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On the Phragmén-Lindelöf theorem and some applications*

In memoriam LEON LICHTENSTEIN

By
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1. One of the most fruitful applications of the Phragmén-Lindelöf theorem occurs in analytic number theory in the discussion of the Riemann zetafunction. In its most frequently used form the theorem is applied to a half-strip and to a single function (cf. LANDAU, Vorlesungen über Zahlentheorie, vol. 2, Satz 404, 405, pp. 48—51). This, however, is not quite general enough for situations encountered e.g. in the theory of character sums and of primes in arithmetic progressions. We need here estimations which are uniform in several parameters for an infinite set of $L(s,\chi)$ or $\zeta(s,\lambda)$-functions, and the whole strip must be considered[1]). It is the purpose of the present paper to provide a suitable refinement of the Phragmén-Lindelöf theorem. The improvement is achieved through the use of certain subharmonic functions. The applications in Theorems 3, 4, 5 concern zetafunctions. But already the results about the Γ-function in Lemmas 1 to 3 are believed to be new.

2. Basic for our arguments is the following

THEOREM 1. *Let a, b, Q, γ, δ be real numbers,*

$$-Q < a < b, \qquad \gamma \leqq \delta. \tag{2.1}$$

Then there exists an analytic function $\varphi(s)=\varphi(s;Q)$, depending also on the parameters a, b, γ, δ, which is regular in the strip

$$S(a,b): \qquad a \leqq \Re(s) \leqq b$$

and such that

$$\begin{cases} |\varphi(a+it;Q)| = |Q+a+it|^{\gamma} \\ |\varphi(b+it;Q)| = |Q+b+it|^{\delta} \end{cases} \tag{2.2}$$

and that for $a \leqq \sigma \leqq b$

$$|\varphi(s;Q)| \geqq |Q+s|^{l(\sigma)} \tag{2.3}$$

where

$$l(\sigma) = \gamma\,\frac{b-\sigma}{b-a} + \delta\,\frac{\sigma-a}{b-a}. \tag{2.31}$$

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[1]) Such a situation is found e.g. in the discussion of the Hecke functions in [5], Hilfssatz 15, pp. 363—365, where, however, a rather crude treatment if sufficient.

Moreover

$$\varphi(s;Q) = O(|t|^c), \quad |t| \to \infty \tag{2.32}$$

for a certain $c > 0$.

REMARK. Because of (2.1) we have $|Q+s| > 0$ so that (2.3) implies

$$\varphi(s;Q) \neq 0. \tag{2.33}$$

It is essential for our further arguments that we have *equality* in (2.2) and that (2.3) holds for all Q with $Q+a>0$.

PROOF. The case $\gamma=\delta$ is trivial since here $\varphi(s;Q)=(Q+s)^\gamma=(Q+s)^\delta$ furnishes the solution. Here, and in similar situations later, we take the principal branch of $(Q+s)^\gamma$ in $S(a,b)$, i.e. that one which is real for positive $Q+s$. We can now assume $\gamma<\delta$.

We construct a harmonic function $u(\sigma,t)$ in the strip $S(a,b)$ which fulfills the boundary conditions

$$\begin{cases} u(a,t) = \gamma \log|Q+a+it| = \frac{1}{2}\gamma \log\left((Q+a)^2+t^2\right) \\ u(b,t) = \delta \log|Q+b+it| = \frac{1}{2}\delta \log\left((Q+b)^2+t^2\right). \end{cases} \tag{2.4}$$

The boundary problem for $\Delta u=0$ in a strip is solved by a transformation of the Poission formula for the circle. Let us put

$$\omega(\sigma,t) = \frac{1}{2}\,\frac{\sin\pi\sigma}{\cosh\pi t - \cos\pi\sigma}. \tag{2.51}$$

Then

$$\begin{aligned} u(\sigma,t) = {} & \frac{1}{b-a}\int_{-\infty}^{\infty}\omega\left(\frac{\sigma-a}{b-a},\ \frac{t-y}{b-a}\right)A(y)\,dy + \\ & + \frac{1}{b-a}\int_{-\infty}^{\infty}\omega\left(\frac{b-\sigma}{b-a},\ \frac{t-y}{b-a}\right)B(y)\,dy \end{aligned} \tag{2.52}$$

satisfies $\Delta(u)=0$ in the interior of $S(a,b)$ with the boundary conditions

$$u(a,t) = A(t), \qquad u(b,t) = B(t), \tag{2.53}$$

where $A(t)$, $B(t)$ are supposed to be continuous and $e^{-\frac{\pi|t|}{b-a}}|A(t)|$, $e^{-\frac{\pi|t|}{b-a}}|B(t)|$ integrable in the interval $(-\infty,\infty)$. (Cf [2], p. 550.)

Taking

$$A(t) = \gamma\log|Q+a+it|, \qquad B(t) = \delta\log|Q+b+it|, \tag{2.6}$$

we have in (2.52) a solution of the boundary problem (2.4).

From (2.51), (2.52) (2.6) it follows by a simple estimation that

$$u(\sigma,t) = O\left(\log(2+|t|)\right) \tag{2.7}$$

in the strip $S(a,b)$ uniformly in σ, although not in Q.

Let now $v(\sigma, t)$ be a conjugate harmonic function to $u(\sigma, t)$. Since $u(\sigma, t)$ in view of the boundary conditions (2.4) is evidently symmetric with respect to the σ-axis, $v(\sigma, t)$ must be constant for $t=0$, and we can normalize $v(\sigma, t)$ through an additive constant so that $v(\sigma, 0)=0$. We put now

$$\varphi(s; Q) = \exp\big(u(\sigma, t) + i v(\sigma, t)\big) \tag{2.8}$$

which is a regular analytic function of $s=\sigma+it$ with the absolute value

$$|\varphi(s; Q)| = \exp u(\sigma, t). \tag{2.81}$$

The boundary conditions (2.4) show that (2.2) is satisfied. Also satisfied is (2.32) because of (2.7).

In order to finish the proof of Theorem 1 we consider the function

$$U(\sigma, t) = (\lambda\sigma + \mu)\log|Q+\sigma+it| = (\lambda\sigma+\mu)\log|Q+s|. \tag{2.9}$$

We have

$$\Delta U = 2\lambda\frac{Q+\sigma}{(Q+\sigma)^2+t^2}, \tag{2.91}$$

which is easily seen, since $U(\sigma, t)=f\cdot g$ and $\Delta U = f\cdot\Delta g + 2(f_\sigma g_\sigma + f_t g_t) + g\cdot\Delta f$, where Δg, Δf and f_t obviously vanish. We choose now the constants λ and μ so that $U(\sigma, t)$ also satisfies the boundary conditions (2.4) i.e.

$$\lambda a + \mu = \gamma, \qquad \lambda b + \mu = \delta$$

or

$$\lambda = \frac{\delta-\gamma}{b-a}, \qquad \mu = \frac{\gamma b - \delta a}{b-a}. \tag{2.92}$$

The conditions (2.1) show then that

$$\lambda > 0.$$

(since we consider here only $\gamma < \delta$), and therefore in virtue of (2.91) and $Q+\sigma>0$

$$\Delta U > 0.$$

In other words, $U(\sigma, t)$ is a *subharmonic* function in the strip $S(a, b)$. It has the same boundary values as the harmonic function $u(\sigma, t)$, and therefore we conclude

$$U(\sigma, t) \leqq u(\sigma, t)$$

in $S(a, b)$. This inequality for the infinite strip $S(a, b)$ is obtained by an application of the Phragmén-Lindelöf argument to the subharmonic function $D(\sigma, t) = U(\sigma, t) - u(\sigma, t)$ which, in view of (2.7) and (2.9), has the property $D(\sigma, t) = O(|t|^c)$, $|t|\to\infty$, in $S(a, b)$. But then it follows from (2.81) that

$$|\varphi(s; Q)| \geqq \exp U(\sigma, t).$$

If we take into account (2.9) and (2.92) we see that this inequality is exactly (2.3). This concludes the proof of Theorem 1.

3. We use this theorem now to prove the following one, which is of the Phragmén-Lindelöf type:

THEOREM 2. *Let $f(s)$ be regular analytic in the strip $S(a, b)$ and satisfy for certain positive constants c, C*

$$|f(s)| < C e^{|t|^c}. \tag{3.1}$$

Suppose moreover that

$$\begin{cases} |f(a+it)| \leqq A|Q+a+it|^{\alpha} \\ |f(b+it)| \leqq B|Q+b+it|^{\beta} \end{cases} \tag{3.2}$$

with

$$Q+a>0, \tag{3.31}$$

$$\alpha \geqq \beta. \tag{3.32}$$

Then in the strip $S(a, b)$

$$|f(s)| \leqq (A|Q+s|^{\alpha})^{\frac{b-\sigma}{b-a}} (B|Q+s|^{\beta})^{\frac{\sigma-a}{b-a}}. \tag{3.4}$$

REMARK. It will be noticed that no constants enter into (3.4) which do not already appear in the conditions (3.2)

PROOF. We consider the function $\varphi(s; Q)$ of Theorem 1 with the parameter values

$$\gamma = -\alpha, \qquad \delta = -\beta, \tag{3.5}$$

which in view of (3.32) satisfy (2.1). Then we form the regular function

$$F(s) = f(s)\,\varphi(s; Q)\,E^{-1} e^{-\nu s}, \tag{3.6}$$

where E and ν are so determined that

$$A = E\,e^{\nu a}, \qquad B = e^{\nu b}. \tag{3.7}$$

We see that because of (3.1) and (2.32) an estimate

$$|F(s)| < C^{|t|^c} \tag{3.8}$$

holds with suitable positive c, C. Now the conditions (3.2), (3.5), (3.7) give together with (2.2) the boundary conditions for $F(s)$:

$$|F(a+it)| \leqq 1, \qquad |F(b+it)| \leqq 1.$$

The well-known Phragmén-Lindelöf argument, admissible in view of (3.8), applied on the rectangle $a \leqq \sigma \leqq b$, $|t| \leqq T$, $T \to \infty$, shows that

$$|F(s)| \leqq 1$$

throughout $S(a, b)$. The definition (3.6) of $F(s)$ shows that this means

$$|f(s)| \leqq E\,e^{\nu\sigma}\,|\varphi(s)|^{-1},$$

which because of (2.3), (2.31), (3.5) and (3.7) yields the assertion (3.4) of the theorem.

4. The case $\alpha<\beta$ can also be treated, but the resulting theorem is less concise, since a factor M depending on a, b, α, β, Q appears. We obtain here

THEOREM 2a. *Under the conditions of Theorem 2, except* (3.32) *to be replaced by*

$$\alpha<\beta, \tag{4.1}$$

we have in the strip $S(a, b)$

$$|f(s)| \leqq M_Q^2 (A|Q+s|^{\alpha})^{\frac{b-\sigma}{b-a}} (B|Q+s|^{\beta})^{\frac{\sigma-a}{b-a}}, \tag{4.2}$$

where

$$M_Q = \operatorname{Max}\left(\left(\frac{a+Q}{b+Q}\right)^{\alpha}, \left(\frac{b+Q}{a+Q}\right)^{\beta}\right). \tag{4.3}$$

The factor M_Q *is a monotone decreasing function of* Q *and*

$$\lim_{Q\to\infty} M_Q = 1. \tag{4.4}$$

PROOF. This theorem is derived from Theorem 2 by the substitutions

$$s' = a+b-s,$$

$$\alpha' = \beta, \qquad \beta' = \alpha, \qquad A' = B, \qquad B' = A.$$

We have again

$$a \leqq \sigma' = \Re(s') \leqq b,$$

and α', β' satisfy condition (3.32) of Theorem 2.

We put

$$g(s') = g(a+b-s) = f(s).$$

Then $g(s')$ satisfies the conditions

$$|g(a+it)| = |f(b-it)| \leqq A'|Q+b+it|^{\alpha'}$$
$$|g(b+it)| = |f(a-it)| \leqq B'|Q+a+it|^{\beta'},$$

which have not yet the form (3.2), a and b being exchanged on the right side of the inequalities. We have, however,

$$\begin{cases} |g(a+it)| \leqq A'|Q+a+it|^{\alpha'} \cdot M_Q \\ |g(b+it)| \leqq B'|Q+b+it|^{\beta'} \cdot M_Q \end{cases} \tag{4.5}$$

with

$$M_Q = \operatorname*{Max}_t \left\{\left|\frac{Q+b+it}{Q+a+it}\right|^{\beta}, \left|\frac{Q+a+it}{Q+b+it}\right|^{\alpha}\right\}.$$

Now

$$1 \leqq \left|\frac{Q+b+it}{Q+a+it}\right| \leqq \frac{Q+b}{Q+a},$$

so that M_Q takes the form (4.3). Application of Theorem 2 on $g(s')$ with conditions (4.5) now yields in $S(a, b)$

$$|g(s')| \leqq M_Q (A'|Q+s'|^{\alpha'})^{\frac{b-\sigma'}{b-a}} (B'|Q+s'|^{\beta'})^{\frac{\sigma'-a}{b-a}}$$

or

$$\text{(4.6)}\quad \begin{cases} |f(s)| \leqq M_Q\left(B\,|\,Q+a+b-s|^{\beta}\right)^{\frac{\sigma-a}{b-a}}\left(A\,|\,Q+a+b-s|^{\alpha}\right)^{\frac{b-\sigma}{b-a}} \\ \qquad = M_Q\left(A\,|\,Q+s|^{\alpha}\right)^{\frac{b-\sigma}{b-a}}\left(B\,|\,Q+s|^{\beta}\right)^{\frac{\sigma-a}{b-a}} P_Q(s) \end{cases}$$

with

$$P_Q(s) = \left|\frac{Q+a+b-s}{Q+s}\right|^{\alpha\frac{b-\sigma}{b-a}+\beta\frac{\sigma-a}{b-a}}.$$

But for $a \leqq \sigma \leqq b$, $-\infty < t < \infty$ we have

$$\left(\frac{Q+a}{Q+b}\right)^2 \leqq \frac{(Q+a)^2+t^2}{(Q+b)^2+t^2} \leqq \frac{(Q+a+b-\sigma)^2+t^2}{(Q+\sigma)^2+t^2} \leqq \frac{(Q+b)^2+t^2}{(Q+a)^2+t^2} \leqq \left(\frac{Q+b}{Q+a}\right)^2$$

and thus

$$\text{(4.7)}\quad \begin{cases} P_Q(s) \leqq \operatorname*{Max}_{a\leqq\sigma\leqq b}\left\{\left(\frac{Q+a}{Q+b}\right)^{\alpha\frac{b-\sigma}{b-a}+\beta\frac{\sigma-a}{b-a}},\ \left(\frac{Q+b}{Q+a}\right)^{\alpha\frac{b-\sigma}{b-a}+\beta\frac{\sigma-a}{b-a}}\right\} \\ \qquad \leqq \operatorname{Max}\left\{\left(\frac{Q+a}{Q+b}\right)^{\alpha},\ \left(\frac{Q+b}{Q+a}\right)^{\beta}\right\} = M_Q. \end{cases}$$

The inequalities (4.6) and (4.7) together prove (4.2) of Theorem 2a. The statement (4.4) es evident from the definition of M_Q.

5. It is fortunate that in the application of our theorems to the theory of Dirichlet series the situation $\alpha \geqq \beta$ of Theorem 2 prevails. We need, however, first some lemmas about Γ-quotients, which are simple consequences of Theorem 2.

LEMMA 1. *For $Q \geqq 0$, $-\frac{1}{2} \leqq \sigma \leqq \frac{1}{2}$ we have*

$$\text{(5.1)}\quad \left|\frac{\Gamma\left(\frac{Q}{2}+\frac{1-s}{2}\right)}{\Gamma\left(\frac{Q}{2}+\frac{s}{2}\right)}\right| \leqq \left(\frac{1}{2}\,|\,Q+1+s|\right)^{\frac{1}{2}-\sigma}.$$

REMARK. Equality holds here for $\sigma=\frac{1}{2}$, any $Q \geqq 0$, and for $\sigma=-\frac{1}{2}$ with $Q=0$.

PROOF. Let us put $a=-\frac{1}{2}$, $b=\frac{1}{2}$ and

$$f(s) = \frac{\Gamma\left(\frac{Q}{2}+\frac{1-s}{2}\right)}{\Gamma\left(\frac{Q}{2}+\frac{s}{2}\right)},$$

which is regular in the strip $S(-\frac{1}{2}, \frac{1}{2})$. We have, because of $|\Gamma(s)|=|\Gamma(\bar{s})|$,

$$\text{(5.2)}\quad \begin{aligned} |f(a+it)| &= \left|\frac{\Gamma\left(\frac{Q}{2}+\frac{1}{2}+\frac{1}{4}-\frac{it}{2}\right)}{\Gamma\left(\frac{Q}{2}-\frac{1}{4}+\frac{it}{2}\right)}\right| = \left|\frac{Q}{2}-\frac{1}{4}+\frac{it}{2}\right| \\ &= \frac{1}{2}\left|Q-\frac{1}{2}+it\right| \leqq \frac{1}{2}\left|Q+\frac{1}{2}+it\right| = \frac{1}{2}\left|Q+1+a+it\right|, \end{aligned}$$

and similarly

$$|f(b+it)| = 1.$$

With $\alpha=1$, $\beta=0$ Theorem 2 is applicable with $Q+1$ instead of Q and yields (5.1).

For $Q \geqq \frac{1}{2}$ this lemma can be sharpened to

LEMMA 2. *For $Q-\frac{1}{2}\geqq 0$, $-\frac{1}{2}\leqq\sigma\leqq\frac{1}{2}$ the inequality*

$$\left|\frac{\Gamma\left(\frac{Q}{2}+\frac{1-s}{2}\right)}{\Gamma\left(\frac{Q}{2}+\frac{s}{2}\right)}\right| \leqq \left(\frac{1}{2}\left|Q+s\right|\right)^{\frac{1}{2}-\sigma} \tag{5.4}$$

holds.

REMARK. Equality in (5.3) on both boundaries of $S(-\frac{1}{2},\frac{1}{2})$ for all $Q\geqq\frac{1}{2}$.

PROOF. The proof is almost the same as the previous one. Only instead of (5.2) we write

$$|f(a+it)| = \tfrac{1}{2}\left|Q-\tfrac{1}{2}-it\right| = \tfrac{1}{2}\left|Q+a+it\right|,$$

which again permits the application of Theorem 2 for $Q+a=Q-\frac{1}{2}>0$. For $Q-\frac{1}{2}=0$ formula (5.3) follows by a passage to the limit.

LEMMA 3. *For $Q\geqq 0$, $-\frac{1}{2}\leqq\sigma\leqq\frac{1}{2}$ we have*

$$\left|\frac{\Gamma(Q+1-s)}{\Gamma(Q+s)}\right| \leqq \left|Q+1+s\right|^{1-2\sigma}. \tag{5.4}$$

REMARK. Equality holds here on the boundaries under the same circumstances as in Lemma 1.

PROOF. We have again $a=-\frac{1}{2}$, $b=\frac{1}{2}$. For

$$f(s) = \frac{\Gamma(Q+1-s)}{\Gamma(Q+s)}$$

we find easily

$$|f(a+it)| = \left|Q+\tfrac{1}{2}+it\right|\cdot\left|Q-\tfrac{1}{2}+it\right| \leqq \left|Q+\tfrac{1}{2}+it\right|^2 = \left|Q+1+a+it\right|^2$$

and

$$|f(b+it)| = 1.$$

Theorem 2 with $\alpha=2$, $\beta=0$ and $Q+1$ instead of Q yields immediately (5.4).

6. We apply our results to some classes of L- and ζ-functions. Let $k>1$ be an integer and $\chi(n)$ a primitive character modulo k. For $\sigma>1$ one defines

$$L(s,\chi) = \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}.$$

This function with its analytic continuation is regular in the whole s-plane. For $\eta>0$ we have

$$|L(1+\eta+it,\chi)| \leqq \sum_{n=1}^{\infty}\frac{1}{n^{1+\eta}} = \zeta(1+\eta). \tag{6.1}$$

On the other hand, $L(s,\chi)$ satisfies the functional equation

$$\left(\frac{\pi}{k}\right)^{-\frac{s}{2}}\Gamma\left(\frac{a+s}{2}\right)L(s,\chi) = \varepsilon(\chi)\left(\frac{\pi}{k}\right)^{-\frac{1-s}{2}}\Gamma\left(\frac{a+1-s}{2}\right)L(1-s,\bar{\chi}),$$

where $\varepsilon(\chi)$ is a certain root of unity depending on χ and $a=0$ or 1 so that

$$\chi(-1)=(-1)^a.$$

We have therefore

$$|L(s,\chi)|=\left(\frac{\pi}{k}\right)^{\sigma-\frac{1}{2}}\left|\frac{\Gamma\left(\frac{a}{2}+\frac{1-s}{2}\right)}{\Gamma\left(\frac{a}{2}+\frac{s}{2}\right)}\right|\,|L(1-s,\bar{\chi})|.$$

For $-\frac{1}{2}\leqq\sigma\leqq\frac{1}{2}$ we apply on the Γ-quotient Lemma 1 or 2 according to the values 0 or 1 of a. This gives

$$|L(s,\chi)|\leqq\left(\frac{2\pi}{k}\right)^{\sigma-\frac{1}{2}}|1+s|^{\frac{1}{2}-\sigma}\,|L(1-s,\bar{\chi})|,$$

and in particular for $0<\eta\leqq\frac{1}{2}$

$$|L(-\eta+it,\chi)|\leqq\left(\frac{k}{2\pi}\right)^{\eta+\frac{1}{2}}|1+s|^{\eta+\frac{1}{2}}\zeta(1+\eta). \tag{6.2}$$

From (6.1) and (6.2) we derive then through Theorem 2 the result:

THEOREM 3. *For $-\frac{1}{2}\leqq-\eta\leqq\sigma\leqq1+\eta\leqq\frac{3}{2}$, for all moduli $k>1$ and all primitive characters χ modulo k the inequality*

$$|L(s,\chi)|\leqq\left(\frac{k|1+s|}{2\pi}\right)^{\frac{1+\eta-\sigma}{2}}\zeta(1+\eta)$$

holds.

It will be noticed that k and t appear here in the same order of magnitude.

7. The Dedekind zetafunction $\zeta_K(s)$ for the algebraic number field K of degree n is defined for $\sigma>1$ as

$$\zeta_K(s)=\sum_{\mathfrak{a}}\frac{1}{N(\mathfrak{a})^s}=\prod_{\mathfrak{p}}\frac{1}{1-N(\mathfrak{p})^{-s}},$$

where $\mathfrak{a}$ runs through all ideals and $\mathfrak{p}$ through all prime ideals of K. Any rational prime number p has the prime ideal decomposition

$$(p)=\mathfrak{p}_1^{e_1}\dots\mathfrak{p}_k^{e_k},$$

$$N(\mathfrak{p}_j)=p^{f_j},$$

so that

$$e_1f_1+\dots+e_kf_k=n.$$

Therefore we can write

$$\zeta_K(s)=\prod_p\left\{\frac{1}{1-p^{-f_1s}}\cdots\frac{1}{1-p^{-f_ks}}\right\},$$

hence

$$|\zeta_K(s)|\leqq\prod_p\left\{\frac{1}{1-p^{-f_1\sigma}}\cdots\frac{1}{1-p^{-f_k\sigma}}\right\}$$

$$\leqq\prod_p\left\{\frac{1}{1-p^{-\sigma}}\right\}^k\leqq\prod_p\left\{\frac{1}{1-p^{-\sigma}}\right\}^n=\zeta(\sigma)^n$$

or, for $\eta > 0$,

(7.1) $$|\zeta_K(1+\eta+it)| \leqq \zeta(1+\eta)^n$$

(which is, of course, a rather crude estimate, since $\zeta_K(s)$ has only a pole of first order at $s=1$). The function $\zeta_K(s)$ satisfies after HECKE the functional equation ([4], p. 74)

(7.11) $$A^s \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \zeta_K(s) = A^{1-s} \Gamma\left(\frac{1-s}{2}\right)^{r_1} \Gamma(1-s)^{r_2} \zeta_K(1-s),$$

where

(7.12) $$A = 2^{-r_2} \pi^{-\frac{n}{2}} \sqrt{|d|},$$

and d is the discriminant of the field K, which has among its conjugates r_1 real and $2r_2$ complex fields, $r_1 + 2r_2 = n$.

We have thus

$$|\zeta_K(s)| = A^{1-2\sigma} \left| \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \right|^{r_1} \left| \frac{\Gamma(1-s)}{\Gamma(s)} \right|^{r_2} |\zeta_K(1-s)|$$

and, from Lemmas 1 and 3, for $-\frac{1}{2} \leqq \sigma \leqq \frac{1}{2}$

$$|\zeta_K(s)| \leqq A^{1-2\sigma} (\tfrac{1}{2}|1+s|)^{r_1(\frac{1}{2}-\sigma)} |1+s|^{r_2(1-2\sigma)} |\zeta_K(1-s)|$$

and in particular for $0 < \eta \leqq \frac{1}{2}$, in view of (7.1)

(7.2) $$|\zeta_K(-\eta+it)| \leqq A^{1+2\eta} 2^{-r_1(\frac{1}{2}+\eta)} |1-\eta+it|^{n(\frac{1}{2}+\eta)} \zeta(1+\eta)^n.$$

Theorem 2 can be applied only to regular functions. We consider therefore $\zeta_K(s)(s-1)$. From (7.1) and (7.2) we obtain

(7.31) $$|\zeta_K(1+\eta+it)(\eta+it)| < \zeta(1+\eta)^n |1+(1+\eta+it)|$$

and

(7.32) $$\left\{ \begin{aligned} &|\zeta_K(-\eta+it)(-\eta+it-1)| \\ &\qquad \leqq \zeta(1+\eta)^n A^{1+2\eta} 2^{-r_1(\frac{1}{2}+\eta)} \frac{1+\eta}{1-\eta} |1-\eta+it|^{n(\frac{1}{2}+\eta)+1} \end{aligned} \right.$$

since

$$|-\eta+it-1| \leqq \frac{1+\eta}{1-\eta} |1-\eta+it|.$$

From (7.31), (7.32) we infer now by means of Theorem 2 that in the strip $-\eta \leqq \sigma \leqq 1+\eta$

$$|\zeta_K(s)(s-1)| \leqq \zeta(1+\eta)^n |1+s| \left\{ A^{1+2\eta} 2^{-r_1(\frac{1}{2}+\eta)} \frac{1+\eta}{1-\eta} |1+s|^{n(\frac{1}{2}+\eta)} \right\}^{\frac{1+\eta-\sigma}{1+2\eta}}.$$

With $1 < \frac{1+\eta}{1-\eta} \leqq 3$ for $0 < \eta \leqq \frac{1}{2}$, and with the meaning of A in (7.12) we obtain thus

THEOREM 4. *In the strip* $-\eta \leqq \sigma \leqq 1+\eta$, $0 < \eta \leqq \frac{1}{2}$, *the Dedekind function* $\zeta_K(s)$ *belonging to the algebraic number field* K *of degree* n *and discriminant* d *satisfies the inequality*

(7.4) $$|\zeta_K(s)| \leqq 3 \left| \frac{1+s}{1-s} \right| \left(|d| \left(\frac{|1+s|}{2\pi} \right)^n \right)^{\frac{1+\eta-\sigma}{2}} \zeta(1+\eta)^n.$$

We know through E. ARTIN and R. BRAUER [1] that RIEMANN'S $\zeta(s)$ divides DEDEKIND'S $\zeta_K(s)$ in case K is normal over the rational field. We can use this fact to obtain a sharper version of the previous theorem.

In R. BRAUER'S notation the quotient, which is an entire function, appears as

$$\frac{\zeta_K(s)}{\zeta(s)} = \prod_\varrho L(s;\omega_\varrho^*)^{c_\varrho} \tag{7.5}$$

where $c_\varrho > 0$ and the functions under the product sign are L-functions for certain congruence characters ω_ϱ^*. Let us put

$$\sum c_\varrho = q = q(K). \tag{7.51}$$

Included in (7.5) is, in view of the class field theory, the case of Abelian K, for which $c_\varrho = 1$, $q \leqq n-1$. Moreover, as Prof. RICHARD BRAUER pointed out to me in a letter of August 10, 1957, the inequality

$$q \leqq n-1 \tag{7.52}$$

holds also for all normal fields. From (7.5) and (6.1) follows then, for $\eta > 0$

$$\left|\frac{\zeta_K(1+\eta+it)}{\zeta(1+\eta+it)}\right| \leqq \zeta(1+\eta)^{n-1}. \tag{7.53}$$

On the other hand, the functional equation (7.11) of $\zeta_K(s)$ together with that of $\zeta(s)$ furnishes

$$\frac{\zeta_K(s)}{\zeta(s)} = A^{1-2s}\pi^{\frac12 - s}\left(\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\right)^{r_1-1}\left(\frac{\Gamma(1-s)}{\Gamma(s)}\right)^{r_2}\frac{\zeta_K(1-s)}{\zeta(1-s)}. \tag{7.6}$$

If $r_1 \geqq 1$ we obtain from Lemmas 1 and 3, for $-\frac12 \leqq \sigma \leqq \frac12$

$$\left|\frac{\zeta_K(s)}{\zeta(s)}\right| \leqq A^{1-2\sigma}\pi^{\frac12-\sigma}\left(\frac{1}{2}|1+s|\right)^{(r_1-1)(\frac12-\sigma)}|1+s|^{r_2(1-\sigma)}\left|\frac{\zeta_K(1-s)}{\zeta(1-s)}\right|$$

and in particular with $0 < \eta \leqq \frac12$

$$\left|\frac{\zeta_K(-\eta+it)}{\zeta(-\eta+it)}\right| \leqq A^{1+2\eta}\pi^{\frac12+\eta}2^{-(r_1-1)(\frac12+\eta)}|1+s|^{(n-1)(\frac12+\eta)}\zeta(1+\eta)^{n-1}. \tag{7.7}$$

From this and (7.53) we obtain just as before by means of Theorem 2 the estimation

$$\left|\frac{\zeta_K(s)}{\zeta(s)}\right| \leqq \left(|d|\left(\frac{|1+s|}{2\pi}\right)^{n-1}\right)^{\frac{1+\eta-\sigma}{2}}\zeta(1+\eta)^{n-1}, \quad -\eta \leqq \Re(s) \leqq 1+\eta. \tag{7.8}$$

If $r_1 = 0$ we write instead of (7.6)

$$\frac{\zeta_K(s)}{\zeta(s)} = A^{1-2s}\pi^{\frac12-s}\left(\frac{\Gamma(1-s)}{\Gamma(s)}\right)^{r_2-1}\frac{\Gamma(1-s)}{\Gamma(s)}\,\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\,\frac{\zeta_K(1-s)}{\zeta(1-s)}$$

$$= (2A)^{1-2s}\pi^{\frac12-s}\left(\frac{\Gamma(1-s)}{\Gamma(s)}\right)^{r_2-1}\frac{\Gamma\left(\frac12+\frac{1-s}{2}\right)}{\Gamma\left(\frac12+\frac{s}{2}\right)}\,\frac{\zeta_K(1-s)}{\zeta(1-s)}.$$

The application of Lemmas 1 and 2 yields also here exactly (7.7), and (7.8) can then be established as before.

We have thus:

THEOREM 4a. *If K is a normal field over the rational field then in the strip $-\eta \leqq \sigma \leqq 1+\eta$, $0<\eta\leqq\frac{1}{2}$ the Dedekind function $\zeta_K(s)$ satisfies*

$$|\zeta_K(s)| \leqq |\zeta(s)| \left(|d| \left(\frac{|1+s|}{2\pi}\right)^{n-1}\right)^{\frac{1+\eta-\sigma}{2}} \zeta(1+\eta)^{n-1}.$$

For Abelian fields the Theorem 4a is a direct consequence of Theorem 3 because of $c_\varrho = 1$ and because the discriminant d is the product of the conductors k_ϱ of the characters ω_ϱ^*.

8. The full strength of the uniformity with respect to the parameter Q which Theorem 2 and Lemma 3 show is established only in their application to HECKE's $\zeta(s, \lambda)$-functions.

Let K be any algebraic number field and $\mathfrak{f}$ an ideal in K. The Hecke functions are then defined as

$$\zeta(s, \lambda) = \sum_{\mathfrak{m}} \frac{\lambda(\mathfrak{m})}{N(\mathfrak{m})^s} = \sum_{(\hat{\mu})} \frac{\lambda(\hat{\mu})}{|N(\hat{\mu})|^s}, \qquad \Re(s) > 1, \tag{8.1}$$

where $\mathfrak{m}$ runs over all ideals of K, $\hat{\mu}$ over all corresponding non-associate ideal numbers, and $\lambda(\mathfrak{m}) = \lambda(\hat{\mu})$ is a "Größencharacter for ideals modulo $\mathfrak{f}$" defined as

$$\lambda(\hat{\mu}) = \prod_{q=1}^{r_1+r_2} |\hat{\mu}^{(q)}|^{-iv_q} \prod_{p=r_1+1}^{n} \left(\frac{\hat{\mu}^{(p)}}{|\hat{\mu}|^{(q)}}\right)^{a_p} \tag{8.2}$$

where the a_p ($p=r_1+1, \ldots, n$) are non-negative integers with the condition $a_p \cdot a_{p+r_2} = 0$, and the v_q ($q=1, \ldots, r_1+r_2$) certain real numbers, whose definition does not concern us here ([3], p. 30 and [5], p. 343).

From (8.1) and (8.2) follows immediately

$$|\zeta(s, \lambda)| \leqq \zeta_K(\sigma), \qquad \sigma > 1. \tag{8.3}$$

We put with HECKE

$$\xi(s, \lambda) = \gamma(\lambda) \Gamma(s, \lambda) A^s \zeta(s, \lambda),$$

where

$$|\gamma(\lambda)| = 1,$$

$$A^2 = \frac{|d|\, N(\mathfrak{f})}{\pi^n 2^{2r_2}},$$

$$\Gamma(s, \lambda) = \prod_{q=1}^{r_1} \Gamma\left(\frac{a_q + s + iv_q}{2}\right) \cdot \prod_{p=r_1+1}^{r_1+r_2} \Gamma\left(\frac{a_p + a_{p+r_2}}{2} + s + \frac{iv_p}{2}\right).$$

Here the a_q for $q=1, \ldots r_1$, which are not explicitly mentioned in (8.2), are 0 or 1 and are determined by the character χ belonging to the congruence group modulo $\mathfrak{f}$ in the narrowest sense and induced by the Größencharacter λ ([3], p. 21). The $\xi(s, \lambda)$-functions with primitive λ-character modulo $\mathfrak{f}$ fulfill a functional equation

$$\xi(s, \lambda) = W(\lambda)\, \xi(1-s, \bar{\lambda}),$$

where $W(\lambda)$ is a certain number of absolute value 1, so that

$$|\zeta(s, \lambda)| = A^{1-2\sigma} \prod_{q=1}^{r_1} \left| \frac{\Gamma\left(\frac{a_q + 1 - s - i v_q}{2}\right)}{\Gamma\left(\frac{a_q + s + i v_q}{2}\right)} \right| \times$$

$$\times \prod_{p=r_1+1}^{r_1+r_2} \left| \frac{\Gamma\left(\frac{a_p + a_{p+r_2}}{2} + 1 - s - \frac{i v_p}{2}\right)}{\Gamma\left(\frac{a_p + a_{p+r_2}}{2} + s + \frac{i v_p}{2}\right)} \right| \cdot |\zeta(1-s, \bar{\lambda})|.$$

If we apply on the first product, where $a_q = 0$ or 1, the Lemmas 1 and 2, and on the second product, where $(a_p + a_{p+r_2})$ are unbounded non-negative integers, the Lemma 3, we obtain for $-\frac{1}{2} \leqq \sigma \leqq \frac{1}{2}$

$$|\zeta(s, \lambda)| \leqq A^{1-2\sigma} \prod_{q=1}^{r_1} \left(\frac{1}{2} |1 + s + i v_q|\right)^{\frac{1}{2}-\sigma} \times$$

$$\times \prod_{p=r_1+1}^{r_1+r_2} \left| \frac{a_p + a_{p+r_2}}{2} + 1 + s + \frac{i v_p}{2} \right|^{1-2\sigma} \cdot |\zeta(1-s, \lambda)|$$

and in particular for $s = -\eta + it$, $0 \leqq \eta \leqq \frac{1}{2}$ in virtue of (8.3),

$$(8.4) \quad \begin{cases} |\zeta(-\eta + it, \lambda)| \leqq A^{1+2\eta} \prod_{q=1}^{r_1} \left(\frac{1}{2} |1 - \eta + i(t + v_q)|\right)^{\frac{1}{2}+\eta} \times \\ \times \prod_{p=r_1+1}^{r_1+r_2} \left| \frac{a_p + a_{p+r_2}}{2} + 1 - \eta + i\left(t + \frac{v_p}{2}\right) \right|^{1+2\eta} \cdot \zeta_K(1+\eta). \end{cases}$$

We have not prepared a theorem which we could apply here directly as we could use Theorem 2 in the previous cases. We construct therefore a function $F(s)$, on which the Phragmén-Lindelöf argument can be applied, out of factors which are furnished by Theorem 1. This theorem yields functions according to the given boundary conditions, and we select, with $a = -\eta$, $b = 1 + \eta$ the two functions

$$\varphi_1(s; 1) \quad \text{with} \quad \gamma = -\tfrac{1}{2} - \eta, \quad \delta = 0$$

and

$$\varphi_2(s; Q) \quad \text{with} \quad \gamma = -1 - 2\eta, \quad \delta = 0$$

which show, in view of (2.3), the properties

$$(8.5) \quad \begin{cases} |\varphi_1(s; 1)| \geqq |1 + s|^{-\frac{1}{2}(1+\eta-\sigma)} \\ |\varphi_2(s; Q)| \geqq |Q + s|^{-(1+\eta-\sigma)} \end{cases}$$

for $-\eta \leqq \sigma \leqq 1 + \eta$, $Q - \eta > 0$.

Then

$$F(s) = \zeta(s, \lambda) \cdot \prod_{q=1}^{r_1} \varphi_1(s + i v_q; 1) \cdot \prod_{p=r_1+1}^{r_1+r_2} \varphi_2\left(s + \frac{i v_p}{2}; 1 + \frac{a_p + a_{p+r_2}}{2}\right) \cdot E e^{\nu s},$$

where E and ν are so determined that

$$E e^{-\nu\eta} = \left(A \cdot 2^{-\frac{r_1}{2}}\right)^{-1-2\eta},$$

$$E e^{\nu(1+\eta)} = 1,$$

has because of (8.3) and (8.4) the properties

$$|F(-\eta + i t)| \leqq \zeta_K(1+\eta)$$

$$|F(1+\eta + i t)| \leqq \zeta_K(1+\eta)$$

and therefore satisfies after PHRAGMÉN and LINDELÖF also

$$|F(s)| \leqq \zeta_K(1+\eta)$$

for $-\eta \leqq \Re(s) \leqq 1+\eta$.

From this we infer, valid in the same strip,

$$|\zeta(s, \lambda)| \leqq \zeta_K(1+\eta) \prod_{q=1}^{r_1} |\varphi_1(s + i v_q; 1)|^{-1} \times$$

$$\times \prod_{p=r_1+1}^{r_1+r_2} \left|\varphi_2\left(s + \frac{i v_p}{2}; 1 + \frac{a_p + a_{p+r_2}}{2}\right)\right|^{-1} \cdot E^{-1} e^{-\nu\sigma}.$$

If we now use (8.5) and take into account also (7.1) we obtain the

THEOREM 5. *Let for the field K of degree n, which is among r_1 real and $2r_2$ imaginary conjugate fields of discriminant d the Hecke function $\zeta(s, \lambda)$ be given by* (8.1) *with primitive Größencharacter $\lambda(\hat{\mu})$ modulo $\mathfrak{f}$ of* (8.2). *Then $\zeta(s, \lambda)$ satisfies in the strip $-\eta \leqq \sigma \leqq 1+\eta$, $0 < \eta \leqq \frac{1}{2}$ the inequality*

$$|\zeta(s, \lambda)| \leqq \zeta(1+\eta)^n \left\{\frac{|d|\, N(\mathfrak{f})}{(2\pi)^n} \prod_{q=1}^{r_1} |1 + s + i v_q| \times \right.$$

$$\left. \times \prod_{p=r_1+1}^{r_1+r_2} \left|1 + \frac{a_p + a_{p+r_2}}{2} + s + \frac{i v_p}{2}\right|^2\right\}^{\frac{1+\eta-\sigma}{2}}.$$

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