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Systems of ideals in partially ordered semigroups

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JAFFARD ([3], Chapt. 1, Sect. 3) has shown that a directed partially ordered group may be imbedded, by means of an ideal extension, in a complete lattice semigroup. In this paper we show how this procedure may be generalised to a wide class of partially ordered semigroups, with, in addition, preservation of existing least upper bounds. In particular, we give a construction which is applicable to a class of semigroups which includes that of all residuated semigroups. Further, we obtain necessary and sufficient conditions under which a partially ordered semigroup may be imbedded in a conditionally complete group.

We introduce in Section 1 the terminology used in the paper, and in the next Section the notion of an ideal extension, in terms of which we find necessary and sufficient conditions that a partially ordered semigroup may be imbedded, with preservation of least upper bounds, in a (conditionally) complete lattice semigroup. It is shown that not every partially ordered semigroup may be so imbedded. We describe the result mentioned above for residuated semigroups, and complete the Section by proving some general results on ideal extensions. In Section 3 relations between different ideal systems are considered, and in the final Section we establish necessary and sufficient conditions under which a partially ordered semigroup may be imbedded in a conditionally complete group.

Section 1. A *partially ordered set* is a set A in which is defined a reflexive, anti-symmetric and transitive relation $\leqq$. Given $a \in A$, the set of all x in A such that $x \leqq a$ will be denoted by $(a)_A$, or simply by (a) if no confusion can arise, and will be called the *principal ideal* generated by a in A.

A *partially ordered semigroup* is a semigroup S (written multiplicatively) in which is defined a partial order $\leqq$ such that for any a, b, x in S,

$$a \leqq b \quad \text{implies} \quad a x \leqq b x \quad \text{and} \quad x a \leqq x b;$$

S need not have an identity element. The *zero* of S is the (unique) element z of S such that: $zx = xz = z \leqq x$, all x in S. The semigroup S is directed $\uparrow$ if for any two elements a and b of S there exists y in S such that $a \leqq y$ and $b \leqq y$. A *residuated semigroup* is a partially ordered semigroup S in which there exist for any couple of elements a and b of S left and right residuals $a \cdot\!. b$ and $a .\!\cdot b$, the greatest elements x and y of S such that, respectively, $x b \leqq a$ and $b y \leqq a$; if $a \cdot\!. b = a .\!\cdot b$, we write $a \cdot\!. b = a .\!\cdot b = a : b$.

A (resp. conditionally) complete lattice semigroup is a (resp. conditionally) complete lattice S in which is defined multiplication such that:

$$a\{\vee X\} = \vee a X \quad \text{and} \quad \{\vee X\} a = \vee X a$$

(wherever the expressions used have a meaning) using $\vee X$ to denote the least upper bound of a subset X of S. To avoid confusion, we write the least upper bound of elements a and b as $a \vee b$, and the set union of sets A and B as $A \cup B$.

A (resp. conditionally) complete semilattice semigroup is a (resp. conditionally) complete $\vee$-semilattice in which multiplication satisfies the above distributive laws (wherever the expressions used have a meaning). A sub-(resp. conditionally) complete lattice semigroup of a (resp. conditionally) complete lattice semigroup S is a subset of S which is also a (resp. conditionally) complete lattice semigroup under the same operations of $\vee$, $\wedge$ and multiplication as S; sub(resp. conditionally) complete semilattice semigroups are defined in an obvious way.

A ($\vee$-complete) homomorphism of a partially ordered semigroup S onto a partially ordered semigroup S' is a mapping ϑ of S onto S' such that:

$$[x y]\vartheta = [x]\vartheta[y]\vartheta \quad \text{and} \quad [\vee X]\vartheta = \vee[X]\vartheta$$

whenever $\vee X$ exists. An isomorphism is a homomorphism ϑ' of S onto S' such that:

$$x \leqq y \quad \text{if and only if} \quad [x]\vartheta' \leqq [y]\vartheta'.$$

Section 2. Let S be a (resp. directed $\uparrow$) partially ordered semigroup in which is defined a mapping $X \to X_r$ of the (resp. non-null, bounded above) subsets of S into the set $\{X_r\}$ of subsets of S which satisfies:

C.1. $X \subset X_r$.

C.2. $X \subset Y_r$ implies $X_r \subset Y_r$.

C.3. For each a in S, $\{a\}_r = (a)$.

C.4. For any a in S, $a X_r \subset (a X)_r$ and $X_r a \subset (X a)_r$.

C.5. If $Y \subset X_r$ and $\vee Y$ exists, $\vee Y \in X_r$.

By C.1. and C.2., $X \to X_r$ is a closure operation on the (resp. non-null, bounded above) subsets of S, and if we call $X \subset S$ an r-closed ideal of S if and only if $X = X_r$, the set (resp. $C'_r(S)$) $C_r(S)$ of all r-closed ideals of S forms a (resp. conditionally) complete (resp. $\vee$-semi-)lattice, partially ordered by inclusion, which we shall call the complete (resp. restricted) ideal system, with, for i in any index set I:

$$\vee X_r^i = (\cup X^i)_r \text{ and in the complete case, } \wedge X_r^i = \cap X_r^i.$$

In the complete case, to the empty set $\emptyset$ corresponds (z) where z is the zero of S, if such exists, and $\emptyset$ itself otherwise.

We may define in (resp. $C'_r(S)$) $C_r(S)$ a law of composition $\circ_r$, or simply $\circ$, if no confusion can arise, by:

$$X_r \circ Y_r = (X\,Y)_r \text{ if } X \neq \emptyset \neq Y, \text{ and by writing, for any subset } W \text{ of } S,$$
$$\emptyset\, W = W\, \emptyset = \emptyset,$$

the empty set is included in this definition.

It is easily seen that if X' and Y' are subsets of S such that $(X')_r = X_r$ and $(Y')_r = Y_r$, then $X_r \circ Y_r = (X'\,Y')_r$. This law of composition is both associative and distributive with respect to $\vee$ as defined above (the verification follows that of [3] Chapt. 1, Sect. 3 and is omitted), and so (resp. $C'_r(S)$) $C_r(S)$ is a (resp. conditionally) complete (resp. semi-)lattice semigroup with, in the complete case, a zero (z) or $\emptyset$ according as S has or has not a zero z. We call (resp. $C'_r(S)$) $C_r(S)$ a (resp. restricted) complete ideal extension of S. Note that in $C_r(S)$, multiplication need not be distributive with respect to intersection; in Example 2 below, $(a) \circ \{(m) \wedge (n)\} = (a)$ while

$$\{(a) \circ (m)\} \wedge \{(a) \circ (n)\} = S \cap S = S.$$

We prove Theorems 1 to 6 for complete r-systems; they, and their corollaries, hold, mutatis mutandis, for restricted systems.

THEOREM 1. *A necessary and sufficient condition that a partially ordered semigroup S may be imbedded in a complete lattice semigroup G, with preservation of existing least upper bounds, is that there can be defined on S an r-ideal system $C_r(S)$. Further, S may be embedded with preservation of existing least upper bounds, greatest lower bounds, and residuals.*

Proof. Let $C_r(S)$ be such a system; we have seen that it forms a complete lattice semigroup. Define the mapping ϑ: $S \to C_r(S)$ by: $[x]\vartheta = (x)$; then ϑ is an isomorphism of S onto the set of principal ideals of S, (a subsemigroup of $C_r(S)$) and preserves existing least upper bounds, etc. For if in S, $a = \wedge x_i$, $i \in I$, then $a \leqq x_i$, all $i \in I$, $(a) \subset (x_i)$ and $(a) \subset \cap (x_i)$; on the other hand, $Y_r \subset \cap (x_i)$, all $i \in I$, implies that every y in Y_r is in each (x_i), whence $y \leqq a$ and $Y_r \subset (a)$. If $b = \vee x_i$, $(b) \supset (x_i)$, all $i \in I$ implies that $(b) \supset (\cup (x_i))_r$, $i \in I$ by C.2., while if $Y_r \supset (\cup (x_i))_r$, all $i \in I$, Y_r contains (b) by C.5. and C.2. Finally, if $c = a \cdot\!\!\cdot b$, $(b) \circ (c) \subset (a)$, while $(b) \circ X_r \subset (a)$ implies that for any x in X_r, $b\,x \in (a)$ whence $x \leqq a \cdot\!\!\cdot b$ and $X_r \subset (c)$.

Conversely, if S is imbedded as above in a complete lattice semigroup G, define, for any subset X of S, $X_r = (a)_G \cap S$, where $a = \vee X$ in G. Conditions C.1., C.2., C.3. and C.5. follow immediately, C.4. on noting that in G, $\vee c X = c\{\vee X\}$.

Note that in any ideal system, restricted or complete,

$$X_r = \big(\bigcup_{x \in X} (x)\big)_r = \vee X \quad \text{in (resp. } C'_r(S)) \; C_r(S).$$

Not every partially ordered semigroup may be imbedded as in Theorem 1; consider the semigroup S whose diagram and multiplication table are:

EXAMPLE 1.

	a	b	c	z
a	a	z	z	z
b	z	z	z	z
c	z	z	z	z
z	z	z	z	z

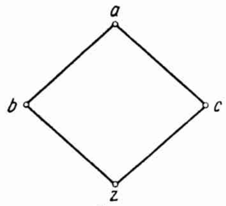

Fig. 1

If there existed G as in Theorem 1, we could define the complete lattice semigroup $C_r(S)$; however, $(a)\circ\{(b)\vee(c)\}=(a)\circ(a)=(a)$, while

$$\{(a)\circ(b)\}\vee\{(a)\circ(c)\}=(z)\vee(z)=(z).$$

Although any (resp. directed $\uparrow$) partially ordered semigroup S may be imbedded in a (resp. conditionally) complete (resp. semi) lattice semigroup if it is not required that existing least upper bounds be preserved, (for (resp. non-null, bounded above) $X\subset S$, let $X_r=\cup(x)$, for x in X; clearly C.1. ... C.4. are satisfied, but not necessarily C.5.) we have the following result.

THEOREM 2. *If a partially ordered semigroup S may be imbedded in a complete lattice semigroup S' as in Theorem 1, it is necessary that if X is a subset of S such that $\vee X$ exists, then for any element a of S, $\vee aX$ and $\vee Xa$ both exist, with:*

$$\vee aX=a\{\vee X\} \quad \textit{and} \quad \vee Xa=\{\vee X\}a.$$

Proof. Let $y=\vee X$; then, with an obvious notation, $y'=\vee X'$ in S', and $a'y'=a'\{\vee X'\}=\vee a'X'$. Since least upper bounds are preserved, and $a'X'$ is the image of aX, it follows that $ay=\vee aX$. Similarly for ya.

COROLLARY 2.1. *If S is a finite semigroup, the condition of Theorem 2 is both necessary and sufficient that S may be imbedded as in Theorem 1 in a complete lattice semigroup.*

Proof. We need only prove sufficiency. For any subset X of S, define $X_1=\cup(x_1)$ where $x_1=\vee Y$ for all subsets Y of X which have a least upper bound, $X_2=(X_1)_1$ etc. Since S is finite, there exists an integer n for which $X_n=X_{n+1}$; we define the closure X_r of X to be X_n for this value of n. Clearly $C_r(S)$ satisfies C.1. ($x=x\vee x$) C.2., C.3. and C.5.; C.4. follows by induction. If $y\in aX_1$, $y\in a(x_1)\subset(a x_1)$ where $x_1=\vee Y$ and $Y\subset X$; then $a x_1=\vee aY\in(aX)_1$ and $aX_1\subset(aX)_1$. Let $y\in aX_{k+1}$ where $aX_k\subset(aX)_k$; then $y\in(a x_{k+1})$ where $x_{k+1}=\vee Y$ and $Y\subset X_k$. Since $a x_{k+1}=\vee aY\in(aX_k)_1\subset((aX)_k)_1=(aX)_{k+1}$ we deduce that $aX_r\subset(aX)_r$ and similarly that $X_r a\subset(Xa)_r$, so completing the proof.

Consider now a (resp. directed $\uparrow$) residuated semigroup S; define the ideal system (resp. $C_v'(S)$) $C_v(S)$ by, for any (resp. non-null, bounded above) subset X of S,

$X_v=$ the set of all lower bounds of all upper bounds of X, if X is bounded,
$=S$ if X is unbounded (cf. BIRKHOFF [1], p. 58, DUBREIL-JACOTIN [2], p. 37).

An equivalent definition is: $X_v = \bigcap (y)$, $(y) \supset X$, if X is bounded, $X_v = S$ if X is unbounded ([3], p. 18).

Clearly $\{X_v\}$ satisfies C.1., C.2., C.3. and C.5.; for C.4., note that if X is unbounded, so is aX, for any a in S, since $y \geqq ax$, all x in X implies that $y .\cdot a$ is an upper bound of X. Further, if y is an upper bound of aX, $y .\cdot a$ is an upper bound of X, therefore of X_v and y is also an upper bound of aX_v. It follows that $(aX_v)_v = (aX)_v$ and similarly that $(X_v a)_v = (Xa)_v$.

Thus any (resp. directed ↑) residuated semigroup may be imbedded in a (resp. conditionally) complete (resp. semi) lattice semigroup with preservation of existing least upper bounds, greatest lower bounds and residuals. In particular, any (resp. directed ↑) partially ordered group may be so imbedded ([3], p. 22) since every partially ordered group is residuated with $a .\cdot b = b^{-1}a$ and $a \cdot. b = ab^{-1}$.

A (resp. directed ↑) partially ordered semigroup S for which the system (resp. $C_v'(S)$) $C_v(S)$ satisfies C.4. will be called (resp. *restrictedly*) *quasi-residuated*, or (resp. r.) q. r. Not every (resp. r.) q. r. semigroup is residuated; in Example 2 below, S is (resp. r.) q. r., the v-ideals being: S, (m), (n), (a), (b), (z) and $\{a, b, z\}$, but neither $a:a$ nor $a:b$ exists.

EXAMPLE 2.

	I	m	n	a	b	z
I	I	I	I	I	I	z
m	I	I	I	I	I	z
n	I	I	I	I	I	z
a	I	I	I	a	a	z
b	I	I	I	a	a	z
z	z	z	z	z	z	z

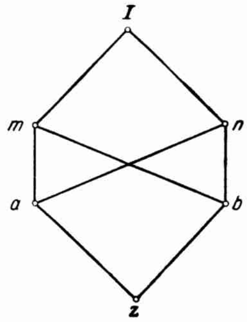

Fig. 2

Further, a (resp. directed ↑) partially ordered semigroup S may contain an ideal system without being (resp. r.) q. r.; in Example 3 define (resp. $C_r'(S)$) $C_r(S)$ as for finite semigroups. The r-ideals are: S, (m), (n), (a), (b), (c), (z), $\{a, b, z\}$, $\{b, c, z\}$, $\{a, c, z\}$, $\{a, b, c, z\}$, and the v-ideals: S, (m), (n), (a), (b), (c), (z), $\{a, b, c, z\}$. Although the r-system satisfies the conditions C.1. ... C.5., the v-system does not satisfy C.4., since if $X = \{a, b\}$, $X_v = \{a, b, c, z\}$ and $cX_v = \{a, c, z\}$ while $(cX)_v = (a)$.

EXAMPLE 3.

	I	m	n	a	b	c	z
I	I	I	I	I	I	I	z
m	I	I	I	I	I	I	z
n	I	I	I	I	I	I	z
a	I	I	I	a	a	a	z
b	I	I	I	a	a	a	z
c	I	I	I	a	a	c	z
z	z	z	z	z	z	z	z

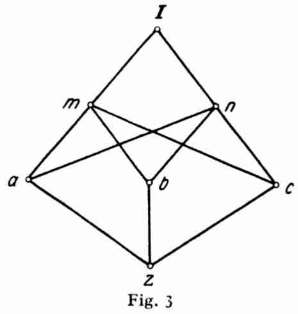

Fig. 3

A given isomorphism ϑ between two (resp. r.) q. r. semigroups S and S' may be extended uniquely to an isomorphism between (resp. $C'_v(S)$) $C_v(S)$ and (resp. $C'_v(S')$) $C_v(S')$. If A is any subset of S, define $A\vartheta'$ by $A\vartheta' = \{a\vartheta\}$, all a in A; noting that y is an upper bound of A if and only if $y' = y\vartheta$ is an upper bound of $A\vartheta'$, it is clear that ϑ' is a $(1-1)$ mapping of (resp. $C'_v(S)$) $C_v(S)$ onto (resp. $C'_v(S')$) $C_v(S')$. Further, $[X_v]\vartheta' = (X\vartheta')_v$ whence ϑ' is an isomorphism. Finally, ϑ' coincides with ϑ on the principal ideals of S, and is the only isomorphism to do so. (Note that $X_v = (\cup(x))_v$, $x\in X$.)

THEOREM 3. *Let S be a partially ordered semigroup which is imbedded as in Theorem 1 in a complete lattice semigroup G. Then if $C_r(S)$ is defined as in Theorem 1, the isomorphism of S into G may be extended uniquely to an isomorphism of $C_r(S)$ onto a sub complete lattice semigroup of G.*

Proof. Define the mapping: $\vartheta: C_r(S)\to G$ by $[X_r]\vartheta = \vee X$ in G; ϑ is single valued, since $\vee X = \vee X_r$. For any two subsets X and Y of G we have:

$$\vee XY = \vee\{\bigcup_{x\in X} xY\} = \bigvee_{x\in X}\{\vee xY\} = \bigvee_{x\in X}\{x\{\vee Y\}\} = \{\bigvee_{x\in X} x\}\{\vee Y\} = \{\vee X\}\{\vee Y\}.$$

It follows that ϑ is a homomorphism, since:

$$[X_r\circ Y_r]\vartheta = [(XY)_r]\vartheta = \vee XY = \{\vee X\}\{\vee Y\} = [X_r]\vartheta[Y_r]\vartheta,$$

and for $i\in I$,

$$[\vee X_r^i]\vartheta = \vee\{\cup X^i\} = \vee\{\vee X^i\} = \vee\{[X_r]\vartheta\},$$

and is $(1-1)$, for:

$$[X_r]\vartheta = [Y_r]\vartheta \text{ implies that } \vee X = \vee Y \text{ whence } X_r = Y_r.$$

Finally, ϑ is the only extension of the isomorphism of S into G, for if $x\to x'$, $[(x)]\vartheta = x'$, while if ψ is another such isomorphism,

$$[X_r]\psi = \bigvee_{x\in X}(x)\psi = \bigvee_{x'\in X'} x' = \vee X.$$

If a partially ordered semigroup S is isomorphic to a partially ordered subsemigroup of the partially ordered semigroup S', write $S \lesssim S'$.

COROLLARY 3.1. *If S is imbedded in G, S' in G', (notation as in Theorem 1), where S and S' are isomorphic and G' is isomorphic to G under an extension α of the isomorphism between S and S', then $C_r(S)$ is isomorphic to $C_{r'}(S')$.*

Proof. There exist isomorphisms $\vartheta: C_r(S)\to G$, $\alpha: G\to G'$ and $\psi: C_{r'}(S')\to G'$; the mapping $\vartheta\circ\alpha\circ\psi^{-1}$ is an isomorphism of $C_r(S)$ onto $C_{r'}(S')$.

COROLLARY 3.2. *If S is imbedded as in Theorem 1 in G and S' is a partially ordered subsemigroup of S, then S' may be imbedded as in Theorem 1 in a complete lattice semigroup $C_{r'}(S')$ such that $C_{r'}(S') \lesssim C_r(S)$.*

Proof. The argument is essentially that of JAFFARD ([3], p. 52) who shows that the system $C_{r'}(S')$ defined by $X_{r'} = X_r \cap S'$ satisfies C.1. ... C.4. (C.5. is obvious) and that the mapping $\vartheta: C_{r'}(S')\to C_r(S)$ given by $[X_{r'}]\vartheta = X_r$ is an isomorphism.

The converse is not necessarily true; the subsemigroup $\{b, c, z\}$ of the semigroup S of Example 1 is q. r., but S does not satisfy the condition of Theorem 2.

DEFINITION. *A (resp. conditionally) complete* $\vee$*-semilattice* G *is a (resp. restricted)* $\vee$*-completion of a subset* S *of* G *if, given* g *in* G, *there exists a (resp. non-null, bounded above) subset* X *of* S *such that* $g = \vee X$ *in* G.

An $\wedge$-completion is defined dually.

In Theorem 1, $C_r(S)$ is a $\vee$-completion of S since $X_r = (\cup (x))_r$, $x \in X$. (Identifying S with the set of principal ideals of S.)

LEMMA. *Let a conditionally complete* $\vee$*-semilattice* A *be a restricted* $\vee$*-completion of* $B \subset A$, *and write* $(a)' = B \cap (a)_A$ *for each* a *in* A. *Then* $(a)'$ *is non-null, bounded above in* B, *and, in* A, $a = \vee (a)'$. *In particular,* B *is directed* $\uparrow$.

Proof. There exists a subset X of B, bounded above in B, such that $a = \vee X$. Then $X \subset (a)'$, which is bounded above in B (by any upper bound of X) and: $a \leqq \vee (a)' \leqq a$. To show that B is directed $\uparrow$, take $a = b \vee c$ (in A) for given b and c in B.

NOTE. We may deduce that if a partially ordered semigroup is imbedded in any manner in a conditionally complete semilattice semigroup G which is a restricted $\vee$-completion of S, then residuals existing in S are preserved in G. For $ax \leqq b$ in G implies, since $x = \vee S \cap (x)_G$, that $x \leqq b \cdot\!\cdot a$ in S.

Clearly S has a restricted completion by ideals which is both $\vee$- and $\wedge$-complete if and only if S is directed $\downarrow$ and r. q. r.

THEOREM 4. *If a partially ordered semigroup* S *is imbedded as in Theorem 1 in a complete lattice semigroup* G *such that* G *is a* $\vee$*-completion of* S, *then* G *is isomorphic to an ideal extension of* S. *Conversely, (as remarked above) every ideal extension of* S *is a* $\vee$*-completion of* S.

Proof. Suppose that G is a $\vee$-completion of S, and that S is imbedded in G with preservation of least upper bounds; define $C_r(S)$ as in Theorem 1. Then the mapping $\vartheta\colon C_r(S) \to G$ of Theorem 3 is an isomorphism of the complete lattice semigroup $C_r(S)$ into G. That ϑ is onto G follows at once from the fact that if $g \in G$ there exists a subset X of S such that $g = \vee X = [X_r]\,\vartheta$.

Section 3. Let $C_r(S)$ and $C_l(S)$ be two (resp. restricted) complete ideal extensions of the (resp. directed $\uparrow$) partially ordered semigroup S such that the r-system is *finer* than the l-system (cf. JAFFARD [3], p. 18). That is, for any (resp. non-null, bounded above) subset X of S, $X_r \subset X_l$. Then $X \subset X_r \subset X_l$ implies that $X_l \subset (X_r)_l \subset (X_l)_l = X_l$ whence $(X_r)_l = X_l$, and similarly, $(X_l)_r = X_l$. Thus every l-ideal is an r-ideal and $C_l(S)$ is a subset of $C_r(S)$.

THEOREM 5. *If the r-system is finer than the l-system,* $C_l(S)$ *is the homomorphic image of* $C_r(S)$ *under the only homomorphism which extends the identity mapping of* S *onto itself* (JAFFARD [3], p. 22).

Proof. The homomorphism ϑ is defined by $[X_r]\vartheta = X_l$. If ψ is another such homomorphism,

$$[X_r]\,\psi = [\vee X]\,\psi = \bigvee_{x\in X}(x)\,\psi \quad = \vee X = X_l.$$

If ϑ is $(1-1)$ (or equivalently, if $X_r = Y_r$ if and only if $X_l = Y_l$), the systems coincide, for $X_l = (X_l)_l$ implies $X_r = (X_l)_r = X_l$.

Define the binary relation $\mathcal{R}$ on $C_r(S)$ by:

$$X_r \equiv Y_r(\mathcal{R}) \quad \text{if and only if} \quad X_l = Y_l.$$

Then $\mathcal{R}$ is a congruence relation on $C_r(S)$, and $C_l(S) \cong C_r(S)/\mathcal{R}$. If $\mathcal{R}$ is the identity relation, $C_r(S) = C_l(S)$. (See above.) Let χ be the $\mathcal{R}$ class of X_r; since $(X_l)_l = (X_r)_l$, $X_l \in \chi$. Further, if $Y_r \in \chi$, $Y_r < (Y_r)_l = Y_l = X_l$, whence $Y_r < X_l$ and each X_l is maximum in its $\mathcal{R}$-class. Thus, defining $A_r .\!\cdot B_r (A_r \cdot\!. B_r)$ as the maximum Z_r such that $B_r \circ Z_r < A_r$, $(Z_r \circ B_r < A_r)$ we have, in the complete case,

$$X \equiv Y(\mathcal{R}) \quad \text{if and only if for any} \quad W_l \in C_l(S), \quad W_l .\!\cdot X_r = W_l .\!\cdot Y_r,$$

$(W_l \cdot\!. X_r = W_l \cdot\!. Y_r)$ ([*5*], Theorem 2). Clearly this holds also in $C'_r(S)$ if $C'_r(S)$ is residuated.

Let S be a (resp. directed $\uparrow$) partially ordered semigroup which may be imbedded as in Theorem 1 in a (resp. conditionally) complete (resp. semi-) lattice semigroup. Then there exists at least one ideal system satisfying C.1. ... C.5.; define the system (resp. $C'_f(S)$) $C_f(S)$ by, for any (resp. non-null, bounded above) subset X of S, $X_f = \cap X_r$ the intersection being over all the ideal extensions (resp. $C'_r(S)$) $C_r(S)$. It is easily verified that (resp. $C'_f(S)$) $C_f(S)$ satisfies C.1. ... C.5.

COROLLARY 5.1. *The system $C_f(S)$ is maximal in the sense that any other ideal extension is the homomorphic image as in Theorem 5 of $C_f(S)$.*

Proof. The result follows immediately from Theorem 5 and the fact that $C_f(S)$ is the finest ideal system.

DEFINITION. *A $\vee$-ideal of a partially ordered set S is a subset I of S such that:*

(i) *$x \in I$ and $t \leqq x$ imply that $t \in I$.*

(ii) *If $Y < I$ and $\vee Y$ exists, then $\vee Y \in I$.*

(Cf. BIRKHOFF [*1*], p. 21.)

In a (resp. directed $\uparrow$) residuated semigroup S define (resp. $C'_{f'}(S)$) $C_{f'}(S)$ by, for any (resp. non-null, bounded above) subset X of S, $X_{f'} = \cap I$, the intersection being over all the $\vee$-ideals of S which contain X. (To each X there corresponds at least one such I, namely, S.)

Clearly (resp. $C'_{f'}(S)$) $C_{f'}(S)$ satisfies C.1., C.2., C.3. and C.5.; suppose that W is a $\vee$-ideal containing aX, where a is any element of S. The set of all $\{(w .\!\cdot a)\}$, where $w \in W$, clearly contains X and satisfies (i); let $b = \vee Y$ where $Y < \{(w .\!\cdot a)\}$. Each $y \in Y$ is in some $(w .\!\cdot a)$, whence $ay \in W$, so that

$ab = \vee a Y$, which exists by Theorem 2, and belongs to W. It follows that $b \in \{(w .\cdot a)\}$, which is therefore a $\vee$-ideal containing X; thus $X_{f'} < \{(w .\cdot a)\}$, $a X_{f'} < W$ and $a X_{f'} < (aX)_{f'}$. Similarly $X_{f'} a < (Xa)_{f'}$; since $X_{f'} < X_f < X_{f'}$, (resp. $C'_{f'}(S)$) $C_{f'}(S)$ is an alternative definition of the finest ideal system.

COROLLARY 5.2. *In a q. r. semigroup S the system $C_v(S)$ is minimal in the sense that if S is imbedded as in Theorem 1 in a complete lattice semigroup G, then $C_v(S)$ is the homomorphic image, as in Theorem 5, of a subcomplete lattice semigroup of G* (cf. LORENZEN [4]).

Proof. The result follows from Theorems 3 and 5 on noting that $C_v(S)$ is the least fine ideal extension.

THEOREM 6. *Let G be a complete lattice semigroup which is a $\vee$-completion of the partially ordered semigroup S in which S is imbedded as in Theorem 1. If S is similarly imbedded in a complete lattice semigroup G' where $G' \lesssim G$ under an isomorphism which preserves the identity mapping of S onto itself, then $C_{r'}(S) = C_r(S) \cong G$.*

Proof. By Theorem 3 and 4, with the usual notation, we have:

$$S \lesssim C_{r'}(S) \lesssim G' \lesssim G \cong C_r(S),$$

so there exists an isomorphism of $C_{r'}(S)$ into $C_r(S)$ which preserves the identity mapping of S onto itself; as in Theorem 5, the only such isomorphism is the identity mapping, whence the result.

It follows that G is minimal in an obvious sense.

Section 4. A conditionally complete $\vee$-semilattice group G is clearly a conditionally complete lattice group, where, for example, $x \wedge y = (x^{-1} \vee y^{-1})^{-1}$, for any x and y in G. Such a group will be called a *cl*-group.

The set $P(S)$ of all subsets of an arbitrary semigroup S, partially ordered by inclusion, and with the obvious binary operation, is a residuated semigroup where, for X and Y in $P(S)$, $X \cdot . Y$ ($X .\cdot Y$) is the set of all elements x in S such that $xY < X$ ($Yx < X$), or $\emptyset$ if this set is empty. If S is itself partially ordered, and has identity element e, write:

$$X^{-1} = (e) \cdot . X, \quad X_t = (X^{-1})^{-1} \quad \text{for each } X < S.$$

THEOREM 7. *A given partially ordered semigroup S, with identity e, can be imbedded as in Theorem 1 in a cl-group G, a restricted $\vee$-completion of S, if and only if:*

(i) *S is directed $\uparrow$.*

(ii) *For any a, b in S, and non-null, bounded above, subset X of S, $\vee a X b = a\{\vee X\} b$ wherever the right hand side exists.*

(iii) *For each X as in* (ii), *X^{-1} is non-null, bounded above, and coincides with $(e) .\cdot X$.*

(iv) *For each a in S, $\{a\}_t = (a)_S$.*

(v) *For each X as in* (ii), *$X_t \cdot . X_t = X_t .\cdot X_t = (e)$.*

Then the sets $\{X_t\}$ form a restricted ideal extension on S, such that $C'_t(S)$ is a group, and any such cl-group G as above is isomorphic to $C'_t(S)$. In particular, this is the only restricted ideal extension on S which can yield a group.

Proof. Suppose there exists a *cl*-group G, a restricted $\vee$-completion of S, such that S is imbedded as in Theorem 1 in G. By the Lemma, S satisfies (i); also, S is abelian ([*1*], p. 234). For each non-null subset X of S, bounded above in S, let $X_r = S \cap (a)_G$, or $(a)'$ (in the notation of the Lemma) where $a = \vee X$ in G. Then the sets $\{X_r\}$ form, by Theorem 1, a restricted ideal extension of S, giving (ii). Further:

$$X_r : X_r = (e)_S \ldots (1).$$

For, given x in S, $x X_r \subset X_r$ means that for any y in $(a)'$, $xy \leqq a$ or: $y \leqq x^{-1} a$ (x^{-1} in G). Hence X is bounded above (in G) by $x^{-1}a$; that is: $x^{-1}a \geqq a$ or $x \leqq e$. Clearly $(e)_S \subset X_r : X_r$. Also, with X and a as above, given y in S, $y \in X^{-1}$ if and only if $x \leqq y^{-1}$, all $x \in X$; that is: $a \leqq y^{-1}$ or $y \leqq a^{-1}$. Hence:

$$X^{-1} = (a^{-1})' \ldots (2).$$

which, by the Lemma, implies both condition (iii) above and that $\vee X^{-1} = a^{-1}$. Hence, (applying (2) to X^{-1} in place of X), $X_t = (a)' = X_r$. So (iv) follows, together with (v), (from (1)).

Conversely, let S by any partially ordered semigroup with identity e, satisfying conditions (i) ... (v). By (iii), the system $\{X_t\}$ obviously satisfies C.1. and C.2. for a restricted ideal system (cf. [*2*], p. 154, noting in particular that $(X^{-1})_t = X^{-1}$). By (iv), C.3. is satisfied. Further, for any non-null, bounded above subset X of S, and any a in S, we have, (with several applications of (iii)), that: $(aX)^{-1} a X \subset (e)$ implies: $X \subset ((aX)^{-1}a)^{-1}$, implying in turn $X_t \subset (((aX)^{-1}a)^{-1})_t = ((aX)^{-1}a)^{-1}$; that is, $(aX)^{-1} a X_t \subset (e)$ or: $a X_t \subset (aX)_t$. Similarly $X_t a \subset (Xa)_t$.

Again, with X as above, let Y be a subset of X_t such that $\vee Y$ exists, equalling b. For any y in X^{-1},

$$by = \vee Yy, \quad \text{by (ii), so} \quad by \leqq e \quad \text{and} \quad b \in X_t.$$

Thus by (i), $C'_t(S)$ is a conditionally complete semilattice semigroup. To verify that this is a group, we use essentially the argument of JAFFARD ([*1*], p. 26, prop. 4). If X is t-closed:

$$\begin{aligned}(XX^{-1})^{-1} &= X_t \cdot\!\cdot X \quad \text{(cf. [2], p. 153)}\\ &= X_t \cdot\!\cdot X_t = (e) \quad \text{by (v).}\end{aligned}$$

Clearly, $(XX^{-1})_t = (e)$, the identity of $C'_t(S)$. Similarly for $(X^{-1}X)_t$.

To complete the proof of the Theorem, we need only note that (with the notation of the initial paragraphs) the mapping $X_r \to a$, or $X_t \to a$, is an isomorphism of $C'_t(S)$ onto G.

COROLLARY 7.1. *A partially ordered semigroup S, with identity e, can be imbedded as in Theorem 1 in a cl-group G, which is both a restricted $\vee$-completion* *and a restricted* $\wedge$*-completion of* S, *if and only if conditions* (i), (iii) *and* (iv) *above hold, together with:*

(ii)′ S *is r. q. r.*

(v)′ *For each non-null, bounded above subset* X *of* S,

$$X_v \cdot\!. X_v = X_v .\!\cdot X_v = (e).$$

Then $C_v'(S)$ *is a group, to which any such group* G *as above is isomorphic.* $C_v'(S)$ *is the only restricted ideal system on* S *which can yield a group.*

Proof. If S is so imbedded, write $c = \vee X$ for each non-null subset X of S, bounded above in S. If x (in S) $\geqq a (= \vee X$ in $G)$, x is an upper bound of X_v; hence $x \geqq c$. That is (by the dual of the Lemma) $a \geqq c$, or: $X_v \subset X_t$. Clearly $X_t \subset X_v$.

Conversely, to see the sufficiency of the conditions of the Corollary, we need only note that if the sets $\{X_t\}$ form a restricted ideal extension on any partially ordered semigroup S, it will be the least fine of them. For, given any restricted ideal extension $C_r(S)$ on S:

$$(X^{-1})_r X_r \subset (X^{-1} X)_r \subset (e);$$

that is:
$$(X^{-1})_r \subset (X_r)^{-1} \subset X^{-1}.$$

So each X^{-1} is r-closed, and hence each X_t. Thus if S is r. q. r., $X_t = X_v$.

COROLLARY 7.2. *If* S *is a residuated semigroup with identity* e *it can be imbedded as in Theorem 7 or as in Corollary 7.1 if and only if:*

(i) S *is directed* $\uparrow$.

(ii)″ S *is a group.*

(v)′ *For each non-null, bounded above subset* X *of* S,

$$X_v \cdot\!. X_v = X_v .\!\cdot X_v = (e).$$

Proof. Condition (iii) of the Theorem is clearly equivalent to:

(iii)′ $e \cdot\!. a = e .\!\cdot a$ for each a in S. That is: e is equiresidual.

Also, for each a in S, $e{:}a$ is clearly the maximum element of $\{a\}^{-1}$ and so $e{:}[e{:}a]$ is the maximum element of $\{a\}_t$. So condition (iv) of the Theorem is implied by:

(iv)′ $a = e : [e : a]$ for each a in S.

Conversely, (iv) implies: $a \geqq e{:}[e{:}a] \geqq a$.

Then, given that S can be imbedded as above, S is abelian and conditions (v)′ and (iv)′ imply that S is integrally closed, with Artin's equivalence reducing to the identity. Hence S is a group ([2], pp. 240—243).

Conversely, conditions (iii)′ and (iv)′ are clearly satisfied if S is a group.

Conditions (i), (iii) ... (v) of Theorem 7 are sufficient that S may be imbedded in a *cl*-group without least upper bounds existing in S being necessarily preserved; hence we have:

COROLLARY 7.3. *Conditions* (i), (iii) ... (v) *of Theorem 7 are sufficient that S be abelian, para-archimedean (or integrally closed in the sense of* BIRKHOFF [*1*], p. 229) *and satisfy the condition:*

$$\text{for } x, y, z \text{ in } S \text{ with } xy \leqq xz, \quad y \leqq z.$$

NOTE 1. It is immediately seen that condition (iii) of Theorem 7 need not hold if S is imbedded in a *cl*-group G with preservation of existing least upper bounds and greatest lower bounds where G is not a $\vee$-completion of S. For example, let S be the semigroup of positive integers and G the group of positive real numbers under multiplication, and ordered by magnitude.

NOTE 2. It can be immediately verified that condition (ii)'' of Corollary 7.2 is necessary. For given a in S, there exists the element a^{-1} in G such that $aa^{-1} = e$. Then:

$$\begin{aligned} a^{-1} &= e : a \text{ (in } G) \text{ (see note to the Lemma in the preceding Section)} \\ &= e : a \text{ in } S. \end{aligned}$$

That is: $a^{-1} \in S$.

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