Werk

Titel: On the Free Product of Associative Rings.

Autor: Cohn, P.M.

Jahr: 1959

PURL: https://resolver.sub.uni-goettingen.de/purl?266833020_0071|log46

Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

COHN, P. M.
Math. Zeitschr. 71, 380–398 (1959)

On the Free Product of Associative Rings

By
P. M. COHN

Introduction

A ring R is said to be the free product of a family of rings R_ι ($\iota \in I$) if the R_ι are subrings of R such that (i) R is generated by the "factors" R_ι, (ii) every relation in R is a consequence of relations in the factors R_ι, and (iii) different factors intersect in the same subring Λ of R. The object of this paper is to give conditions under which the free product of associative rings exists.

In order to describe these conditions we require the notions of "flatness" and "purity". Given any associative ring Λ with a unit-element, let U be a right Λ-module with a submodule U' and let V be a left Λ-module; then we may form the (not necessarily exact) sequence

$$0 \to U' \otimes_\Lambda V \to U \otimes_\Lambda V. \tag{0.1}$$

If, for a given V, (0.1) is exact for all U, U', then V is said to be (left) *flat*; if U and U' are given and (0.1) is exact for all V, then U' is said to be (right) *pure* in U. The definitions of "right flat" and "left pure" are similar (cf. § 2). For brevity we shall call a ring R with Λ as subring (with the same unit-element as R) a Λ-ring (cf. § 1). It is then clear that R may be considered as a Λ-bimodule in a natural way. Now we have the following conditions for the existence of free products.

I. Given any family of Λ-rings R_ι ($\iota \in I$), if each R_ι/Λ is left flat, then the free product R of the R_ι exists and R/Λ is again left flat. A similar assertion holds with "left" replaced by "right" throughout.

II. Given a Λ-ring R, if for every Λ-ring S the free product of R and S exists, then Λ is both right pure and left pure in R.

The sufficient condition I is proved in § 4 (Theorem 4.5) and the necessary condition II in § 5 (Theorem 5.2 Cor.). These sections also contain some further conditions and a result on the structure of the free product (Theorem 4.6). It seems likely that more precise conditions for the existence of the free product could be established in terms of the Tor groups of the factors, but a criterion of this sort might be rather complicated, and hence more difficult to apply.

A number of special cases of I and II are perhaps worth mentioning. From I and a theorem of M. HARADA and M. AUSLANDER [*1*] we deduce that the free product of Λ-rings always exists if Λ is a regular ring. This includes the elementary (and probably well known) case where Λ is a division ring.

From II, together with certain conditions for purity given in § 5, examples are readily constructed of rings whose free product does not exist. Such rings may be obtained e.g. by taking Λ to be the ring of polynomials in one variable over a field.

In § 3 we prove some general theorems on free products and universal products of rings which have for the most part analogues in group theory[1]). Indeed many could be formulated in the general theory of algebraic systems and are no doubt well known, but no convenient reference seems to be available[2]).

In the final section (§ 6) we consider the case where Λ is without a unit-element, but we confine our attention throughout to the case were all the intersections of pairs of factors coincide, corresponding to the case of the product of groups with a single amalgamated subgroup. This case was first treated by O. SCHREIER [*14*], who showed that such a free product always exists. More recently it has been shown by Z. E. DIDIDSE [*8*] that in the case of non-associative linear algebras[3]) the free product with one amalgamated subalgebra always exists, in contrast to the situation in the associative case, treated here.

1. Λ-rings

Let Λ be a ring[4]) with a unit-element, denoted by 1. A ring R is said to be a *Λ-ring*, if it is a unital Λ-bimodule such that[5])

$$(1.1)\quad \lambda(xy)=(\lambda x)y, \qquad (x\lambda)y=x(\lambda y), \qquad (xy)\lambda=x(y\lambda) \qquad (x,y\in R,\lambda\in\Lambda).$$

The Λ-rings form a class of algebras[6]), the operations being the ring-operations together with the unary operations of multiplying by the elements of Λ (on the right or left). In this sense we understand by a subring of a Λ-ring a subring admitting Λ, and homomorphisms, isomorphisms etc. of Λ-rings are understood to be Λ-homomorphisms, Λ-isomorphisms etc.

If the Λ-ring R has a unit-element e, say, and if $\lambda e=e\lambda$ for all $\lambda\in\Lambda$, then it is easily seen, by (1.1), that the mapping $\varphi:\lambda\to e\lambda$ is a (ring-)homomorphism of Λ into R. If moreover Λ acts faithfully on R, then φ is a monomorphism, and the embedding so defined allows us to identify Λ with a subring of R. The unit-elements of Λ and R then coincide and both may be denoted by 1. Conversely, if R is any ring with a unit-element 1, and Λ any subring of R which contains 1, then R may be regarded as a Λ-ring in a natural way.

---

[1]) Cf. e.g. B. H. NEUMANN [*12*].

[2]) R. SIKORSKI [*15*] treats the problem of the existence of the free product of general algebraic systems, but his work does not overlap ours in any way.

[3]) Cf. also KUROŠ [*11*].

[4]) All rings and algebras are understood to be associative.

[5]) In the more usual case of a Λ-algebra the operator domain is essentially commutative (BOURBAKI [*3*] p. 101). The above definition is wider in that it allows any ring with 1 as unit-element and containing Λ as a subring to be regarded as a Λ-ring.

[6]) Cf. BIRKHOFF [*2*] p. vii.

The Λ-ring R will be said to be a Λ-ring *in the strict sense*, if R has a unit-element and Λ acts faithfully on R (qua set of right multiplications), to distinguish it from a Λ-ring *in the wide sense*, which also has a unit-element, but where Λ need not act faithfully. We shall only be dealing with Λ-rings with a unit-element, and they will nearly always be Λ-rings in the strict sense; we therefore drop the epithet in this case, so that from now on "Λ-ring" stands for "Λ-ring in the strict sense". The above remarks show that a Λ-ring is then essentially no more than a ring with Λ as a subring. The advantage of the terminology lies in the fact that a Λ-ring contains Λ in a canonical fashion. We note further that in requiring our rings to have a unit-element 1, we admit as subrings only the rings containing 1 (and hence containing Λ), and a homomorphism from R to S must map the 1 of R on the 1 of S. In other words, we regard 1 as a 0-ary operator on our class of algebras. From this point of view Λ itself may be regarded as the free Λ-ring on 0 free generators. Similarly a Λ-ring in the wide sense is essentially a ring containing a homomorphic image of Λ as a subring in a canonical fashion.

2. Flat modules and pure submodules

For the rest of this paper, except § 6, Λ will be a fixed (but arbitrary) ring with 1. Unless the contrary is stated, all tensor products are taken over Λ, and we shall write Tor for Tor_1^Λ (cf. CARTAN & EILENBERG [5] p. 107).

A left Λ-module V is said to be *flat* if for every right Λ-module X, $\mathrm{Tor}(X, V) = 0$. Clearly this is equivalent to the definition given in the introduction (cf. [5] p. 122). A corresponding definition applies for right Λ-modules. It is easily verified that (i) every free module is flat, (ii) every direct summand of a flat module is flat, and (iii) if V has a submodule V' such that both V' and V/V' are flat, then so is V. The last property follows from the exact homology sequence of the tensor product ([5] p. 39). The homology sequence can also be used to show that a submodule of a flat module is not necessarily flat. A further property which will be used is given in

LEMMA 2.1. *Let S be any ring with 1. If U is an (S, Λ)-bimodule*[7]) *which is flat, qua left S-module, and V is a flat left Λ-module, then $U \otimes_\Lambda V$, qua left S-module, is flat.*

For let

$$0 \to X' \to X \tag{2.1}$$

be an exact sequence of right S-modules. Since U is a flat left S-module, the sequence

$$0 \to X' \otimes_S U \to X \otimes_S U$$

is exact, and since V is a flat left Λ-module, the sequence

$$0 \to X' \otimes_S U \otimes_\Lambda V \to X \otimes_S U \otimes_\Lambda V$$

is exact. The result now follows from the associativity of the tensor product.

[7]) Cf. CARTAN & EILENBERG [5] p. 22.

Given a right Λ-module U with a submodule U', we defined U' to be *pure* in U in the introduction, if the sequence

$$0 \to U' \otimes V \to U \otimes V \tag{2.2}$$

is exact for all left Λ-modules V. Equivalently we may define U' to be pure in U if the natural homomorphism $\mathrm{Tor}(U, V) \to \mathrm{Tor}(U/U', V)$ is an epimorphism. In particular, U' is pure in U if U/U' is flat (but not conversely). More generally, by considering the exact sequence

$$\mathrm{Tor}(U', V) \to \mathrm{Tor}(U, V) \to \mathrm{Tor}(U/U', V)$$

for the case where U' is flat, we obtain

THEOREM 2.2. *Let U be a right Λ-module with submodule U' and suppose that* (i) *U' is flat and* (ii) *U' is pure in U. Then for every left Λ-module V, the natural homomorphism*

$$\mathrm{Tor}(U, V) \to \mathrm{Tor}(U/U', V)$$

is an isomorphism. In particular, under the given conditions U is flat if and only if U/U' is flat.

In order to obtain a criterion for purity we require a lemma.

LEMMA 2.3. *If V is a finitely presented left Λ-module, with generators $v_1^*, \ldots, v_n^*$ and relations $\sum \lambda_{ij} v_j^* = 0$ $(i = 1, \ldots, m)$, and if for some set of elements $u_1, \ldots, u_n$ in a right Λ-module U, we have*

$$\sum u_j \otimes v_j^* = 0 \qquad \text{in } U \otimes V,$$

then $u_j = \sum a_i \lambda_{ij}$, for some elements $a_i \in U$.

Proof. Let F be the free left Λ-module on $v_1, \ldots, v_n$ as free generators, and let R be the submodule of F generated by $\sum \lambda_{ij} v_j$ $(i = 1, \ldots, m)$. The mapping $v_j \to v_j^*$ may be extended to a homomorphism π of F into V; from the definition of V it follows that π is an epimorphism with kernel R. We therefore have the exact sequence

$$0 \to R \xrightarrow{\iota} F \xrightarrow{\pi} V \to 0,$$

which induces the exact sequence

$$U \otimes R \xrightarrow{\iota'} U \otimes F \xrightarrow{\pi'} U \otimes V \to 0. \tag{2.3}$$

We are given $(\sum u_j \otimes v_j)\pi' = 0$, hence by the exactness of (2.3),

$$\sum u_j \otimes v_j = (\sum a_i \otimes \lambda_{ij} v_j)\, \iota',$$

for some elements $a_i \in U$. Now ι' is the homomorphism induced by the inclusion mapping $\iota: R \to F$, and F is free, hence equating the coefficients of v_j, we obtain $u_j = \sum a_i \lambda_{ij}$, as we wished to show.

We can now state a necessary and sufficient condition for a Λ-submodule to be pure.

THEOREM 2.4. *Let U be a right Λ-module. Then a submodule U' of U is pure in U if and only if, for any finite sets of elements $u_i \in U$, $a_j \in U'$, $\lambda_{ij} \in \Lambda$ $(i = 1, \ldots, m;\ j = 1, \ldots, n)$, the relations*

$$a_j = \sum u_i \lambda_{ij} \tag{2.4}$$

imply the existence of elements $u_i' \in U'$ such that $a_j = \sum u_i' \lambda_{ij}$. Moreover, it is sufficient for the purity of U' in U that the condition hold when the u_i are chosen from a fixed generating set of U.

The corresponding criterion for left modules follows by symmetry and will not be stated separately.

Proof. If U' is pure in U and $a_j = \sum u_i \lambda_{ij}$, let V be the left Λ-module with generators $v_1, \ldots, v_n$ and defining relations $\sum \lambda_{ij} v_j = 0$. Then we have

$$\sum a_j \otimes v_j = \sum u_i \otimes \lambda_{ij} v_j = 0 \qquad \text{in } U \otimes V.$$

From the purity of U' in U we conclude that $\sum a_j \otimes v_j = 0$ in $U' \otimes V$. But this means that $a_j = \sum u_i' \lambda_{ij}$ for some $u_i' \in U'$, by Lemma 2.3.

Conversely, suppose that U' satisfies the condition of the enunciation for all finite sets u_i contained in a fixed generating set of U, and let V be any finitely presented left Λ-module, with generators v_j and relations $\sum \lambda_{ij} v_j = 0$ $(i = 1, \ldots, m;\ j = 1, \ldots, n)$. If in $U \otimes V$

$$\sum a_r \otimes b_r = 0,$$

where $a_r \in U'$, $b_r = \sum \alpha_{rj} v_j \in V$, then $\sum a_r \alpha_{rj} \otimes v_j = 0$, hence by Lemma 2.3,

$$\sum a_r \alpha_{rj} = \sum u_\nu \beta_{\nu i} \lambda_{ij}$$

for some elements $\sum u_\nu \beta_{\nu i} \in U$. Now $\sum a_r \alpha_{rj} \in U'$; by hypothesis there exist $u_\nu' \in U'$ such that

$$\sum a_r \alpha_{rj} = \sum u_\nu' \beta_{\nu i} \lambda_{ij}$$

and hence we have in $U' \otimes V$,

$$\begin{aligned} \sum a_r \otimes b_r = \sum a_r \alpha_{rj} \otimes v_j &= \sum u_\nu' \beta_{\nu i} \lambda_{ij} \otimes v_j \\ &= \sum u_\nu' \beta_{\nu i} \otimes \lambda_{ij} v_j \\ &= 0. \end{aligned}$$

This shows that for any finitely presented left Λ-module V, the sequence (2.2) is exact. Now any left Λ-module may be expressed as a direct limit of finitely presented modules, and since direct limits preserve exact sequences, (2.2) is exact for all V and the proof is complete.

From Theorem 2.4 it is not hard to deduce that in the case where Λ is a commutative principal ideal domain, the definition of a pure submodule reduces to the usual one (KAPLANSKY [*10*], p. 14): a submodule U' is pure in U if and only if $U' \cap U\lambda = U'\lambda$ for all $\lambda \in \Lambda$.

3. Free products and universal products

DEFINITION. A Λ-ring R is said to be the *free product* of the Λ-rings R_ι $(\iota \in I)$ if the R_ι form a family of subrings of R such that

(i) $R_\iota \cap R_\varkappa = \Lambda$ for $\iota \neq \varkappa$,

(ii) if X_ι is a set of generators of R_ι $(\iota \in I)$ then $\bigcup X_\iota$ is a set of generators of R,

(iii) if C_ι is a set of defining relations of R_ι $(\iota \in I)$ (in terms of the set of generators X_ι) then $\bigcup C_\iota$ is a set of defining relations of R.

This corresponds to the definition of a free product of groups with one amalgamated subgroup (cf. B. H. NEUMANN [*12*]); but since we are using Λ-rings, the ring Λ is embedded canonically in each factor, so that the isomorphisms between the copies of Λ in the different factors are specified automatically.

Given a family of Λ-rings R_ι, there may or may not exist a Λ-ring R which is the free product of the R_ι; in order to discuss conditions for the existence of the free product we shall introduce the "universal product" of the R_ι, which unlike the free product always exists.

THEOREM 3.1. *Let R_ι $(\iota \in I)$ be any family of Λ-rings in the wide sense, then there exists a Λ-ring in the wide sense R, with homomorphisms $\varphi_\iota : R_\iota \to R$ such that*

(i) *R is generated by the $R_\iota \varphi_\iota$,*

(ii) *given any Λ-ring in the wide sense S and any family of homomorphisms $\psi_\iota : R_\iota \to S$, there exists a homomorphism $\psi : R \to S$ satisfying*

$$\psi_\iota = \varphi_\iota \psi .$$

Moreover, R is determined up to isomorphism by (i) *and* (ii).

The Λ-ring in the wide sense R so defined is called the *universal product* of the R_ι, with the *canonical homomorphisms* φ_ι. The construction of R is an instance of BOURBAKI's "problème (*U*)" (BOURBAKI [*4*], p. 131; SAMUEL [*13*]), and we therefore only outline the proof.

We define R as the Λ-ring in the wide sense generated by the elements[8]) of all the rings R_ι, with the defining relations of each R_ι as the defining relations. If the element of R corresponding to an element x of R_ι is denoted by $x\varphi_\iota$, then φ_ι is easily seen to be a homomorphism of R_ι into R, and the φ_ι satisfy (i) by definition. If S is any Λ-ring in the wide sense and $\psi_\iota : R_\iota \to S$ any family of homomorphisms, then the elements

$$x_\iota \varphi_\iota \qquad (x_\iota \in R_\iota,\ \iota \in I) \tag{3.1}$$

form a set of generators of R, the elements

$$x_\iota \psi_\iota \qquad (x_\iota \in R_\iota,\ \iota \in I) \tag{3.2}$$

form a subset of S and by definition every relation holding between the elements (3.1) follows from the defining relations in the different R_ι, and therefore also holds between the elements (3.2). Hence the mapping $\psi : x_\iota \varphi_\iota \to x_\iota \psi_\iota$ of R into S may be extended to a homomorphism of R into S, again denoted by ψ; clearly ψ satisfies (ii). If S, ψ_ι also satisfy (i), (ii), then applying (ii)

[8]) Note that the unit-elements of the different rings all count as the same element here.

with R and S interchanged we obtain a homomorphism $\varphi: S \to R$ such that $\psi_\iota \varphi = \varphi_\iota$. In particular, for every $x_\iota \in R_\iota$, $x_\iota \varphi_\iota \psi \varphi = x_\iota \varphi_\iota$. Since the $x_\iota \varphi_\iota$ ($x_\iota \in R_\iota$, $\iota \in I$) generate R, we have $\psi\varphi = 1$ on a set of generators of R, and hence $\psi\varphi = 1$ on the whole of R. By symmetry $\varphi\psi = 1$ on S, so that $S \cong R$. This establishes the theorem.

Suppose now that the R_ι in Theorem 3.1 are Λ-rings (in the strict sense). Then the free product of the R_ι, when it exists, clearly satisfies (i) and (ii) and is therefore isomorphic to the universal product. Since the universal product is unique up to isomorphism, we have the

COROLLARY. *The free product of any family of Λ-rings, when it exists, is determined up to isomorphism.*

By examining the proof of Theorem 3.1 further, we derive

THEOREM 3.2. *Let R_ι ($\iota \in I$) be a family of Λ-rings and R their universal product. Then the free product of the R_ι exists if and only if*

(i) *the canonical homomorphism $\varphi_\iota: R_\iota \to R$ is a monomorphism,*

(ii) $R_\iota \varphi_\iota \cap R_\varkappa \varphi_\varkappa = \Lambda$ *for* $\iota \neq \varkappa$.

When these conditions hold, R is in fact the free product of the R_ι.

More generally we have the following criterion for the existence of free products. The proof is analogous to that of the (more general) criterion for the existence of generalised free products of groups, by HANNA NEUMANN, and will therefore be omitted (cf. B. H. NEUMANN [*12*], p. 521).

THEOREM 3.3. *Let R_ι ($\iota \in I$) be a family of Λ-rings. If there exists a Λ-ring S with monomorphisms $\psi_\iota: R_\iota \to S$ such that $R_\iota \psi_\iota \cap R_\varkappa \psi_\varkappa = \Lambda$, for $\iota \neq \varkappa$, then the free product of the R_ι exists.*

We shall find it more convenient to have a criterion in terms of modules; this may be obtained as follows.

Let R be any Λ-ring. By a *right R-module* V we shall understand a unital right Λ-module which is also a right R-module (in the usual sense) in such a way that the right Λ-module structure of V coincides with the right Λ-module structure defined on V by restricting R to Λ. The terms *left R-module*, *R-bimodule* are defined analogously. The module criterion for the existence of free products is now given by

THEOREM 3.4. *Let R_ι ($\iota \in I$) be a family of Λ-rings. If there exists a right Λ-module V such that for each $\iota \in I$, V is a right R_ι-module containing R_ι as submodule*[9]*, and $R_\iota \cap R_\varkappa = \Lambda$ for $\iota \neq \varkappa$, then the free product of the R_ι exists.*

Proof. For each $x \in R_\iota$, let $x\varrho_\iota$ be the endomorphism of V defined by x. If $x \in \Lambda$, then the resulting endomorphism is independent of ι, by definition; if $x \notin \Lambda$, then x belongs to at most one R_ι, by hypothesis, and hence ι is in this case determined by x. Now the right multiplications of V by elements of Λ form a ring isomorphic to Λ, which is a subring of the ring $\mathfrak{E}(V)$ of all endomorphisms of V (qua abelian group). This allows us to regard $\mathfrak{E}(V)$ as a Λ-ring. The mapping $\varrho_\iota: R \to \mathfrak{E}(V)$ is easily seen to be a monomorphism, since $1 \cdot x\varrho_\iota = x$, and so $x\varrho_\iota = 0$ implies $x = 0$. This completes the proof.

[9]) R_ι is regarded as right R_ι-module with respect to right multiplication by R_ι.

Finally we prove a reduction theorem, which shows that when establishing conditions for the existence of free products we may limit ourselves to the case of a finite number of factors[10]).

THEOREM 3.5. *Let R_ι $(\iota \in I)$ be any family of Λ-rings. Then the free product of the R_ι $(\iota \in I)$ exists if and only if the free product of the R_ι $(\iota \in I_0)$ exists for every finite subset I_0 of I.*

Proof. If the free product R of all the R_ι $(\iota \in I)$ exists, then by the definition of the free product, for any subset K of I, the subring of R generated by the $R_\varkappa$ $(\varkappa \in K)$ is the free product of the $R_\varkappa$ $(\varkappa \in K)$.

Conversely, suppose that for every finite subset F of I the free product R_F of the R_ι $(\iota \in F)$ exists. By the first part of the proof, if G is any subset of F, then there is a monomorphism

$$\varrho_G^F : R_G \to R_F,$$

which satisfies the coherence conditions $\varrho_F^F = 1$, $\varrho_H^G \varrho_G^F = \varrho_H^F$ for $H \leqq G \leqq F$. Let R be the direct limit of the system (R_F, ϱ_G^F). Since each ϱ_G^F is a monomorphism, the "projections" $\varrho_F : R_F \to R$ are monomorphisms (cf. EILENBERG & STEENROD [9], p. 221). This defines a canonical embedding of R_ι in R, for each $\iota \in I$. Clearly R satisfies the conditions (i) and (ii) in the definition of the free product. To verify (iii) we note that any relation in R involves only elements from a finite number of rings R_ι and therefore holds in some R_F. Since R_F is a free product, the relation must be a consequence of the relations in the different R_ι $(\iota \in F)$. Thus condition (iii) is also satisfied, and the proof is complete.

4. Sufficient conditions for the existence of the free product

In order to establish the existence of the free product of a given family of Λ-rings R_ι it is enough, by Theorem 3.4, to construct a right Λ-module V which for each $\iota \in I$ is an R_ι-module and contains R_ι as submodule, and such that $R_\iota \cap R_\varkappa = \Lambda$ for $\iota \neq \varkappa$. Moreover, by Theorem 3.5, we may limit ourselves to the case of a finite number of factors. Our first aim (Theorem 4.4) will be to prove the existence, under suitable conditions, of the free product of two factors. The construction is carried out in two steps: First we form a Λ-bimodule U in which R_1 and R_2 are contained as submodules (Lemma 4.1) and then we extend U so as to make it into a right R_i-module $(i = 1, 2)$ (Lemma 4.3). Finally in Lemma 4.2 we obtain a condition which will be used to show that in certain cases the free product is flat.

Throughout this section the Λ-rings occurring in contexts like $R_1 \otimes R_2$ are considered as Λ-bimodules. Further, if R is any Λ-ring, we shall denote the Λ-bimodule R/Λ by $\overline{R}$. If V is any Λ-bimodule, the phrase "V is flat" is ambiguous unless we specity whether V is to be regarded as right or left Λ-module. A similar ambiguity attaches to the expression "the submodule V'

[10]) Cf. DIDIDSE [8], where a corresponding theorem is proved for amalgams of (non-associative) algebras.

is pure in V". We shall remove this ambiguity by saying "V is right flat", "V is left pure in V" etc.; on the rare occasions when there is another coefficient domain S, we distinguish between S-flat modules and Λ-flat modules, but whenever the coefficient domain is not mentioned explicitly, it is understood to be Λ. All this applies in particular when V is a Λ-ring R or the quotient module $\overline{R}$, which is again a Λ-bimodule.

LEMMA 4.1. *Given any Λ-rings R_1 and R_2, put $U = R_1 \otimes R_2$ and define a homomorphism $\vartheta_i : R_i \to U$ $(i = 1, 2)$ by the rule*

$$x\vartheta_1 = x \otimes 1 \qquad (x \in R_1),$$
$$y\vartheta_2 = 1 \otimes y \qquad (y \in R_2).$$

Then ϑ_1, ϑ_2 are monomorphisms and $R_1\vartheta_1 \cap R_2\vartheta_2 = \Lambda$ if and only if the sequence

$$\operatorname{Tor}(R_1, R_2) \to \operatorname{Tor}(\overline{R}_1, \overline{R}_2) \to 0 \tag{4.1}$$

induced by the natural homomorphisms $R_i \to \overline{R}_i$ $(i = 1, 2)$ is exact. In particular, the conclusions hold if $\overline{R}_1$ is right flat, or if $\overline{R}_2$ is left flat.

Proof. By applying the exact homology sequence to the exact sequences

$$0 \to \Lambda \to R_i \to \overline{R}_i \to 0$$

(right modules for $i = 1$, left modules for $i = 2$) and remembering that Λ is free (as right and as left Λ-module) and therefore flat, we obtain the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \operatorname{Tor}(R_1, R_2) & \xrightarrow{\alpha} & \operatorname{Tor}(R_1, \overline{R}_2) & \to & R_1 & \xrightarrow{\vartheta_1} & R_1 \otimes R_2 \\
 & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\delta}\downarrow & & {\scriptstyle\pi}\downarrow & & \downarrow \\
0 & \to & \operatorname{Tor}(\overline{R}_1, R_2) & \xrightarrow{\gamma} & \operatorname{Tor}(\overline{R}_1, \overline{R}_2) & \to & \overline{R}_1 & \xrightarrow{\vartheta_1'} & \overline{R}_1 \otimes R_2 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & R_2 & \to & \overline{R}_2 & \to & 0 & & \\
 & & {\scriptstyle\vartheta_2}\downarrow & & \downarrow & & & & \\
 & & R_1 \otimes R_2 & \to & R_1 \otimes \overline{R}_2 & & & &
\end{array}$$

with exact rows and columns. In terms of this diagram the homomorphism φ in (4.1) is $\alpha\delta = \beta\gamma$, and if this is an epimorphism, the diagram shows that it must be an isomorphism and moreover, $\alpha, \beta, \gamma, \delta$ are all isomorphisms. It follows by exactness that ϑ_1 and ϑ_2 are then monomorphisms. If further,

$$x \otimes 1 = 1 \otimes y \qquad (x \in R_1,\ y \in R_2),$$

then $x\pi \otimes 1 = 0$ and by the exactness of the second row, $x\pi = 0$, i.e. $x \in \Lambda$, therefore $1 \otimes (x - y) = 0$, $x = y \in \Lambda$, which proves that $R_1\vartheta_1 \cap R_2\vartheta_2 = \Lambda$.

Conversely, if ϑ_1, ϑ_2 are monomorphisms then α and β are isomorphisms and so $\alpha\delta = \beta\gamma$ is an isomorphism if and only if ϑ_1' is a monomorphism. Thus if $\beta\gamma$ is not an isomorphism, there exists an $x \in R_1$ such that $x \notin \Lambda$, i.e. $x\pi \neq 0$, but $x\pi \otimes 1 = 0$; in other words we have $x \otimes 1 = \lambda \otimes y = 1 \otimes \lambda y$ $(\lambda \in \Lambda,\ y \in R_2)$. Thus $x \otimes 1 \in R_1\vartheta_1 \cap R_2\vartheta_2$, but $x \notin \Lambda$. This completes the proof.

By Lemma 4.1, if e.g. $\overline{R}_2$ is left flat, we may regard Λ as a submodule of the Λ-bimodule $U = R_1 \otimes R_2$. With this convention we have the following extension of Lemma 4.1.

LEMMA 4.2. *Let R_1 and R_2 be Λ-rings, S a Λ-subring of R_1 and put $U = R_1 \otimes R_2$. If R_1/S is left S-flat and $\overline{R}_2$ is left Λ-flat, then U/R_1 is left S-flat.*

For we may regard R_1 as an (S, Λ)-bimodule and R_2 as a left Λ-module in a natural way. Then U is a left S-module and since $\overline{R}_2$ is left Λ-flat, Λ is left pure in R_2 and we find that

$$U/R_1 \cong R_1 \otimes \overline{R}_2$$

by tensoring the exact sequence

$$0 \to \Lambda \to R_2 \to \overline{R}_2 \to 0$$

on the left with R_1. Thus it remains to show that $R_1 \otimes \overline{R}_2$ is left S-flat, and this follows from Lemma 2.1, since $\overline{R}_2$ is left Λ-flat by hypothesis, and R_1 is left S-flat because both R_1/S and S are. This establishes the lemma.

LEMMA 4.3. *Let U be an (S, Λ)-bimodule, where S is a Λ-ring, and let U' be any submodule of U, with quotient $U'' = U/U'$. Further, let R be a Λ-ring such that the sequence*

$$\operatorname{Tor}(U, R) \to \operatorname{Tor}(U'', \overline{R}) \to 0 \tag{4.2}$$

induced by the natural homomorphisms is exact, and suppose that U' is an (S, R)-bimodule. Then there exists an (S, R)-bimodule

$$W = W(U, U'; R)$$

with an (S, Λ)-bimodule monomorphism $\vartheta : U \to W$ such that the restriction $\vartheta | U'$ is an (S, R)-bimodule monomorphism. If, moreover, the sequence

$$\operatorname{Tor}(U, R) \to \operatorname{Tor}(U, \overline{R}) \to 0 \tag{4.3}$$

is also exact, then

$$W/U \cong U'' \otimes \overline{R}. \tag{4.4}$$

Proof. From the exact sequences

$$0 \to U' \to U \to U'' \to 0 \qquad \text{(right } \Lambda\text{-modules)},$$
$$0 \to \Lambda \to R \to \overline{R} \to 0 \qquad \text{(left } \Lambda\text{-modules)}$$

and the fact that Λ is left flat, we obtain the commutative diagram

$$\begin{array}{ccccccc}
0 \to \operatorname{Tor}(U, R) & \to & \operatorname{Tor}(U, \overline{R}) & & & & \\
\downarrow \alpha & & \downarrow & & & & \\
0 \to \operatorname{Tor}(U'', R) & \xrightarrow{\beta} & \operatorname{Tor}(U'', \overline{R}) & \to & U'' & \to & U'' \otimes R \\
\downarrow & & & & & & \\
U' \otimes R & & & & & & \\
\downarrow & & & & & & \\
U \otimes R & & & & & &
\end{array}$$

with exact rows and columns. By hypothesis $\alpha\beta$ is an epimorphism, hence β is an epimorphism and therefore an isomorphism; it follows that α is also an epimorphism. By exactness we deduce that the sequences

$$0 \to U'' \to U'' \otimes R, \tag{4.5}$$

$$0 \to U' \otimes R \to U \otimes R \tag{4.6}$$

are both exact. Thus we may regard $U' \otimes R$ as a submodule of $U \otimes R$. Because U' is a right R-module, there is a mapping $\varphi: U' \otimes R \to U'$ defined by

$$(u \otimes x)\varphi = ux \qquad (u \in U',\ x \in R),$$

and this is clearly an (S, R)-bimodule homomorphism. Let V be the kernel of φ, so that we have the inclusions $V \subseteq U' \otimes R \subseteq U \otimes R$, qua (S, R)-bimodules. We obtain the commutative diagram[11])

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & V & \to & U' \otimes R & \to & U' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & V & \to & U \otimes R & \xrightarrow{\alpha} & W & \to & 0 \\
 & & \downarrow & & \downarrow \beta & & \downarrow & & \\
 & & 0 & \to & U'' \otimes R & \to & U'' \otimes R & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

with exact rows and columns. Here $W = (U \otimes R)/V$ is an (S, R)-bimodule since both $U \otimes R$ and V are (S, R)-bimodules. Combining the homomorphism $u \to u \otimes 1$ of U into $U \otimes R$ with the homomorphism $\alpha: U \otimes R \to W$ given by the diagram, we obtain a homomorphism $\vartheta: U \to W$, whose restriction to U' is an (S, R)-bimodule monomorphism, again from the diagram. To show that ϑ itself is a monomorphism, let $u\vartheta = 0$ $(u \in U)$, then $(u \otimes 1)\alpha = 0$, hence $u \otimes 1$ is the image of an element from V and the diagram shows that $(u \otimes 1)\beta = 0$. If $u \to u^*$ is the natural homomorphism $U \to U''$, then this may be written $u^* \otimes 1 = 0$, whence by (4.5), $u^* = 0$, i.e. $u \in U'$. Since $\vartheta | U'$ has already been shown to be a monomorphism, the result follows.

Now suppose that (4.3) holds; then the sequence

$$0 \to U \to U \otimes R \tag{4.7}$$

is exact, and combining this with (4.6), we obtain the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & U' & \to & U' \otimes R & \to & U' \otimes \overline{R} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & U & \to & U \otimes R & \to & U \otimes \overline{R} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & U'' & \to & U'' \otimes R & \to & U'' \otimes \overline{R} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

[11]) CARTAN & EILENBERG [5], p. 16, ex. 1.

with exact rows and columns. From it we obtain the exact sequence

$$0 \to U + U' \otimes R \to U \otimes R \to U'' \otimes \overline{R} \to 0.$$

By (4.7) we may regard U as embedded in $U \otimes R$; then the fact that ϑ is a monomorphism means that $U \cap V = 0$, while $U + V = U + U' \otimes R$, as is easily verified. It follows that

$$U'' \otimes \overline{R} \cong (U \otimes R)/(U + V) \cong (U \otimes R)/V/(U + V)/V \cong W/U,$$

and this establishes the lemma.

We note that if in Lemma 4.3, $\overline{R}$ is left flat, then both the sequences (4.2) and (4.3) are exact; if moreover U'' is left S-flat then W/U is left S-flat. This is seen by applying Lemma 2.1 to the result (4.4).

A set of sufficient conditions for the existence of the free product of two factors can now be given.

THEOREM 4.4. *Let R_1 and R_2 be any Λ-rings and let S be any Λ-subring of R_1. If* (i) *$\overline{R}_i = R_i/\Lambda$ is left flat ($i = 1, 2$) and* (ii) *R_1/S is left S-flat, then the free product R of R_1 and R_2 exists, with S as Λ-subring, such that R/S is again left S-flat.*

Proof. First we form $U = R_1 \otimes R_2$; by Lemma 4.1, each R_i is embedded in a natural way in U and

$$R_1 \cap R_2 = \Lambda \qquad \text{in } U. \tag{4.8}$$

Put $W_1 = R_1$ and $W_2 = U$, then W_2 is an (S, Λ)-bimodule and W_1 a submodule of W_2 which is also a right R_1-module. Now let $n > 2$ and suppose that an ascending chain of (S, Λ)-bimodules

$$W_1 \subseteq W_2 \subseteq \cdots \subseteq W_{n-1}$$

has already been defined, where W_r is an R_1-module if r is odd and an R_2-module if r is even. Then we put

$$W_n = W(W_{n-1}, W_{n-2}; R_i) \qquad (i \equiv n \pmod 2),$$

where the module on the right is defined as in Lemma 4.3. This lemma also shows that there is a natural embedding of W_{n-1} in W_n (because $\overline{R}_i$ is left flat), that W_n is an R_i-module with W_{n-2} as submodule, and that

$$W_n/W_{n-1} \cong (W_{n-1}/W_{n-2}) \otimes \overline{R}_i. \tag{4.9}$$

Let V be the union of the ascending sequence of modules W_n, then V is an (S, Λ)-bimodule and qua Λ-module, V contains R_1 and R_2 as submodules such that $R_1 \cap R_2 = \Lambda$, by (4.8). If we can show that V is an R_i-module with R_i as submodule, for $i = 1, 2$, then we may apply Theorem 3.4 and conclude that the free product of R_1 and R_2 exists. Let $v \in V$, $x \in R_i$, then $v \in W_n$ for some n, and hence $v \in W_\nu$ for all $\nu > n$. If we choose ν to be of the form $2k + i$, for some k, then W_ν is an R_i-module with R_i as submodule, and therefore vx is defined in W_ν. The definition is independent of the choice of k, since W_μ is a submodule of W_ν, qua R_i-module, whenever $\nu - \mu = 2k > 0$. The product

vx is therefore uniquely defined in V and it is easily verified that this definition makes V an R_ι-module. This establishes the existence of the free product, and hence the first part of the theorem.

For the remaining part we note that $W_2/W_1 = U/R_1$ is left S-flat, by Lemma 4.2. Using (4.9) we find, by induction on n, that W_n/W_{n-1} is left S-flat for all n, and hence $W_n/W_1 = W_n/R_1$ is left S-flat; since R_1/S is S-flat by hypothesis, W_n/S is S-flat, and going to the limit we find that V/S is S-flat. Now V is the free right R-module on one free generator, namely the unit-element of Λ. It follows from the equation

$$s1 \cdot x = 1 \cdot sx \qquad (s \in S,\ x \in R)$$

that V and R are isomorphic, qua left S-modules, and moreover S itself, which is a submodule of both, corresponds elementwise to itself in the isomorphism. Hence $V/S \cong R/S$ and this proves that R/S is left S-flat. This completes the proof of the theorem.

If in the theorem we take $S = \Lambda$ or $S = R_1$, the hypothesis that R_1/S be left S-flat is certainly verified, and we obtain the following corollaries:

COROLLARY 1. *If R_1 and R_2 are any Λ-rings such that $\overline{R}_i$ is left flat ($i = 1, 2$), then the free product R of R_1 and R_2 exists and $\overline{R}$ is left flat.*

COROLLARY 2. *If R_1, R_2 and R are as in Cor. 1, then R/R_1 is left R_1-flat.*

From Cor. 1 we can easily deduce the existence of the free product of any family of Λ-rings R_ι such that each $\overline{R}_\iota$ is left (or right) flat. First let $R_1, \ldots, R_n$ be any finite set of Λ-rings such that each $\overline{R}_\iota$ is left flat, and suppose that it has already been shown that the free product T of $R_1, \ldots, R_{n-1}$ exists and that $\overline{T}$ is left flat. By Theorem 4.4 Cor. 1, the free product R of T and R_n exists and $\overline{R}$ is left flat. Now it is easily verified, using Theorem 3.1 Cor., that R is the free product of $R_1, \ldots, R_n$. It follows by induction that the free product of any finite family of rings R_ι satisfying the condition that $\overline{R}_\iota$ be left flat, exists and satisfies the same condition. We may now apply Theorem 3.5 to prove the existence of the free product of any family of rings satisfying the above condition. Since flatness is preserved under direct limits, $\overline{R}$ is again left flat. Of course the corresponding assertion with "left" replaced by "right" also holds, by symmetry. Thus we have proved

THEOREM 4.5. *Let R_ι ($\iota \in I$) be any family of Λ-rings. If for each $\iota \in I$, the module R_ι/Λ is left flat, then the free product R of the R_ι exists and R/Λ is again left flat. A similar assertion holds with "left" replaced by "right" throughout.*

The further assertion of Theorem 4.4, concerning S, could also be extended to arbitrary families of rings, but this may be left to the reader. We note one further consequence of Theorem 4.4, giving some information about the structure of the free product.

THEOREM 4.6. *If R_1 and R_2 are any Λ-rings and R_i/Λ is left flat ($i = 1, 2$) so that the free product R of R_1 and R_2 exists, then R has a filtration*

$$A_1 \subseteq A_2 \subseteq \cdots, \quad \cup A_n = R \qquad (A_i A_j \subseteq A_{i+j}),$$

where the A_i are Λ-bimodules and

$$A_n/A_{n-1} \cong R_1 \otimes \overline{R}_2 \otimes \overline{R}_1 \otimes \cdots \otimes \overline{R}_i \qquad (n \text{ factors, } i \equiv n \pmod 2.)$$

This follows by taking A_n to be the submodule of R corresponding to W_n in the isomorphism of V and R established in the proof of Theorem 4.4, and applying (4.9).

It is known that any Λ-module is flat provided that Λ is a regular ring, i.e. if for any $\lambda \in \Lambda$, the equation

$$\lambda \xi \lambda = \lambda$$

has a solution in Λ (AUSLANDER [1]). We therefore obtain the following corollary to Theorem 4.5.

THEOREM 4.7. *Let Λ be a regular ring; then for any family of Λ-rings the free product exists.*

In particular the free product of Λ-rings always exists if Λ is a semisimple ring. By further specialising Λ we obtain the

COROLLARY. *The free product of any family of Λ-rings exists if Λ is a division ring.*

5. Necessary conditions for the existence of free products

The problem of obtaining conditions for the existence of the free product can be treated on three levels. We may study (i) conditions on the Λ-rings R_ι ($\iota \in I$) for their free product to exist, (ii) conditions on the Λ-ring R for the free product of R with any Λ-ring S to exist, and (iii) conditions on Λ for the free product of any family of Λ-rings to exist. Whereas the sufficient conditions given in the last section were of the types (iii) and (i), we shall confine ourselves to the cases (ii) and (iii) in establishing necessary conditions[12]). We begin with a necessary condition of type (ii).

THEOREM 5.1. *Let R be any Λ-ring. If the free product of R with any Λ-ring exists, then for any right Λ-module U and any left Λ-module V, the homomorphism $\varphi: U \otimes V \to U \otimes R \otimes V$ defined by*

$$(u \otimes v)\varphi = u \otimes 1 \otimes v \tag{5.1}$$

is a monomorphism.

Proof. Suppose that the condition is not satisfied and let U, V be a right and left Λ-module respectively with elements $u_i \in U$ and $v_i \in V$ such that

$$\sum u_i \otimes v_i \neq 0 \quad \text{in } U \otimes V, \quad \text{but} \quad (\sum u_i \otimes v_i)\varphi = 0. \tag{5.2}$$

Put ${}_\Lambda U = \Lambda \otimes_Z U$, $V_\Lambda = V \otimes_Z \Lambda$, where Z is the ring of integers, and let $W = {}_\Lambda U \oplus V_\Lambda$, so that W is a Λ-bimodule. Now let S be the tensor algebra[13]) over Λ on the Λ-bimodule W. The terms of degree two in S form a module $W \otimes W$ which contains ${}_\Lambda U \otimes V_\Lambda$ as a direct summand. Since S is graded, it

[12]) Cf. however Lemma 4.1, which contains a necessary condition of type (i).

[13]) Cf. e.g. CHEVALLEY [6]; the discussion there is confined to Λ-algebras, but may be extended to Λ-rings without difficulty.

contains $W \otimes W$ as a direct summand, and hence ${}_\Lambda U \otimes V_\Lambda$ is a direct summand in S. It follows that

$$\sum u_i \otimes v_i \neq 0 \quad \text{in } S. \tag{5.3}$$

Now suppose that the free product P of R and S exists. Then there is a trilinear mapping of $S \times R \times S$ into P, defined by $(x, y, z) \to xyz$ ($y \in R$, $x, z \in S$). Hence there is a module homomorphism of $S \otimes R \otimes S$ into P, defined by $x \otimes y \otimes z \to xyz$. By (5.2), $\sum u_i \otimes 1 \otimes v_i = 0$ and so $\sum u_i v_i = 0$ in P; a comparison with (5.3) shows that the natural homomorphism $S \to P$ is not an embedding. Therefore the free product of R and S does not exist, and the theorem follows.

The condition of Theorem 5.1 may be reformulated as follows: Suppose that U and V are finitely presented modules, with generating sets u_r, v_s and defining relations $\sum u_r \mu_{rj} = 0$, $\sum \lambda_{is} v_s = 0$ respectively. Let $w = \sum u_r \xi_{rs} \otimes v_s$ be an element of the kernel of φ, given by (5.1), then by Lemma 2.3, since the u_r generate U, we have

$$\sum u_r \xi_{rs} \otimes 1 = \sum u_r \otimes a_{ri} \lambda_{is} \quad \text{in } U \otimes R,$$

for some $a_{ri} \in R$. Applying Lemma 2.3 again, this time to U, we find that

$$\xi_{rs} = \sum a_{ri} \lambda_{is} + \sum \mu_{rj} b_{js}, \tag{5.4}$$

for some $b_{js} \in R$. Conversely, if (5.4) holds, then $(\sum u_r \xi_{rs} \otimes v_s)\varphi = 0$, as is easily seen. On the other hand, the equation

$$\sum u_r \xi_{rs} \otimes v_s = 0$$

holds in $U \otimes V$ if and only if there exist $\alpha_{ri}, \beta_{js} \in \Lambda$ such that

$$\xi_{rs} = \sum \alpha_{ri} \lambda_{is} + \sum \mu_{rj} \beta_{js} \tag{5.5}$$

as follows by taking $R = \Lambda$ in the preceding argument. Therefore if φ is to be a monomorphism, any equation of the form (5.4) must imply the corresponding equation (5.5) in Λ. This proves

THEOREM 5.2. *Let R be a Λ-ring such that the free product of R with any Λ-ring exists. Then for any finite sets of elements ξ_{rs}, $\lambda_{is}, \mu_{rj} \in \Lambda$ and $a_{ri}, b_{js} \in R$, the equation*

$$\xi_{rs} = \sum a_{ri} \lambda_{is} + \sum \mu_{rj} b_{js}$$

implies the existence of $\alpha_{ri}, \beta_{js} \in \Lambda$ such that

$$\xi_{rs} = \sum \alpha_{ri} \lambda_{is} + \sum \mu_{rj} \beta_{js}.$$

COROLLARY. *Given a Λ-ring R, if the free product of R with any Λ-ring exists, then Λ is right pure and left pure in R.*

The corollary follows by taking in turn $\lambda_{is} = 0$ and $\mu_{rj} = 0$ in the theorem and applying Theorem 2.4. Of course it also follows directly from Theorem 5.1.

From the corollary we readily obtain a necessary condition on Λ. We could of course use the possibly more general condition of Theorem 5.2, but

this is rather more difficult to handle, and we shall have no more to say about it.

THEOREM 5.3. *Given any ring Λ with 1, if any free product of Λ-rings exists, then Λ must be right pure and left pure in any Λ-ring.*

For brevity we shall say that Λ is *right pure*, if Λ is right pure in every Λ-ring; the expression "left pure" is defined similarly. When Λ is an algebra over a field, it will follow from Theorem 5.5 that Λ is right (left) pure if and only if it is right (left) pure in every Λ-bimodule containing Λ as a submodule, but we shall not make use of this result.

Our next object is to give a criterion for a ring to be left pure which will enable us to construct rings which are not left pure. Of course the same considerations apply to the right purity of a ring. We shall confine our attention to the case where Λ is an algebra over a field F. It is clear that any Λ-ring may then also be considered as an algebra over F. The argument could be carried through in the case where Λ has a right operator domain with respect to which it is completely reducible, but we shall not enter into this refinement. We begin with a theorem which is of some interest in itself.

THEOREM 5.4. *Let Λ be an algebra with 1 over a field F. Given any elements $\lambda_{ij}, \alpha_i \in \Lambda$ $(i = 1, \ldots, m;\ j = 1, \ldots, n)$, there exists a Λ-ring R with elements t_j $(j = 1, \ldots, n)$ such that*

$$\sum \lambda_{ij} t_j = \alpha_i \qquad (i = 1, \ldots, m), \tag{5.6}$$

if and only if, for all sets of elements $\xi_i \in \Lambda$,

$$\sum \xi_i \lambda_{ij} = 0 \quad (j = 1, \ldots, n) \quad \textit{implies} \quad \sum \xi_i \alpha_i = 0. \tag{5.7}$$

Proof. It is clear that condition (5.7) is necessary. In order to prove the sufficiency we require a lemma.

LEMMA. *If (5.7) holds for all $\xi_i \in \Lambda$, then there exists a Λ-ring R with an element t such that for all $\xi_i \in \Lambda$,*

$$\sum \xi_i \lambda_{ij} = 0 \quad (j = 2, \ldots, n) \quad \textit{implies} \quad \sum \xi_i (\alpha_i - \lambda_{i1} t) = 0.$$

To prove the lemma, we put

$$I = \{\textstyle\sum \xi_i \lambda_{i1} \mid \xi_i \in \Lambda \text{ such that } \sum \xi_i \lambda_{ij} = 0 \text{ for } j = 2, \ldots, n\}.$$

Thet set I is easily seen to be a left ideal of Λ. Let I' be a complementary subspace of I in Λ, and define a linear transformation t on Λ as follows: given any $\mu \in \Lambda$, let

$$\mu = \sum \xi_i \lambda_{i1} + \mu' \qquad (\mu' \in I',\ \textstyle\sum \xi_i \lambda_{i1} \in I). \tag{5.8}$$

By definition of I' such a decomposition is possible, and μ' is uniquely determined by μ. The elements ξ_i are not uniquely determined by μ, but if we also have

$$\mu = \sum \xi_i' \lambda_{i1} + \mu'$$

then $\sum(\xi_i-\xi_i')\lambda_{i1}=0$, and since $\sum(\xi_i-\xi_i')\lambda_{ij}=0$ for $j=2,\ldots,n$, by definition of I, it follows from (5.7) that

$$\sum\xi_i\alpha_i-\sum\xi_i'\alpha_i=0.$$

Hence if we put

$$\mu t=\sum\xi_i\alpha_i+\mu', \tag{5.9}$$

the element μt is uniquely determined by μ. Since I and I' are subspaces it follows that $\mu\to\mu t$ is a linear transformation.

Now consider the ring $\mathfrak{L}=\mathfrak{L}(\Lambda)$ of all linear transformations of Λ qua space over F. The right multiplications by elements of Λ form a subring of $\mathfrak{L}$ isomorphic with Λ, and identifying the right multiplications with the elements of Λ, we may regard Λ as a subring of $\mathfrak{L}$ and hence $\mathfrak{L}$ as a Λ-ring. If $\mu=\sum\xi_i\lambda_{i1}$ is any element of I, then

$$\mu t=(\sum\xi_i\lambda_{i1})t=\sum\xi_i\alpha_i,$$

by (5.9), hence $\sum\xi_i(\alpha_i-\lambda_{i1}t)=0$ in $\mathfrak{L}$, and this establishes the lemma.

We now complete the proof of Theorem 5.4 by induction on n. For $n=0$, the hypothesis states that $\sum\xi_i\alpha_i=0$ for all $\xi_i\in\Lambda$. Taking $\xi_i=\delta_{ik}$, the Kronecker delta, we find that $\alpha_i=0$ $(i=1,\ldots,m)$ and hence (5.6) holds with $t_j=0$. Now let $n>0$; by the lemma there is a Λ-ring R_1 containing an element t_1 such that

$$\sum\xi_i\lambda_{ij}=0 \qquad (j=2,\ldots,n) \quad \text{implies} \quad \sum\xi_i(\alpha_i-\lambda_{i1}t_1)=0.$$

Moreover, R_1 is again an algebra over F. Applying the induction hypothesis to R_1 we find a ring R containing R_1 and elements $t_2,\ldots,t_n$ such that

$$\alpha_i-\lambda_{i1}t_1=\sum_{j=2}^{n}\lambda_{ij}t_j \qquad (i=1,\ldots,m),$$

in other words,

$$\alpha_i=\sum_{j=1}^{n}\lambda_{ij}t_j.$$

This completes the proof of the theorem.

As an illustration of Theorem 5.4, we note that if R is any algebra with 1 and $a\in R$ is not a right zero-divisor, then a right inverse of a may be adjoined to R.

We use Theorem 5.4 to establish the following criterion for an algebra to be left pure.

THEOREM 5.5. *An algebra Λ with 1 is left pure if and only if, given any $\lambda_{ij},\alpha_i\in\Lambda$ $(i=1,\ldots,m;\ j=1,\ldots,n)$ such that for all $\xi_i\in\Lambda$,*

$$\sum\xi_i\lambda_{ij}=0 \quad \textit{implies} \quad \sum\xi_i\alpha_i=0, \tag{5.10}$$

there exist $\beta_j\in\Lambda$ such that

$$\alpha_i=\sum\lambda_{ij}\beta_j. \tag{5.11}$$

Proof. Suppose that the condition is satisfied. Given any Λ-ring R, let

$$\alpha_i = \sum \lambda_{ij} t_j \qquad (\alpha_i, \lambda_{ij} \in \Lambda,\ t_j \in R). \tag{5.12}$$

Then it clear that (5.10) holds for all $\xi_i \in \Lambda$, and so (5.11) holds for some $\beta_j \in \Lambda$. But this just means that Λ is left pure in R, by Theorem 2.4. Conversely, suppose that Λ is left pure in every Λ-ring R, and let (5.10) be satisfied. Then by Theorem 5.4, there is a Λ-ring R with elements t_j satisfying (5.12), and since Λ is left pure in R, (5.11) follows. Therefore the condition of the theorem holds and the proof is complete.

The remark following Theorem 5.4, together with Theorem 5.5 shows that in a left pure algebra Λ any element which is not a right zero-divisor must have a right inverse. Thus e.g. the ring $F[x]$ of polynomials in a variable x over a field F is not left pure. Now Theorem 5.3 shows that there are $F[x]$-rings whose free product does not exist.

6. Rings without a unit-element

Up to this point we have dealt only with the class of rings with a unit-element. By regarding the unit-element as a constant (i.e. 0-ary) operator we obtained a product in which the unit-elements of all rings coincide. We shall now indicate briefly the modifications necessary in the general case. It should be emphasised that some or all of the rings occurring may still contain a unit-element, but these unit-elements will not be identified in the free product, unless they belong to the common subring Λ.

Given any ring R, we shall denote by R^1 the ring obtained by adjoining a unit-element 1 to R. This construction may be carried out whether or not R has a unit-element; it is to be understood that any unit-element already existing in R is denoted by a symbol other than 1. If the unit-element of R is e say, then $e1 = e + 1$, so that e is not unit-element in R^1. In order to recover R from R^1 we make the following

DEFINITION. A *1-ring* is a ring T with unit-element 1 and an ideal R such that $T/R \cong Z$, where Z is the ring of integers. R is called the *ideal associated with* T.

Clearly, for any ring R, R^1 is a 1-ring with associated ideal R, and if T is any 1-ring with associated ideal R, then $T \cong R^1$ (cf. [7]).

If Λ is any ring and R is any ring containing Λ as a subring, then it is clear that R^1 is a Λ^1-ring. Therefore if we have a family R_ι $(\iota \in I)$ of rings each containing Λ as a subring, then we can apply the theorems of §§ 3—5 to the rings R_ι^1. All that is needed is a proof that we can drop the 1 from the universal product. This follows from

THEOREM 6.1. *If T_ι $(\iota \in I)$ is any family of Λ^1-rings, and each T_ι is a 1-ring, then the universal product of the T_ι is a 1-ring.*

Proof. Let R_ι be the ideal associated with T_ι, and let T be the universal product of the T_ι. We define the ring Z of integers as a Λ^1-ring in the wide sense by putting

$$(m + \lambda) n = n (m + \lambda) = mn \qquad (m, n \in Z,\ \lambda \in \Lambda).$$

Then the natural homomorphism $\psi_\iota: T_\iota \to Z$ is a Λ^1-homomorphism and hence there is a homomorphism $\psi: T \to Z$ such that $\psi_\iota = \varphi_\iota \psi$, where φ_ι is the canonical homomorphism $T_\iota \to T$. Since each ψ_ι is an epimorphism, so is ψ, and the theorem follows.

It is clear from the proof that the ideal R associated with the universal product T contains the images $R_\iota \varphi_\iota$ of the ideals associated with the T_ι. We may therefore regard R as the universal product of the R_ι; moreover, the free product of the R_ι (with Λ as common subring) exists if and only if the free product of the R_ι^1 qua Λ^1-rings exists.

Bibliography

[1] AUSLANDER, M.: On regular group rings. Proc. Amer. Math. Soc. **8**, 658–664 (1957). — [2] BIRKHOFF, G.: Lattice theory. New York 1948. — [3] BOURBAKI, N.: Algèbre, chap. II. Paris 1947. — [4] BOURBAKI, N.: Algèbre, chap. III, appendice III. Paris 1948. — [5] CARTAN, H., and S. EILENBERG: Homological algebra. Princeton 1956. — [6] CHEVALLEY, C.: The construction and study of certain important algebras. Tokyo 1955. — [7] COHN, P. M.: On a class of simple rings. Mathematika **5**, 103–117 (1958). — [8] DIDIDSE, Z. E.: Non-associative free sums of algebras with amalgamated subalgebras (Russian). Mat. Sbornik **43** (85), 379–396 (1957). — [9] EILENBERG, S., and N. STEENROD: Foundations of algebraic topology. Princeton 1952. — [10] KAPLANSKY, I.: Infinite abelian groups. Ann Arbor 1954. — [11] KUROŠ, A. G.: Non-associative free algebras and free products of algebras. Rec. Math. (Mat. Sbornik) **20** (62), 239–262 (1947). — [12] NEUMANN, B. H.: An essay on free products of groups with amalgamations. Phil. Trans. Roy. Soc., Ser. A **246**, 503–554 (1954). — [13] SAMUEL, P.: On universal mappings and free topological groups. Bull. Amer. Math. Soc. **54**, 591–598 (1948). — [14] SCHREIER, O.: Die Untergruppen der freien Gruppen. Abh. math. Sem. Hamburg **5**, 161–183 (1927). — [15] SIKORSKI, R.: Products of abstract algebras. Fund. Math. **39**, 211–228 (1953).

The University, Manchester, England

(Eingegangen am 26. Januar 1959)