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On n-th Roots of Normal Operators[1])

By

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1. Introduction and notations

Throughout this paper $X=\{x, y, \ldots\}$ will denote a Hilbert space, $L=\{T, S, Q, \ldots\}$ the set of all linear continuous mappings of X into X, endowed with the usual structure of a Banach space, $\sigma(T)$ and $\sigma_p(T)$ will denote the spectrum of $T\in L$ and the point spectrum of T respectively. The symbol $\dot{+}$ ($\oplus$) will denote the direct (orthogonal) sum of spaces or operators. All operators considered are elements of L. An operator T is called regular if the origin is not an element of $\sigma(T)$ and we say that two operators T_1, T_2 are similar if there exists a regular operator S such that $T_2=S^{-1}T_1S$. By E we denote the identity operator and by $Q, Q_0, \ldots$ a bounded, regular, positive definite selfadjoint operator. Finally, by N we denote a normal operator and by n a natural number.

An operator T is called an n-th root of N if $T^n=N$ holds. Also, S is called a logarithm of N if $\exp S=N$. The main result of this paper (Theorem 1) is that any n-th root of a regular normal operator N is similar to a normal operator R which is an n-th root of N. This is a generalisation of results of [7] and [4] to an infinite dimensional space. Furthermore in Theorem 2 we prove that every logarithm of N is similar to a normal logarithm of N.

In 3 (Theorems 4, 5), generalising the result of C. R. Putnam [6], we find a sufficient condition, in terms of the numerical range of T, for normality of an n-th root T of a normal operator.

The author uses this opportunity to express his gratitude to the referee for his helpful suggestions and especially for giving us, as presented in the text, the short proof of Lemma 2, the original proof of which was much more complicated.

2. A characterisation of the n-th roots of a normal operator

Theorem 1. *If T is an n-th root of a regular normal operator N, then there exists a regular operator Q such that:*

1. $QN=NQ$,
2. $R=Q^{-1}TQ$ *is normal and*
3. $R^n=N$,

i.e. every n-th root of N is similar to a normal n-th root of N.

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For the proof of this theorem we need two lemmas.

LEMMA 1. *If N and T are the same operators as in Theorem 1, then*

$$T = N_0 W = W N_0,$$

where $W \in L$ is an n-th root of E and

$$N_0 = \int \lambda^{1/n} dE(\lambda).$$

Here $E(\lambda)$ is the spectral resolution of the identity which belongs to N, and $\lambda^{1/n}$ is the n-th root of λ the argument of which belongs to $(-\pi/n, \pi/n]$.

PROOF. Since $T^n = N$, the operator T commutes with N. This implies that T commutes with $E(\lambda)$ ([*1*], [*2*]) and therefore T also commutes with N_0^{-1}. Hence the operator $W = T N_0^{-1}$ possesses all required properties.

LEMMA 2. *If W is an n-th root of E, then there exists a Q_0 such that*

$$W_0 = Q_0^{-1} W Q_0 \tag{1}$$

is a unitary operator, $W_0^n = E$ and

$$\sigma(W_0) \subseteq \Gamma = \{1, \varepsilon, \ldots, \varepsilon^{n-1}\} \tag{2}$$

with $\varepsilon = \exp(2\pi i/n)$ (Cf. [*3*], Theorem 1).

PROOF. The spectral mapping theorem and

$$W^n = E \tag{3}$$

imply $\sigma(W) \subseteq \Gamma$ so that W is regular. But then (3) implies that the group

$$\{W^k,\ k = 0, \pm 1, \pm 2, \ldots\}$$

is uniformly bounded. Now, the theorem of B. SZ.-NAGY [*10*], applied to the group $\{W^k\}$, implies the existence of a Q_0 with the property that the operator W_0, defined by (1), is unitary. Plainly (1) and (3) lead to $W_0^n = E$ from which, by use of the spectral mapping theorem, (2) follows.

PROOF OF THEOREM 1. According to Lemma 1 we have $T = N_0 W = W N_0$, where N_0 is a normal operator, and by Lemma 2 $W = Q_0 W_0 Q_0^{-1}$ with

$$W_0 = \sum_{k=0}^{n-1} \varepsilon^k E_k.$$

Here E_k is the spectral resolution of W_0 and some of the projections E_k may vanish. If Y_k denotes the range of E_k, then X is the orthogonal sum of the subspaces Y_k. If we set $X_k = Q_0 Y_k$, then

$$X = \sum_{k=0}^{n-1} \dot{+}\, X_k, \qquad W = \sum_{k=0}^{n-1} \dot{+}\, \varepsilon^k P_k,$$

where P_k is the restriction to X_k of the operator $Q_0 E_k Q_0^{-1}$. If in each X_k we take an orthonormal basic set, then we get a basic set in X. The elements of this set we denote by e_γ (γ runs through some set of indices which depends on the dimension of X only). Let e''_γ be an orthonormal basic set in X. The linear operator S, defined by the relation

$$e''_\gamma = S\,e_\gamma$$

has X as its domain. If x is a unit vector in X, then

$$x = \sum_{j=0}^{n-1} \lambda_j f_j,$$

where $f_j \in X_j$ are unit vectors and λ_j are numbers. Since the Gram matrix $G = \big((f_k, f_j)\big)$ is a positive definite matrix of the order n, every eigenvalue of G is less than the trace of G which equals n. Hence we have:

$$\|S\,x\|^2 = \sum_{j=0}^{n-1} |\lambda_j|^2 > \frac{1}{n} \sum_{j=0}^{n-1} \lambda_j \bar{\lambda}_k (f_j, f_k) = \frac{1}{n} \|x\|^2 = \frac{1}{n}.$$

Thus S^{-1} is a bounded operator. Now, we write: $S^{-1} = H\,U$, where U is a unitary operator and H is a bounded, positive definite selfadjoint operator $\big([9]$, p. 332$\big)$. For any $x \in X$ we have $\|H^{-1} x\| = \|U\,S\,x\| = \|S\,x\| < +\infty$, i.e. the domain of the selfadjoint operator H^{-1} is X. Thus $Q = H^{-1}$ is a bounded, regular, positive definite selfadjoint operator with the property that

$$e'_\gamma = Q\,e_\gamma$$

is an orthonormal basic set in X.

Since W commutes with N_0 it also commutes with N_0^*. This and the fact that $x \in X_k$ if and only if $W x = \varepsilon^k x$ implies that the X_k's are invariant for N_0 and N_0^*. Let R_k be the restriction of N_0 to X_k. Obviously R_k, as an operator in X_k, is normal and

$$T = \sum_{k=0}^{n-1} \dot{+}\, \varepsilon^k R_k, \qquad N_0 = \sum_{k=0}^{n-1} \dot{+}\, R_k, \qquad N_0^* = \sum_{k=0}^{n-1} \dot{+}\, R_k^* \tag{4}$$

hold. These relations imply

$$N = \sum_{k=0}^{n-1} \dot{+}\, (R_k)^n \quad \text{and} \quad N^* = \sum_{k=0}^{n-1} \dot{+}\, (R_k^*)^n. \tag{5}$$

A matrix of an operator A in the basic set e_γ we denote by $A(e)$. From (4) and (5) we see that $T(e)$, $N(e)$ and $(N^*)(e)$, as a direct sum of normal matrices, are normal. Moreover $[R_k(e)]^* = (R_k^*)(e)$ and (5) imply $(N^*)(e) = [N(e)]^*$ [2]. Let R be the normal operator to which, in the basic set e'_γ, the matrix $R(e') = T(e)$ belongs. Then

$$[R(e')]^n = [T(e)]^n = N(e)$$

which implies

$$[R^*(e')]^n = N^*(e).$$

Furthermore we have:

$$T\,Q^{-1} e'_\gamma = T\,e_\gamma = \sum_\alpha [T(e)]_{\alpha\gamma} e_\alpha = Q^{-1} \sum_\alpha [T(e)]_{\alpha\gamma} e'_\alpha = Q^{-1} \sum_\alpha [R(e')]_{\alpha\gamma} e'_\alpha = Q^{-1} R\,e'_\gamma$$

[2]) Notice that $(T^*)(e)$ is in general different from $[T(e)]^*$.

and also

$$N^* Q^{-1} e'_\gamma = Q^{-1} (R^*)^n e'_\gamma .$$

Thus

(6) $$T = Q^{-1} R Q , \quad N^* = Q^{-1} M^* Q ,$$

where $M = R^n$ is a normal operator. By $T^n = N$ and the conjugation of the second equation in (6) we get:

(7) $$N = Q^{-1} M Q , \quad N = Q M Q^{-1} .$$

From (7) we find

$$Q^2 M = M Q^2 .$$

Since the bounded operator M commutes with Q^2 and since $Q > 0$ we conclude, ([1], [2]), that M also commutes with Q, which, together with (7), implies $N = M$. Thus R is an n-th root of N, it is normal and T is similar to R.

COROLLARY 1. *Any n-th root of a unitary operator V is similar to a unitary n-th root of V.*

COROLLARY 2. *Any square root of a Q is similar to a selfadjoint square root of Q.*

These corollaries are direct consequences of Theorem 1. Corollary 2 can be directly proved in a very simple way. Indeed, suppose that $T^2 = Q$ and by Q_1 denote the unique positive definite selfadjoint square root of Q. We have

$$\exp i t T = \cos t\, Q_1 + i\, T\, Q_1^{-1} \sin t\, Q_1$$

which implies

$$\sup_{t \in C} \| \exp i t T \| < + \infty ,$$

where C is the set of real numbers. By the theorem of B. SZ.-NAGY [10], there exists a Q_0 such that $Q_0^{-1} \exp i t T Q_0$ is a group of unitary operators. Thus $H = Q_0^{-1} T Q_0$ is selfadjoint. From here we get

$$H^2 = Q_0^{-1} Q Q_0 \quad \text{and} \quad Q_0^{-1} Q Q_0 = Q_0 Q Q_0^{-1} .$$

Thus

$$Q_0 Q = Q Q_0 \quad \text{and} \quad H^2 = Q .$$

THEOREM 2. *If $N = \exp T$ is a normal operator, then a Q exists such that:*

1. $Q N = N Q$,
2. $R = Q^{-1} T Q$ *is a normal operator and*
3. $\exp R = N$,

i.e. every bounded logarithm of a normal operator N is similar to a normal logarithm of N[3]).

PROOF. Since $N = \exp T$ is normal and $T \in L$, we have $T = N_0 + 2 \pi i W$, where N_0 is a normal operator, $\exp N_0 = N$ and a bounded operator W commutes with N_0. Moreover, there exists a Q_0 such that $W_0 = Q_0^{-1} W Q_0$ is a selfadjoint

[3]) Observe that in the proofs of Theorems 1, 2 the group property of functions λ^n and $\exp \lambda$ are essentially used.

operator with the property that $\sigma(W_0)$ is contained in the set of all integers ([5], Theorem 1). Using this result, in the same way as in the proof of Theorem 1, we prove Theorem 2.

REMARK 1. The condition of Theorem 1 that N is regular is essential, because the square of the matrix

$$T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

is a normal matrix, but T is not similar to a normal matrix. On the other hand, if we assume that the origin is an isolated point of $\sigma(N)$, then $X = X_0 \oplus X_1$ with X_0 as the null space of N. The equation $T^n = N$ reduces to $T_0^n = 0$ and $T_1^n = N_1$, where T_j, N_j ($j = 0, 1$; $N_0 = 0$) are restrictions of T, N to X_j, and the operators N_1 and T_1 satisfy all conditions of Theorem 1 (Cf. [4]). The case in which the origin is not an isolated point of $\sigma(N)$ seems to be difficult. But if N is a compact (completely continuous) normal operator, then we have:

THEOREM 3. *Let X be a separable Hilbert space and $T \in L$. Suppose that there exists an entire function*

$$f(\lambda) = \sum_{n=1}^{\infty} \alpha_n \lambda^n$$

such that

$$N = f(T) = \sum_{n=1}^{\infty} \alpha_n T^n$$

is a compact normal operator with the property that zero is not an eigenvalue of N and that $f'(\lambda) \neq 0$ for any $\lambda \in \sigma_p(T)$. Then there exists a normal operator R and a positive definite selfadjoint operator H such that:

1. *The origin is not an eigenvalue of H,*
2. *$HN = NH$,*
3. *$HR = TH$ and*
4. *$f(R) = N$.*

PROOF. The assumption about N implies:

$$N = \sum_{k=1}^{\infty} \oplus \lambda_k E_k, \qquad X = \sum_{k=1}^{\infty} \oplus X_k, \qquad \lambda_k \neq 0$$

where E_k is the identity operator defined on a finite dimensional subspace X_k. Since T and T^* commute with N we have

$$T = \sum_{k=1}^{\infty} \oplus T_k,$$

where T_k is the restriction of T to X_k. Now $f(T) = N$ implies $f(T_k) = \lambda_k E_k$, $k = 1, 2, \ldots$. The fact that $f'(\lambda) \neq 0$ for $\lambda \in \sigma_p(T)$ implies that the Jordan form of S_k is diagonal. Hence, there exists an operator $T_k: X_k \to X_k$ such that $M_k = S_k^{-1} T_k S_k$ is a normal operator. Taking S_k in a suitable way we conclude that

$$M = \sum_{k=1}^{\infty} \oplus M_k \quad \text{and} \quad S = \sum_{k=1}^{\infty} \oplus S_k$$

are bounded operators, $f(M)=N$ and $SM=TS$. On the other hand, from the definition of S it follows that $SN=NS$. If we write $S=HU$ with a unitary operator U and a bounded positive definite selfadjoint operator H, then we find that H and $R=UMU^*$ satisfy all assertions of Theorem 3.

REMARK 2. The assumption about f' in Theorem 3 is essential, because $f(T)=T^2-2T$ for

$$T=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$$

is normal, but T is not similar to a normal matrix. Moreover, for any operator T in a finite dimensional unitary space, there is a polynomial $f \neq 0$ such that $f(T)$ is a normal operator and $f(0)=0$.

3. A sufficient condition for normality of an n-th root of a normal operator

THEOREM 4. *Suppose that T is an n-th root of N and N regular and normal as in Theorem 1, and that there exists a real number α such that for every unit vector $x \in X$*

$$\arg(Tx, x) \in (\alpha, \alpha + 2\pi/n)$$

holds, where arg *a denotes the argument of a number a. Then, T is a normal operator. (For the case $n=2$ see [6].)*

PROOF. The operator $T_1 = T \exp i(-\alpha - \pi/n)$ has the property that T_1^n is normal and $\arg(T_1 x, x) \in (-\pi/n, \pi/n)$. Therefore it is sufficient to prove that T_1 is normal. Because of this, in the sequel we assume $T_1 = T$, i.e. $\alpha = -\pi/n$. Denote by Ω the open set of all complex numbers a such that $\arg a \in (-\pi/n, \pi/n)$. Obviously for any $x \in X$, $x \neq 0$ we have

$$(Tx, x) \in \Omega. \tag{8}$$

The normal n-th root of N introduced in Lemma 1 possesses the property that

$$(N_0 x, x) \in \overline{\Omega} \tag{9}$$

for any x, where $\overline{\Omega}$ denotes the closed set determined by Ω. Now, (4) implies

$$(Tx, x) = \varepsilon^k (N_0 x, x) \tag{10}$$

for any $x \in X_k$. From (10), (9) and (8) we conclude that $k=0$, i.e. $T=N_0$. It is also seen that N_0 is a unique root of N with the property that (8) holds.

In the proof of Theorem 4 it was essential that the set Ω be open. In the next theorem we prove a similar theorem assuming less about Ω but more about N. We have:

THEOREM 5. *Let N be a compact normal operator and T an n-th root of N. Suppose that there exists a real number α such that for every unit vector $x \in X$*

$$\arg(Tx, x) \in [\alpha, \alpha + 2\pi/n]$$

holds. Then T is normal.

For the proof of this theorem we need the following

LEMMA 3. *If P is an operator with the property that*

1. $\mathrm{Re}(Px, x) \geqq 0$ *for all* $x \in X$ *and*

2. $P^n = 0$, *then* $P = 0$. Here $\mathrm{Re}\,\lambda$ denotes the real part of λ.

PROOF. Suppose that Lemma 3 does not hold and by $m \geqq 2$ denote the natural number m with the property that $P^m = 0$ and $P^{m-1} \neq 0$. Let X_0 be the null space of P and $X_1 = X \ominus X_0$. If $e \in X_0$ and $e' \in X_1$, then for any complex number a we have that

$$\bigl(P(a\,e + e'), a\,e + e'\bigr) = \bar{a}(P\,e', e) + (P\,e', e')$$

is in the right half plane. But this is possible only if $(Pe', e) = 0$. Hence X_1 is invariant for P. If P_1 denotes the restriction of P to X_1, then $P_1^m = 0$ and also $P_1^{m-1} \neq 0$. There exists, therefore, an $x \in X$ such that $e' = P_1^{m-1} x \neq 0$ and $P_1 e' = 0$. This implies $Pe' = 0$, $e' \in X_1$, $e' \neq 0$ which contradicts the definition of X_0. Thus $P = 0$.

PROOF OF THEOREM 5. The restriction of T to the null space of N satisfies the conditions of Lemma 3 and therefore it is a normal operator. Applying the same considerations as in the proof of Theorem 3 to the restriction of N and of T to the orthogonal complement of the null space of N, we conclude that it is sufficient to prove that:

If X is an m-dimensional unitary space and $T: X \to X$ an operator such that $(Tx, x) \in \bar{\Omega}$ for every $x \in X$ and $T^n = \lambda E \neq 0$, then T is normal. In order to prove this, denote by $e_1, \ldots, e_m$ an orthonormal basic set in X with the property that

$$t_{ij} = [T(e)]_{ij} = 0 \quad \text{if} \quad 1 \leqq i < j \leqq m,$$

where $T(e)$ is the matrix of T in this basic set [8]. If

$$x = \lambda_k e_k + \lambda_j e_j, \qquad 1 \leqq k < j \leqq m,$$

then

$$(Tx, x) = t_{kk}|\lambda_k|^2 + t_{jj}|\lambda_j|^2 + t_{jk}\lambda_k\bar{\lambda}_j \in \bar{\Omega} \tag{11}$$

for all complex numbers λ_k, λ_j. If λ possesses an n-th root which is in the interior of $\bar{\Omega}$, then this root is unique and we have $t_{11} = \cdots = t_{mm} = t$. Since T possesses a single eigenvalue t and since it is similar to a normal operator (Theorem 1), it is a normal operator. If λ does not possess an n-th root in the interior of $\bar{\Omega}$, then it has two roots on the boundary of $\bar{\Omega}$; we denote them by

$$\alpha = \varrho\exp(-i\pi/n), \qquad \beta = \varrho\exp i\pi/n, \qquad \varrho = |\lambda|^{1/n}. \tag{12}$$

Suppose that $t_{kk} = t_{jj} = \beta$, and take λ_k, λ_j such that $\arg t_{jk} + \arg\lambda_k - \arg\lambda_j = \pi$. Then (11) becomes:

$$\beta\bigl(|\lambda_k|^2 + |\lambda_j|^2\bigr) - |t_{jk}\lambda_k\lambda_j| \in \bar{\Omega}$$

which is possible only if $t_{jk} = 0$. The same holds if $t_{kk} = t_{jj} = \alpha$. Now, suppose that $t_{kk} = \alpha$, $t_{jj} = \beta$ and $t_{jk} \neq 0$. Set $\mu = 2\varrho/|t_{jk}|$, $\lambda_j = \mu$ and $\lambda_k = \exp i(\pi - \pi/n - \arg t_{jk})$. Then (11) becomes: $-\alpha + \beta\mu^2$ which is not in $\bar{\Omega}$ if $n > 2$. If $n = 2$, then we take $\lambda_k = \exp i(\pi - \pi/n - \arg t_{jk} + \varepsilon)$ where $0 < \varepsilon < \pi/2$. Using the fact that in this case $\alpha = -i\varrho$ and $\beta = i\varrho$, (11) gives: $-i\varrho(1 - 2\exp i\varepsilon) + \beta\mu^2 \in \overline{\Omega}$ which is impossible because the real part of this number is $-2\varrho \sin\varepsilon < 0$. Therefore the assumption $t_{jk} \neq 0$ leads to a contradiction. In the same way $t_{kk} = \beta$, $t_{jj} = \alpha$ leads to $t_{jk} = 0$. Thus the matrix $T(e)$ is diagonal which, by virtue of the fact that $e_1, \ldots, e_m$ is an orthonormal basic set, implies that T is a normal operator.

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