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PURL: https://resolver.sub.uni-goettingen.de/purl?266833020_0063|log30

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Math. Zeitschr. Bd. 63, S. 331—337 (1955)

On the characteristic frequencies of a symmetric membrane*).

Dedicated to the memory of I. SCHUR.

By
G. PÓLYA.

1. *Notation.* We consider a simply connected open domain $\mathfrak{D}$ in the z-plane, $z=x+iy$; we let $\mathfrak{C}$ denote the (sufficiently smooth) boundary curve of $\mathfrak{D}$. The function $u=u(x, y)$ is defined in $\mathfrak{D}$ and satisfies the differential equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\nu^2 u=0 \quad \text{in } \mathfrak{D}. \tag{1.1}$$

We consider two, very different, boundary conditions:

$$u=0 \quad \text{on } \mathfrak{C}, \tag{1.2}$$

$$\frac{\partial u}{\partial n}=0 \quad \text{on } \mathfrak{C}; \tag{1.3}$$

n stands for the normal to $\mathfrak{C}$. In both cases, there exists an infinity of characteristic frequencies ν; let $\lambda_1, \lambda_2, \ldots$ denote the values of ν with the condition (1.2), and $\mu_1, \mu_2, \ldots$ with the condition (1.3);

$$0<\lambda_1<\lambda_2\leqq\lambda_3\leqq\cdots, \tag{1.4}$$

$$0=\mu_1<\mu_2\leqq\mu_3\leqq\cdots. \tag{1.5}$$

Let $\dot{r}$ denote the *maximum inner conformal radius of* $\mathfrak{D}$, defined as follows. We map the interior of $\mathfrak{D}$ conformally onto the interior of a circle so that the point a of $\mathfrak{D}$ corresponds to the center of the circle and the linear magnification at the point a is equal to 1; let the radius of the circle so obtained be r_a. The value of r_a varies with the point a, and $\dot{r}$ is the maximum of r_a.

We say that the domain $\mathfrak{D}$ possesses *symmetry of order* n if $\mathfrak{D}$ coincides with itself after a rotation through an angle $2\pi/n$ about a certain point O, called the center of symmetry if $n\geqq 2$. Any domain possesses symmetry of order 1, a square has symmetry of order 4, a circle symmetry of every order.

2. *Aim.* The products $\lambda_n\dot{r}$ and $\mu_n\dot{r}$ depend only on the shape of the domain $\mathfrak{D}$, not on its size. G. SZEGÖ and the present author have shown that $\lambda_1\dot{r}$ attains its maximum when $\mathfrak{D}$ becomes a circle [*3*, p. 97—98][1]). M. SCHIFFER and the present author have shown that, for any n,

$$\left(\frac{1}{\lambda_1^2}+\frac{1}{\lambda_2^2}+\cdots+\frac{1}{\lambda_n^2}\right)\frac{1}{\dot{r}^2}$$

*) Sponsored by the Office of Naval Research.

[1]) Numerals in square brackets refer to the Bibliography at the end of the paper.

attains its minimum when $\mathfrak{D}$ becomes a circle [2, p. 304—307; attention to the difference of notation]. It follows from a theorem recently proved by G. SZEGÖ that also $\mu_2 \dot{r}$ attains its maximum for the circle [4]. Yet, for higher values of n, we can not generally assert that $\lambda_n \dot{r}$ or $\mu_n \dot{r}$ attains its maximum for the circle. In fact, Table 1 shows that both assertions are wrong for $n = 5$.

If, however, we restrict the choice of $\mathfrak{D}$ and admit only convex domains with sufficiently high symmetry, both $\lambda_n \dot{r}$ and $\mu_n \dot{r}$ will be maximum for the circle. This is expressed more precisely by the following theorem.

Table 1.

n	$\lambda_n \dot{r}$		$\mu_n \dot{r}$	
	Circle	Square	Circle	Square
1	2.405	2.396	0.000	0.000
2	3.832	3.789	1.841	1.695
3	3.832	3.789	1.841	1.695
4	5.135	4.793	3.054	2.396
5	5.135	5.358	3.054	3.389

Table 2.

n	$2n+1$	$l(n)$	$m(n)$
1	3	3	3
2	5	6	6
3	7	10	8
4	9	15	12
5	11	19	17

There are two increasing functions $l(n)$ and $m(n)$ of the positive integer n, with the following property: If the domain $\mathfrak{D}$ is convex and possesses a symmetry of order not lower than $2n+1$ but is not a circle, each of the $l(n)+m(n)-1$ quantities

$$\lambda_1 \dot{r}, \lambda_2 \dot{r}, \ldots, \lambda_{l(n)} \dot{r}, \quad \mu_2 \dot{r}, \ldots, \mu_{m(n)} \dot{r}$$

belonging to $\mathfrak{D}$ will be inferior to the corresponding quantity belonging to a circle.

Table 2 contains a few values of $l(n)$ and $m(n)$.

As it can be seen from the full definition in Section 4(a)

$$l(n+1) - l(n) \geqq 2, \quad m(n+1) - m(n) \geqq 2. \tag{2.1}$$

For example, let us consider a square. It has symmetry of order 4, which is higher than 3. Now, as Table 2 shows, $l(1) = m(1) = 3$, and so our theorem asserts that each of the values

$$\lambda_1 \dot{r}, \lambda_2 \dot{r}, \lambda_3 \dot{r}, \mu_2 \dot{r}, \mu_3 \dot{r}$$

belonging to the square must be less than the corresponding value belonging to the circle; see Table 1. Yet for $\lambda_5 \dot{r}$ or $\mu_5 \dot{r}$ such an assertion is wrong, as Table 1 shows.

The principal aim of the present paper is to prove the theorem here formulated. I stated that part of the theorem that is concerned with λ_n some time ago [1]. Additional remarks are presented in Section 5.

3. *Lemma.* It is convenient to define the frequencies $\lambda_1, \lambda_2, \ldots, \mu_1, \mu_2, \ldots$ by their extremal properties which are connected with the RAYLEIGH ratio

$$R[f] = \frac{\iint |\operatorname{grad} f|^2 \, dx \, dy}{\iint f^2 \, dx \, dy}. \tag{3.1}$$

The function $f=f(z)$ can be considered also as a function of the two real variables x and y; f is defined in the domain $\mathfrak{D}$, which is also the domain of integration of both double integrals arising in (3.1); of course

$$|\operatorname{grad} f|^2 = f_x^2 + f_y^2.$$

The most convenient is, in our case as in many other cases, to use a lemma due to POINCARÉ [2, p. 275—278] which we formulate here in detail only for one of the two cases with which we are concerned.

Suppose that the functions $g_1, g_2, \ldots, g_l$, defined, realvalued, and sufficiently smooth in $\mathfrak{D}$ and on $\mathfrak{C}$, are linearly independent and that

$$g_1 = g_2 = \cdots = g_l = 0 \text{ on } \mathfrak{C}. \tag{3.2}$$

Suppose also that the quadratic forms $\Sigma\Sigma\, a_{ik}x_ix_k$ and $\Sigma\Sigma\, b_{ik}x_ix_k$, the latter of which is positive definite, are such that for all values of $x_1, x_2, \ldots, x_l$ (0, 0, ..., 0, of course, excepted)

$$R[x_1g_1 + x_2g_2 + \cdots + x_lg_l] \leqq \frac{\Sigma\Sigma\, a_{ik}x_ix_k}{\Sigma\Sigma\, b_{ik}x_ix_k}. \tag{3.3}$$

Let the roots of the characteristic equation

$$\begin{vmatrix} a_{11} - x\,b_{11} & \ldots & a_{1l} - x\,b_{1l} \\ \cdots & \cdots & \cdots \\ a_{l1} - x\,b_{l1} & \ldots & a_{ll} - x\,b_{ll} \end{vmatrix} = 0 \tag{3.4}$$

be $x=\lambda_1'^2, \lambda_2'^2, \ldots, \lambda_l'^2$, increasingly ordered so that

$$0 < \lambda_1' \leqq \lambda_2' \leqq \cdots \leqq \lambda_l'. \tag{3.5}$$

Then

$$\lambda_1 \leqq \lambda_1', \quad \lambda_2 \leqq \lambda_2', \ldots, \lambda_l \leqq \lambda_l'. \tag{3.6}$$

It is convenient to call $g_1, \ldots, g_l$ the "trial functions". We shall need two corollaries of this lemma.

(a) *If the case of equality is excluded in* (3.3), *it is necessarily also excluded in each of the l resulting relations* (3.6). The proof given l. c. [2, p. 277] yields also this corollary.

(b) *If*

$$a_{ik} = \iint \operatorname{grad} g_i \cdot \operatorname{grad} g_k \, dx\, dy$$
$$b_{ik} = \iint g_i g_k \, dx\, dy$$

(so that the inequality (3.3) *becomes an equation) and the trial functions $g_1, g_2, \ldots, g_l$ coincide with the first l eigenfunctions of $\mathfrak{D}$, all the l inequalities* (3.6) *become equations.* Also this corollary is proved l. c. [2, p. 278].

4. *Proof.* It will be enough to prove what the theorem of Section 2 states about λ_n; the proof of the similar assertion dealing with μ_n is strictly parallel and presents no additional difficulty.

(a) The eigenfunctions of the equation (1.1) under the boundary condition (1.2) are well known when the domain is the unit circle. They are of the form

$$J_p(\lambda\varrho)\cos p\vartheta, \quad J_p(\lambda\varrho)\sin p\vartheta \tag{4.1}$$

where ϱ and ϑ are the polar coordinates of an arbitrary point ζ inside the unit circle; that is,

$$\zeta = \varrho e^{i\vartheta} \tag{4.2}$$

with $0 \leqq \varrho < 1$, $-\pi \leqq \vartheta \leqq \pi$. The corresponding characteristic frequency is λ, and p is an integer that we shall call the *type* of the eigenfunction. If the type is 0, only the first of the two expressions (4.1) is an eigenfunction, the second vanishes identically.

We define $l(n)$ as the greatest integer with the property that none of the eigenfunctions belonging to the characteristic frequencies

$$\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{l(n)} \tag{4.3}$$

is of a type superior to n. We call the eigenfunctions corresponding to the frequencies (4.3)

$$\chi_1(\zeta), \chi_2(\zeta), \chi_3(\zeta), \ldots, \chi_{l(n)}(\zeta), \tag{4.4}$$

respectively.

(b) We consider the domain $\mathfrak{D}$ mentioned by our theorem. Thus $\mathfrak{D}$ is not a circle, but it is convex and symmetric of order q, where

$$q \geqq 2n + 1. \tag{4.5}$$

The domain $\mathfrak{D}$ lies in the z-plane. Let the origin be the center of symmetry of $\mathfrak{D}$.

Let the power series

$$z = c_0\zeta + c_1\zeta^{q+1} + c_2\zeta^{2q+1} + \cdots \tag{4.6}$$

express the conformal mapping of the unit circle in the ζ-plane onto $\mathfrak{D}$ in the z-plane; (4.6) maps the center of the circle into the center of symmetry of $\mathfrak{D}$. Since $\mathfrak{D}$ is convex, we can assert [3, p. 22, 161—163] that

$$|c_0| = r. \tag{4.7}$$

(c) We wish to apply the lemma of Section 3. We construct the trial functions $g_k(z)$ by *transplanting* the eigenfunctions of the unit circle into $\mathfrak{D}$. That is, we define $g_k(z)$ by postulating that

$$g_k(z) = \chi_k(\zeta) \tag{4.8}$$

when z and ζ are connected by (4.6). Since $\chi_k(\zeta)$ vanishes on the boundary of the unit circle, $g_k(z)$ must vanish on the boundary of $\mathfrak{D}$, and so it fulfills the condition (3.2) of the lemma.

We set $l(n) = l$ and, in view of (4.8), we set

$$\begin{cases} f(z) = x_1 g_1(z) + \cdots + x_l g_l(z) \\ \quad = x_1\chi_1(\zeta) + \cdots + x_l\chi_l(\zeta) = \varphi(\zeta). \end{cases} \tag{4.9}$$

We transform the integrals in the numerator and denominator of the RAYLEIGH ratio $R[f(z)]$, see (3.1), by introducing the polar coordinates ϱ, ϑ in the ζ-plane instead of the rectangular coordinates x, y in the z-plane; we have to use (4.9), (4.6), and (4.2) for this purpose. Observing that the numerator of the RAYLEIGH ratio (the DIRICHLET integral) is a conformal invariant [3, p. 95], we obtain

$$R[f(z)] = \frac{\iint |\operatorname{grad} \varphi(\zeta)|^2 \varrho \, d\vartheta \, d\varrho}{\iint [\varphi(\zeta)]^2 \left|\frac{dz}{d\zeta}\right|^2 \varrho \, d\vartheta \, d\varrho} . \tag{4.10}$$

It is understood that for these double integrals (and for the other double integrals that will be considered in the present section) the domain of integration is the unit circle, so that ϱ and ϑ vary from 0 to 1, and from $-\pi$ to π, respectively.

Set

$$\left|\frac{dz}{d\zeta}\right|^2 = \frac{dz}{d\zeta}\,\overline{\frac{dz}{d\zeta}} = \sum_{\nu=0}^{\infty} (\nu q + 1)^2 |c_\nu|^2 \varrho^{2\nu q} + H(\zeta) . \tag{4.11}$$

According to (4.6), $H(\zeta)$ is an absolutely convergent double infinite series consisting of terms of the form

$$C \varrho^{h'} e^{ih\vartheta} \quad \text{with} \quad |h| \geq q \geq 2n + 1 , \tag{4.12}$$

see (4.5); C is a constant, h' and h are integers, $h' > 0$. On the other hand, according to (4.9), the definition of $l = l(n)$, and (4.1), $[\varphi(\zeta)]^2$ is the sum of a finite number of terms which are all of the form

$$\psi(\varrho) e^{ik\vartheta} \quad \text{with} \quad |k| \leq 2n ; \tag{4.13}$$

$\psi(\varrho)$ depends on ϱ alone (not on ϑ) and k is an integer. Integrating the product of the terms (4.12) and (4.13) with respect to ϑ from $-\pi$ to π, we get 0, and we conclude hence that

$$\iint [\varphi(\zeta)]^2 H(\zeta) \varrho \, d\vartheta \, d\varrho = 0 . \tag{4.14}$$

It follows from (4.9), (4.10), (4.11), (4.14), and (4.7) that

$$\left\{ \begin{aligned} R[x_1 g_1 + x_2 + \cdots + x_l g_l] &= \frac{\iint |\operatorname{grad} \varphi(\zeta)|^2 \varrho \, d\vartheta \, d\varrho}{\iint [\varphi(\zeta)]^2 \sum_{\nu=0}^{\infty} (\nu q + 1)^2 |c_\nu|^2 \varrho^{2\nu q + 1} \, d\vartheta \, d\varrho} < \\ &< \frac{\iint |\operatorname{grad} \varphi(\zeta)|^2 \varrho \, d\vartheta \, d\varrho}{\dot{r}^2 \iint [\varphi(\zeta)]^2 \varrho \, d\vartheta \, d\varrho} . \end{aligned} \right. \tag{4.15}$$

In fact, we can exclude the case of equality in the last step under (4.15), since $\mathfrak{D}$ is not a circle and so some of the coefficients $c_1, c_2, c_3, \ldots$ must be different from zero.

The last line of (4.15) is, by (4.9), the ratio of two quadratic forms in the variables $x_1, \ldots, x_l$. The roots of the corresponding characteristic equation are, as it is easily seen from Section 3 (b), $\lambda_1^{0\,2} \dot{r}^{-2}, \ldots, \lambda_l^{0\,2} \dot{r}^{-2}$ where $\lambda_1^0, \ldots, \lambda_l^0$

stand for the first l characteristic frequencies of the unit circle. It follows from (4.15), by virtue of the lemma of Section 3 and its corollary (a), that

$$\lambda_1 < \lambda_1^0 \dot{r}^{-1}, \ldots, \lambda_l < \lambda_l^0 \dot{r}^{-1}$$

which is the desired theorem.

5. *Additional remarks.* (a) *If the simply connected domain $\mathfrak{D}$ varies without any restriction, $\lambda_n \dot{r}$ possesses a finite upper bound that depends on n alone.*

This theorem, which I stated some time ago [*1*], can be proved by the method of Section 4. In fact, for the present purpose, we do not transplant all the successive eigenfunctions of the circle into $\mathfrak{D}$, as we did in Section 4(c), but only those of type 0; we use the terminology of Section 4(a). We obtain so easily the upper bounds, of which the first yields a "best" result [*3*, p. 97—98]:

$$\lambda_1 \dot{r} \leqq 2.4048, \quad \lambda_2 \dot{r} \leqq 5.5201, \quad \lambda_3 \dot{r} \leqq 8.6537, \ldots^{2}).$$

The method of proof works also for the μ_n and yields the bounds:

$$\mu_2 \dot{r} \leqq 3.8317, \quad \mu_3 \dot{r} \leqq 7.0156, \quad \mu_4 \dot{r} \leqq 10.1735, \ldots$$

Just the existence of a bound for $\mu_n \dot{r}$ follows from that for $\lambda_n \dot{r}$ since, as it is well known, $\mu_n < \lambda_n$.

(b) M. SCHIFFER found a very different method to prove the fact stated under (a). Indeed, if $\dot{r}$ is the maximum inner radius of the domain $\mathfrak{D}$, then $\mathfrak{D}$ contains, as is well known, a full circle of radius $\dot{r}/4$. Now the λ_n are monotonically decreasing domain functionals, and so each λ_n of $\mathfrak{D}$ is less than the corresponding λ_n of the circle with radius $\dot{r}/4$, which yields:

$$\lambda_1 \dot{r} \leqq 9.619, \quad \lambda_2 \dot{r} \leqq 15.328, \quad \lambda_3 \dot{r} \leqq 15.328, \ldots$$

At the beginning, the bounds just obtained are higher than those obtained under (a), yet at the end, for sufficiently large n, the opposite is true.

(c) We do not know, for example, the least upper bound of $\lambda_2 \dot{r}$, or for which shape of $\mathfrak{D}$ this bound is attained or approximated. It may be worthwhile to mention that we can solve a similar problem.

Let A denote the area of $\mathfrak{D}$. Then

$$\lambda_2^2 A > 2j^2 \pi \tag{5.1}$$

where j, approximately 2.4048, *is the first root of $J_0(x)$; the right hand side of* (5.1) *is the greatest lower bound of the left hand side.*

This result is due to PETER SZEGÖ. It is based on the well known theorem, discovered by Lord RAYLEIGH and proved by G. FABER and E. KRAHN [*3*, p. VI, 191, 275],

$$\lambda_1^2 A \geqq j^2 \pi. \tag{5.2}$$

[2]) Of course, the numerical values are only approximate and should be replaced by the appropriate zeros of the appropriate Bessel function.

We shall use (5.2) in its sharpest form: equality is attained only if $\mathfrak{D}$ is a circle.

We consider a more general situation: we assume that the nodal lines of the eigenfunction that corresponds to the characteristic frequency λ of $\mathfrak{D}$ dissect $\mathfrak{D}$ into s subdomains $\mathfrak{D}_1, \mathfrak{D}_2, \ldots, \mathfrak{D}_s$ of area $A_1, A_2, \ldots, A_s$, respectively; of course, $s \geqq 2$ and

$$A_1 + A_2 + \cdots + A_s = A. \tag{5.3}$$

The eigenfunction considered vanishes along the boundary of each of the subdomains $\mathfrak{D}_1, \mathfrak{D}_2, \ldots, \mathfrak{D}_s$, and so λ is a characteristic frequency of each. Applying (5.2) s times, we obtain

$$\lambda^2 A_1 \geqq j^2 \pi, \ldots, \lambda^2 A_s \geqq j^2 \pi. \tag{5.4}$$

Equality cannot be valid in each of the s relations (5.4): otherwise each of the s subdomains would be an open circular disk which obviously could not be fitted together into one *connected* open domain $\mathfrak{D}$. Hence, adding the inequalities (5.4) and using (5.3) we obtain

$$\lambda^2 A > s j^2 \pi \tag{5.5}$$

which for $s = 2$ yields the desired (5.1). In fact, only the eigenfunction corresponding to λ_1 is without nodal lines, and so for λ_2 we must have $s \geqq 2$. (Indeed, we have $s = 2$ for λ_2 by a theorem of COURANT.) This proof suggests also the shape of $\mathfrak{D}$ that renders the difference of the two sides of (5.1) infinitesimal: the union of two equal circles with given radius the common chord of which is of infinitesimal length.

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(Eingegangen am 4. Juli 1955.)