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On Non-Linear Wave Equations

By

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Introduction.

One of the non-linear partial differential equations of greatest interest in the recent development of theoretical physics is the non-linear wave equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + k u^3 = 0 \tag{1}$$

of importance in quantum field theory. In a recent paper [2], K. JÖRGENS has given a proof of the solvability of the Cauchy initial value problem for this equation in three space variables, as well as for a more general class of equations of the form

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + F'(|u|^2)\, u = 0 \tag{2}$$

where F' is the derivative of a function F satisfying certain growth and definiteness conditions. The techniques used by JÖRGENS in [2] are concrete-analytical and fitted to the particular equation on hand, and they give rise to the natural question of finding a more general framework within which existence theorems for non-linear equations of a "wave equation type" can be established.

It is our purpose in the present paper to establish an abstract existence theorem from which the results of JÖRGENS will follow as a special case. Our discussion is therefore completely operator-theoretical in character, but we should point out that the significance of the results obtained is rooted in the possibility of verifying the abstract hypotheses in particular concrete cases. Section 1 is devoted to the general discussion for operator equations. Section 2 specializes the general results to obtain Jörgens' theorem, as well as theorems for non-linear wave equations of the form (2) in R^n as well as R^3.

Let us note that in a footnote to [2], it is remarked that J. L. LIONS (unpublished) has obtained weak solutions for non-linear wave equations generalizing Jörgens' results. Our results establish the existence and uniqueness of strong solutions or more precisely, of strict solutions of second-order operator differential equations in the time variable t. Non-linear wave equations have been studied for a slightly different purpose by W. STRAUSS in his M.I.T. Doctoral Dissertation [4].

Section 1. Let H be a Hilbert space with inner product denoted by $(\cdot,\cdot)$ and norm by $\|\cdot\|$. Suppose that A is a positive densely defined selfadjoint linear operator in H, $A^{\frac{1}{2}}$ its positive square root. If $R^+=\{t: t>0\}$, we consider the operator differential equation

$$\frac{d^2u}{dt^2}+Au+M(u)=0, \qquad t\in R^+; \tag{1.1}$$

with the Cauchy initial conditions

$$u(0+)=\varphi, \frac{du}{dt}(0+)=\psi. \tag{1.2}$$

Here $M(u)$ is a (possibly) non-linear function from $D(A^{\frac{1}{2}})$ to H, concerning which we shall make some assumptions below. We shall consider *strict* solutions of our initial value problem, i.e. the function u: $R^+\to H$ will be said to be a solution provided that $u(t)\in D(A)$ for all $t>0$, Au and d^2u/dt^2 are uniformly continuous functions into H on every interval of the form $[0, T]$ with $T>0$, $A^{\frac{1}{2}}u(t)$ and $A^{\frac{1}{2}}\frac{du}{dt}$ are uniformly continuous on $[0, T]$ and the equations (1.1) and (1.2) are satisfied. (All derivatives throughout this paper are strong derivatives.)

We impose upon the linear subset $D(A^{\frac{1}{2}})$ a Hilbert space structure which turns it into a Hilbert space W by identifying $D(A^{\frac{1}{2}})$ with the graph of $A^{\frac{1}{2}}$. Thus

$$\|u\|_W^2=\|A^{\frac{1}{2}}u\|^2\geqq c\|u\|^2.$$

Assumptions upon M:

(I) *Given any $C>0$, there exists $k_C>0$ such that*

$$\|M(u)\|\leqq k_C,$$
$$\|M(u_1)-M(u)\|\leqq k_C\|u-u_1\|_W$$

provided that $\|u\|_W+\|u_1\|_W\leqq C$.

(II) *There exists a real number k_0 such that for every strongly differentiable function u from R^+ to H with du/dt uniformly continuous from $[0, T]$ to W, we have*

$$\mathrm{Re}\left\{\int_0^t\left(M(u(s)), \frac{du}{ds}(s)\right)ds\right\}\geqq -k_0\left\{1+\int_0^t\|u(s)\|_W^2\,ds\right\}$$

for all $t>0$.

(III) *If C is any positive constant, there exists $k_C>0$ such that for every continuously differentiable function v from R^+ to W,*

$$\left\|\frac{d}{dt}\{M(v(t))\}\right\|\leqq k_C\left\|\frac{dv}{dt}\right\|_W, \qquad 0\leqq t\leqq T$$

provided that

$$\|v(t)\|_W\leqq C, \qquad 0\leqq t\leqq T.$$

(IV) *For any positive constant C, there exists $k_C'>0$ such that for any pair of continuously differentiable functions v and v_1 from R^+ into W with*

$$\|v(t)\|_W + \|v_1(t)\|_W + \left\|\frac{dv}{dt}\right\|_W + \left\|\frac{dv_1}{dt}\right\|_W \leqq C, \qquad 0 \leqq t \leqq T,$$

we have

$$\left\|\frac{d}{dt}\{M(v(t)) - M(v_1(t))\}\right\| \leqq k'_C\Big\{\|v(t) - v_1(t)\|_W + $$
$$+ \left\|\frac{dv}{dt}(t) - \frac{dv_1}{dt}(t)\right\|_W\Big\}, \quad \textit{for} \quad 0 \leqq t \leqq T.$$

Theorem 1. *If A and M satisfy the conditions* (I) *through* (IV) *above and if $\varphi \in D(A)$, $\psi = D(A^{\frac{1}{2}})$, then there exists a solution u of the equation*

$$\frac{d^2 u}{dt^2} + Au + M(u) = 0 \tag{1.1}$$

satisfying the initial conditions

$$u(0) = \varphi, \qquad \frac{du}{dt}(0) = \psi. \tag{1.2}$$

Theorem 2. (a) *Under the hypotheses of Theorem 1, for each T and $C > 0$ and for each pair of initial data $[\varphi, \psi]$ and $[\varphi_1, \psi_1]$ with*

$$\|\varphi\|_W + \|\psi\| \leqq C, \qquad \|\varphi_1\|_W + \|\psi_1\| \leqq C,$$

we have for corresponding solution u and u_1 of equation (1.1) *on the interval* $[0, T]$,

$$\|u(t) - u_1(t)\|_W + \left\|\frac{du_1}{dt}(t) - \frac{du}{dt}(t)\right\| \leqq k(T, C)\{\|\varphi - \varphi_1\|_W + $$
$$+ \|\psi - \psi_1\|\}, \qquad 0 \leqq t \leqq T.$$

(b) *Similarly if*

$$\|A\varphi\| + \|\varphi\|_W + \|\psi\|_W \leqq C, \qquad \|A\varphi_1\| + \|\varphi_1\|_W + \|\psi_1\|_W \leqq C,$$

then

$$\left\|\frac{du_1}{dt}(t) - \frac{du}{dt}(t)\right\|_W + \|Au_1(t) - Au(t)\| \leqq k(T, C)\{\|A\varphi - A\varphi_1\| + $$
$$+ \|\varphi - \varphi_1\|_W + \|\psi - \psi_1\|_W\}, \qquad 0 \leqq t \leqq T.$$

Corollary to Theorem 2. *The solution of the initial value problem for the equation* (1.1) *is uniquely determined by the initial data $[\varphi, \psi]$.*

We shall obtain the proofs of Theorems 1 and 2 from a series of Lemmas which hold under the hypotheses of Theorem 1.

Lemma 1. *Let u be a solution of equation* (1.1) *on $[0, T]$ with initial data given by* (1.2). *Then for all $t > 0$, we have*

$$\left\|\frac{du}{dt}(t)\right\|^2 + \|A^{\frac{1}{2}}u(t)\|^2 \leqq \{\|\varphi\|_W^2 + \|\psi\|^2\} e^{2k_0 t}.$$

Proof of Lemma 1. Taking the inner product of both sides of equation (1.1) with du/dt, we obtain

$$\frac{1}{2}\frac{d}{dt}\left\{\left\|\frac{du}{dt}\right\|^2 + \|A^{\frac{1}{2}}u\|^2\right\} + \mathrm{Re}\left(M(u(t)), \frac{du}{dt}\right) = 0.$$

Integrating from 0 to t and applying assumption (II) on M, we see that

$$\left\| \frac{du}{dt}(t) \right\|^2 + \|A^{\frac{1}{2}} u(t)\|^2 - \|\psi\|^2 - \|A^{\frac{1}{2}} \varphi\|^2 \leqq 2k_0 \left\{ 1 + \int_0^t \|u(s)\|_W^2 \, ds \right\}.$$

Since $\|A^{\frac{1}{2}}\varphi\|^2 = \|\varphi\|_W^2$, the inequality of the Lemma is obtained by applying the following lemma.

Lemma 2. *Let $h(t)$ be a continuous realvalued function for $t \geqq 0$ such that*

$$h(t) \leqq k + k_1 \int_0^t h(s)\, ds.$$

Then:

$$h(t) \leqq k e^{k_1 t}.$$

Proof of Lemma 2. Let $g(t) = \int_0^t h(s)\, ds$. Then $g(t)$ is absolutely continuous, $g(0) = 0$, and for $t > 0$,

$$g'(t) \leqq k + k_1 g(t).$$

Multiplying by $e^{-k_1 t}$, we have

$$\frac{d}{dt}\{g(t) e^{-k_1 t}\} \leqq k e^{-k_1 t}.$$

Integrating, we obtain

$$g(t) \leqq k(k_1)^{-1}\{e^{k_1 t} - 1\}.$$

If we substitute in the inequality for $g'(t)$ and note that $g' = h$, we obtain the desired conclusion.

Lemma 3. *Let u be a solution of equation* (1.1) *on* $[0, T]$ *with initial conditions* (1.2). *Given $C > 0$, $T > 0$ there exists a constant $k(T, C)$ such that*

$$\left\| \frac{du}{dt} \right\|_W^2 + \left\| \frac{d^2 u}{dt^2} \right\|^2 \leqq k(T, C)$$

if $\|\varphi\|_W^2 + \|A\varphi\|^2 + \|\psi\|_W^2 \leqq C$, $0 \leqq t \leqq T$.

Proof of Lemma 3. It follows from Lemmas 1 and 2 that

$$\|u(t)\|_W^2 + \left\| \frac{du}{dt}(t) \right\|^2 \leqq k_C e^{2k_0 t}, \qquad k_C > 0.$$

Let $\delta_h(u)(t) = h^{-1}\{u(t+h) - u(t)\}$. We propose to estimate various norms of $\delta_h(u)$ for h small. Since u satisfies equation (1.1), it follows that for fixed $h > 0$,

$$\frac{d^2}{dt^2}[\delta_h(u)](t) + A\,\delta_h(u)(t) + \frac{M(u(t+h)) - M(u(t))}{h} = 0.$$

By Assumption (III) about M, we may estimate the last term by

$$\left\| \frac{M(u(t+h)) - M(u(t))}{h} \right\| \leqq k_{C,T} \|\delta_h(u)(t)\|_W + \varepsilon(h), \qquad t \leqq T.$$

where $\varepsilon(h) \to 0$ as $h \to 0$.

Set $v=\delta_h(u)$. Then v is a solution of the equation

$$\frac{d^2}{dt^2}v(t)+Av(t)=j(t)$$

where $\|j(t)\|\leqq k\|v(t)\|_W+\varepsilon(h)$ for $0\leqq t\leqq T$. Taking the inner product of the last equality with dv/dt, we obtain

$$\frac{1}{2}\frac{d}{dt}\left\{\left\|\frac{dv}{dt}\right\|^2+\|A^{\frac{1}{2}}v(t)\|^2\right\}\leqq k\|v(t)\|_W\left\|\frac{dv}{dt}\right\|+\varepsilon(h)\left\|\frac{dv}{dt}\right\|, \qquad 0\leqq t\leqq T.$$

Hence

$$\frac{1}{2}\frac{d}{dt}\left\{\left\|\frac{dv}{dt}\right\|^2+\|v(t)\|_W^2\right\}\leqq\frac{k'}{2}\left\|\frac{dv}{dt}\right\|_W^2+\varepsilon_1(h).$$

Hence $g(t)=\|v(t)\|_W^2+\left\|\frac{dv}{dt}(t)\right\|^2$ satisfies the inequality

$$g'(t)\leqq k'g(t)+\varepsilon_1(h), \qquad 0\leqq t\leqq T.$$

Therefore, $g(t)$ satisfies the inequality

$$g(t)\leqq\{g(0)+\varepsilon_1(h)\}e^{k't}\leqq k(T,C)\{g(0)+\varepsilon_1(h)\}, \qquad 0\leqq t\leqq T$$

with $k(T,C)$ independent of h. Translating this latter inequality in terms of our solution u of equation (1.1), it becomes

$$\left\|h^{-1}\left\{\frac{du}{dt}(t+h)-\frac{du}{dt}(t)\right\}\right\|^2+\|h^{-1}\{u(t+h)-u(t)\}\|_W^2$$
$$\leqq k(T,C)\left\{\|A\varphi+M(\varphi)\|^2+\|\psi\|_W^2+\varepsilon_1(h)\right\}, \qquad 0\leqq t\leqq T$$

where $\varepsilon_1(h)\to 0$ as $h\to 0$. The terms in each of the norms at the left of the last inequality approach d^2u/dt^2 in H and du/dt in W, respectively, as $h\to 0$. Hence we have the inequality

$$\left\|\frac{d^2u}{dt^2}\right\|^2+\left\|\frac{du}{dt}(t)\right\|_W^2\leqq k(T,C), \qquad 0\leqq t\leqq T,$$

and the proof of Lemma 3 is complete.

Proof of Theorem 2(a). Let u and u_1 be solutions of equation (1.1) corresponding to the sets of initial data $[\varphi,\psi]$ and $[\varphi_1,\psi_1]$. Then $w=u-u_1$ is a solution of the equation

$$\frac{d^2}{dt^2}w(t)+Aw(t)+M(u(t))-M(u_1(t))=0$$

with initial conditions

$$w(0)=\varphi-\varphi_1,\ \frac{dw}{dt}(0)=\psi-\psi_1.$$

Taking the inner product of the equation for w with dw/dt, we obtain

$$\frac{1}{2}\frac{d}{dt}\left\{\left\|\frac{dw}{dt}(t)\right\|^2+\|A^{\frac{1}{2}}w(t)\|^2\right\}\leqq\left\|\frac{dw}{dt}\right\|\,\|M(u(t))-M(u_1(t))\|.$$

The last factor on the right can be estimated using assumption (I) on M to obtain

$$\|M(u(t))-M(u_1(t))\|\leqq k_{C,T}\|w(t)\|_W,\qquad 0\leqq t\leqq T,$$

an estimation which is possible because of the apriori estimate on $\|u(t)\|_W$ and $\|u_1(t)\|_W$ for $0\leqq t\leqq T$ in terms of T and C given by Lemmas 1 and 2.

Thus we have

$$\frac{1}{2}\frac{d}{dt}\left\{\left\|\frac{dw}{dt}(t)\right\|^2+\|w(t)\|_W^2\right\}\leqq k_{C,T}\left\|\frac{dw}{dt}(t)\right\|\cdot\|w(t)\|_W.$$

Setting $g(t)=\left\|\frac{dw}{dt}(t)\right\|^2+\|w(t)\|_W^2$, we obtain

$$g'(t)\leqq k'g(t),\qquad 0\leqq t\leqq T.$$

Hence

$$g(t)\leqq g(0)e^{k't}\leqq k(T,C)g(0),\qquad 0\leqq t\leqq T.$$

Evaluating g in terms of u and u_1, we obtain the conclusion of Theorem 2(a).

Proof of Theorem 2(b). We again set $w(t)=u(t)-u_1(t)$ and by analogy with the proof of Lemma 3, consider

$$\delta_h(w)(t)=h^{-1}\{w(t+h)-w(t)\}.$$

The function $\delta_h(w)$ satisfies the equation

$$\frac{d^2}{dt^2}\big(\delta_h(w)\big)(t)+A\,\delta_h(w)(t)+\frac{M(u(t+h))-M(u(t))}{h}-\frac{M(u_1(t+h))-M(u_1(t))}{h}=0.$$

Since u and u_1 are by assumption continuously differentiable maps of R^+ into W with uniformly continuous derivatives on $[0,T]$, it follows that

$$\left\|\{M(u(t+h))-M(u(t))\}h^{-1}-\frac{d}{dt}\big(M(u(t))\big)\right\|=\varepsilon_1(h),$$

$$\left\|\{M(u_1(t+h))-M(u_1(t))\}h^{-1}-\frac{d}{dt}\big(M(u_1(t))\big)\right\|=\varepsilon_2(h),$$

$$\left\|\frac{du}{dt}(t)-\delta_h(u)(t)\right\|=\varepsilon_3(h),$$

$$\left\|\frac{du_1}{dt}(t)-\delta_h(u_1)(t)\right\|=\varepsilon_4(h),$$

where all the $\varepsilon_j(h)\to 0$ as $h\to 0$. It follows then from Assumption (IV) about the operation M as well as these last inequalities that

$$\|h^{-1}\{M(u(t+h))-M(u(t))\}-h^{-1}\{M(u_1(t+h))-M(u_1(t))\}\|\leqq k'_{T,C}\{\|\delta_h(w)(t)\|_W+\|w(t)\|_W\},$$

where the possibility of the estimation depends upon the a priori bound for $\|du/dt\|_W$ and $\|du_1\|/dt$ derived in Lemma 3, as well as the bound for $\|u\|_W$ and $\|u_1\|_W$ derived in Lemmas 1 and 2.

Taking the inner product of both sides of the differential equation for $v=\delta_h(w)$ with dv/dt and applying the inequality for the last term in that equation derived above, we find

$$\frac{d}{dt}\left\{\left\|\frac{dv}{dt}(t)\right\|^2+\|v(t)\|_W^2\right\}\leqq k\left\{\left\|\frac{dv}{dt}(t)\right\|^2+\|v(t)\|_W^2\right\}+\|w(t)\|_W^2+\varepsilon(h).$$

Setting $g(t)=\|v(t)\|_W^2+\left\|\frac{dv}{dt}\right\|^2+\|w(t)\|_W^2$, we obtain

$$g'(t)\leqq kg(t)+\varepsilon(h), \qquad 0\leqq t\leqq T,$$

and therefore

$$g(t)\leqq[g(0)+\varepsilon(h)]e^{kt}\leqq k(T,C)[g(0)+\varepsilon(h)], \qquad 0\leqq t\leqq T.$$

Replacing g by its value in terms of u and u_1 and letting $h\to 0$, we obtain the inequality of Theorem 2(b). Q.E.D.

Thereby the proof of Theorem 2 is complete, and thereby the Corollary to Theorem 2 is valid, namely, the solution $u(t)$ of equation (1.1) on $[0, T]$ is uniquely determined by its initial data $[\varphi, \psi]$. As a consequence of this last fact, we have the following:

Lemma 4. *In order to prove Theorem 1, it suffices to show that for each $C>0$, there exists an interval $T_C>0$ such that equation (1.1) has a solution $u(t)$ on the interval $[0, T_C]$ for every set of initial data $[\varphi, \psi]$ with*

$$\|A\varphi\|+\|\varphi\|_W+\|\psi\|_W\leqq C.$$

Proof of Lemma 4. It follows by the Corollary to Theorem 2 which has already been established that if a solution u existed on every interval $[0, T]$ for every given set of data $[\varphi, \psi]$, then these various solutions would coincide on the overlap of their domains of definition and could be assembled together to give a solution of the initial value problem defined over all of R^+. If Theorem 1 were false, it would have to be true that for some $\varphi\in D(A)$ and $\psi\in W$ and for some $T>0$, no solution u would exist on the interval $[0, T]$ with the given initial data.

By the assumptions of the Lemma, however, a solution u would exist over some interval $(0, T_1]$ and hence by the uniqueness theorem over a largest such interval. By Lemmas 1, 2, and 3, which do not depend for their validity upon u being a solution outside of the interval $(0, T_1)$, we know that

$$\left\|\frac{d^2u}{dt^2}(t)\right\|+\left\|\frac{du}{dt}\right\|_W+\|u(t)\|_W\leqq k_{C,T_1}, \qquad 0\leqq t\leqq T_1,$$

where $k_{C\ T}$ depends only upon the initial data $[\varphi, \psi]$ and upon T_0.

Furthermore, since u is a solution of equation (1.1) for which Au may be expressed in terms of d^2u/dt^2 and $M(u)$, it follows from Assumption I on M that $Au(t)$ is uniformly bounded in H for $0 \leqq t < T_1$.

Let

$$C_1 = \sup_{0 \leqq t \leqq T_1} \left\{ \|A u(t)\| + \|u(t)\|_W + \left\| \frac{du}{dt}(t) \right\|_W \right\},$$

and let T_0 be the interval length corresponding to C_1 by the assumption of Lemma 4. The initial value problem for equation (1.1) with initial data $\left[u(T_2), \frac{du}{dt}(T_2)\right]$ taken at $T_2 = T_1 - \frac{1}{2} T_0$ (rather than 0) has a solution $u_1(t)$ on the interval $[T_2, T_1 + \frac{1}{2} T_0]$. By the uniqueness result which is a corollary of Theorem 2, $u(t)$ and $u_1(t)$ which have the same initial data at $T = T_2$ must coincide on the interval $[T_2, T_1]$. Hence u can be extended to a solution of equation (1.1) on the interval $(0, T_1 + \frac{1}{2} T_0)$ contradicting the maximality of the interval $(0, T_1)$. Hence we have a contradiction from the assumption that Theorem 1 is false, and Lemma 4 follows.

To complete the proof of Theorem 1 by establishing the hypothesis of Lemma 4, we shall carry out the construction of a solution for equation (1.1) on an interval $[0, T]$ with initial data (1.2) by successive approximations, or more precisely by using the contraction principle of Picard in a suitable complete metric space.

Let φ, ψ, and T be given. We consider the metric space X, depending upon φ, ψ, and T, whose elements consist of all functions u from $[0, T]$ to H such that u is twice continuously differentiable into H from $[0, T]$, u is once continuously differentiable into W from $[0, T]$, $u(t)$ lies in $D(A)$ for all t in $[0, T]$ and Au is continuous into H from that interval, and finally,

$$u(0) = \varphi, \frac{du}{dt}(0) = \psi.$$

We impose upon X the metric $\varrho(u, u_1) = \|u - u_1\|_X$, where

$$\|u\|_X = \sup_{0 \leqq t \leqq T} \left\{ \left\| \frac{d^2 u}{dt^2}(t) \right\| + \|A u(t)\| + \left\| \frac{du}{dt}(t) \right\|_W + \|u(t)\|_W \right\}.$$

It follows by standard arguments that X is complete with respect to this metric.

The elements φ and ψ which are given satisfy the conditions: $\varphi \in D(A)$, $\psi \in D(A^{\frac{1}{2}})$. We define

$$\|[\varphi, \psi]\| = \|A\varphi\| + \|\varphi\|_W + \|\psi\|_W.$$

We suppose that $\|[\varphi, \psi]\| \leqq C$, and propose to choose $T > 0$ and a non-empty closed set $\|v\|_X \leqq C_1$ in the corresponding space X, with C_1 and T depending only upon C and not otherwise on φ and ψ, such that the equation (1.1) has a solution in that set on the interval $[0, T]$. To ensure the latter fact, we construct a mapping S of X into itself whose fixed points will coincide with solutions of equation (1.1) and show that S has a fixed point because on a given set $\{v \mid \|v\|_X \leqq C_1\}$ which is carried into itself by S,

$$\|Sv - Sv_1\|_X \leqq c\|v - v_1\|_X, \qquad c < 1.$$

Let $v \in X$. Then $u = Sv$ is given for an arbitrary choice of T as the solution of the equation

$$\frac{d^2u}{dt^2} + Au = -M(v(t)), \qquad 0 < t \leqq T, \tag{1.3}$$

with initial conditions

$$u(0) = \varphi, \frac{du}{dt}(0) = \psi. \tag{1.4}$$

Remark. The linear equation

$$\frac{d^2u}{dt^2} + Au = f(t), \qquad 0 < t \leqq T \tag{1.5}$$

with the initial conditions (1.4) has an unique solution if $\int \|f(t)\|\, dt < +\infty$ with the solution given by

$$u(t) = \cos(A^{\frac{1}{2}}t)\,\varphi + \sin(A^{\frac{1}{2}}t)\,(A^{-\frac{1}{2}}\psi) + \int_0^t \sin(A^{\frac{1}{2}}(t-s)) \cdot (A^{-\frac{1}{2}}f)(s)\,ds. \tag{1.6}$$

Lemma 5. (a) *The solution u of the equation* (1.5) *with initial data* $[\varphi, \psi]$ *will lie in X provided that $\varphi \in D(A)$, $\psi \in D(A^{\frac{1}{2}})$, and f is a continuously differentiable function from* $[0, T]$ *into H.*

(b) *For the solution u of equation* (1.5) *with initial data* $[\varphi, \psi]$, *we have*

$$\|u\|_X \leqq k\left\{\|[\varphi, \psi]\| + \|f(0)\| + T \sup_{0 \leqq t \leqq T}\|f(t)\| + T \sup_{0 \leqq t \leqq T}\|f'(t)\|\right\}.$$

with the constant k independent of φ, ψ, f, and T.

Proof of Lemma 5. We set $u = u_1 + u_2 + u_3$, with

$$u_1(t) = \cos(A^{\frac{1}{2}}t)\,\varphi,$$
$$u_2(t) = \sin(A^{\frac{1}{2}}t)\,(A^{-\frac{1}{2}}\psi),$$
$$u_3(t) = \int_0^t \sin(A^{\frac{1}{2}}(t-s))\,(A^{-\frac{1}{2}}f)(s)\,ds.$$

If $\varphi \in D(A)$,

$$u_1'(t) = -\sin(A^{\frac{1}{2}}t)\,(A^{\frac{1}{2}}\varphi),$$
$$u_1''(t) = -\cos(A^{\frac{1}{2}}t)\,(A\varphi),$$

and u_1' and u_1'' are both continuous into H. Furthermore

$$A^{\frac{1}{2}}u_1'(t) = -\sin(A^{\frac{1}{2}}t)\,(A\varphi)$$

is also continuous into H. Hence $u_1 \in X$, and

$$\|u_1\|_X \leqq k\{\|A\varphi\| + \|\varphi\|_W\}.$$

For u_2, we know that if $\psi \in D(A^{\frac{1}{2}})$,

$$u_2'(t) = \cos(A^{\frac{1}{2}}t)(\psi),$$
$$u_2''(t) = -\sin(A^{\frac{1}{2}}t)(A^{\frac{1}{2}}\psi),$$

so that both u_2' and u_2'' are continuous into H. So is

$$A^{\frac{1}{2}}u_2'(t) = \cos(A^{\frac{1}{2}}t)(A^{\frac{1}{2}}\psi).$$

Hence $u_2 \in X$, and

$$\|u_2\|_X \leqq k\|\psi\|_W.$$

We turn now to the component u_3. We remark that

$$u_3'(t) = \int_0^t \cos\left(A^{\frac{1}{2}}(t-s)\right) f(s)\, ds.$$

Since u_3 is a solution of equation (1.5), it suffices, in order to estimate $A u_3(t)$, to estimate d^2u_3/dt^2. For the latter, we have

$$\begin{aligned} u_3''(t) &= \frac{d}{dt}\int_0^t \cos\left(A^{\frac{1}{2}}(t-s)\right) f(s)\, ds \\ &= f(t) + \int_0^t \frac{d}{dt}\left[\cos(A^{\frac{1}{2}}(t-s))\right] f(s)\, ds \\ &= f(t) - \int_0^t \frac{d}{ds}\left[\cos(A^{\frac{1}{2}}(t-s))\right] f(s)\, ds \\ &= f(t) + \int_0^t \cos\left(A^{\frac{1}{2}}(t-s)\right) f'(s)\, ds - f(t) + \cos(A^{\frac{1}{2}}t) f(0). \\ &= \cos(A^{\frac{1}{2}}t) f(0) + \int_0^t \cos\left(A^{\frac{1}{2}}(t-s)\right) f'(s)\, ds. \end{aligned}$$

Therefore, $\|u_3''(t)\| \leqq \|f(0)\| + T \sup_{0 \leqq t \leqq T} \|f'(t)\|$, and u_3' is a continuous function into H. The same inequality holds for $\sup_{0 \leqq t \leqq T} \|f(t)\|$. A similar calculation yields the same result for $A^{\frac{1}{2}}u_3'(t)$. Hence $u_3 \in X$, and

$$\|u_3\|_X \leqq k\left\{\|f(0)\| + T\sup\|f'(t)\| + T\sup\|f(t)\|\right\}.$$

Thus u, being the sum of three elements of X, lies in X itself and satisfies the inequality of part (b) of Lemma 5.

Lemma 6. *Let $C > 0$ be given. Suppose that $\|[\varphi, \psi]\| \leqq C$. Then there exist constant $C_1 > C$, $T > 0$, and $c < 1$, depending only upon C such that:*

(a) For $\|v\|_X \leqq C_1$, $\|Sv\|_X \leqq C_1$.

(b) If $\|v\|_X \leqq C_1$, $\|v_1\|_X \leqq C_1$, then

$$\|Sv - Sv_1\|_X \leqq c\|v - v_1\|_X.$$

Proof of Lemma 6. If $u = Sv$, u is the solution of the equation

$$u''(t) + A u(t) = -M(v(t)), \qquad 0 < t \leqq T,$$

with initial data $[\varphi, \psi]$. By Lemma 5 with $f(t) = -M(v(t))$, we know that

$$\|u\|_X \leqq kC + k\|M(\varphi)\| + kT \sup_{0 \leqq t \leqq T} \|M(v(t))\| + kT \sup_{0 \leqq t \leqq T} \left\| \frac{d}{dt} M(v(t)) \right\|.$$

Since $\|v\|_X \leqq C_1$, it follows from Assumption (III) on the operator M that

$$\sup_{0 \leqq t \leqq T} \left\| \frac{d}{dt} M(v(t)) \right\| \leqq k_{C_1},$$

and similarly it follows from Assumption (I) that

$$\sup_{0 \leqq t \leqq T} \|M(v(t))\| \leqq k_{C_1}.$$

Thus

$$\|u\|_X \leqq k'_{C_1} T.$$

We choose C_1 so large that $C_1 > 2k'C$, and then T so small that

$$T \leqq C_1 (2k'_{C_1})^{-1}.$$

Then the inequality of Lemma 6 (a) is verified, i.e.

$$\|u\|_X \leqq C_1.$$

To prove Lemma 6 (b), let $u = Sv$, $u_1 = Sv_1$, $w = Sv - Sv_1$. Then w is the solution of the equation

$$w''(t) + A w(t) = M(v_1(t)) - M(v(t))$$

with initial data $[0, 0]$. Thus if we set $f(t) = M(v_1(t)) - M(v(t))$, as in Lemma 5, we have $f(0) = M(\varphi) - M(\varphi) = 0$. By Assumption (IV) on M, we have

$$\left\| \frac{d}{dt} \{M(v_1(t)) - M(v(t))\} \right\| \leqq k_{C_1} \{\|v_1'(t) - v'(t)\|_W + \|v_1(t) - v(t)\|_W\}.$$

By Assumption (I) on M, we know moreover that

$$\|M(v_1(t)) - M(v(t))\| \leqq k_{C_1} \|v_1(t) - v(t)\|_W.$$

It follows from Lemma 5 that

$$\|w\|_X \leqq k_{C_1} T \|v - v_1\|_X.$$

Choosing $T - (2k_{C_1})^{-1}$, $c = \frac{1}{2}$, we find the proof of Lemma 6 to be complete.

Proof of Theorem 1. By Lemma 6, we can choose T so small that on the set $\|v\|_X \leqq C_1$ which is carried into itself by S, S contracts distances by a ratio $c < 1$. Applying the contraction principle, we know that S has a fixed point in this set. Obviously, however, a fixed point of S is a solution of equation (1.1) with initial data $[\varphi, \psi]$ on the interval $[0, T]$ with the length of T depending only upon $\|[\varphi, \psi]\|$. By Lemma 4, this suffices for the proof of Theorem 1. Q.E.D.

Section 2. To specialize the results of Section 1 to obtain solutions of non-linear wave equations in the ordinary sense, we let $H=L^2(E^n)$. (More generally, we could consider $H=L^2(G)$ for suitably smooth open sets G of E^n and consider realizations of strongly elliptic operators A under suitable boundary conditions.) Let A_0 be a linear elliptic partial differential operator of order $2m$ with real coefficients on E^n. We suppose that A_0 is formally self-adjoint and uniformly elliptic on E^n, all of its coefficients are uniformly bounded, and the top-order coefficients are uniformly continuous. We know then that

$$(A_0\varphi, \varphi) \geqq -k_0(\varphi, \varphi)$$

for φ in $C_c^\infty(E^n)$, the set of C^∞ functions with compact support in E^n.

Consider the unique self-adjoint realization A_0 of the differential operator A_0 in $L^2(E^n)$. It follows from the results of [1] that $A=A_0+(k_0+1)$ is a positive operator and that

$$\sum_{|\alpha|\leqq m} \|D^\alpha u\|^2 \leqq k\|A^{\frac{1}{2}} u\|^2.$$

If we use the $W^{m,2}$-norm defined by

$$\|u\|_{m,2}^2 = \sum_{|\alpha|\leqq m} \|D^\alpha u\|^2,$$

it follows also that the W norm corresponding to the graph norm of $A^{\frac{1}{2}}$ is equivalent to the $W^{m,2}$ norm.

By the Sobolev Imbedding Theorem (e.g. Lemma 5 of [1]), $W^{m,2}(E^n) \subset L^p(E^n)$ for any $p \geqq 2$ for which $\frac{1}{p} \geqq \frac{1}{2} - \frac{m}{n}$, ($p<+\infty$ if the latter quantity equals zero) with a continuous imbedding. We specialize the operator M to be the non-linear multiplication operator

$$M(u) = F'(|u|^2)\, u,$$

where $F'(r)$ is the derivative of function $F(r)$, $r \geqq 0$, which we normalize by setting $F(0)=0$. Let us consider what assumptions we may make on F, F', F'' and F''' which will cause Assumptions (I) to (IV) on the operator M to hold. Let us consider Assumption (II) first.

We have

$$\int_0^t (M(u(s)), u'(s))\, ds = \int_0^t \frac{d}{dt} \int_{E^n} F(|u(s,x)|)^2\, dx\, ds$$

$$= \int_{E^n} F(|u(t,x)|^2)\, dx - \int_{E^n} F(|u(t,x)|^2)\, dx.$$

If we assume for example that $F(r) \geqq 0$, then the function $G(u, t)$ defined by

$$G(u, t) = \int_{E^n} F(|u(t, x)|^2) dx$$

will be non-negative whenever it makes sense and Assumption (II) will be satisfied. More generally, if $G(u, t) \geqq -k_0$, Assumption (II) will hold.

Theorem 3. *Let $H = L^2(E^n)$, A the unique self-adjoint realization of a formally self-adjoint, linear strongly elliptic partial differential operator of order $2m$ with the boundedness and uniformity conditions stated above. Let $F(r)$ be a twice-differentiable function of the real variable r, $0 \leqq r < +\infty$, such that*

(i) $F(0) = 0$, $F(r) \geqq 0$ *for* $r > 0$. (Or more generally $G(u, t) \geqq -k_0$.)

(ii) *If $n > 2m$, let $q = m(n-2m)^{-1}$, and suppose that*

$$|F'(r)| \leqq k r^q, \quad r \geqq 1$$
$$|F''(r)| \leqq k r^{q-1}, \quad r \geqq 1$$
$$|F'''(r)| \leqq k r^{q-2}, \quad r \geqq 1.$$

(iii) *If $n \leqq 2m$, suppose that for some $q < +\infty$,*

$$|F'(r)| \leqq k r^q, \quad r \geqq 1$$
$$|F''(r)| + |F'''(r)| \leqq k r^q, \quad r \geqq 1.$$

Then the equation

$$u''(t) + A u(t) + F'(|u(t)|^2) u(t) = 0 \tag{2.1}$$

has an unique solution defined on R^+ for initial data $[\varphi, \psi]$ with φ in $W^{2m,2}(E^n)$, ψ in $W^{m,2}(E^n)$. The corresponding continuity properties follow from Theorem 2.

Proof of Theorem 3. We need only to verify Assumptions (I), (III), and (IV) on the operator $M(u) = F'(|u|^2) u$.

(1) We have

$$\|M(u)\|^2 = \int_{E^n} |F'(|u|^2(x)) u(x)|^2 dx \leqq k \int_{E^n} |u|^{4q+2} dx \leqq k \|u\|_{L^{4q+2}}^{4q+2}.$$

In case (ii) we see that

$$\|u\|_{4q+2} = \|u\|_{2n(n-2m)^{-1}} \leqq k \|u\|_{W^{m,2}(E)^n},$$

and the same inequality obviously holds in case (iii). Thus

$$\|M(u)\| \leqq f(\|u\|_W).$$

For the estimate of the Lipschitz constant in (I), we observe that $M(u) - M(u_1)$ can be estimated by

$$\|M(u) - M(u_1)\| \leqq \|F'(|u|^2)(u - u_1)\| + \|F'(|u|^2) - F'(|u|^2) u_1\|.$$

For the first of these terms, we have by Holder's inequality,

$$\|F'(|u|^2)(u-u_1)\|^2 = \int_{E^n} |F'(|u|^2)(x)|\cdot|u-u_1|^2\,dx$$
$$\leqq k\int_{E^n} |u|^{4q}|u-u_1|^2\,dx \leqq k\|u\|_{4q+2}^{4q}\|u-u_1\|_{4q+2}^2.$$

For the second term, we have

$$\|F'(|u|^2)-F'(|u_1|^2)u_1\|^2 = \int_{E^n} |F''(\vartheta(x)|u(x)|^2+(1-\vartheta(x))|u_1(x)|^2)|^2\times$$
$$\times|u_1(x)+u(x)|^2\cdot|u(x)-u_1(x)|^2\,dx \leqq \int_{E^n}\{|u(x)|^2+|u_1(x)|^2\}^{4q-2}|u_1|^2$$
$$|u(x)-u_1(x)|^2\,dx \leqq k\{\|u\|_{4q+2}^{4q}+\|u_1\|_{4q+2}^{4q}\}\|u-u_1\|_{4q+2}^2.$$

Hence

$$\|M(u)-M(u_1)\| \leqq k'(\|u\|_W)\,\|u-u_1\|_W.$$

Therefore Assumption (I) on M has been verified.

(2) We wish to estimate $\left\|\frac{d}{dt}M(v(t))\right\|$. We find that

$$\frac{d}{dt}M(v(t)) = \frac{d}{dt}F'(|u(t)|^2)u(t) = F'(|u(t)|^2)u'(t)+2F''(|u(t)|^2)|u(t)|^2u'(t).$$

Estimating as above, we have

$$\|F'(|u(t)|^2)u'(t)\| \leqq k(\|u(t)\|_W)\,\|u'(t)\|_W.$$

For the second term, we have

$$\|F''(|u(t)|^2)|u(t)|^2u'(t)\|^2 \leqq k\int_{E^n}|u(t)|^{4q-4}|u(t)|^4|u'(t)|^2\,dx$$
$$\leqq \int_{E^n}|u(t)|^{4q}|u'(t)|^2\,dx \leqq k(\|u\|_W)\cdot\|u'\|_W^2.$$

A similar estimate holds for the first term, and thereby we have

$$\left\|\frac{d}{dt}M(u(t))\right\| \leqq k(\|u(t)\|_W)\,\|u'(t)\|_W.$$

(3) To obtain the estimate for Assumption (IV), consider the function

$$\frac{d}{dt}M(u(t)) - \frac{d}{dt}M(u_1(t)) = F'(|u(t)|^2)(u'(t)-u_1'(t)) +$$
$$+F'(|u(t)|^2)-F'(|u_1(t)|^2)u_1'(t)+2F''(|u(t)|^2)|u(t)|^2(u'-u_1')+$$
$$+2F''(|u(t)|^2)(|u(t)|^2-|u_1(t)|^2)u_1'(t)+$$
$$+2|u_1(t)|^2u_1'(t)(F''(|u(t)|^2)-F''(|u_1(t)|^2)).$$

All the terms except the last are estimated as before. For the last, we need our assumption on d^3F/dr^3. We have

$$\|[u_1(t)]^2|u_1'(t)|[F''(|u(t)|^2)-F''(|u_1(t)|^2)]\|^2$$
$$\leqq k\int|u_1(t)|^4|u_1'(t)|^2[|u(t)|+|u_1(t)|]^{4q-8+2}|u(t)-u_1(t)|^2\,dx$$
$$\leqq k(\|u\|_W+\|u_1\|_W)\,k_1(\|u'(t)\|_W)\,\|(u-u_1)(t)\|_W,$$

by Holder's inequality since

$$\frac{4q-2}{4q+2}+\frac{2}{4q+2}+\frac{2}{4q+2}=1.$$

Hence

$$\left\|\frac{d}{dt} M\left(u(t)\right)-M\left(u_1(t)\right)\right\| \leq k\left(\|u(t)\|_W+\|u_1(t)\|_W+\right.$$

$$\left.+\left\|\frac{du}{dt}(t)\right\|_W+\left\|\frac{du_1}{dt}(t)\right\|_W\right)\left\{\|u_1(t)-u(t)\|+\|u_1'(t)-u'(t)\|\right\}.$$

Thus the proof of Theorem 3 is complete.

For the case where A is the Laplace operator on E^3, $q=1$, and the hypotheses of Theorem 3 are satisfied by

$$F'(r)=kr^2+k_1 r, \qquad (k, k_1 \geqq 0).$$

Section 3. We finish our discussion with the following remarks. The assumption on F''' in Theorem 3 which was not made by JÖRGENS in [2] can be eliminated in the present discussion if one is willing to consider weak solutions rather than strict solutions. More precisely one can prove the existence of weak solutions with merely the Assumptions (I) and (II) on the operator M in Theorem 1 and 2. We have avoided the detailed discussion of this possibility in the earlier part of this paper in order to avoid the complication of discussing the concept of weak solution. Let us outline this concept briefly at this point and indicate the character of the existence proof under the weaker hypotheses without (III) and (IV).

Let X' be the complete metric space of functions u from $[0, T]$ to H which are once continuously differentiable into H and continuous into W with $u(0)=\varphi\in D(A^{\frac{1}{2}})$, $\psi\in H$, and with the metric defined in X' by $\varrho(u, u_1)=\|u-u_1\|_{X'}$, where

$$\|u\|_{X'}=\sup_{0\leqq t\leqq T}\left\{\|u(t)\|_W+\left\|\frac{du}{dt}\right\|\right\}.$$

We may define a mapping S of X' into itself by analogy with the definition given for S on X by letting $u=Sv$ be the solution of

$$u''(t)+Au(t)=-M\left(v(t)\right), \qquad v\in X', \tag{3.1}$$

with $u(0)=\varphi$, $u'(0)=\psi$. In this case, the solution u of equation (3.1) is a weak solution since $u''(t)$ will not exist in general. We may define it without ambiguity by setting

$$u(t)=\cos\left(A^{\frac{1}{2}}t\right)\varphi+\sin\left(A^{\frac{1}{2}}t\right)\left(A^{-\frac{1}{2}}\right)+\int_0^t \sin\left(A^{\frac{1}{2}}(t-s)\right)\left(A^{-\frac{1}{2}}f\right)(s)\,ds. \tag{3.2}$$

Analogous of Lemma 1, 2, 4, 5, and 6 may be established for weak solutions under Assumptions (I) and (II) for φ in $D(A^{\frac{1}{2}})$, $\psi\in H$. Corresponding versions of Theorems 1 and 2 then follow.

Finally let us note that since only "energy" estimates are basic to the argument, self-adjointness is not necessary for A with suitable modifications of the definitions (along lines described in [3]). As we have noted before, we may also specialize to self-adjoint realizations A of strongly elliptic operators under general self-adjoint boundary conditions on smooth open subsets G of E^n.

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