Approximation theorems for generalized almost periodic functions[1]).

By

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§ 1.

Introduction.

The theory of continuous almost periodic functions was founded by Harald Bohr[2]). He defines a *displacement number* of a function $f(x)$ corresponding to ε as a number $\tau(\varepsilon)$ such that, for all values of x,

$$(1) \qquad |f(x+\tau)-f(x)| \leqq \varepsilon .$$

A (complex) function of the real variable x is then said to be *almost periodic* if

a) it is continuous for all values of x, $-\infty < x < \infty$, and

b) for every positive ε, there exists a length $L(\varepsilon)$, such that every interval of length L contains at least one displacement number for the function corresponding to ε.

Bohr shows that for these ap C functions[3]) the mean value

$$(2) \qquad M\{f(t)\, e^{-i\lambda t}\} = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t)\, e^{-i\lambda t}\, dt = \alpha(\lambda, f)$$

exists for all real λ, but is only distinct from zero for an enumerable number of values of λ, λ_j, so that there is an analogue of the Fourier development for periodic functions

$$(3) \qquad f(x) \sim \sum \alpha(\lambda, f)\, e^{i\lambda x}.$$

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1) Presented to the American Mathematical Society, January, 1928.

2) H. Bohr, Acta Mathematica (1924–1925) **45**, pp. 29–121; **46**, pp. 101–214.

3) We write ap C for almost periodic continuous.

An analogue of Parseval's theorem holds, and also an approximation theorem, which states the possibility of approximating any ap C function uniformly by a finite linear combination of the exponentials appearing in its Fourier expansion (3). This leads to the alternative definition of the ap C functions as *those functions which may be approximated uniformly for* $-\infty < x < \infty$ *to any degree of exactness by a linear combination of exponentials* [4]).

Several generalizations of the original definition have been given [5]). The modifications consist in weakening the continuity requirement, a) and making a corresponding weakening in the equation which defines a displacement number (1). As the goal in all these generalizations has been the proof of an analogue of the Parseval relation; the possibility of finding approximation theorems and proving that such approximation properties suffice to define the functions under consideration has been overlooked except for the functions of Weyl[5a]). For these, the approximation theorem coincides with the Parseval relation.

It is the object of the present paper to show that for all the classes of generalized ap functions which have been defined, there exist approximation theorems, the sense in which the linear combination of exponentials approximates the function being the same as the sense in which $f(x+\tau)$ approximates $f(x)$ in the definition of a displacement number. Furthermore, these approximation properties will be found to be sufficient as well as necessary conditions for the different types of ap functions, so that they give rise to alternative definitions. The point of view of approximation properties leads us to the definition and discussion of some new generalizations.

§ 2.

The almost periodic functions with summable square.

We begin with the generalization which applies to functions with summable square, S sq. These ap S sq functions [6]) are defined as (com-

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[4]) For a short proof of the properties just quoted, as well as some others used later, cf. H. Weyl, Math. Annalen **97** (1927), pp. 338–356.

[5]) N. Wiener, Math. Zeitschr. **24** (1925), pp. 575–616. — W. Stepanoff, Math. Annalen **95** (1926), pp. 472–498. — A. S. Besicovitch, Paris C. R. **181** (1925), part 2, pp. 394–397. — H. Weyl, loc. cit. pp. 353–354.

[5a]) I have recently been informed by Professor Harald Bohr that he has developed many of the approximation theorems of this paper. His approach is somewhat different from mine. cf. A. Besicovitch and H. Bohr, Det Kgl. Danske Videnskabernes Selskab. Math.-fys. Meddelelser. VIII, 5. pp. 1—31 (added in proof).

[6]) They are the pseudoperiodic functions of Wiener, loc. cit., and the functions ap III of Stepanoff, loc. cit.

plex) functions of a real variable x, $f(x)$ such that, in every finite interval $f(x)$ is measurable and $|f(x)|^2$ is summable, satisfying condition b) provided $\tau(\varepsilon)$ is defined as a displacement number for ε when, for a fixed d and all values of a,

$$\frac{1}{d}\int_a^{a+d} |f(x+\tau)-f(x)|^2\,dx \leqq \varepsilon^2. \tag{4}$$

Stepanoff shows that if this definition holds for a particular d, it holds for all values of d. As the L of b) varies with d, when we think of d as variable we write $L(\varepsilon, d)$. We wish to prove

Theorem I. *A necessary and sufficient condition for a (measurable) function $f(x)$ to be almost periodic with summable square is the existence of a sequence of linear combinations of exponentials,*

$$P_n(x) = \sum_{k=0}^{N(n)} c_{nk}\, e^{-i\lambda_{nk}x}, \tag{5}$$

approximating it in the sense that:

$$\lim_{n\to\infty} \int_a^{a+d} |f(x)-P_n(x)|^2\,dx = 0, \tag{6}$$

uniformly in a, for one and hence for all d.

For the sufficiency of the condition [7]) we note that for any positive ε, we will have for some $n(\varepsilon)$ in view of (6)

$$\frac{1}{d}\int_a^{a+d} |f(x)-P_n(x)|^2\,dx < \frac{\varepsilon^2}{9}. \tag{7}$$

But, as $P_n(x)$ is ap C, we may write for any $\tau\left(\frac{\varepsilon}{3}\right)$:

$$\frac{1}{d}\int_a^{a+d} |P_n(x+\tau)-P_n(x)|^2\,dx \leqq \frac{\varepsilon^2}{9}. \tag{8}$$

On combining (7) as applied to x and $x+\tau$ with (8) by means of

$$\left(\sum_1^3 A_s\right)^2 \leqq \sum_1^3 A_s^2 \cdot \sum_1^3 1^2 = 3\sum_1^3 A_s^2, \tag{9}$$

we see that $f(x)$ satisfies (4) and we may use $L\left(\frac{\varepsilon}{3}\right)$ for P_n, as the $L(\varepsilon)$ for $f(x)$. As $|f(x)|^2$ is summable because of (6), $f(x)$ is an ap S sq function.

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[7]) cf. Stepanoff, loc. cit. p. 493, or Wiener, loc. cit. p. 607 for related theorems.

To prove the necessity of the condition, we form the function

$$F(x)=\frac{1}{h}\int_{x}^{x+h} f(t)\,dt=\frac{1}{h}\int_{0}^{h} f(t+x)\,dt, \tag{10}$$

where h is a constant to be specified later. This is an ap C function, and hence may be uniformly approximated by a linear combination of exponentials, $P_n(x)$ for which we will have, for all y,

$$\frac{1}{d}\int_{y}^{y+d} |F(x)-P_n(x)|^2\,dx<\frac{\varepsilon}{4}. \tag{11}$$

But

$$F(x)-f(x)=\frac{1}{h}\int_{0}^{h} f(x+t)\,dt-\frac{1}{h}\int_{0}^{h} f(x)\,dt, \tag{12}$$

from which we find

$$\begin{aligned}|F(x)-f(x)|^2&=\left|\frac{1}{h}\int_{0}^{h}[f(x+t)-f(x)]\,dt\right|^2\\&\leqq\frac{1}{h}\int_{0}^{h}|f(x+t)-f(x)|^2\,dt,\end{aligned} \tag{13}$$

by the Schwartz inequality. Hence we have[8])

$$\begin{aligned}\frac{1}{d}\int_{y}^{y+d}|F(x)-f(x)|^2\,dx&\leqq\frac{1}{d}\int_{y}^{y+d}dx\,\frac{1}{h}\int_{0}^{h}|f(x+t)-f(x)|^2\,dt\\&\leqq\frac{1}{h}\int_{0}^{h}dt\,\frac{1}{d}\int_{y}^{y+d}|f(x+t)-f(x)|^2\,dx.\end{aligned} \tag{14}$$

Now recall that $f(x)$ is an ap Ssq function, so that there exists an $L(\varepsilon, d)$ with the property b). In particular, there is a $\tau\left(\frac{\varepsilon}{6}\right)$ in the interval $y-L<\tau<y$, so that if

$$x'=x+y-\tau,\qquad 0<x'<L+d \quad\text{as}\quad 0<x<d. \tag{15}$$

Using (4) and (15), we find

$$\frac{1}{d}\int_{0}^{d} f(x+y)-f(x')|^2\,dx<\frac{\varepsilon^2}{36}, \tag{16}$$

$$\frac{1}{d}\int_{0}^{d}|f(x+y+t)-f(x'+t)|^2\,dx<\frac{\varepsilon^2}{36}. \tag{17}$$

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[8]) For the inversion of the order of integration, cf. e. g. L. Schlesinger und A. Plessner, Lebesguesche Integrale und Fouriersche Reihen (1926), p. 179.

But, by a property[9]) of the Lebesgue integral of a function with summable square over a *finite* interval, here $0 < x' < L + d + h$, we must have

$$(18) \qquad \frac{1}{d}\int_0^d |f(x'+t) - f(x')|^2 \, dx \leqq \frac{1}{d}\int_0^{L+d} |f(x'+t) - f(x')|^2 \, dx' < \frac{\varepsilon^2}{36},$$

provided $t < \eta(\varepsilon)$.

On combining (16), (17) and (18) by means of (9), we obtain

$$(19) \qquad \frac{1}{d}\int_y^{y+d} |f(x+t) - f(x)|^2 \, dx = \frac{1}{d}\int_0^d |f(x+y+t) - f(x+y)|^2 \, dx$$

$$< \frac{\varepsilon^2}{4}, \quad \text{for} \quad t < \eta(\varepsilon).$$

Consequently, if we take $h < \eta(\varepsilon)$, we will have from (14)

$$(20) \qquad \frac{1}{d}\int_y^{y+d} |F(x) - f(x)|^2 \, dx \leqq \frac{\varepsilon^2}{4}.$$

Finally we combine (11) and (20) to obtain

$$(21) \qquad \frac{1}{d}\int_y^{y+d} |f(x) - P_n(x)|^2 \, dx \leqq \varepsilon^2,$$

which is equivalent to the condition demanded by the theorem.

Corollary 1. *The frequencies appearing in the approximating linear combinations may be restricted to be frequencies of the function.*

By the frequencies of the function, $f(x)$, we mean the values of λ for which

$$(22) \qquad \alpha(\lambda, f) = M\{f(t)\, e^{-i\lambda t}\} \neq 0.$$

Stepanoff has shown (cf. (81) below) that they are related to the frequencies of $F(x)$ by

$$(23) \qquad \alpha(\lambda, F) = \alpha(\lambda, f)\frac{e^{i\lambda h} - 1}{i\lambda h},$$

from which he deduces that they are enumerable in number. As (23) shows that every frequency of $F(x)$ is a frequency of $f(x)$, and as the frequencies appearing in the $P_n(x)$ of (11) may be selected from those of $F(x)$, the corollary is proved.

In particular, if we have a sequence of $P_n(x)$ of this kind for which (6) holds, we may supply any exponential appearing in a given $P_m(x)$

[9]) cf. Schlesinger und Plessner, loc. cit. p. 215.

in all the following $P_n(x)$, $n > m$ in which it does not occur with zero coefficients, and write in place of (5),

$$P_n(x) = \sum_{k=0}^{N(n)} c_{nk} e^{-i\lambda_k x}, \tag{24}$$

where the λ_k are the frequencies of $f(x)$, enumerated once for all.

As a converse to corollary 1, we will establish

Corollary 2. *For any given approximating linear combination* (5), *such that, for all a,*

$$\frac{1}{d}\int_a^{a+d} |f(x) - P_n(x)|^2 dx < \varepsilon^2, \tag{25}$$

the coefficients satisfy the relation

$$|c_{nk} - \alpha(\lambda_{nk}, f)| < \varepsilon, \tag{26}$$

so that any frequency which appears in $P_n(x)$ with a coefficient in absolute value greater than ε is a frequency of $f(x)$.

From the existence of the limits (2), (22) which define the $\alpha(\lambda, f)$ for ap C or ap S sq functions, we may put $T = Kd$, K integral, and write

$$\alpha(\lambda, f) - \alpha(\lambda, P_n) = \lim_{K\to\infty} \frac{1}{Kd}\int_0^{Kd} [f(x) - P_n(x)] e^{-i\lambda x} dx. \tag{27}$$

But we have

$$\left|\frac{1}{Kd}\int_0^{Kd} [f(x) - P_n(x)] e^{-i\lambda x} dx\right| \leqq \frac{1}{Kd}\int_0^{Kd} |f(x) - P_n(x)| dx \tag{28}$$

$$\leqq \sqrt{\frac{1}{Kd}\int_0^{Kd} |f(x) - P_n(x)|^2 dx}$$

by the Schwartz inequality.

Finally, we notice that

$$\frac{1}{Kd}\int_0^{Kd} |f(x) - P_n(x)|^2 dx = \frac{1}{K}\left\{\frac{1}{d}\int_0^d + \frac{1}{d}\int_d^{2d} + \cdots + \frac{1}{d}\int_{(K-1)d}^{Kd}\right\} \leqq \varepsilon^2 \tag{29}$$

by (25), and on using (27), (28) and (29) we see that

$$\alpha(\lambda, f) - \alpha(\lambda, P_n) \leqq \varepsilon, \tag{30}$$

from which (26) follows since $\alpha(\lambda, P_n) = c_{nk}$.

We note further that the $\alpha(\lambda, f)$ may be defined in terms of any sequence of $P_n(x)$ satisfying (6) by

$$\alpha(\lambda, f) = \lim_{n\to\infty} c_n(\lambda), \tag{31}$$

where $c_n(\lambda)$ is the coefficient of $e^{i\lambda x}$ in $P_n(x)$, i. e. c_{nk} if λ is a λ_{nk} and otherwise zero.

§ 3.

The almost periodic summable functions.

We shall now treat the generalization which applies to summable functions, S. These ap S functions[10]) are defined as (complex) functions of a real variable x, $f(x)$ summable in every finite interval, satisfying condition b) provided $\tau(\varepsilon)$ is defined as a displacement number for ε when, for a fixed d and all values of a,

$$\frac{1}{d}\int_a^{a+d} |f(x+\tau) - f(x)|\, dx \leqq \varepsilon. \tag{32}$$

Here, also, if the definition holds for one d, it holds for all d. As the L of b) varies with d, when we wish to indicate this dependence we write $L(\varepsilon, d)$. Our theorem of approximation is here

Theorem II. *A necessary and sufficient condition for a (measurable) function $f(x)$ to be a summable almost periodic function is the existence of a sequence of linear combinations of exponentials, $P_n(x)$, (5), approximating it in the sense that*

$$\lim_{n\to\infty} \frac{1}{d}\int_a^{a+d} |f(x) - P_n(x)|\, dx = 0, \tag{33}$$

uniformly in a, for one and hence for all d.

We prove the sufficiency of the condition by showing that, if n is selected so that

$$\frac{1}{d}\int_a^{a+d} |f(x) - P_n(x)|\, dx < \frac{\varepsilon}{3}, \tag{34}$$

any $L\left(\frac{\varepsilon}{3}\right)$ for this $P_n(x)$ is an $L(\varepsilon)$ for $f(x)$. The argument is entirely parallel to that used for theorem I. The summability of $f(x)$ follows from (33).

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[10]) These are the functions ap II of Stepanoff, loc. cit.

The necessity may also be proved by reasoning analogous to that used for theorem I. We define the function $F(x)$ by (10), which again is ap C, and approximate it uniformly, and hence in the sense we are using here, by a $P_n(x)$. By using the property[11]) of the Lebesgue integral of a summable function over a *finite* interval which necessitates

$$(35) \qquad \frac{1}{d}\int_0^{L+d} |f(x'+t)-f(x')|\,dx', \quad \text{if } t<\eta(\varepsilon),$$

which we extend to any interval of length d by using condition b) as for (19) above and inverting a double integral as in (14) we show that $F(x)$, and hence $P_n(x)$, approximates $f(x)$ in the present sense.

We also prove as before

Corollary 1. *The frequencies appearing in the approximating linear combinations may be restricted to be frequencies of the function.*

Here, again, we may replace (5) by (24).

The converse is here

Corollary 2. *For any given approximating linear combination* (5), *such that, for all a*

$$(36) \qquad \frac{1}{d}\int_a^{a+d} |f(x)-P_n(x)|\,dx$$

the coefficients satisfy the relation

$$(37) \qquad |c_{nk}-\alpha(\lambda_{nk}, f)|<\varepsilon,$$

so that any frequency which appears in $P_n(x)$ with a coefficient in absolute value greater than ε is a frequency of $f(x)$.

Since (30) holds, the $\alpha(\lambda, f)$ may be defined for any ap S function in terms of any sequence of $P_n(x)$ satisfying (33) by means of (31).

§ 4.

The almost periodic measurable functions.

We have come now to the widest generalization, that which applies to functions merely measurable, M. These ap M functions[12]) are defined as (complex) functions of a real variable x, $f(x)$, measurable in every finite interval satisfying condition b) provided $\tau(\varepsilon)$ is defined as a displacement number for ε when, for a fixed d,

$$(38) \qquad |f(x+\tau)-f(x)|\leqq\varepsilon,$$

[11]) cf. Schlesinger und Plessner, loc. cit. p. 195.

[12]) These are the functions ap I of Stepanoff, loc. cit.

for all x *except* those on a set $\{x\}$ whose exterior measure in every interval of length d is $< d\varepsilon$, or whose density on such an interval is $< \varepsilon$. The definition remains essentially unchanged if d is regarded as variable, in which case the L of b) varies with d, and we write $L(\varepsilon, d)$.

For the ap M functions, we shall prove:

Theorem III. *A necessary and sufficient condition for a (measurable) function $f(x)$ to be a measurable almost periodic function is the existence of a sequence of linear combinations of exponentials, $P_n(x)$, (5), approximating it in the sense that*

$$\lim \varepsilon_n = 0, \qquad |f(x) - P_n(x)| < \varepsilon_n \tag{39}$$

for all x except those on a set $\{x\}$ of density less than ε_n in every interval of length d, i. e. of exterior measure less than $d\varepsilon_n$ in any such interval.

We first prove the condition sufficient. Select an n for which $\varepsilon_n < \frac{\varepsilon}{3}$. For any $L\left(\frac{\varepsilon}{3}\right)$ for the ap C function $P_n(x)$, we will have:

$$|P_n(x+\tau) - P_n(x)| \leqq \frac{\varepsilon}{3}, \text{ for all } x. \tag{40}$$

But, by (39), we have

$$|f(x) - P_n(x)| < \frac{\varepsilon}{3}, \text{ except on a set } \{x_1\}, \tag{41}$$

and hence

$$|f(x+\tau) - P_n(x+\tau)| < \frac{\varepsilon}{3}, \text{ except on a set } \{x_2\}. \tag{42}$$

The sets $\{x_1\}$ and $\{x_2\}$ are both of density less than $\frac{\varepsilon}{3}$ in every interval of length d. From the last three inequalities, it follows that

$$|f(x+\tau) - f(x)| < \varepsilon, \text{ except on } \{x_1\} + \{x_2\}. \tag{43}$$

As the set $\{x_1\} + \{x_2\}$ is at most of density $\frac{2\varepsilon}{3} < \varepsilon$, in every interval of length d, τ is a displacement number of $f(x)$ for ε, and condition b) is satisfied by taking $L\left(\frac{\varepsilon}{3}\right)$ for $P_n(x)$ as $L(\varepsilon)$.

We must now prove the condition necessary. We recall a theorem of Stepanoff which states every ap M function is bounded, $|f(x)| < G(\varepsilon)$, *except* for a set $\{x\}$ of density less than ε on every interval of length d. Consequently, if we define a function $g(x)$ by

$$\begin{aligned} g(x) &= f(x), & |f(x)| &\leqq G\left(\frac{\varepsilon}{2}\right), \\ g(x) &= G\frac{f(x)}{|f(x)|} & |f(x)| &\geqq G\left(\frac{\varepsilon}{2}\right), \end{aligned} \tag{44}$$

we shall have

(45) $$g(x) - f(x) = 0 \quad \text{except for a set } \{x_1\}$$

of density less than $\frac{\varepsilon}{2}$ on every interval of length d. We easily see from (44) that, for any τ, and all x,

(46) $$|g(x+\tau) - g(x)| \leqq |f(x+\tau) - f(x)|.$$

In particular, if τ is a displacement number of $f(x)$ for $\frac{\varepsilon}{(2G+1)}$, the right, and hence the left member will be less than this except on a set of this density on any interval of length d, so that:

(47) $$\frac{1}{d}\int_a^{a+d} |g(x+\tau) - g(x)|\, dx \leqq \frac{\varepsilon}{2G+1} + \frac{2G\varepsilon}{2G+1} \leqq \varepsilon,$$

since the integrand is at most $2G$ on the excepted set. Thus $g(x)$ is an ap S function, since a bounded measurable function is necessarily summable, and condition b) is satisfied with L an $L\left(\frac{\varepsilon}{(2G+1)}\right)$ for $f(x)$. By theorem II we may find a linear combination of exponentials, $P_n(x)$, such that for all a,

(48) $$\frac{1}{d}\int_a^{a+d} |g(x) - P_n(x)|\, dx < \frac{\varepsilon^2}{4}.$$

It follows from this that if

(49) $$|g(x) - P_n(x)| < \frac{\varepsilon}{2} \text{ except on a set } \{x_2\},$$

this set is of density less than $\frac{\varepsilon}{2}$ in any interval of length d. For, if the measure of $\{x_2\}$ included in the interval a to $a+d$ is kd, we have from (48) and (49)

(50) $$\frac{1}{d}\cdot\frac{\varepsilon}{2}\cdot kd < \frac{\varepsilon^2}{4}, \quad \text{or} \quad k < \frac{\varepsilon}{2}.$$

Combining (45) and (49) we see that

(51) $$|f(x) - P_n(x)| < \varepsilon \text{ except on a set } \{x\},$$

included in the set $\{x_1\} + \{x_2\}$, and hence of density less than ε in every interval of length d. This completes the proof.

Before we obtain analogues of the corollaries previously proved, we must define the frequencies of an ap M function. If such a function is not uniformly summable, the limits used in the earlier cases to define the $a(\lambda, f)$ will not exist in general. We may, however, define an enumerable set of frequencies for $f(x)$ as follows.

We select a sequence of increasing values, G_j such that

(52) $$\lim_{j=\infty} G_j = \infty, \quad \text{but} \quad G_{j+1} - G_j < K, \qquad j = 1, 2, \ldots$$

K being any fixed bound. We might, e. g., take $G_j = j$. For each of these we define a function $g_j(x)$ by means of (44). These are all ap S functions, and hence have frequencies defined in the usual way, by the coefficients $\alpha(\lambda, g_j)$. The totality of these frequencies gives an enumerable set. If we throw out all those λ's for which

(53) $$\lim_{j=\infty} \alpha(\lambda, g_j) = 0,$$

we obtain a set which we define as the frequencies of $f(x)$. For these the limit indicated in (53) either fails to exist, or gives a nonzero value.

To show the validity of this definition, we must prove that the same set are obtained from any other choice of G_j' satisfying (52). Let, for a particular G_s', for wich the modified function is $g_s'(x)$,

(54) $$G_j \leqq G_s' \leqq G_{j+1}.$$

If k_j is the maximum density of the point set where $|f(x)| \geqq G_j$ on the intervals $0 < x < T$, for all T, we will have:

(55) $$\left| \frac{1}{T} \int_0^T g_s'(x) e^{-i\lambda x} dx - \frac{1}{T} \int_0^T g_j(x) e^{-i\lambda x} dx \right| \leqq (G_s' - G_j) k_j \leqq K k_j,$$

by (44), (54) and (52). On taking the limit for $T \to \infty$, we find:

(56) $$|\alpha(\lambda, g_s') - \alpha(\lambda, g_j)| \leqq K k_j.$$

Now let s increase. As $s \to \infty$, $G_j \to \infty$, and since, by the theorem of Stepanoff we quoted above, $k_j \leqq \varepsilon$, when $G_j > G(\varepsilon)$, the right member of (56) approaches zero. This shows that if λ is not a frequency of $f(x)$ as determined by (53) using the g_j, it will not be when the g_j' are used, and as the roles of these quantities may be interchanged, the totality of frequencies obtained in either case is the same.

As Stepanoff has shown that an ap M function is also ap S if it is uniformly summable, in order to be useful a definition of frequencies should reduce to the usual one in this case. Consideration of the definition of the Lebesgue integral for non-bounded functions shows that, if

(57) $$\alpha(\lambda, f) = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(t) e^{-i\lambda d} dt$$

exists,

(58) $$\lim_{j \to \infty} \alpha(\lambda, g_j)$$

will also exist and have the same value. The existence of (58) does not, of course, imply that of (57). In consequence of the relation just stated, whenever for a particular λ_1 the mean value used before to define $a(\lambda_1, f)$ exists, the previous definition of λ being a frequency agrees with the present one, and in particular, if the ap M function is an ap S function, its characteristic frequencies are the same whether defined by (53) or the mean value definition.

Our definition immediately leads to:

Corollary 1. *For any almost periodic measurable function we may define an enumerable set of characteristic frequencies. The frequencies appearing in the approximating linear combinations may be restricted to be members of this set.*

§ 5.

Asymptotically almost periodic functions.

The definition of an ap C function is a restriction on the values of the function at all points, in the sense that, if the values of such a function be changed on any point set included in a finite interval, the resulting function can not be ap C. If sets of measure zero be disregarded, this property is shared by the three generalizations we have discussed above. The close relation of these classes of functions is further brought out by Stepanoff, who has shown that each ap C, ap S sq, ap S or ap M function belongs to all the later types in the series, and if it possesses the requisite continuity (uniform continuity, with square uniformly summable, uniformly summable respectively) belongs to the corresponding earlier type.

The generalizations of Weyl and Besicovitch are essentially different from these, in as much as they merely restrict the values of the functions in question as to their asymptotic behavior. With Weyl, these asymptotically almost periodic functions with summable square, or Aap S sq functions may be defined as (complex) functions of a real variable x, $f(x)$ measurable and with $|f(x)|^2$ summable in every finite interval satisfying condition b) provided $\tau(\varepsilon)$ is defined as a displacement number for ε when for $T > J(\varepsilon)$;

$$\frac{1}{T}\int_a^{a+T} |f(x+\tau) - f(x)|^2\, dx < \varepsilon^2 \tag{59}$$

so that as Weyl shows

$$\mathrm{M}\{|f(x+\tau) - f(x)|^2\} = \lim_{T\to\infty} \frac{1}{T}\int_a^{a+T} |f(x+\tau) - f(x)|^2\, dx$$

exists uniformly in a and is less than ε^2. Weyl has shown that these functions are the widest generalization for which Bohr's analogue of the Parseval relation holds, in as much as a necessary and sufficient condition for a function to satisfy the completeness relation is that it be Aap S sq. This result may be restated as an approximation theorem, in

Theorem IV. *A necessary and sufficient condition for a (measurable) function $f(x)$ to be an asymptotically almost periodic function with summable square is the existence of a sequence of linear combinations of exponentials* (5) *approximating it in the sense that:*

$$(60) \qquad \lim \varepsilon_n = 0, \qquad \lim_{T\to\infty} \frac{1}{T}\int_a^{a+T} |f(x) - P_n(x)|^2\,dx < \varepsilon_a^2,$$

where the last limit exists uniformly in a.

The sufficiency follows as for theorem I, and the necessity is a part of Weyl's theorem.

Corollary 1 to theorem I applies here, as well as an obvious analogue of Corollary 2 and the remarks which follow.

While the Aap S sq exhaust the generalizations from the standpoint of the completeness relation, from the point of view of approximation theorems we are led to investigate asymptotically ap functions of the other types. As the Aap C functions have no properties distinct from those Aap S sq, and the Aap M functions seem to be too weakly restricted to be of much interest, we shall confine our attention to functions Aap S. We define these as (complex) functions of a real variable x, $f(x)$ summable in every finite interval, satisfying condition b) provided $\tau(\varepsilon)$ is defined as a displacement number for ε when for all a and

$$(61) \qquad T > J(\varepsilon), \qquad \frac{1}{T}\int_a^{a+T} |f(x+\tau) - f(x)|\,dx < \varepsilon$$

we evidently may select $L(\varepsilon) = J(\varepsilon)$. Our approximation theorem is here

Theorem V. *A necessary and sufficient condition for a (measurable) function $f(x)$ to be an asymptotically almost periodic summable function is the existence of a sequence of linear combinations of exponentials* (5) *approximating it in the sense that:*

$$(62) \qquad \lim \varepsilon_n = 0, \qquad \lim_{T\to\infty} \frac{1}{T}\int_a^{a+T} |f(x) - P_n(x)|\,dx < \varepsilon_n,$$

where the last limit exists uniformly in a.

The proof of the sufficiency is entirely analogous to that used for Theorem II.

For the proof of the necessity, we must form certain auxiliary functions. We begin with

$$(63) \qquad F(x) = \frac{1}{h}\int_x^{x+h} f(t)\,dt = \frac{1}{h}\int_0^h f(t+x)\,dt.$$

We shall show that this, like $f(x)$ is also Aap S. We have

$$(64) \qquad F(x+\tau) - F(x) = \frac{1}{h}\int_0^h [f(\tau+t+x) - f(t+x)]\,dt$$

$$(65) \quad \frac{1}{T}\int_a^{a+T} |F(x+\tau) - F(x)|\,dx \leqq \frac{1}{T}\int_a^{a+T} dx\,\frac{1}{h}\int_0^h |f(\tau+t+x) - f(t+x)|\,dt$$

$$\leqq \frac{1}{h}\int_0^h dt\,\frac{1}{T}\int_a^{a+T} f(\tau+t+x) - f(t+x)\,dx.$$

As the inner integral is less than ε for $T > J(\varepsilon)$, depending on ε but not on a, for τ a displacement number for ε for the Aap S function $f(x)$, while the second one is an average, it follows that the left member of (65) is $\leqq \varepsilon$, and $F(x)$ is Aap S with the same $\tau(\varepsilon)$ and $J(\varepsilon)$ as $f(x)$.

We must next show that, for h small, $F(x)$ approximates $f(x)$ asymptotically in the mean. By inverting a double integral, we find

$$(66) \quad \frac{1}{T}\int_y^{y+T} |F(x) - f(x)|\,dx \leqq \frac{1}{h}\int_0^h dt\,\frac{1}{T}\int_y^{y+T} |f(x+t) - f(x)|\,dx.$$

Now recall that $f(x)$ is an Aap S function, so that there exists an $L(\varepsilon)$ with the property b). In particular, there is a $\tau\left(\frac{\varepsilon}{24}\right)$ in the interval $y - L < \tau < y$, so that if

$$(67) \qquad x' = x + y - \tau, \qquad 0 < x' < L + T_0 \text{ as } 0 < x < T_0.$$

By use of (61) and (67) we find for $T_0 > J\left(\frac{\varepsilon}{24}\right)$

$$(68) \qquad \frac{1}{T_0}\int_0^{T_0} |f(x+y) - f(x')|\,dx \leqq \frac{\varepsilon}{24},$$

$$(69) \qquad \frac{1}{T_0}\int_0^{T_0} |f(x+y+t) - f(x'+t)|\,dx \leqq \frac{\varepsilon}{24}.$$

If we recall a property[11]) of the Lebesgue integral over a *finite* interval, here $0 < x' < L + T_0 + h$, we see that

$$(70) \quad \frac{1}{T_0}\int_0^{T_0} |f(x'+t) - f(x')|\,dx \leqq \frac{1}{T_0}\int_0^{L+T_0} |f(x'+t) - f(x')|\,dx' \leqq \frac{\varepsilon}{24},$$

provided $t < \eta(\varepsilon)$. These inequalities give

$$(71) \quad \frac{1}{T_0}\int_y^{y+T_0} |f(x+t) - f(x)|\,dx = \frac{1}{T_0}\int_0^{T_0} |f(x+y+t) - f(x+y)|\,dx \leqq \frac{\varepsilon}{8}.$$

Thus, if h is taken less than $\eta(\varepsilon)$ we shall have by (66),

$$(72) \qquad \frac{1}{T_0}\int_y^{y+T_0} |F(x) - f(x)|\,dx \leqq \frac{\varepsilon}{8},$$

for a *particular* T_0. But, for any $T > T_0$, we have $T = mT_0 + \theta T_0$ and

$$(73) \quad \frac{1}{T}\int_y^{y+T} |F(x) - f(x)|\,dx = \frac{1}{T}\left\{\int_y^{y+T_0} + \int_{y+T_0}^{y+2T_0} + \cdots \int_{y+(m-1)T_0}^{y+mT_0} + \int_{y+mT_0}^{T}\right\}$$
$$\leqq \frac{(m+1)T_0\,\varepsilon}{8T}.$$

As a consequence of this,

$$(74) \qquad \lim_{T\to\infty} \frac{1}{T}\int_y^{y+T} |F(x) - f(x)|\,dx \leqq \frac{\varepsilon}{4},$$

where the existence of the limit uniformly with respect to a easily follow from (61) and (65).

We have now shown that in a certain sense, $F(x)$ approximates $f(x)$ when h is small, and also that it is Aap S. Since it is defined as an integral, it is continuous. Using this fact, we shall now prove that it may be approximated by a function which is Aap S sq. To do this, define for it a $g(x)$ by (44), where $f(x)$ is replaced by $F(x)$ and the quantity G used in the definition is to be more precisely specified later. The function $g(x)$ is obviously continuous and bounded, and hence, being Aap S may readily be proved to be Aap S sq, since

$$(75) \quad \frac{1}{T}\int_a^{a+T} |g(x+\tau) - g(x)|^2\,dx \leqq 2G\cdot\frac{1}{T}\int_a^{a+T} |g(x+\tau) - g(x)|\,dx.$$

We must now show that $g(x)$ approximates $F(x)$. On selecting the $L\left(\frac{\varepsilon}{16}\right)$ of condition b), (61), we shall have

$$(76) \qquad \frac{1}{T}\int_0^T |F(x+\tau)-F(x)|\,dx < \frac{\varepsilon}{16}, \qquad T > J\left(\frac{\varepsilon}{16}\right).$$

The same relation will hold for $g(x)$, and also for $F(x)-g(x)$ with twice the right member.

In particular, take for G the maximum value of $|F(x)|$ in the interval $0 < x < L + T_0$. In this interval the function $F(x)-g(x)$ will be zero, and as (cf. (67)) we can find a displacement number taking any interval of length T_0 into an interval of the same length included in this zero-valued interval, we may write

$$(77) \qquad \frac{1}{T_0}\int_y^{y+T_0} |F(x)-g(x)|\,dx < \frac{\varepsilon}{8}$$

for a *particular* T_0, and hence reasoning as in (73), for any $T > T_0$,

$$(78) \qquad \frac{1}{T}\int_y^{y+T} |F(x)-g(x)|\,dx < \frac{\varepsilon}{4}$$

which is the approximating relation we sought.

Finally, by applying Theorem V to $g(x)$, and using the Schwartz inequality, we obtain

$$(79) \qquad \frac{1}{T}\int_y^{y+T} |g(x)-P_n(x)|\,dx \leqq \sqrt{\frac{1}{T}\int_y^{y+T} |g(x)-P_n(x)|^2\,dx} \leqq \frac{\varepsilon}{2}.$$

On combining the relations (74), (78) and (79) we find

$$(80) \qquad \frac{1}{T}\int_y^{y+T} |f(x)-P_n(x)|\,dx \leqq \varepsilon,$$

which completes the proof of the necessity.

For a discussion of the frequencies in $P_n(x)$, we note that

$$(81) \qquad \frac{1}{T}\int_a^{T+a} dx\, e^{-i\lambda x}\frac{1}{h}\int_x^{x+h} f(t)\,dt = \frac{1}{T}\int_a^{T+a} dx\, e^{-i\lambda x}\frac{1}{h}\int_0^h f(x+t)\,dt$$

$$= \frac{1}{h}\int_0^h e^{i\lambda t}\,dt\,\frac{1}{T}\int_a^{T+a} f(t+x)\,e^{-i\lambda(x+t)}\,dx$$

$$= \frac{1}{h}\int_0^h e^{i\lambda t}\,dt\,\frac{1}{T}\int_{a+t}^{T+a+t} f(z)\,e^{-i\lambda z}\,dz.$$

This, in view of the uniform convergence with respect to a of the inner average, shows that

$$(82) \qquad M\{F(x)e^{-i\lambda x}\} = M\{f(x)e^{-i\lambda x}\}\frac{e^{i\lambda h}-1}{i\lambda h},$$

so that every frequency of $F(x)$ is a frequency of $f(x)$. Also, if we take two incommensurable values of h, and form the corresponding functions $F_1(x)$ and $F_2(x)$, every frequency of $f(x)$ will be a frequency of at least one of these functions, so that there are only an *enumerable* number of frequencies belonging to $f(x)$, for which

$$\alpha(\lambda, f) = M\{f(x)e^{-i\lambda x}\} \neq 0.$$

We may also formulate

Corollary. *For any given approximating linear combination* (5) *such that,*

$$(83) \qquad \lim_{T\to\infty}\frac{1}{T}\int_a^{a+T}|f(x) - P_n(x)|\,dx < \varepsilon_n,$$

the coefficients satisfy the relation

$$(84) \qquad |c_{nk} - \alpha(\lambda_{nk}, f)| < \varepsilon_n$$

so that any frequency which appears in $P_n(x)$ with a coefficient greater than ε is a frequency of $f(x)$.

This follows from the relations

$$(85) \qquad |\alpha(\lambda, f) - \alpha(\lambda, P_n)| = \left|\lim_{T\to\infty}\frac{1}{T}\int_0^T [f(x) - P_n(x)]\,e^{-i\lambda x}\,dx\right|$$
$$\leqq \lim_{T\to\infty}\frac{1}{T}\int_0^T |f(x) - P_n(x)|\,dx.$$

As this also shows that every frequency of $f(x)$ with a coefficient greater than ε_n must appear in $P_n(x)$, we see that the number of frequencies of $f(x)$ with coefficients greater than any given ε is finite.

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(Eingegangen am 8. Januar 1928.)