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The generalized Legendre functions with special relations between the parameters

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1. Introduction

In the present note we derive properties of the generalized Legendre functions $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$, introduced by KUIPERS and the author in [1], for special values of the parameters k, m and n. On the crosscut we restrict $z=x$ to lie in the interval $-1<x<1$, and we define:

$$(1)\qquad \begin{cases} P_k^{m,n}(x) = \frac{1}{2}\{e^{\frac{1}{2}\pi i m} P_k^{m,n}(x+0i) + e^{-\frac{1}{2}\pi i m} P_k^{m,n}(x-0i)\} \\ Q_k^{m,n}(x) = \frac{1}{2}e^{-\pi i m}\{e^{-\frac{1}{2}\pi i m} Q_k^{m,n}(x+0i) + e^{\frac{1}{2}\pi i m} Q_k^{m,n}(x-0i)\}, \end{cases}$$

where $f(x\pm 0i)$ means $\lim\limits_{\varepsilon\to 0} f(x\pm i\varepsilon)$, $\varepsilon>0$.

It is wellknown that the associated Legendre functions $R_k^m(z)$ (R denotes both P and Q) satisfy the differential equation:

$$(2)\qquad R_k^m(z) = (z^2-1)^{\frac{1}{2}m}\frac{d^m R_k(z)}{dz^m}\ (m \text{ is an integer} \geqq 0),$$

valid over the whole plane of z with the crosscut, and which gives the connection between $R_k^m(z)$ and $R_k(z)$.

In section 2 we shall give an extension of (2), representing a similar relation between $R_k^{m,n}(z)$ and $R_k^{\frac{1}{2}(m+n)}(z)$.

In section 3 we shall show that $R_k^{m,n}(z)$ with $k=\pm\frac{1}{2}(m\pm n)$ can be expressed in terms of incomplete betafunctions.

In [5], 76 ROBIN proved that $R_k(x)$ $(-1<x<1)$ with k half an odd integer can be expressed in terms of the elliptic integrals

$$E(s)=\int_0^{\frac{1}{2}\pi}(1-s^2\sin^2\varphi)^{\frac{1}{2}}\,d\varphi \quad\text{and}\quad K(s)=\int_0^{\frac{1}{2}\pi}(1-s^2\sin^2\varphi)^{-\frac{1}{2}}\,d\varphi.$$

In section 4 we prove that this is also true for $R_k^m(x)$ (k is half an odd integer and m is an integer). For the sake of brevity we put

$$\alpha=k+\tfrac{1}{2}(m+n),\quad \beta=k-\tfrac{1}{2}(m-n),\quad \gamma=k+\tfrac{1}{2}(m-n),\quad \delta=k-\tfrac{1}{2}(m+n).$$

In section 5 we consider $R_k^{m,n}(z)$, in which one of the four numbers $\alpha,\beta,\gamma,\delta$ is integral.

Finally in section 6 we derive extensions of CHRISTOFFEL's summation-formula.

2. Extension of formula (1)

In [6], (5) has been shown that if $\frac{1}{2}(n-m)$ is a positive integer or 0, we have

$$(3) \qquad P_k^{m,n}(z) = \frac{(z+1)^{\frac{1}{2}n}\Gamma(\gamma+1)\,2^{\frac{1}{2}(n-m)}}{(z-1)^{\frac{1}{2}m}\Gamma(\beta+1)}\left(\frac{d}{dz}\right)^{\frac{1}{2}(n-m)}\left\{\left(\frac{z-1}{z+1}\right)^{\frac{1}{2}(m+n)} P_k^{\frac{1}{2}(m+n)}(z)\right\}.$$

Starting from the expansion (see [2], (8))

$$Q_k^{m,n}(z) = \frac{e^{\pi i m}\,2^{-m+n-1}\,\Gamma(\alpha+1)\,\Gamma(\gamma+1)\,\Gamma(-m)\,(z+1)^{-\frac{1}{2}n}(z-1)^{\frac{1}{2}m}}{\Gamma(\beta+1)\,\Gamma(\delta+1)}\times$$
$$\times F\{-\beta,\gamma+1;1+m;\tfrac{1}{2}(1-z)\}$$
$$+\tfrac{1}{2}e^{\pi i m}\,\Gamma(m)\,(z+1)^{\frac{1}{2}n}(z-1)^{-\frac{1}{2}m}F\{-\gamma,\beta+1;1-m;\tfrac{1}{2}(1-z)\},$$

and employing the relations (see [9], 2.8):

$$\frac{\Gamma(c)}{\Gamma(c-n)}z^{c-1-n}(1-z)^{a+b-c-n}F\{a-n,b-n;c-n;z\}$$
$$=\left(\frac{d}{dz}\right)^n\{z^{c-1}(1-z)^{a+b-c}F\{a,b;c,;z\}\}$$

and

$$\left(\frac{d}{dz}\right)^n F\{a,b;c;z\} = \frac{\Gamma(a+n)\,\Gamma(b+n)\,\Gamma(c)}{\Gamma(a)\,\Gamma(b)\,\Gamma(c+n)}F\{a+n,b+n;c+n;z\},$$

it is easily found that $Q_k^{m,n}(z)$ also satisfies (3), so that we have:

$$R_k^{m,n}(z) = \frac{(z+1)^{\frac{1}{2}n}\Gamma(\gamma+1)\,2^{\frac{1}{2}(n-m)}}{(z-1)^{\frac{1}{2}m}\Gamma(\beta+1)}\left(\frac{d}{dz}\right)^{\frac{1}{2}(n-m)}\left\{\left(\frac{z-1}{z+1}\right)^{\frac{1}{2}(m+n)} R_k^{\frac{1}{2}(m+n)}(z)\right\}$$

[$\frac{1}{2}(n-m)$ is an integer $\geqq 0$].

Employing the definitions (1) of the functions for $z=x=\cos\vartheta$ $(0<\vartheta<\pi)$ we deduce

$$R_k^{m,n}(\cos\vartheta) = \frac{2^{n-m}\cos^n\frac{1}{2}\vartheta\,\Gamma(\gamma+1)}{\sin^m\frac{1}{2}\vartheta\,\Gamma(\beta+1)}\left(\frac{d}{dx}\right)^{\frac{1}{2}(n-m)}\left\{\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}(m+n)} R_k^{\frac{1}{2}(m+n)}\right\}.$$

3. $R_k^{m,n}(z)$ for $k=\pm\frac{1}{2}(m\pm n)$

a) For $|1-z|<2$, z not lying on the crosscut, we have:

$$P_{\frac{1}{2}(m+n)}^{m,n} = \frac{1}{\Gamma(1-m)}\frac{(z+1)^{\frac{1}{2}n}}{(z-1)^{\frac{1}{2}m}}F\left\{-m,n+1;1-m;\frac{1}{2}(1-z)\right\}.$$

This series may be summed up by means of the formula

$$(4) \qquad B_z(a,b) = a^{-1}z^aF\{a,1-b;a+1;z\},$$

valid for $\operatorname{Re} a>0$ (see [9], (2.5.3)), and where $B_z(a,b)$ represents the incomplete betafunction:

$$B_z(a,b) = \int_0^z t^{a-1}(1-t)^{b-1}\,dt.$$

Thus we find for $\operatorname{Re} m<0$:

$$P_{\frac{1}{2}(m+n)}^{m,n} = \frac{(z+1)^{\frac{1}{2}n}}{\Gamma(1-m)\,(z-1)^{\frac{1}{2}m}}(-m)\left(\frac{1-z}{2}\right)^m B_{\frac{1}{2}(1-z)}(-m,-n)$$
$$= \frac{e^{\mp\pi i m}2^{-m}}{\Gamma(-m)}(z-1)^{\frac{1}{2}m}(z+1)^{\frac{1}{2}n}B_{\frac{1}{2}(1-z)}(-m,-n),$$

the upper or lower sign being taken in the exponential, according as $\operatorname{Im}(z)$ is positive or negative.

In a similar way we find for $\operatorname{Re} m < 0$:

$$(5) \qquad P^{m,n}_{\frac{1}{2}(m+n)}(z) = \frac{2^{-m}}{\Gamma(-m)} (z-1)^{\frac{1}{2}m} (z+1)^{\frac{1}{2}n} B_{\frac{z-1}{z+1}}(-m, m+n+1).$$

For $\operatorname{Re} m \geqq 0$ we transform (5):

$$B_{\frac{z-1}{z+1}}(-m, m+n+1) = B_1(m+n+1, -m) - B_{\frac{2}{z+1}}(m+n+1, -m),$$

where the second term is meaningful for $\operatorname{Re}(m+n) > -1$.

Since

$$B_1(m+n+1, -m) = \frac{\Gamma(m+n+1)\,\Gamma(-m)}{\Gamma(n+1)},$$

it follows

$$P^{m,n}_{\frac{1}{2}(m+n)}(z) = (z-1)^{\frac{1}{2}m} (z+1)^{\frac{1}{2}n}\, 2^{-m} \times$$
$$\times \left\{ \frac{\Gamma(m+n+1)}{\Gamma(n+1)} - \frac{1}{\Gamma(-m)} B_{\frac{2}{z+1}}(m+n+1, -m) \right\}.$$

These formulas hold good in the whole plane of z with the crosscut by analytical continuation.

For $z = x = \cos\vartheta\ (0 < \vartheta < \pi)$ we obtain if $\operatorname{Re} m < 0$:

$$P^{m,n}_{\frac{1}{2}(m+n)}(\cos\vartheta) = \frac{2^{\frac{1}{2}(n-m)} \cos^n \frac{1}{2}\vartheta \sin^m \frac{1}{2}\vartheta}{\Gamma(-m)} B_{\sin^2 \frac{1}{2}\vartheta}(-m, -n).$$

For $\operatorname{Re} m \geqq 0$ we may use the analytical continuation of the function $B_z(a, b)$. Applying the formula:

$$B_z(a, b) = \Gamma(a)\,\Gamma(b) \left\{ \sum_{q=0}^{p-1} \frac{z^{a+q}(1-z)^{b-q-1}}{\Gamma(a+q+1)\,\Gamma(b-q)} + \frac{B_z(a+p, b-p)}{\Gamma(a+p)\Gamma(b-p)} \right\}$$

we find, if $p - 1 \leqq \operatorname{Re}(m) < p$, p a positive integer:

$$P^{m,n}_{\frac{1}{2}(m+n)}(\cos\vartheta) = 2^{\frac{1}{2}(n-m)}\, \Gamma(-n) \cos^n \frac{1}{2}\vartheta \sin^m \frac{1}{2}\vartheta \times$$
$$\times \left\{ \sum_{q=0}^{p-1} \frac{(\sin\frac{1}{2}\vartheta)^{-2(m-q)} (\cos\frac{1}{2}\vartheta)^{-2(n+q+1)}}{\Gamma(-m+q+1)\,\Gamma(-n-q)} + \frac{B_{\sin^2\frac{1}{2}\vartheta}(-m+p, -n-p)}{\Gamma(-m+p)\,\Gamma(-n-p)} \right\}.$$

From [2] (8) we have for $|1-z| < 2$:

$$Q^{m,n}_{\frac{1}{2}(m+n)}(z) = \frac{e^{\pi i m} 2^{-m+n-1} \Gamma(m+n+1)\,\Gamma(m+1)\,\Gamma(-m)\,(z+1)^{-\frac{1}{2}n} (z-1)^{\frac{1}{2}m}}{\Gamma(n+1)} \times$$
$$\times F\left\{-n, m+1; m+1; \frac{1}{2}(1-z)\right\} +$$
$$+ \frac{1}{2} e^{\pi i m} \Gamma(m) (z+1)^{\frac{1}{2}n} (z-1)^{-\frac{1}{2}m} F\left\{-m, n+1; 1-m; \frac{1}{2}(1-z)\right\}.$$

Because of

$$F\{-a, b; b; -z\} = (1+z)^a$$

we obtain:

$$F\{-n, m+1; m+1; \tfrac{1}{2}(1-z)\} = (z+1)^n 2^{-n}.$$

Furthermore from (4) for $\operatorname{Re} m<0$:

$$F\{-m, n+1; 1-m; \tfrac{1}{2}(1-z)\} = -m\, e^{\mp\pi i m} 2^{-m} (z-1)^m B_{\frac{1}{2}(1-z)}(-m, -n),$$

so that for $\operatorname{Re} m<0$ and z not lying on the crosscut we have:

$$Q^{m,n}_{\frac{1}{2}(m+n)}(z) = \frac{e^{\pi i m} 2^{-m-1}\pi}{\sin m\pi} (z+1)^{\frac{1}{2}n} (z-1)^{\frac{1}{2}m} \times$$
$$\times \left\{-\frac{\Gamma(m+n+1)}{\Gamma(n+1)} + \frac{e^{\mp\pi i m}}{\Gamma(-m)} B_{\frac{1}{2}(1-z)}(-m, -n)\right\}$$

and

$$Q^{m,n}_{\frac{1}{2}(m+n)}(z) = \frac{e^{\pi i m} 2^{-m-1}\pi}{\sin m\pi} (z+1)^{\frac{1}{2}n} (z-1)^{\frac{1}{2}m} \times$$
$$\times \left\{-\frac{\Gamma(m+n+1)}{\Gamma(n+1)} + \frac{1}{\Gamma(-m)} B_{\frac{z-1}{z+1}}(-m, m+n+1)\right\}.$$

For $\operatorname{Re}(m+n) > -1$ we have:

$$Q^{m,n}_{\frac{1}{2}(m+n)}(z) = e^{\pi i m} 2^{-m-1} \Gamma(m+1)\, (z+1)^{\frac{1}{2}n} (z-1)^{\frac{1}{2}m} B_{\frac{2}{z+1}}(m+n+1, -m).$$

For $z=x=\cos\vartheta$ $(0<\vartheta<\pi)$ we obtain if $\operatorname{Re}(m+n)>-1$:

$$Q^{m,n}_{\frac{1}{2}(m+n)}(\cos\vartheta) = 2^{\frac{1}{2}(n-m)-1}\Gamma(m+1) \cos^n\tfrac{1}{2}\vartheta \sin^m\tfrac{1}{2}\vartheta\, B_{\frac{1}{\cos^2\frac{1}{2}\vartheta}}(m+n+1, -m).$$

b) For $|1-z|<2$, z not lying on the crosscut, we have:

$$P^{m,n}_{-\frac{1}{2}(m+n)}(z) = \frac{(z+1)^{\frac{1}{2}n}}{\Gamma(1-m)\,(z-1)^{\frac{1}{2}m}} F\left\{1-m, n; 1-m; \frac{1}{2}(1-z)\right\}$$
$$= \frac{2^n}{\Gamma(1-m)} (z+1)^{-\frac{1}{2}n} (z-1)^{-\frac{1}{2}m},$$

and for $z=\cos\vartheta$ $(0<\vartheta<\pi)$:

$$P^{m,n}_{-\frac{1}{2}(m+n)}(\cos\vartheta) = \frac{2^{\frac{1}{2}(n-m)} \cos^{-n}\frac{1}{2}\vartheta \sin^{-m}\frac{1}{2}\vartheta}{\Gamma(1-m)}.$$

For $|1+z|<2$, z not lying on the crosscut we have (see [2], (9)):

$$Q^{m,n}_{-\frac{1}{2}(m+n)}(z) = \frac{e^{\pi i m} 2^{-m} \Gamma(1-n)\,(z+1)^{m+\frac{1}{2}n-1} (z-1)^{-\frac{1}{2}m}}{\Gamma(-m-n+2)} \times$$
$$\times F\left\{-m-n+1, 1-m; -m-n+2; \frac{2}{1+z}\right\}.$$

Hence if $\operatorname{Re}(m+n)<1$:

$$Q^{m,n}_{-\frac{1}{2}(m+n)}(z) = \frac{e^{\pi i m} 2^{n-1} \Gamma(1-n)}{\Gamma(-m-n+1)} (z+1)^{-\frac{1}{2}n} (z-1)^{-\frac{1}{2}m} B_{\frac{2}{z+1}}(-m-n+1, m),$$

and if $\operatorname{Re} m>0$:

$$Q^{m,n}_{-\frac{1}{2}(m+n)}(z) = e^{\pi i m} 2^{n-1} (z+1)^{-\frac{1}{2}n} (z-1)^{-\frac{1}{2}m} \times$$
$$\times \left\{\Gamma(m) - \frac{\Gamma(1-n)}{\Gamma(-m-n+1)} B_{\frac{z-1}{z+1}}(m, -m-n+1)\right\}.$$

For $z=\cos\vartheta$ $(0<\vartheta<\pi)$ we obtain if $\mathrm{Re}(m+n)<1$:

$$Q^{m,n}_{-\frac{1}{2}(m+n)}(\cos\vartheta)=\frac{2^{\frac{1}{2}(n-m)-1}\Gamma(1-n)\cos m\pi}{\Gamma(-m-n+1)}\times$$
$$\times\cos^{-n}\frac{1}{2}\vartheta\sin^{-m}\frac{1}{2}\vartheta\, B_{\frac{1}{\cos^2\frac{1}{2}\vartheta}}(-m-n+1,m).$$

c) The expressions for $R^{m,n}_{\pm\frac{1}{2}(m-n)}(z)$ are easily found by changing n into $-n$ in the results of a and b, and using the formulas:

$$R_k^{m,-n}(z)=2^{-n}R_k^{m,n}(z). \tag{6}$$

4. $R_k^m(\cos\vartheta)$ if k is half an odd integer and m is an integer

a) In [7] (3) the following recurrence relation is deduced:

$$\begin{cases}\gamma P^{m,n+1}_{k-\frac{1}{2}}(z)-(z+1)^{-\frac{1}{2}}\{(2k+1)z+2k-2n+1\}\times\\ \times P_k^{m,n}(z)+2(\delta+1)P^{m,n-1}_{k+\frac{1}{2}}(z)=0,\end{cases}\tag{7}$$

valid in the whole z plane including the crosscut $(-1<z<1)$. Now we have for $0<\vartheta<\pi$:

$$P^{0,-1}_{-1}(\cos\vartheta)=P_0^{0,-1}(\cos\vartheta)=\frac{2^{-\frac{1}{2}}}{\cos\frac{1}{2}\vartheta}F\left\{-\frac{1}{2},\frac{1}{2};1;\sin^2\frac{1}{2}\vartheta\right\}$$
$$=\frac{2^{\frac{1}{2}}}{\pi\cos\frac{1}{2}\vartheta}E\left(\sin\frac{1}{2}\vartheta\right),$$
$$P^{0,0}_{-\frac{1}{2}}(\cos\vartheta)=P_{\frac{1}{2}}(\cos\vartheta)=F\left\{\frac{3}{2},-\frac{1}{2};1;\sin^2\frac{1}{2}\vartheta\right\}$$
$$=\frac{2}{\pi}\left\{2E\left(\sin\frac{1}{2}\vartheta\right)-K\left(\sin\frac{1}{2}\vartheta\right)\right\}.$$

From (7) we find for instance:

$$P^{0,-2}_{-\frac{1}{2}}(\cos\vartheta)=\frac{1}{2}P^{0,0}_{-\frac{1}{2}}(\cos\vartheta)+\frac{2^{\frac{1}{2}}\sin^2\frac{1}{2}\vartheta}{\cos\frac{1}{2}\vartheta}P^{0,-1}_{-1}(\cos\vartheta)$$
$$=\frac{2}{\pi}\left\{\frac{E(\sin\frac{1}{2}\vartheta)}{\cos^2\frac{1}{2}\vartheta}-\frac{1}{2}K\left(\sin\frac{1}{2}\vartheta\right)\right\}.$$

Continuing this process it is clear that if p is an integer the function $P^{0,-p-2}_{\frac{1}{2}(p-1)}(\cos\vartheta)$ can be expressed in terms of E and K functions (for abbrevation we say: $P^{0,-p-2}_{\frac{1}{2}(p-1)}(\cos\vartheta)$ is an E, K function). Applying the recurrence relation [7] (9a):

$$\begin{cases}\delta(-\beta\cos\vartheta+\alpha+m+2)P_k^{m,n}(\cos\vartheta)+4\sin^2\frac{1}{2}\vartheta\, P_k^{m+2,n}(\cos\vartheta)-\\ -(m+1)\gamma\, 2^{\frac{1}{2}}\cos\frac{1}{2}\vartheta\, P^{m,n+1}_{k-\frac{1}{2}}(\cos\vartheta)=0\end{cases}\tag{8}$$

with $m=0$, we can express $P^{2,-p-2}_{\frac{1}{2}(p-1)}(\cos\vartheta)$ in terms of $P^{0,-p-2}_{\frac{1}{2}(p-1)}(\cos\vartheta)$ and $P^{0,-p-1}_{\frac{1}{2}(p-2)}(\cos\vartheta)$, so that $P^{2,-p-2}_{\frac{1}{2}(p-1)}(\cos\vartheta)$ is an E, K function. From the recurrence relation [7], 4a:

$$\begin{cases}m\left\{(m^2-1)\frac{3+\cos\vartheta}{1-\cos\vartheta}-(2k+1)^2+n^2\right\}P_k^{m,n}(\cos\vartheta)\\ =(m+1)\alpha(\beta+1)\gamma(\delta+1)P_k^{m-2,n}(\cos\vartheta)+4(m-1)P_k^{m+2,n}(\cos\vartheta)=0\end{cases}\tag{9}$$

it follows that for integral p and q the function $P^{2q,-p-2}_{\frac{1}{2}(p-1)}(\cos\vartheta)$ is an E, K function.

In particular, if p is an even integer $=2q-2$ the assertion above holds good for the associated Legendre function $P^{-p-2}_{\frac{1}{2}(p-1)}(\cos\vartheta)$, and thus if $p=-4$ for $P^{2}_{-\frac{5}{2}}(\cos\vartheta)$. ROBIN [5], p. 177 proved that $P^{0}_{-\frac{5}{2}}(\cos\vartheta)$ is an E, K function, so that applying the recurrence relations:

$$P_k^{m+1}(\cos\vartheta)+2m\operatorname{cotg}\vartheta\, P_k^m(\cos\vartheta)+(k+m)\,(k-m+1)\,P_k^{m-1}(\cos\vartheta)=0$$

and

$$(k+m)\sin\vartheta\, P_k^{m-1}(\cos\vartheta)-\cos\vartheta\, P_k^m(\cos\vartheta)+P_{k+1}^m(\cos\vartheta)=0$$

successively, we find that for integral m $P^m_{-\frac{5}{2}}(\cos\vartheta)$, $P^m_{-\frac{3}{2}}(\cos\vartheta)$ and generally $P_k^m(\cos\vartheta)$ (k half an odd integer) is an E, K function.

b) On emplying the relation [8] (11):

$$Q_k^{n,m}(-\cos\vartheta)=-\frac{2^{m-n}\,\Gamma(\beta+1)}{\Gamma(\gamma+1)}\left\{\cos\alpha\pi\, Q_k^{m,n}(\cos\vartheta)+\frac{1}{2}\pi\sin\alpha\pi\, P_k^{m,n}(\cos\vartheta)\right\}$$

we deduce for integral p and q:

$$(10)\quad Q^{-p-2,\,2q}_{\frac{1}{2}(p-1)}(\cos\vartheta)=\frac{(-1)^{q+1}\,\pi\, 2^{2q+p+1}\,\Gamma(-\frac{1}{2}-q)}{\Gamma(p+q+\frac{3}{2})}\,P^{2q,\,-p-2}_{\frac{1}{2}(p-1)}(-\cos\vartheta).$$

Now we have

$$Q^{-\frac{1}{2},0}_{-1}(\cos\vartheta)=2\pi\, P^{0,\,-1}_{-1}(-\cos\vartheta)=\frac{2^{\frac{3}{2}}}{\sin\frac{1}{2}\vartheta}\,E\left(\cos\frac{1}{2}\vartheta\right),$$

$$Q^{0,0}_{-\frac{1}{2}}(\cos\vartheta)=-\,2E(\cos\tfrac{1}{2}\vartheta)+K(\cos\tfrac{1}{2}\vartheta).$$

From (7) we derive the recurrence relation:

$$\left(p+q+\frac{1}{2}\right)P^{2q,\,-p-1}_{\frac{1}{2}(p-2)}(\cos\vartheta)-\frac{2^{-\frac{1}{2}}}{\cos\frac{1}{2}\vartheta}\,(p\cos\vartheta+3p+4)\,P^{2q,\,-p-2}_{\frac{1}{2}(p-1)}(\cos\vartheta)+$$
$$+\,2\left(p-q+\frac{3}{2}\right)P^{2q,\,-p-3}_{\frac{1}{2}p}(\cos\vartheta)=0,$$

therefore from (10):

$$Q^{-p-1,\,2q}_{\frac{1}{2}(p-2)}(\cos\vartheta)-\frac{2^{-\frac{3}{2}}}{\sin\frac{1}{2}\vartheta}\,(-p\cos\vartheta+3p+4)\,Q^{-p-2,\,2q}_{\frac{1}{2}(p-1)}(\cos\vartheta)+$$
$$+\,\frac{1}{2}\left(p-q+\frac{3}{2}\right)\left(p+q+\frac{3}{2}\right)Q^{-p-3,\,2q}_{\frac{1}{2}p}(\cos\vartheta)=0.$$

We find for instance if $p=0$, $q=0$:

$$Q^{-2,0}_{-\frac{1}{2}}(\cos\vartheta)=16\left\{\frac{E(\cos\frac{1}{2}\vartheta)}{\sin^2\frac{1}{2}\vartheta}-\frac{1}{2}\,K\left(\cos\frac{1}{2}\vartheta\right)\right\}.$$

With the aid of (10) it is possible to transform the relations (8) and (9) in recurrence relations between the special Q functions, so that, following the same argument as in a, $Q^{-p-2,\,2q}_{\frac{1}{2}(p-1)}(\cos\vartheta)$ is an E, K function, and further that for m is an integer and k is half an odd integer $Q_k^m(\cos\vartheta)$ is an E, K function.

5. $R_k^{m,n}(z)$ with one of the four numbers α, δ, γ, β integral

a) *α is a non-negative integer.*

We interchange the parameters and consider the function $P_\alpha^{-m,-n}(z)$ where k is an integer $\geqq 0$. In this case we have for $|1-z|<2$:

$$(11)\quad \begin{cases} P_\alpha^{-m,-n}(z) = \dfrac{1}{\Gamma(1+m)}(z+1)^{-\frac{1}{2}n}(z-1)^{\frac{1}{2}m}\times \\ \qquad \times F\left\{k+m+1,\,-k-n;\,1+m;\,\dfrac{1}{2}(1-z)\right\} \\ \qquad = \dfrac{2^{-n}}{\Gamma(1+m)}(z+1)^{\frac{1}{2}n}(z-1)^{\frac{1}{2}m}\times \\ \qquad \times F\left\{-k,\,k+m+n+1;\,1+m;\,\dfrac{1}{2}(1-z)\right\}. \end{cases}$$

Hence the function $(z+1)^{-\frac{1}{2}n}(z-1)^{-\frac{1}{2}m}P_\alpha^{-m,-n}(z)$ is a polynomial in z of the degree k.

On the crosscut $(0<\vartheta<\pi)$ we have:

$$P_\alpha^{-m,-n}(\cos\vartheta) = \frac{2^{\frac{1}{2}(m-n)}\sin^m\frac{1}{2}\vartheta\cos^n\frac{1}{2}\vartheta}{\Gamma(1+m)}F\left\{-k,\,k+m+n+1;\,1+m;\,\sin^2\frac{1}{2}\vartheta\right\}.$$

This polynomial is related to the Jacobi polynomial $P_k^{(m,n)}(\cos\vartheta)$. According to [10], p. 170 (16) we have:

$$P_\alpha^{-m,-n}(\cos\vartheta) = \frac{2^{\frac{1}{2}(m-n)}\Gamma(k+1)\sin^m\frac{1}{2}\vartheta\cos^n\frac{1}{2}\vartheta}{\Gamma(k+m+1)}P_k^{(m,n)}(\cos\vartheta).$$

For $|1+z|<2$ we find from [3] (21) and (11) or from [2], (2):

$$P_\alpha^{-m,-n}(z) = \frac{(-1)^k\Gamma(k+n+1)\,2^{-n}}{\Gamma(k+m+1)\Gamma(n+1)}(z+1)^{\frac{1}{2}n}(z-1)^{\frac{1}{2}m}\times$$
$$\times F\left\{-k,\,k+m+n+1;\,1+n;\,\frac{1}{2}(1+z)\right\},$$

and on the crosscut:

$$P_\alpha^{-m,-n}(\cos\vartheta) = \frac{(-1)^k\Gamma(k+n+1)\,2^{\frac{1}{2}(m-n)}}{\Gamma(k+m+1)\Gamma(n+1)}\cos^n\frac{1}{2}\vartheta\sin^m\frac{1}{2}\vartheta\times$$
$$\times F\left\{-k,\,k+m+n+1;\,1+n;\,\cos^2\frac{1}{2}\vartheta\right\}.$$

From [2], (7) we have for $|1-z|>2$:

$$Q_\alpha^{-m,-n}(z) = \frac{e^{-\pi i m}\,2^{k+m}\Gamma(k+1)\Gamma(k+n+1)}{\Gamma(2k+m+n+2)}(z+1)^{-\frac{1}{2}n}(z-1)^{-k-\frac{1}{2}m-1}\times$$
$$\times F\left\{k+1,\,k+m+1;\,2k+m+n+2,\,\frac{2}{1-z}\right\},$$

which may be written as

$$Q_\alpha^{-m,-n}(z) = \frac{e^{-\pi i m}\,2^{-n}\Gamma(k+1)}{\Gamma(k+m+1)}(z+1)^{\frac{1}{2}n}(z-1)^{\frac{1}{2}m}Q_k^{(m,n)}(z),$$

where $Q_\alpha^{(m,n)}$ denotes the Jacobi function of the second kind. Using the definition of $Q_k^{(m,n)}(x)$ on the crosscut:

$$Q_k^{(m,n)}(x) = \tfrac{1}{2}\{Q_k^{(m,n)}(x+0i) + Q_k^{(m,n)}(x-0i)\}$$

we find for $x = \cos\vartheta$ $(0<\vartheta<\pi)$:

$$Q_\alpha^{m,n}(\cos\vartheta) = \frac{2^{\frac{1}{2}(m-n)}\Gamma(k+m+n+1)\sin^m\frac{1}{2}\vartheta\cos^n\frac{1}{2}\vartheta}{\Gamma(k+n+1)} Q_k^{(m,n)}(\cos\vartheta).$$

b) *δ is a negative integer.*

This case may be reduced to that of α, on account of

$$P_k^{m,n}(z) = P_{-k-1}^{m,n}(z),$$

and

$$\sin\alpha\pi\sin\gamma\pi\, Q_k^{m,n}(z) = \tfrac{1}{2}\pi e^{\pi i m}\sin 2k\pi\, P_k^{m,n}(z).$$

c) *γ is a non-negative integer.*

We interchange the parameters and consider the function $P_\beta^{m,n}(z)$ where k is an integer $\geqq 0$. If we replace m by $-m$ and apply formula (6), we may reduce this case to that of *a*.

d) *β is a non-negative integer.*

In the same way as the results in *b* follow from those in *a*, the results in *d* may be deduced from those in *c*.

REMARK. Further relations between the generalized Legendre associated functions and the Jacobi polynomials and Jacobi functions are given by KUIPERS and ROBIN in [*4*].

6. Extension of CHRISTOFFEL'S summationformula

From [*11*], (16) with $(-m, -n)$ in stead of (m, n) we have

$$\{2k(k+1)(2k+1)z_1 + \tfrac{1}{2}(m^2-n^2)(2k+1)\}P_k^{-m,-n}(z_1)$$
$$= 2k(\gamma+1)(\alpha+1)P_{k+1}^{-m,-n}(z_1) + 2(k+1)\beta\delta P_{k-1}^{-m,-n}(z_1),$$

$$\{2k(k+1)(2k+1)z_2 + \tfrac{1}{2}(m^2-n^2)(2k+1)\}P_k^{-m,-n}(z_2)$$
$$= 2k(\gamma+1)(\alpha+1)P_{k+1}^{-m,-n}(z_2) + 2(k+1)\beta\delta P_{k-1}^{-m,-n}(z_2).$$

Hence if $k \neq 0$:

$$(k+1)(2k+1)(z_1-z_2)P_k^{-m,-n}(z_1)P_k^{-m,-n}(z_2)$$
$$= (\gamma+1)(\alpha+1)\Delta P_k^{-m,-n}(z_1,z_2) - \frac{k+1}{k}\beta\delta\Delta P_{k-1}^{-m,-n}(z_1,z_2),$$

where

$$\Delta P_k^{-m,-n}(z_1,z_2) = P_{k+1}^{-m,-n}(z_1)P_k^{-m,-n}(z_2) - P_k^{-m,-n}(z_1)P_{k+1}^{-m,-n}(z_2).$$

Thus

$$\frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\beta+1)\Gamma(\delta+1)}(2k+1)(z_1-z_2)P_k^{-m,-n}(z_1)P_k^{-m,-n}(z_2)$$
$$= \frac{\Gamma(\gamma+2)\Gamma(\alpha+2)}{\Gamma(\beta+1)\Gamma(\delta+1)(k+1)}\Delta P_k^{-m,-n}(z_1,z_2) - \frac{\Gamma(\gamma+1)\Gamma(\alpha+1)}{\Gamma(\beta)\Gamma(\delta)k}\Delta P_{k-1}^{-m,-n}(z_1,z_2).$$

Hence by giving k the values: $\frac{1}{2}(m+n)$, $\frac{1}{2}(m+n)+1, \ldots, \frac{1}{2}(m+n)+p$, ($p$ a non-negative integer) and summing we have:

$$(z_1 - z_2) \sum_{k=\frac{1}{2}(m+n)}^{\frac{1}{2}(m+n)+p} \frac{\Gamma(\alpha+1)\,\Gamma(\gamma+1)}{\Gamma(\beta+1)\,\Gamma(\delta+1)} (2k+1)\, P_k^{-m,\,-n}(z_1)\, P_k^{-m,\,-n}(z_2)$$
$$= \frac{\Gamma(m+p+2)\,\Gamma(m+n+p+2)}{\{\frac{1}{2}(m+n)+p+1\}\,\Gamma(n+p+1)\,\Gamma(p+1)}\, \Delta_{\frac{1}{2}(m+n)+p}^{-m,\,-n}(z_1, z_2),$$

or, changing the notation:

$$(12)\quad \begin{cases} (z_1 - z_2) \displaystyle\sum_{k=0}^{p} \frac{\Gamma(k+m+n+1)\,\Gamma(k+m+1)}{\Gamma(k+n+1)\,\Gamma(k+1)} (2k+m+n+1)\, P_\alpha^{-m,\,-n}(z_1) \times \\ \times P_\alpha^{-m,\,-n}(z_2) = \dfrac{\Gamma(m+p+2)\,\Gamma(m+n+p+2)}{\{\frac{1}{2}(m+n)+p+1\}\,\Gamma(n+p+1)\,\Gamma(p+1)}\, \Delta_{\frac{1}{2}(m+n)+p}^{-m,\,-n}(z_1, z_2). \end{cases}$$

This formula holds good for every z_1 and z_2 provided that m and n are not negative integers.

If in (12) we change m into $-m$, we have, employing (6):

$$(13)\quad \begin{cases} (z_1 - z_2) \displaystyle\sum_{k=0}^{p} \frac{\Gamma(k-m+n+1)\,\Gamma(k-m+1)}{\Gamma(k+n+1)\,\Gamma(k+1)} (2k-m+n+1)\, P_\beta^{m,\,n}(z_1) \times \\ \times P_\beta^{m,\,n}(z_2) = \dfrac{\Gamma(-m+p+2)\,\Gamma(-m+n+p+2)}{\{-\frac{1}{2}(m-n)+p+1\}\,\Gamma(n+p+1)\,\Gamma(p+1)}\, \Delta_{-\frac{1}{2}(m-n)+p}^{m,\,n}(z_1, z_2). \end{cases}$$

(12) and (13) are generalizations of CHRISTOFFEL's first summation-formula.

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