Werk

Titel: The Bilinear Generating Function for Hermite Polynomials in Several Variables.

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The bilinear Generating Funktion for Hermite Polynomials in Several Variables

By
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1. BURCHNALL [2] used the formula

$$(D-2x)^n=\sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}H_{n-r}(x)\cdot D_r, \tag{1}$$

where $D=d/dx$ and

$$e^{2xt-t^2}=\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n,$$

to give an elegant proof of the identity (due to NIELSEN [6; p. 31])

$$H_{m+n}(x)=\sum_{r=0}^{\min(m,n)}(-1)^r\binom{m}{r}\binom{n}{r}2^r r!\,H_{m-r}(x)\,H_{n-r}(x). \tag{2}$$

It may be of interest to point out that (1) can also be used to give an elementary proof of MEHLER'S formula

$$\sum_{n=0}^{\infty}\frac{H_n(x)\,H_n(y)}{n!}\left(\frac{z}{2}\right)^n=\frac{1}{(1-z^2)^{\frac{1}{2}}}\exp\left\{\frac{2xyz-(x^2+y^2)z^2}{1-z^2}\right\}; \tag{3}$$

for other proofs of (3) see for example [7].

Indeed if $F=F(x,y,z)$ stands for the right member of (3) and $D_x=\partial/\partial x$, $D_y=\partial/\partial y$, it is easily verified that

$$(D_x-2x)F=\frac{2yz-2x}{1-z^2}F,\qquad (D_y-2y)F=\frac{2xz-2y}{1-z^2}F,$$

$$(D_x-2x)(D_y-2y)F=\left\{\frac{(2yz-2x)(2xz-2y)}{(1-z^2)^2}+\frac{2z}{1-z^2}\right\}F. \tag{4}$$

On the other hand we find that

$$\frac{\partial F}{\partial z}=\left\{2\frac{(x-yz)(y-xz)}{(1-z^2)^2}+\frac{z}{1-z^2}\right\}F. \tag{5}$$

Comparing (4) and (5) we get

$$2\frac{\partial F}{\partial z}=(D_x-2x)(D_y-2y)F. \tag{6}$$

If now we put

$$F(x,y,z)=\sum_{n=0}^{\infty}A_n(x,y)\frac{z^n}{n!}\qquad (A_0(x,y)=1),$$

it is evident that (6) implies

$$2A_n(x,y) = (D_x - 2x)(D_y - 2y)A_{n-1}(x,y);$$

since the operators $D_x - 2x$, $D_y - 2y$ commute, this yields

$$2^n A_n(x,y) = (D_x - 2x)^n (D_y - 2y)^n \cdot 1.$$

Using (1) we have at once

$$2^n A_n(x,y) = H_n(x) H_n(y),$$

which evidently proves (3).

2. If we put

$$\varphi(x) = \sum_{i,j=1}^{n} a_{ij} x_i x_j,$$

a non-singular quadratic form, then, following APPELL and KAMPÉ DE FÉRIET [1]

$$e^{\xi_1 t_1 + \cdots + \xi_n t_n - \frac{1}{2}\varphi(t)} = \sum_{m_i=0}^{\infty} \frac{t_1^{m_1} \ldots t_n^{m_n}}{m_1! \ldots m_n!} H_{m_1 \ldots m_n}(x_1, \ldots, x_n), \tag{7}$$

where

$$\xi_i = \frac{1}{2}\frac{\partial \varphi(x)}{\partial x_i} = \sum_j a_{ij} x_j. \tag{8}$$

ERDÉLYI [3] has proved the following extension of MEHLER's formula (see KOSCHMIEDER [4] for $n=2$):

$$\left\{ \begin{aligned} &\sum_{m_i=0} \frac{t_1^{m_1} \ldots t_n^{m_n}}{m_1! \ldots m_n!} H_{m_1 \ldots m_n}(x_1, \ldots, x_n) H_{m_1 \ldots m_n}(y_1, \ldots, y_n) \\ &\qquad = \frac{1}{t_1 \ldots t_n (\Delta_1 \Delta_2)^{\frac{1}{2}}} \exp\left\{\frac{1}{4}\psi_1(\xi + \eta) - \frac{1}{4}\psi_2(\xi - \eta)\right\}, \end{aligned} \right. \tag{9}$$

where

$$\varphi_1(x) = \sum_1^n \frac{x_i^2}{t_i} + \varphi(x), \qquad \varphi_2(x) = \sum_1^\infty \frac{x_i^2}{t} - \varphi(x), \tag{10}$$

Δ_1, Δ_2 are the discriminants of φ_1, φ_2, respectively and $\psi_1(x), \psi_2(x)$ denote the quadratic forms inverse to φ_1 and φ_2; ξ_i is defined by (8) and

$$\eta_i = \frac{1}{2}\frac{\partial \varphi(y)}{\partial y_i} = \sum a_{ij} y_j. \tag{11}$$

We now prove (9) by the method used above in proving (3). In the first place it follows readily from (7) that

$$E_i H_{m_1 \ldots m_n}(x_1, \ldots, x_n) = (\xi_i - D_{x_i}) H_{m_1 \ldots m_n}(x_1, \ldots, x_n), \tag{12}$$

where $D_{x_i} = \partial/\partial x_i$ and the left member denotes the H polynomial with m_i replaced by $m_i + 1$.

Put

$$\psi_1(x) = \sum_{i,j} \beta_{ij} x_i x_j, \qquad \psi_2(x) = \sum_{ij} \gamma_{ij} x_i x_j,$$

so that, in view of (10),

$$\sum_j \Big(\frac{\delta_{ij}}{t_i} + a_{ij}\Big)\beta_{jk} = \delta_{ik} = \sum_j \Big(\frac{\delta_{ij}}{t_i} - a_{ij}\Big)\gamma_{jk},$$

which gives

$$\sum_j a_{ij}\beta_{jk} = \delta_{ik} - \frac{1}{t_i}\beta_{ik}, \qquad \sum_j a_{ij}\gamma_{jk} = -\delta_{ik} + \frac{1}{t_i}\gamma_{ik}. \tag{13}$$

If we put

$$F = \tfrac{1}{4}\psi_1(\xi+\eta) - \tfrac{1}{4}\psi_2(\xi-\eta),$$

it is easily verified that

$$\frac{\partial F}{\partial x_i} = \xi_i - \frac{1}{2t_i}(X_i + Y_i + X_i' - Y_i'), \tag{14}$$

$$\frac{\partial F}{\partial y_i} = \eta_i - \frac{1}{2t_i}(X_i + Y_i - X_i' + Y_i'), \tag{15}$$

where

$$X_i = \sum_k \beta_{ik}\xi_k, \quad Y_i = \sum_k \beta_{ik}\eta_k, \quad X_i' = \sum_k \gamma_{ik} x_k, \quad Y_i' = \sum_k \gamma_{ik} y_k. \tag{16}$$

We have also from (14) or (15)

$$\frac{\partial^2 F}{\partial x_i \partial y_i} = a_{ii} - \frac{1}{2t_i}\sum_k (\beta_{ik}a_{ki} - \gamma_{ik}a_{ki}).$$

Using (13) this reduces to

$$\frac{\partial^2 F}{\partial x_i \partial y_i} = a_{ii} - \frac{1}{t_i} + \frac{\beta_{ii}+\gamma_{ii}}{2t_i^2}. \tag{17}$$

If Φ denotes the right member of (9), it follows from (14), (15), (17) that

$$(D_{x_i} - \xi_i)(D_{y_i} - \eta_i)\Phi = \Big\{\frac{(X_i+Y_i)^2 - (X_i'-Y_i')^2}{4t_i^2} - \frac{1}{t_i} + \frac{\beta_{ii}+\gamma_{ii}}{2t_i^2}\Big\}\Phi. \tag{18}$$

To compute $\partial\Phi/\partial t_r$ we need $\partial\beta_{ij}/\partial t_r$ and $\partial\gamma_{ij}/t_k$. Using (13) we find that

$$\frac{\partial\beta_{ij}}{\partial t_k} = \frac{\beta_{ik}\beta_{jk}}{t_k^2}, \qquad \frac{\partial\gamma_{ij}}{\partial t_k} = \frac{\gamma_{ik}\gamma_{jk}}{t_k^2}. \tag{19}$$

We have also

$$\frac{1}{\Delta_1}\frac{\partial\Delta_1}{\partial t_k} = -\frac{\beta_{kk}}{t_k^2}, \qquad \frac{1}{\Delta_2}\frac{\partial\Delta_2}{\partial t_k} = -\frac{\gamma_{kk}}{t_k^2}. \tag{20}$$

Now employing (19) and (16) we get

$$\frac{\partial\Phi}{\partial t_k} = \Big\{\frac{(X_k+Y_k)^2 - (X_k'-Y_k')^2}{4t_k^2} - \frac{1}{t_k} + \frac{\beta_{kk}+\gamma_{kk}}{2t_k^2}\Big\}\Phi. \tag{21}$$

Comparing (21) with (18) we see that

$$\frac{\partial \Phi}{\partial t_k} = (D_{x_k} - \xi_k)(D_{y_k} - \eta_k)\Phi. \tag{22}$$

If we put

$$\Phi = \sum_{m_i=0}^{\infty} A_{m_1 \dots m_n} \frac{t_1^{m_1} \dots t_n^{m_n}}{m_1! \dots m_n!},$$

clearly (22) implies

$$E_i A_{m_1 \dots m_n} = (D_{x_i} - \xi_i)(D_{y_i} - \eta_i) A_{m_1 \dots m_n}, \tag{23}$$

where E_i has the same meaning as in (12). The operators $D_{x_i} - \xi_i$ and $D_{y_j} - \eta_j$ obviously commute; the same is also true of $D_{x_i} - \xi_i$ and $D_{x_j} - \eta_j$ as well as of $D_{y_i} - \eta_i$ and $D_{y_j} - \eta_j$. Thus repeated use of (23) yields

$$A_{m_1 \dots m_n} = \prod_{i=1}^{n} (D_{x_i} - \xi_i)^{m_i} \cdot \prod_{j=1}^{n} (D_{y_j} - \eta_j)^{m_j} \cdot 1.$$

But by (12)

$$H_{m_1 \dots m_n}(x_1, \dots, x_n) = (-1)^{m_1 + \dots + m_n} \prod_{i=1}^{n} (D_{x_i} - \xi_i) \cdot 1. \tag{24}$$

Hence we get

$$A_{m_1 \dots m_n} = H_{m_1 \dots m_n}(x_1, \dots, x_n)\, H_{m_1 \dots m_n}(y_1, \dots, y_n),$$

which evidently completes the proof of (9).

Incidentally by means of (12) we can prove that

$$\left\{ \prod_{i=1}^{n} (D_{x_i} - \xi_i)^{m_i} = \sum_{r_i=0}^{m_i} (-1)^{m_1 + \dots + m_n - r_1 - \dots - r_n} \binom{m_1}{r_1} \dots \binom{m_n}{r_n} \times \right. \tag{25}$$
$$\left. \times H_{m_1 - r_1, \dots, m_n - r_n}(x_1, \dots, x_n)\, D_{x_1}^{r_1} \dots D_{x_n}^{r_n}, \right.$$

which in particular contains (24). Using (25) we can obtain a generalization of (2). Indeed (12) and (25) imply

$$\left\{ H_{m_1 + p_1, \dots, m_n + p_n}(x_1, \dots, x_n) \right. \tag{26}$$
$$= \sum_{r_i=0}^{m_i} (-1)^{r_1 + \dots + r_n} \binom{m_1}{r_1} \dots \binom{m_n}{r_n} H_{m_1 - r, \dots, m_n - r_n}(x_1, \dots, x_n) \times$$
$$\left. \times D_{x_1}^{r_1} \dots D_{x_n}^{r_n} H_{p_1 \dots p_n}(x_1, \dots, x_n). \right.$$

Using (7) we get

$$D_{x_i} H_{p_1 \dots p_n}(x_1, \dots, x_n) = \sum_j a_{ij} p_j E_j^{-1} H_{p_1 \dots p_n}(x_1, \dots, x_n)$$

so that

$$D_{x_1}^{r_1} \dots D_{x_n}^{r_n} H_{p_1 \dots p_n}(x_1, \dots, x_n)$$

can be expressed as a linear combination of H's. However, the resulting form of (26) seems too complicated to be of interest.

3. Closely related to the polynomials H is a second set of polynomials defined by

$$\sum_{m_i=0}^{\infty} \frac{t_1^{m_1} \dots t_n^{m_n}}{m_1! \dots m_n!} G_{m_1 \dots m_n}(x_1, \dots, x_n) = e^{x_1 t_1 + \dots + x_n t_n - \frac{1}{2}\psi(t)} \tag{27}$$

where $\psi(t)$ denotes the quadratic form inverse to $\varphi(t)$. From (27) we get

$$E_i G_{m_1 \dots m_n}(x_1, \dots, x_n) = (x_i - D_{\xi_i}) G_{m_1 \dots m_n}(x_1, \dots, x_n), \tag{28}$$

analogous to (12). This formula yields

$$\left\{ \begin{aligned} \prod_{i=1}^{n} (x_i - D_{\xi_i})^{m_i} = \sum_{r_i=0}^{m_i} (-1)^{r_1 + \dots + r_n} \binom{m_1}{r_1} \dots \binom{m_n}{r_n} \times \\ \times G_{m_1 - r_1, \dots, m_n - r_n}(x_1, \dots, x_n) D_{\xi_1}^{r_1} \dots D_{\xi_n}^{r_n} \end{aligned} \right. \tag{29}$$

and in particular

$$\prod_{i=1}^{n} (x_i - D_{\xi_i})^{m_1} \cdot 1 = G_{m_1 \dots m_n}(x_1, \dots, x_n); \tag{30}$$

(29) and (30) may be compared with (25) and (26). Here again by means of (29) we get a complicated generalization of (2) which we shall not state. However, we may note the following results:

$$\left\{ \begin{aligned} \prod_{i=1}^{n} (\xi_i - D_{x_i})^{m_i} \cdot G_{p_1 \dots p_n}(x_1, \dots, x_n) \\ = \sum_{r_i=0}^{m_i} (-1)^{r_1 + \dots + r_n} \binom{m_1}{r_1} \dots \binom{m_n}{r_n} \binom{p_1}{r_1} \dots \binom{p_n}{r_n} r_1! \dots r_n! \times \\ \times H_{m_1 - r_1, \dots, m_n - r_n}(x_1, \dots, x_n) G_{p_1 - r_1, \dots, p_n - r_n}(x_1, \dots, x_n), \end{aligned} \right. \tag{31}$$

$$\left\{ \begin{aligned} \prod_{i=1}^{n} (x_i - D_{\xi_i})^{m_i} \cdot H_{p_1 \dots p_n}(x_1, \dots, x_n) \\ = \sum_{r_i=0}^{m_i} (-1)^{r_1 + \dots + r_n} \binom{m_1}{r_1} \dots \binom{m_n}{r_n} \binom{p_1}{r_1} \dots \binom{p_n}{r_n} r_1! \dots r_n! \times \\ \times G_{m_1 - r_1, \dots, m_n - r_n}(x_1, \dots, x_n) H_{p_1 - r_1, \dots, p_n - r_n}(x_1, \dots, x_n). \end{aligned} \right. \tag{32}$$

The inverses of (31) and (32) are readily obtained. Note also that (31) and (32) imply

$$\prod_{1}^{n} (\xi_i - D_{x_i})^{m_1} \cdot G_{p_1 \dots p_n}(x_1, \dots, x_n) = \prod_{1}^{n} (x_i - D_{\xi_i})^{p_i} \cdot H_{m_1, \dots, m_n}(x_1, \dots, x_n). \tag{33}$$

By means of (28) we can also obtain the analog of (9) involving the G's. However, we shall not take the space to state the result. More interesting is the following mixed formula proved by ERDÉLYI [3] for general n and by KOSCHMIEDER [5] for $n = 2$.

Let

$$\Phi(w) = \sum_{r=1}^{n} \frac{u_r^2 + v_r^2}{t_r} + \frac{1}{2}\varphi(u + iv) + \frac{1}{2}\psi(u - iv),$$

a quadratic form in the $2n$ indeterminates

$$w_1 = u_1, \ldots, w_n = u_n, \qquad w_{n+1} = v_1, \ldots, w_{2n} = v_n;$$

let Δ' denote the discriminant of Φ and $\Psi(w)$ the inverse form. Also put

$$z_1 = \xi_1 + y_1, \ldots, z_n = \xi_n + y_n, \qquad z_{n+1} = i(\xi_1 - y_1), \ldots, z_{2n} = i(\xi_n - y_n).$$

Then we have

$$(34) \qquad \begin{cases} \displaystyle\sum_{m_i=0}^{\infty} \frac{t_1^{m_1} \ldots t_n^{m_n}}{m_1! \ldots m_n!} H_{m_1 \ldots m_n}(x_1, \ldots, x_n)\, G_{m_1, \ldots, m_n}(y_1, \ldots, y_n) \\ \qquad = \dfrac{1}{t_1 \ldots t_n \Delta'^{\frac{1}{2}}} \exp\left\{\frac{1}{4}\Psi(z)\right\}. \end{cases}$$

Let W denote the right member of (34). Then as in the proof of (9), we can show that

$$\frac{\partial W}{\partial t_k} = (D_{x_k} - \xi_k)(D_{\eta_k} - y_k)\Phi.$$

Using (24) and (30) we at once get (34).

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