Werk

Titel: Some Formulae for the Product of Bessel and Legendre Functions.

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Some Formulae for the Product of Bessel and Legendre Functions

By
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Introduction. The basic formula to be proved is

$$(1)\quad \left\{\begin{aligned} &\int_0^\infty \lambda^{\gamma-1}(1+\lambda)^{-\sigma} E\left\{p;\alpha_r:q;\varrho_t:\frac{z}{1+\lambda}\right\} {}_2F_1\left(\begin{matrix}\alpha,\beta;\ -\lambda\\ \gamma\end{matrix}\right) d\lambda \\ &\quad = \Gamma(\gamma)\frac{\sin\pi\sigma\sin\pi(\alpha+\beta-\gamma+\sigma)}{\sin\pi(\alpha-\gamma+\sigma)\sin\pi(\beta-\gamma+\sigma)}\times \\ &\qquad \times E\left(\begin{matrix}\alpha_1,\ldots,\alpha_p,\ 1-\sigma,\ 1-\alpha-\beta+\gamma-\sigma:z\\ \varrho_1,\ldots,\varrho_q,\ 1-\alpha+\gamma-\sigma,\ 1-\beta+\gamma-\sigma\end{matrix}\right)+ \\ &\qquad + \sum_{\alpha,\beta}\left[\frac{\sin\pi(\gamma-\alpha)\sin\pi\beta}{\sin\pi(\gamma-\alpha-\sigma)\sin\pi(\beta-\alpha)}\Gamma(\gamma)\,z^{-(\alpha-\gamma+\sigma)}\times\right. \\ &\qquad \left.\times E\left(\begin{matrix}\alpha_1+\alpha-\gamma+\sigma,\ldots,\alpha_p+\alpha-\gamma+\sigma,\ 1+\alpha-\gamma,\ 1-\beta:z\\ \varrho_1+\alpha-\gamma+\sigma,\ldots,\varrho_q+\alpha-\gamma+\sigma,\ 1+\alpha-\gamma+\sigma,\ 1+\alpha-\beta\end{matrix}\right)\right] \end{aligned}\right.$$

where $R(\gamma)>0$, $R(\alpha-\gamma+\sigma+\alpha_r)>0$, $R(\beta-\gamma+\sigma+\alpha_r)>0$ $r=1,2,3,\ldots,p$, and $p\geqq q+1$. For other values of p and q, the formula holds if the integral converges. In (1), z is taken for simplicity to be real and positive and $\sum\limits_{\alpha,\beta}$ denotes that to the expression following it the corresponding expression with α,β interchanged is to be added. For the definitions and properties of the E-functions the reader is referred to [*1*] p. 352. Integrals involving the product of Legendre and Bessel functions are deduced from (1) by substituting for the hypergeometric and E-functions from the following formulae respectively

$$(2)\qquad P_n^m(1+2\lambda)=\frac{1}{\Gamma(1-m)}\{\lambda(1+\lambda)\}^{-\frac{1}{2}m}F\left(\begin{matrix}-m-n,\ n-m+1;\ -\lambda\\ 1-m\end{matrix}\right)$$

$$(3)\qquad \cos\nu\pi\, E\left(\frac{1}{2}+\nu,\frac{1}{2}-\nu::\frac{2z}{1+\lambda}\right)=\left(\frac{2\pi z}{1+\lambda}\right)^{\frac{1}{2}}\exp\left(\frac{z}{1+\lambda}\right)K_\nu\left(\frac{z}{1+\lambda}\right)$$

and

$$(4)\qquad E\left(\nu+1:\frac{z}{1+\lambda}\right)=\left(\frac{z}{1+\lambda}\right)^{\frac{1}{2}\nu}I_\nu\left\{2\sqrt{\left(\frac{1+\lambda}{z}\right)}\right\}$$

which are to be found in [*1*] pp. 262, 351 and 414 respectively. The following formulae are required in the proof of (1):

$$(5)\quad \left\{\begin{aligned} &\int_0^\infty \lambda^{\gamma-1}(1+\lambda)^{-\sigma}E\{p;\alpha_r:q;\varrho_t:z(1+\lambda)\}\,{}_2F_1\left(\begin{matrix}\alpha,\beta;\ -\lambda\\ \gamma\end{matrix}\right)d\lambda \\ &\quad = \Gamma(\gamma)\,E\left(\begin{matrix}p;\alpha_r,\alpha-\gamma+\sigma,\beta-\gamma+\sigma:z\\ q;\varrho_t,\sigma,\alpha+\beta-\gamma+\sigma\end{matrix}\right) \end{aligned}\right.$$

where $R(\gamma)>0$, $R(\alpha-\gamma+\sigma)>0$, $R(\beta-\gamma+\sigma)>0$, $R(z)>0$ see [*2*] p. 133 formula (*7*).

If $p \geq q+1$,

$$(6)\quad \begin{cases} E(p;\alpha_r:q;\varrho_t:z) = \pi^{p-q-1}\sum_{r=1}^{p}\prod_{t=1}^{q}\sin\{(\varrho_t-\alpha_r)\pi\}\left\{\prod_{t=1}^{p}{}'\sin(\alpha_t-\alpha_r)\pi\right\}^{-1}\times \\ \qquad \times z^{\alpha_r}E\left\{\begin{matrix}\alpha_r,\alpha_r-\varrho_1+1,\ldots,\alpha_r-\varrho_q+1:\\ \alpha_r-\alpha_1+1,\ldots,\alpha_r-\alpha_p+1\end{matrix}\ \frac{e^{\pm i\pi(p-q-1)}}{z}\right\} \end{cases}$$

([1] pp. 352, 353 formulae (21), (22), (33)).

Proof of the formula. Consider the special case of (1) with $p=1$, $q=0$; then the left hand side becomes

$$\int_0^\infty \lambda^{\gamma-1}(1+\lambda)^{-\sigma}E\left\{\alpha_1::\frac{z}{1+\lambda}\right\}{}_2F_1\left(\begin{matrix}\alpha,\beta;\\ \gamma\end{matrix}\ -\lambda\right)d\lambda$$

$$= z^{-\alpha_1}\int_0^\infty \lambda^{\gamma-1}(1+\lambda)^{-\alpha_1-\sigma}E\left\{a_1::\frac{1+\lambda}{z}\right\}{}_2F_1\left(\begin{matrix}\alpha,\beta;\\ \gamma\end{matrix}\ -\lambda\right)d\lambda$$

$$= z^{-\alpha_1}\Gamma(\gamma)\,E\left(\begin{matrix}\alpha_1,\alpha-\gamma+\alpha_1+\sigma,\beta-\gamma+\alpha_1+\sigma:\\ \alpha+\beta-\gamma+\alpha_1+\sigma,\alpha_1+\sigma\end{matrix}\ 1/z\right)$$

by (5). Here expand the last E-function by means of (6) and so obtain (1) with the special case $p=1$, $q=0$. The general case can then be deduced in the usual way (see [3] p. 193).

Particular case: In (1) take $p=2$, $q=0$, $\alpha_1=\frac{1}{2}+\nu$, $\alpha_2=\frac{1}{2}-\nu$, $\alpha=-m-n$, $\beta=n-m+1$, $\gamma=1-m$, write $2z$ for z; then apply (2) and (3) and so obtain the following result:

$$(7)\quad \begin{cases} \int_0^\infty \lambda^{-\frac{1}{2}m}(1+\lambda)^{\frac{1}{2}m-\sigma-\frac{1}{2}}\exp\left(\frac{z}{1+\lambda}\right)K_\nu\left(\frac{z}{1+\lambda}\right)P_n^m(1+2\lambda)\,d\lambda \\ = \frac{\cos\nu\pi}{\sqrt{(2\pi z)}}\cdot\frac{\sin(\pi\sigma)\sin\pi(\sigma-m)}{\sin\pi(\sigma-n)\sin\pi(n+\sigma)}E\left\{\begin{matrix}\frac{1}{2}+\nu,\frac{1}{2}-\nu,1-\sigma,1+m-2\sigma:\\ 2+n-\sigma,1-n-\sigma\end{matrix}\ 2z\right\}+ \\ \quad + \frac{\cos\nu\pi}{\sqrt{(2\pi z)}}\cdot\frac{\sin(\pi n\sin\pi(m-n)}{\sin\pi(\sigma-n)\sin(2n\pi)}(2z)^{-(-n-1+\sigma)}\times \\ \quad \times E\left\{\begin{matrix}\nu-\frac{1}{2}-n+\sigma,-\nu-\frac{1}{2}-n+\sigma,-n,m-n:\\ -n+\sigma,-2n\end{matrix}\ 2z\right\}+ \\ \quad + \frac{\cos\nu\pi}{\sqrt{(2\pi z)}}\cdot\frac{\sin(\pi n)\sin\pi(m+n)}{\sin\pi(n+\sigma)\sin\pi(2n+1)}(2z)^{-(n+\sigma)}\times \\ \quad \times E\left\{\begin{matrix}\frac{1}{2}+\nu+n+\sigma,\frac{1}{2}-\nu+n+\sigma,1+n,1+m+n:\\ 1+n+\sigma,2+2n\end{matrix}\ 2z\right\} \end{cases}$$

where $R(z)>0$, $R(1-m)>0$, $R(-\frac{1}{2}-n+\sigma\pm\nu)>0$.

References

[1] MacRobert, T. M.: Functions of a complex variable, 4th ed. Glasgow 1954. — [2] Rathie, C. B.: Some results involving hypergeometric and E-functions. Proc. Glasgow Math. Assoc. **2**, 132–138 (1955). — [3] Ragab, F. M.: Integrals of E-functions expressed in terms of E-functions. Proc. Glasgow Math. Assoc. **1**, 192–195 (1953).

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