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Measure preserving functions on locally compact spaces

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1. Let M and N be locally compact metric spaces with Borel measures (or completions of Borel measures) μ and ν, respectively. A function $f: M \to N$ will be called *mearure preserving* if:

(1) f is measurable, i.e., for each measurable set Y in N, $f^{-1}(Y)$ is a measurable set; and

(2) whenever X and $f(X)$ are each measurable, then $\mu(X) = \nu(f(X))$.

This note is concerned with the nature of measure-preserving functions, and particularly the extent to which they differ from homeomorphisms. If $f: M \to N$ is continuous, M is separable, and μ and ν are Borel, then (1) is necessarily satisfied. For one-to-one functions, this definition agrees with the usual definition of a measure preserving transformation [*4*; 164]. Otherwise, we shall be consistent with the terminology of [*4*].

The set $\{p \in M: f^{-1}(f(p)) \neq \{p\}\}$ will be called the *singular set* of f and will be denoted by F.

1.1. THEOREM. *There exists a continuous Lebesgue measure preserving map f of the closed unit square onto itself which is not a homeomorphism.*

PROOF. The map of the square $|x| \leqq 1$, $|y| \leqq 1$ onto the square $|u \pm v| \leqq 2^{\frac{1}{2}}$ is given by

1.2.
$$u = \sqrt{2}\left(1 - [1 - |x|]^{\frac{1}{2}}\right) \cdot \operatorname{sign} x,$$
$$v = \sqrt{2}[1 - |x|]^{\frac{1}{2}} \cdot y.$$

Since the Jacobian determinant is identically 1 except for $x = 0$, 1, and -1, and since these segments have measure zero, the map is measure preserving.

The authors are indebted to the referee for this example, which replaces a geometric construction. The map 1.2 is a homeomorphism on the *open* square, while the original example is also a counter-example for that case. In addition, the original example can be extended to a map which is not a local homeomorphism at any point. Since its construction is quite complicated and technical, we omit it.

Theorem 1.1 answers a question D. G. BOURGIN asked in conversation.

2. The rest of this note discusses measure preserving functions f from one locally compact metric space to another. Some examples are given to

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show that several plausible extensions of these results fail. The next result is the basis for most of the remainder of the note.

2.1. THEOREM. *Let M and N be separable, locally compact metric spaces with Borel measures μ and ν, respectively, and let $f\colon M\to N$ be continuous and measure preserving. Then the singular set $F=\{p\in M: f^{-1}(f(p))\neq\{p\}\}$ is a Borel set, $\mu(F)=0$, and the restriction $f|(M-F)$ is (of course) one-to-one. If M is compact, then $f|(M-F)$ is a homeomorphism onto its image.*

A slightly weaker version of this theorem follows directly from [2; 343]: take $M=G$, $N=Z$, $G_\zeta=f^{-1}(\zeta)$, $Z'=f(M)$, and use the third paragraph of the following proof. In general, however, the set of measure zero thus found properly contains F. The other version given in (3.1) does not seem to follow from [2; 343], except as it follows from this theorem.

PROOF. First, suppose that M is compact. For each natural number n, let $\{U_{n,j}: j=1, 2, \ldots, k_n\}$ be a finite cover of M by open sets of diameter less than $1/n$. Let $V_{n,j}=U_{n,j}-\bigcup_{i=1}^{j-1} U_{n,i}$ ($j=1, 2, \ldots, k_n$; $n=1, 2, \ldots$), and let $F_{n,i,j}$ be the Borel sets $f(V_{n,i})\cap f(V_{n,j})$. If for $j_1\neq j_2$ $\nu(V_{n,j_1,j_2})=a>0$, then $\mu(f^{-1}(F_{n,j_1,j_2})\cap V_{n,j_i})=a$ also ($i=1, 2$); since these two sets are disjoint, the measure preserving property of f is contradicted. Let F^* be the union of $\{f^{-1}(F_{n,j_1,j_2}): j_1, j_2=1, 2, \ldots, k_n;\ j_1\neq j_2;\ n=1, 2, \ldots\}$; then $\mu(F^*)=0$.

Suppose that $p, q\in M$, and that $f(p)=f(q)$. There exists n such that the distance $\varrho(p,q)<\frac{1}{n}$, and there exist j_1 and j_2 ($j_1, j_2=1, 2, \ldots, k_n$) such that $p\in V_{n,j_1}$ and $q\in V_{n,j_2}$. Therefore, $p, q\in f^{-1}(F_{n,j_1 j_2})$, which is contained in F^*; thus $F=F^*$. By definition of F, f is one-to-one on $M-F$.

To prove that the inverse function g (on $f(M)-f(F)$) is continuous, we use the sequential definition of continuity. Let y_n and y be points of $f(M)-f(F)$ ($n=1, 2, \ldots$), and let $y_n\to y$. If $g(y_n)$ does not converge to $g(y)$, there exists an $\varepsilon>0$ and a subsequence $\{n_m\}$ such that $\varrho(g(y_{n_m}), g(y))\geqq\varepsilon$. This subsequence has a subsequence converging to a point y^* in M, and by the continuity of f, $f(y^*)=y$. This contradicts the fact that f is one-to-one on $M-F$.

If M is not compact, let U_n be a basis for the topology of M such that $\overline{U}_n$ is compact ($n=1, 2, \ldots$). Let $M_n=\bigcup_{i=1}^{n}\overline{U}_i$, and let $f_n\colon M_n\to f(M_n)$ be the restriction of f. By (2.1) $\mu(F_n)=0$, where $F_n=\{p\in M_n: f_n^{-1}(f_n(p))\neq\{p\}\}$. Since each pair of points p and q is contained in some M_n, $F=\bigcup_{n=1}^{\infty}F_n$ and $\mu(F)=0$.

2.2. REMARKS. The restriction of f to $M-F$ need not be a homeomorphism onto $f(M-F)$. Indeed, using techniques similar to those of (1.1), one may construct a continuous function f of the closed unit square onto itself such that the image of the boundary is the entire square, but the restriction of f to the open square M preserves measure. If the restriction of f to $M-F$ were a homeomorphism (into), this fact would contradict [3; 92, 6.11], where $E=Y=\overline{M}$ and $\vartheta=f$.

If M and N are $I=[0, 1]$, then f is a homeomorphism; if μ and ν are (the Borel measure which generates) Lebesgue measure, then $f(x)=x$ or $1-x$. For an interesting general result in this direction, see [*1*].

It is known that there exist one-to-one measure preserving functions of I^m onto I^n for all m and n. The authors do not know, in general, if there exist continuous, measure preserving (not one-to-one) functions of I^m onto I^n for $m \neq n$; the answer is negative for $n=1$, however.

2.3. THEOREM. *Under the hypotheses of* (2.1), *if* μ *and* ν *are positive on open sets, then* F *and* $f(F)$ *are* F_σ *sets of the first category.*

PROOF. First, suppose that M is compact. Let $F_n=\left\{x\in N : \operatorname{diam}(f^{-1}(x)) \geqq \frac{1}{n}\right\}$ $(n=1, 2, \ldots)$; we now show that F_n is closed. Let x_k be distinct points in F_n, x_k converging to x_0 and $x_0 \in N$. There exist sequences y_k and z_k such that $f(y_k)=f(z_k)=x_k$ and the distance $\varrho(y_k, z_k) \geqq \frac{1}{n}$. Let y_0 and z_0 be limit points (in M) of these two sequences; then $\varrho(y_0, z_0) \geqq \frac{1}{n}$, and $f(y_0)=f(z_0)=x_0$. Thus $x_0 \in F_n$, and F_n is closed.

Since μ is positive on open sets, and $f^{-1}(F_n) \subset F$, $f^{-1}(F_n)$ is nowhere dense in M; since $\nu(f(F))=0$, F_n is nowhere dense in N. Now $F=\bigcup_{n=1}^{\infty} f^{-1}(F_n)$, and thus F and $f(F)$ are F_σ sets of the first category. The extension to the locally compact case is done as in (2.1).

3. A slightly weaker statement results if f is not necessarily continuous.

3.1. THEOREM. *Let* M *and* N *be locally compact metric spaces with Borel measures* μ *and* ν, *respectively, and let* $f: M \to N$ *be measure preserving.*

(1) *If, for each compact subset* C *of* M, $f(C)$ *is separable, then* f *is one-to-one on* $M-E$, *where* E *is some set whose intersection with each Borel set has measure zero.*

(2) *If* f *is continuous, then* E *may be chosen as the singular set* F.

For the proof we need a generalization of Lusin's Theorem [*6*; 35, Theorem 1]:

3.2. THEOREM (SCHAERF). *Let* K *and* L *be locally compact metric spaces with Borel measures* μ *and* ν, *respectively. Let* L *be separable, and let* $f: K \to L$ *be measurable. If* $A \subset K$ *is a measurable set of finite measure, and* $\varepsilon > 0$ *is arbitrary, then there exists a compact set* $B \subset A$ *such that* $\mu(A-B) < \varepsilon$ *and* $f|B$ *(the restriction of* f *to* B*) is continuous.*

A family $\mathfrak{A}$ of open sets in N is called discrete [*5*; 127] if each point in N has a neighborhood which intersects at most one member of $\mathfrak{A}$. A family is σ-discrete if it can be written as a union of a countable number of discrete subfamilies. Since N is locally compact, the open sets with compact closures constitute a cover. By [*5*; 129] this cover has a σ-discrete refinement, call it $\mathfrak{A}=\bigcup_{n=1}^{\infty} \mathfrak{A}_n$, where each $\mathfrak{A}_n$ is a discrete subfamily.

For each U in $\mathfrak{A}$, let $V_U=f^{-1}(\overline{U})$; V_U is a Borel set with finite measure. Hence, by SCHAERF's Theorem, there exist compact sets $C_{m,U}$ $(m=1, 2, \ldots)$

such that $\mu(V_U - C_{m,U}) < \frac{1}{m}$ and $f | C_{m,U}$ is continuous: let $D_{m,U} = \bigcup_{k=1}^{m} C_{k,U}$. Then $f(D_{m,U}) \subset \overline{U}$, and $f | D_{m,U}$ satisfies the hypothesis of (2.1); thus there exists a set $F_{m,U}$ such that $\mu(F_{m,U}) = 0$ and the restriction $f | (D_{m,U} - F_{m,U})$ is one-to-one. Let

$$E_U = (\bigcup_{m=1}^{\infty} F_{m,U}) \cup (V_U - \bigcup_{m=1}^{\infty} D_{m,U}),$$

and let $E = \bigcup_{U \in \mathfrak{A}} E_U$.

Since f is one-to-one on each set $V_U - E_U$, and since $V_U = f^{-1}(\overline{U})$ $(U \in \mathfrak{A})$, $f | (M - E)$ is one-to-one.

Although each E_U is a Borel set, E, in general, is not Borel (measurable); we now show that the intersection of E with any Borel set B has measure zero. Let U in $\mathfrak{A}_n$ and m $(m \neq n;\ m = 1, 2, \ldots)$ be given. Suppose that there exists a sequence of open sets U_j in $\mathfrak{A}_m$ such that $\overline{U}_j$ meets $\overline{U}$, say in p_j. Then the points p_j are distinct and have a limit point p in $\overline{U}$, contradicting the definition of $\mathfrak{A}_m$. Thus for each m $(m = 1, 2, \ldots)$ only a finite number of sets $U_{m,j}$ $(j = 1, 2, \ldots, k_m)$ of $\mathfrak{A}_m$ meet $\overline{U}$. Since

$$V_U \cap E = V_U \cap (\bigcup_{m=1}^{\infty} \bigcup_{j=1}^{k_m} E_{U_{m,j}}),$$

it is a Borel set; since $\mu(E_U) = 0$ for each U in $\mathfrak{A}$, $\mu(V_U \cap E) = 0$.

It now suffices to prove that each Borel set B in M can be covered by a countable union of sets V_U $(U \in \mathfrak{A})$, together with a set of measure zero. The set B can be covered by a countable union of compact sets C_n; since $f(C)$ is separable, it is contained in a Borel set. By SCHAERF's Theorem, there exist compact sets $C_{n,m} \subset C_n$ such that $\mu(C_n - C_{n,m}) < \frac{1}{m}$ and $f | C_{n,m}$ is continuous $(m, n = 1, 2, \ldots)$. Since $f(C_{n,m})$ is a compact subset of N, it can be covered by a finite number of sets $U_{n,m,i}$ of $\mathfrak{A}$. Then $\bigcup_{n,m,i} V_{U_{n,m,i}}$ covers B, except possibly for $\bigcup_{n=1}^{\infty} (C_n - \bigcup_{m=1}^{\infty} C_{n,m})$, which has μ measure zero.

If f is continuous, we proceed as above, defining the σ-discrete family $\mathfrak{A}$ of subsets of N and the family V_U $(U \in \mathfrak{A})$. Since each V_U is a Borel set, it is contained in the countable union of compact sets and is thus separable. Hence $f | V_U$ satisfies (2.1); let the associated set be designated by F_U, and let $F = \bigcup_{U \in \mathfrak{A}} F_U$. The properties of F_U and F follow from simplifications of the arguments about E_U and E.

4. EXAMPLE. The preceding results can not be extended to non-metric compact sets; let $M = N = \prod_{\alpha \in \Omega} I^2$, an uncountable direct product of unit squares. Let μ and ν be the product measures generated on all Baire sets by Lebesgue measure in each factor. Let $\boldsymbol{f}\colon M \to N$ be defined on each factor as the function $f\colon I^2 \to I^2$ given by Theorem 1.1. Then $\boldsymbol{f}$ is a continuous measure preserving function from M onto N. Let F_α be the α-cylinder consisting of F (the singular set of f) in the α-th coordinate. Then $\boldsymbol{F} = \bigcup_{\alpha \in \Omega} F_\alpha$ consists of all the points where $\boldsymbol{f}$ is not one-to-one. We will show that $\boldsymbol{F}$ is not measurable in the completion of this Baire measure; indeed, that its only Baire measurable cover is M, and that its only Baire measurable kernels are sets of measure zero. Let A be a Baire measurable cover; then there exists a countable set J of indices such that A is a J-cylinder. Since F_α, for $\alpha \in J$, is an α-cylinder contained in A, $A = M$. Let B be a Baire measurable kernel. If $B \neq \delta$, it must be a J-cylinder for some countable set J. Thus $B \subset \bigcup_{\alpha \in J} F_\alpha$, which has measure zero.

5. It is natural to ask: under what additional hypothesis will a measure preserving function be one-to-one?

5.1. THEOREM. *Let M and N be compact metric spaces with metrics ϱ and $\bar{\varrho}$ and with Borel measures μ and ν, respectively. Suppose that μ and ν are positive on open sets, and that $f: M \to N$ is measure preserving. Then f is a homeomorphism into if and only if there exists a metric ϱ^* equivalent to ϱ on M such that, for all m in M and Borel sets B such that $f(B)$ is also Borel,*

$$\int_B \varrho^*(m, x)\, d\mu(x) = \int_{f(B)} \bar{\varrho}(f(m), y)\, d\nu(y).$$

Two metrics are equivalent if they define the same open sets. A stronger converse can be stated: If M is locally compact and separable, and if f is measure preserving, then the integral condition implies that f is one-to-one.

Note the similarity of these integrals to moments of inertia.

PROOF. If f is a homeomorphism (into), let $\varrho^*(a, b) = \varrho(f(a), f(b))$; ϱ^* has the desired properties.

Conversely, suppose that the integral condition is satisfied and that there exist two points m_1 and m_2 such that $f(m_1) = f(m_2)$; let $A = \frac{1}{2}\varrho^*(m_1, m_2)$. Since μ is finite (on compact sets) and positive on open sets, there exists an open neighborhood V of m_1 such that $A' = \varrho^* - \operatorname{diam} V < A$ and $0 < \mu(V_i) < \infty$ Let W be an open set about $f(m_1)$ having diameter less that A', and having finite, but positive, measure. Let $W_j = V \cap f^{-1}(W)$, and let $W_2 = f^{-1}(W) - V$. Then W_1 and W_2 are both Borel sets, and at least one has positive measure. If $\mu(W_i) > 0$, let C_i be a compact subset given by SCHAERF's Theorem such that $f|C_i$ is continuous and $\mu(C_i) > 0$. Suppose $\mu(W_1) > 0$. Then

$$\int_{C_1} \varrho^*(m_2, x)\, d\mu(x) \geqq A'\mu(C_1)$$

and

$$\int_{f(C_1)} \bar{\varrho}(f(m_2), y)\, d\nu(y) < A'\nu(f(C_1)),$$

contradicting the hypothesis. If $\mu(W_2) > 0$, a contradiction results from an interchange of the indices 1 and 2. Thus, f is one-to-one.

Suppose that f is not continuous; then there exist sequences $m_{i,k} \to m_0$ such that $f(m_{i,k}) \to n_i$ $(i = 1, 2)$ and $n_1 \neq n_2$. Let $A = \frac{1}{2}\bar{\varrho}(n_1, n_2)$, and let W be an open neighborhood of n_1 with $\operatorname{diam} W < A$. Let C be a compact set, given by SCHAERF's Theorem, such that $C \subset f^{-1}(W)$, $\mu(C) > 0$, and $f|C$ is continuous. Since

$$\int_C \varrho^*(m_{i,k}, x)\, d\mu(x) = \int_{f(C)} \bar{\varrho}(f(m_{i,k}), y)\, d\nu(y)$$

$(i=1, 2;\ k=1, 2, \ldots)$ by hypothesis,

$$\int_C \varrho^*(m_0, x)\, d\mu(x) = \int_{f(C)} \bar{\varrho}(n_i, y)\, d\nu(y) = I_i.$$

Thus $I_1 < A \cdot \nu(f(C))$ and $A \cdot \nu(f(C)) < I_2$, and a contradiction results.

6. Some remarks. Let M and N be open subsets of E^u, and let $f: M \to N$ be Lebesgue measure preserving and continuous.

(1) If f is open, then it is a homeomorphism. To prove this statement, it is sufficient to prove that f is one-to-one. Suppose $f(p) = f(q)$, for distinct p and q. Let U and V be disjoint open sets of finite measure about p and q, respectively. Then $f(U) \cap f(V)$ is an open set with the same measure as $U \cap f^{-1}(f(U) \cap f(V))$ and $V \cap f^{-1}(f(U) \cap f(V))$, contradicting the hypothesis.

(2) It is well-known that if f has continuous first partial derivatives, then f is a homeomorphism (into) with Jacobian J_f identically 1 or -1. In particular, if M and N are the open unit disk in the complex plane, and f is analytic, then f is a rotation. (By the previous remark, $J_f \equiv 1$; but $J_f(z) = |f'(z)|^2$. Since a non-constant analytic map is open, $f'(z) \equiv e^{i\theta}$).

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