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Math. Zeitschr. 81, 99—107 (1963)

On self-reciprocal functions involving infinite series

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1. Introduction. The discussion of self-reciprocity of infinite series of the type

$$f(x) - f(3x) + f(5x) - f(7x) + \cdots \tag{1.1}$$

was introduced by WATSON [*10*a]. Here $f(x)$ is any function integrable in $(0, \infty)$.

BHATNAGAR [*1*a] has defined the kernel $\omega_{n_1, n_2, \ldots, n_m}(x)$ as

$$\omega_{n_1, n_2, \ldots, n_m}(x) = \sqrt{x} \int_0^\infty \cdots \int_0^\infty J_{n_1}(t_1) \cdots J_{n_{m-1}}(t_{m-1}) \, J_{n_m}(x/t_1 \cdots t_{m-1}) \frac{dt_1 \ldots dt_{m-1}}{t_1 \cdots t_{m-1}} \tag{1.2}$$

where the n's can be permuted amongst themselves and $n_k + \frac{1}{2} > 0$, for $k = 1, 2, \ldots, m$. The kernel in (1.2) is a generalisation of the kernel of the Hankel transform and also of $\omega_{\mu,\nu}(x)$ defined by WATSON [*10*a], in the form

$$\omega_{\mu,\nu}(x) = \sqrt{x} \int_0^\infty J_\mu(t) \, J_\nu(x/t) \, dt/t. \tag{1.3}$$

The kernels of (1.2) and (1.3) play the role of transforms under suitable conditions. Let $f(x)$ and $g(x)$ satisfy

$$g(x) = \int_0^\infty \omega_{n_1, n_2, \ldots, n_m}(xy) \, f(y) \, dy \tag{1.4}$$

then the equation (1.4) also holds good when $f(x)$ and $g(x)$ are interchanged. If $g(x) = f(x)$, then $f(x)$ is said to be self-reciprocal in $\omega_{n_1, n_2, \ldots, n_m}$ transform and is denoted by $R_{n_1, n_2, \ldots, n_m}$. Functions satisfying the integral equation

$$g(x) = \int_0^\infty \sqrt{xy} \, J_\nu(xy) \, f(y) \, dy \tag{1.5}$$

are said to be reciprocal in the Hankel transform of order ν. If $g(x) = f(x)$ then $f(x)$ is said to be self-reciprocal in the Hankel transform of order ν and is denoted by R_ν. For $\nu = \pm 1/2$, (1.5) reduces to sine or cosine transform according as the sign is positive or negative. A function self-reciprocal in the sine or cosine transform is denoted by R_s or R_c respectively. Also we know that ([*1*], (*a*))

$$\omega_{\nu,\nu-1}(x) = J_{2\nu-1}(2\sqrt{x}). \tag{1.6}$$

The object of this paper is to establish a result (theorem 1) on self-reciprocity of functions, involving the series (1.1), in the generalised transform (1.2). Some particular cases of this arise with the transforms in (1.3) and (1.5). The following results believed to be new are obtained by the application of theorem 1.

(i) $\sum_{r=1}^{\infty}(-1)^{r-1}(\overline{2r-1}\,x\sqrt{\pi/2})^{-\nu}[I_\nu(\overline{2r-1}\,x\sqrt{\pi/2})-L_\nu(\overline{2r-1}\,x\sqrt{\pi/2})]$

is $R_{2\nu+1/2}$.

(ii) $\sum_{r=1}^{\infty}(-1)^{r-1}(\overline{2r-1}\,x\sqrt{\pi/2})^{\nu}[H_{-\nu}(\overline{2r-1}\,x\sqrt{\pi/2})-Y_{-\nu}(\overline{2r-1}\,x\sqrt{\pi/2})]$

is $R_{2\nu-1/2}$ $(0\leqq\nu<1/2)$.

(iii) $\sum_{r=1}^{\infty}(-1)^{r-1}(\overline{2r-1}\,x\sqrt{\pi/2})^{3/2-\mu}\times$

$\times E[\mu+1,\mu+\nu+1,\mu/2+5/4::(\overline{r-1/2}\,x\sqrt{\pi/2})^2]$

is $R_{2\nu+\mu}$.

Theorem 1 can be used for establishing the self-reciprocal property of the series (1.1) involving well known functions of Mathematical Physics. Some known results [*6*] can also be obtained by the application of this theorem.

2. HARDY and TITCHMARSH [*4*] have defined a class $A(\omega, a)$ of functions where $0<\omega\leqq\pi$, $a<1/2$. They are (i) analytic functions of $x=re^{i\vartheta}$ regular in the angle A defined by $r>0$, $|\vartheta|<\omega$,

(ii) $O(|x|^{-a-\delta})$ for small x

(iii) $O(|x|^{a-1+\delta})$ for large x

for every positive δ and uniformly in any angle $|\vartheta|\leqq\omega-\eta<\omega$, where η is a small positive quantity.

3. Let $M(s)$ be the Mellin transform of $f(x)$, and let

$$F(x)=\sum_{n=1}^{\infty}(-1)^{n-1}f(\overline{2n-1}\,x), \tag{3.1}$$

then

$$\int_0^\infty F(x)\,x^{s-1}dx=L(s)\int_0^\infty f(x)\,x^{s-1}dx=L(s)\,M(s),$$

where

$$L(s)=\sum_{n=1}^{\infty}(-1)^{n-1}(2n-1)^{-s}.$$

Hence the Mellin transform of $F(x)$ is $M(s)L(s)$. Let $f(x)$ belong to $A(\omega, a)$ and be $R_{\mu_1,\mu_2,\dots,\mu_n}$; then [*1*a] $M(s)$ is of the form

$$M(s)=2^{ns/2}\,\Gamma(s/2+\mu_1/2+1/4)\dots\Gamma(s/2+\mu_n/2+1/4)\,\psi(s) \tag{3.2}$$

where $\psi(s)$ is regular in $a<\sigma<1-a$, $(s=\sigma+it)$; i.e.

$$(3.3)\qquad \psi(s) \text{ is } O\left(\exp\,(n\pi/4-\omega+\eta)\,|t|\right),$$

and satisfies

$$(3.4)\qquad \psi(s)=\psi(1-s).$$

Consider

$$\varphi(x)=\int_0^\infty F(x/y\cdot\sqrt{\pi}/2)/(1+y^2)^{\mu_n/2+5/4}\,dy.$$

Substituting Mellin's inversion formula for $F(x)$, we have

$$(3.5)\quad \varphi(x)=(1/2\pi i)\int_0^\infty dy/(1+y^2)^{\mu_n/2+5/4}\int_{c-i\infty}^{c+i\infty} M(s)\,L(s)\,(x/y\cdot\sqrt{\pi}/2)^{-s}\,ds.$$

On inverting the order of integrations, we get the right hand side

$$=(1/2\pi i)\int_{c-i\infty}^{c+i\infty} M(s)\,L(s)\,(x\sqrt{\pi}/2)^{-s}\,ds\int_0^\infty y^s/(1+y^2)^{\mu_n/2+5/4}\,dy.$$

On evaluating the y-integral with the help of ([2], p. 349) and substituting for $M(s)$ from (3.2), we get

$$(3.6)\ \varphi(x)=(1/2\pi i)\int_{c-i\infty}^{c+\infty} 2^{(n-1)s/2}\,\Gamma\left(\frac{s+\mu_1+1/2}{2}\right)\cdots\Gamma\left(\frac{s+\mu_{n-1}+1/2}{2}\right)\chi(s)\,x^{-s}\,ds$$

where

$$(3.7)\ \chi(s)=\frac{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{\mu_n+s+1/2}{2}\right)\Gamma\left(\frac{\mu_n-s+3/2}{2}\right)L(s)\,\psi(s)\,(\sqrt{\pi}/2)^{-s}}{2\Gamma(\mu_n/2+5/4)}=\chi(1-s)$$

by the functional equation for $L(s)$, viz. ([8], p. 66)

$$(2/\sqrt{\pi})^s\,\Gamma\left(\frac{s+1}{2}\right)L(s)=(2/\sqrt{\pi})^{1-s}\,\Gamma(1-s/2)\,L(1-s).$$

Also as $|t|\to\infty$, we have

$$(3.8)\qquad \Gamma(A+it/2)\sim k\,e^{-(\pi/4)|t|}\,|t/2|^{A-1/2}$$

where k is some constant factor. On using (3.8) and (3.3) we have

$$(3.9)\qquad \chi(s) \text{ is } O(\exp(\overline{n-3}\,\pi/4-\omega+\eta)\,|t|) \quad\text{as}\quad |t|\to\infty.$$

Further putting $x=re^{i\vartheta}$, where $|\vartheta|<\omega$ and $r>0$, we see that the modulus of the integrand on the right of (3.6) does not exceed a constant multiple of

$$(3.10)\ r^{-c}\left|\Gamma\left(\frac{c+\mu_1+1/2}{2}+i\,t/2\right)\right|\cdots\left|\Gamma\left(\frac{c+\mu_{n-1}+1/2}{2}+i\,t/2\right)\right|\,|\chi(c+it)|\,e^{|\vartheta|\,|t|}.$$

It follows from the application of (3.8) and (3.9) that the expression in (3.10) is $O(\exp-(\pi/2+\omega-|\vartheta|-\eta)\,|t|)$ as $|t|\to\infty$. Hence the integral on the right of (3.6) is uniformly and absolutely convergent, in any domain of x for which $|\vartheta|<\omega$ and in particular for $\vartheta=0$. Consequently $\varphi(x)$ belongs to the class $A(\omega, a)$ ([8], Theorem 31, p. 47).

Now the necessary and sufficient condition that a function of $A(\omega, a)$ be $R_{\mu_1, \mu_2, \ldots, \mu_n}$ is that its Mellin transform should be of the form given in (3.2) and both the conditions (3.3) and (3.4) should be satisfied ([*1*], (*a*)). From (3.6) it is seen that the Mellin transform of $\varphi(x)$ is of the same form, as (3.2) with $(n-1)$ in place of n. Also the other conditions are satisfied, vide (3.7) and (3.9). Hence $\varphi(x)$ is $R_{\mu_1, \mu_2, \ldots, \mu_{n-1}}$. We now proceed to examine the permissibility of changing the order of integrations in (3.5). We see that the right hand side of (3.5) is not greater than a constant multiple of

$$(3.11) \qquad I = x^{-c} \int_0^\infty |1/(1+y^2)^{\mu_n/2+5/4}|\, y^c\, dy \int_{-\infty}^\infty |M(c+it)|\, |L(c+it)|\, dt.$$

Hence the inversion in the order of integrations in (3.5) is justifiable by De la Vallée Poussin's conditions ([*3*], p. 456), if the two integrals in (3.11) exist.

Now the y-integral exists if $-1<c<\mu_n+3/2$. Also by hypothesis $f(x)$ belongs to $A(\omega, a)$, therefore its Mellin transform $M(s)$ is $O(\exp -(\omega-\eta)\,|t|)$ as $|t|\to\infty$. Hence the t-integral also converges. Hence

Theorem 1. *Let*

(i) $f(x)$ *be continuous in* $x \geqq 0$,

(ii) $\sum_{r=1}^\infty (-1)^{r-1} f(\overline{2r-1}\,x)$ *converge uniformly in* $(0, \infty)$ *to* $F(x)$,

(iii) $f(x)$ *belong to* $A(\omega, a)$ *and be* $R_{\mu_1, \mu_2, \ldots, \mu_n}$; *then*

$$\varphi(x) = \int_0^\infty F(x/y \cdot \sqrt{\pi/2})/(1+y^2)^{\mu_n/2+5/4}\, dy$$

is $R_{\mu_1, \mu_2, \ldots, \mu_{n-1}}$.

Example 1. $f(x) = x^{-\nu} J_\nu(x)$ is $R_{2\nu+1/2, -1/2}$ [*1a*]. In the theorem, let $n=2$, $\mu_1 = 2\nu+1/2$, $\mu_2 = -1/2$; then we get

$$\varphi(x) = \int_0^\infty \frac{\sum_{r=1}^\infty (-1)^{r-1} (\overline{2r-1}\, x/y \cdot \sqrt{\pi/2})^{-\nu} J_\nu(\overline{2r-1}\, x/y \cdot \sqrt{\pi/2})}{(1+y^2)}\, dy \text{ is } R_{2\nu+1/2}.$$

Writing $1/y$ for y and inverting the order of integration and summation, which can be justified on the lines of ([*3*], Art. 176C, p. 455), we get

$$\varphi(x) = \sum_{r=1}^\infty (-1)^{r-1} (\overline{2r-1}\, x\sqrt{\pi/2})^{-\nu} \int_0^\infty \cdot\, y^{-\nu} J_\nu(\overline{2r-1}\, xy\sqrt{\pi/2})/(1+y^2)\, dy.$$

Evaluating the integral with the help of ([*10b*] p. 425), we get

$$\varphi(x) = (\pi/2) \sum_{r=1}^\infty (-1)^{r-1} (\overline{2r-1}\, x\sqrt{\pi/2})^{-\nu} \times$$
$$\times [I_\nu(\overline{2r-1}\, x\sqrt{\pi/2}) - L_\nu(\overline{2r-1}\, x\sqrt{\pi/2})]$$

is $R_{2\nu+1/2}$.

If we put $\nu=0$, we fall back upon a known result, [*7*].

Example 2. $f(x)=x^{\nu}K_{\nu}(x)$ is $R_{2\nu-1/2,\,-1/2}$ [*1b*]. Let $n=2$ and μ_2 of the theorem $=-1/2$, then we have

$$\varphi(x)=\int_0^{\infty}\sum_{r=1}^{\infty}(-1)^{r-1}\left(\overline{2r-1}\,x/y\cdot\sqrt{\pi/2}\right)^{\nu}K_{\nu}\left(\overline{2r-1}\,x/y\cdot\sqrt{\pi/2}\right)/(1+y^2)\,dy$$

is $R_{2\nu-1/2}$.

On inverting the order of integration and summation and integrating term-by-term, which is permissible, we get

$$\varphi(x)=(\pi^2/2\cos\nu\pi)\sum_{r=1}^{\infty}(-1)^{r-1}\left(\overline{2r-1}\,x\sqrt{\pi/2}\right)^{\nu}\times$$
$$\times\left[H_{-\nu}\left(\overline{2r-1}\,x\sqrt{\pi/2}\right)-Y_{-\nu}\left(\overline{2r-1}\,x\sqrt{\pi/2}\right)\right]$$

is $R_{2\nu-1/2}$ $(0\leqq\nu<1/2)$.

Example 3. Let $f(x)=2e^{-x}$, $n=2$ and $\mu_2=-1/2$, then we have

(a) $$\varphi(x)=2\int_0^{\infty}\left[\sum_{r=1}^{\infty}(-1)^{r-1}e^{-\overline{2r-1}\,x/y\cdot\sqrt{\pi/2}}/(1+y^2)\right]dy$$

is R_s.

On summing up the series, we get, after a slight change in variables

$$\varphi(x)=\int_0^{\infty}\left[x\,\mathrm{sech}\,(y\sqrt{\pi/2})/(x^2+y^2)\right]dy$$

as a function of R_s ([*8*], p. 268).

(b) Let $\mu_2=1/2$, we get, after a slight change in variables,

$$\varphi(x)=\int_0^{\infty}\left[\mathrm{sech}\,(x\,y\sqrt{\pi/2})\cdot y/(1+y^2)^{3/2}\right]dy$$

as a function of R_c.

Since $\mathrm{sech}\,(y\sqrt{\pi/2})$ is R_c and $y/(1+y^2)^{3/2}=K(y)=K(1/y)/y$, the above follows at once ([*8*], Rule 3, p. 270).

Example 4. $f(x)=x^{\mu+\nu+1/2}K_{\nu}(x)$ is $R_{\mu,\mu+2\nu}$, $\nu\geqq0$, $\mu\geqq-1/2$ ([*1*b], p. 180). Let $n=2$, $\mu_1=2\nu+\mu$, $\mu_2=\mu$; then we get

$$\varphi(x)=\int_0^{\infty}\sum_{r=1}^{\infty}(-1)^{r-1}\left(\overline{2r-1}\,x/y\cdot\sqrt{(\pi/2)}\right)^{\mu+\nu+1/2}\times$$
$$\times K_{\nu}\left(\overline{2r-1}\,x/y\cdot\sqrt{\pi/2}\right)/(1+y^2)^{\mu/2+5/4}\,dy$$

is $R_{2\nu+\mu}$.

Integrating term-by-term with the help of the result [*9*], viz.

$$\int_0^{\infty}x^{(\alpha+\beta-2)/2}K_{\beta-\alpha}(2\sqrt{x})/(p+x)^{m+1}\,dx=E(\alpha,\beta,m+1::p)/2\,\underline{|m}\,p^{m+1}$$

($\alpha \geqq \beta > m-1$, m should be a positive integer), we get

$$\varphi(x) = 2^{\nu+2\mu}/\Gamma(\mu/2+5/4)\cdot\sum_{r=1}^{\infty}(-1)^{r-1}\left(\overline{2r-1}\,x\sqrt{\pi/2}\right)^{3/2-\mu}\times$$
$$\times E\left[\mu+1, \mu+\nu+1, \mu/2+5/4 :: (\overline{r-1/2}\,x\sqrt{\pi/2})^2\right]$$

is $R_{2\nu+\mu}$ ($\mu/2+1/4$ should be a positive integer, $\nu \geqq 0$, $\mu > 5/2$). To justify the inversion in the order of integration and summation, I divide the range of integration into two parts from 0 to 1 and 1 to ∞ and use ([*3*], art. 176 B, p. 453).

(A) Let $0 \leqq y \leqq 1$, then we have:

(i) Since $K_\nu(x) \sim O\{(\pi/2x)^{1/2}e^{-x}\}$ for large x [*10 b*],

$$(r/y)^{\mu+\nu+5/2}K_\nu(r/y) \sim O\left(e^{-rz}(rz)^{\mu+\nu+2}\right)$$

whenever either r or z or both become large, where $z = 1/y$. Hence

$$v_r(y) = \{(2r-1)\,x/y\}^{\mu+\nu+5/2}K_\nu(\overline{2r-1}\,x/y)$$

is uniformly bounded for $0 \leqq y \leqq 1$ and r arbitrary. It is a positive monotonic decreasing function of r, and $v_r \to 0$ uniformly as $r \to \infty$ for all values of y in $0 \leqq y \leqq 1$; and

$$\sum a_r = \sum_{r=1}^{\infty}(y/x)^2/(2r-1)^2 \leqq \sum_{r=1}^{\infty} 1/x^2(2r-1)^2$$

(for fixed x) is uniformly convergent in (0, 1) by Weierstrass's M-test [*3*].

Hence the uniform convergence of the series

$$G(x,y) = \sum_{r=1}^{\infty}(\overline{2r-1}\,x/y)^{\mu+\nu+1/2}K_\nu(\overline{2r-1}\,x/y)$$

follows by Dirichlet's test ([*3*], Art. 44, (3), p. 114).

(ii) Also convergence of $\sum_{r=1}^{\infty}\left|(\overline{2r-1}\,x/y)^{\mu+\nu+1/2}K_\nu(\overline{2r-1}\,x/y)\right|$ follows by the ratio test.

(iii) $\int_0^1 \left|1/(1+y^2)^{\mu/2+5/4}\right| dy$ is convergent.

(iv) $$\sum_{n+1}^{n+p}\int_0^1 \left|(\overline{2r-1}\,x/y)^{\mu+\nu+1/2}K_\nu(\overline{2r-1}\,x/y)/(2+\pi^2y^2)^{\mu/2+5/4}\right| dy$$
$$\leqq A\sum_{n+1}^{n+p}\left|(\overline{2r-1}\,x/y)^{\mu+\nu+1/2}K_\nu(\overline{2r-1}\,x/y)\right|\int_0^1\left|1/(2+\pi^2y^2)^{\mu/2+5/4}\right| dy$$
$$< \varepsilon/A'\cdot\int_0^1\left|1/(2+\pi^2y^2)^{\mu/2+5/4}\right| dy < \varepsilon,$$

where A and A' are constants.

Hence the order of integration and summation for the range of integration 0 to 1 may be interchanged.

(B) Let $1 \leqq y \leqq \alpha$, where α may be arbitrary, $\mu > 5/2$ and $\nu \geqq 0$, then we have:

(i) $$H(y) = \sum_{r=1}^{\infty} \left| \overline{2r-1} / \{1 + (\overline{2r-1}\, y)^2\}^{\mu/2+5/4} \right| \leqq \sum_{r=1}^{\infty} 1/(2r-1)^{\mu+3/2}$$

for all values of y in the interval.

Hence by Weierstrass's M-test [3] the series $H(y)$ is uniformly convergent in $1 \leqq y \leqq \alpha$.

(ii) $$\left| (x/y)^{\mu+\nu+1/2} (2r-1)\, K_\nu(x/y) / \{2 + (\overline{2r-1}\, y \sqrt{\pi})^2\}^{\mu/2+5/4} \right|$$
$$\leqq \left| (x/y)^{\mu+\nu+1/2} K_\nu(x/y) \right| / (2r-1)^{\mu+3/2}$$

$$\therefore \sum_{r=1}^{\infty} \int_1^{\infty} \left| (x/y)^{\mu+\nu+1/2} (2r-1)\, K_\nu(x/y) / \{2 + (\overline{2r-1}\, y \sqrt{\pi})^2\}^{\mu/2+5/4} \right| dy$$
$$\leqq \sum_{r=1}^{\infty} 1/(2r-1)^{\mu+3/2} \cdot \int_1^{\infty} \left| (x/y)^{\mu+\nu+1/2} K_\nu(x/y) \right| dy.$$

Since both

$$\sum_{r=1}^{\infty} 1/(2r-1)^{\mu+3/2} \quad \text{and} \quad \int_1^{\infty} \left| (x/y)^{\mu+\nu+1/2} K_\nu(x/y) \right| dy$$

are convergent; the series on the left of the above inequality also converges.

Hence for the range of integration $(1, \infty)$, the order of integration and summation may be interchanged.

Combining the two results term-by-term integration is justifiable.

It now remains to be seen whether

$$I = \int_0^{\infty} \sqrt{xy}\, J_{2\nu+\mu}(xy)\, \varphi(y)\, dy$$

exists.

I divide the range of integration into two parts from 0 to 1 and 1 to ∞.

Let $\nu \geqq 0$, $\mu > 5/2$, then for the range 0 to 1 we note that:

(i) $J_{2\nu+\mu}(x) \sim O(x^{2\nu+\mu})$ for small x [10b].

(ii) $|J_{2\nu+\mu}(x y \overline{2r-1})| \leqq B$, a constant for all values of r and $\to 0$ as $r \to \infty$ if $2\nu + \mu > 0$.

(iii) $E(\alpha, \beta, \gamma :: x) \sim O(x^\alpha + x^\beta + x^\gamma)$ for small x [5].

Hence

$$|I'| = \left| \int_0^1 \sqrt{xy}\, J_{2\nu+\mu}(xy)\, \varphi(y)\, dy \right|$$
$$\leqq A \sum_{r=1}^{\infty} \left| \sqrt{x}\, J_{2\nu+\mu}(x y \overline{2r-1}) / (2r-1)^{3/2} \right| \times$$
$$\times \int_0^1 \left| y^{2-\mu} E(\mu+1, \mu+\nu+1, \mu/2+5/4 :: y^2) \right| dy$$

where A is some constant,

$$\leqq A' \sum_{r=1}^{\infty} 1/(2r-1)^{3/2} \cdot \int_0^1 |y^{2-\mu} E(\mu+1, \mu+\nu+1, \mu/2+5/4 :: y^2)|\, dy.$$

Since the integral on the right of the inequality converges if $\mu+5>0$, $\mu+2\nu+5>0$, the integral I' converges. For the range 1 to ∞, we note that:

(i) $\sqrt{(xy)}\, J_{2\nu+\mu}(xy)$ is bounded for all values of x, y.

(ii) $E(\alpha, \beta, \gamma :: x) \to a$ constant as $x \to \infty$ [5]. Hence $E(\alpha, \beta, \gamma :: rx)$ is uniformly bounded for all values of r and x in $(1, \infty)$.

(iii) Let

$$I'' = \int_1^{\infty} \sqrt{xy}\, J_{2\nu+\mu}(xy)\, \varphi(y)\, dy.$$

We have

$$\sum_{r=1}^{\infty} |E[\mu+1, \mu+\nu+1, \mu/2+5/4 :: (\overline{r-1/2}\, y\sqrt{\pi/2})^2] / (\overline{2r-1}\, y\sqrt{\pi/2})^{\mu-3/2}| \times$$
$$\times |\sqrt{(xy)}\, J_{2\nu+\mu}(xy)|$$
$$\leqq k \sum_{r=1}^{\infty} [\overline{2r-1}\, \sqrt{(\pi/2)}]^{3/2-\mu} |\sqrt{(xy)}\, J_{2\nu+\mu}(xy)|\, y^{3/2-\mu}$$

where k is some finite constant. Hence

$$|I''| \leqq \int_1^{\infty} |\varphi(y)|\, |\sqrt{(xy)}\, J_{2\nu+\mu}(xy)|\, dy$$
$$\leqq k' \sum_{r=1}^{\infty} (2r-1)^{3/2-\mu} \int_1^{\infty} y^{3/2-\mu} |\sqrt{(xy)}\, J_{2\nu+\mu}(xy)|\, dy$$

where k' is another constant.

Since

$$\sum_{r=1}^{\infty} (2r-1)^{3/2-\mu} \quad \text{and} \quad \int_1^{\infty} y^{3/2-\mu} |\sqrt{xy}\, J_{2\nu+\mu}(xy)|\, dy$$

both converge if $\mu > 5/2$, the integral I'' converges. Combining the above two results we see that the integral I exists and $\varphi(x)$ is $R_{2\nu+\mu}$.

In conclusion I express my thanks to Dr. S. C. Mitra for his helpful suggestions.

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(Received July 15, 1962)