valid if the characteristic of $K$ is zero.

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