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Boundary problems of the second order with an indefinite weight-function*)

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1. Introduction

The boundary problems under consideration here are of the form:

$$Au = \lambda wu \tag{1. 1}$$

where: Ω is a bounded domain in R^n; $u \in \mathring{W}^{1,2}(\Omega)$; either $A = -\Delta + q$, with Δ the usual Laplacean, or $A = -\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)$ for $n=1$. In (1. 1) we assume that w is a sign indefinite function in Ω, while A has some negative spectrum. For technical simplicity of presentation, we assume unless otherwise stated $q, w \in L^\infty(\Omega)$ and that there exists a ball $B \subset \Omega$ such that $\mu(B \cap \{x \mid w(x) = 0\}) = 0$. The latter condition implies, as shown below, that the identities $Av \equiv wv \equiv 0$ in Ω imply $v \equiv 0$, so that the operators A and w do not have a common nontrivial vector in their kernel. We remark that many of our proofs hold without change not only if these regularity assumptions on the coefficients are weakened but also in the more general case: $Au = \lambda Bu$ with A, B operators acting in a Hilbert space. To be definite, however we shall assume the explicitly given conditions unless otherwise stated.

Since both the operator A and the operator generated by the function w have both positive and negative spectrum, problem (1. 1) is called indefinite (or non definite). For a survey of boundary problems of this type we refer the reader to [14] from which much of the terminology used in this paper derives.

In Section 2, after some preliminary definition, terminology and recall of earlier results, we rewrite problem (1. 1) as a two parameter problem, viz:

$$L_1 u = \lambda wu + ku \tag{1. 2}$$

in the parameters (λ, k) where $L_1 = L + k_0$, k_0 being chosen so that $L_1 > 0$. Observe that the real eigenvalues of (1. 1) correspond to the eigenpairs (λ, k_0) of (1. 2). For fixed λ we analyze the definite problem (1. 2) and thus obtain a family of eigencurves $(\lambda, k_j(\lambda))$,

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where $k_j(\lambda)$ is the j^{th} eigenvalue of (1. 2) as given by the Courant min.-max. theorem. We conclude, in particular, that the real spectrum of (1. 1) consists of a countable collection of eigenvalues having only $\pm\infty$ as accumulation points. It is then shown that all sufficiently large real eigenvalues λ_j of (1. 1) have eigenfunctions u_j for which sign $\{\lambda_j(wu_j, u_j)\} > 0$. This extends a corresponding one dimensional result [17] to the multidimensional setting. It follows that we may always assume that (1. 1) does not have eigenvalue zero (by a shift in λ), a common hypothesis which simplifies many of the proofs, see e.g. [7]. After the introduction of a suitable labeling of the (i, j)th eigenpair (λ, k) of (1. 2), we show that λ lies between two (possibly different) max.-min. expressions of the Courant types. We show that in some cases there is a classical max.-min. theorem for these, provided λ is sufficiently large, while we also observe that no max.-min. theorem may hold if λ is sufficiently near zero. This represents an extension to the indefinite case of the classical max.-min. theorem for problems where some form of positivity was assumed (see, e.g. [13], [20], [25]). Finally, in this section, we present sufficient conditions for the existence of exactly one complex pair $(\lambda, \bar{\lambda})$ of eigenvalues of (1. 1) in the case, for example, that $w(x, y) = -w(-x, y)$ and $q = -k \in R$ with Ω, k suitably chosen in a range independent of w (compare with [2]). We believe these to be the first explicit sufficiency criteria for a class of partial differential equations.

While in Section 2 we discuss uniformly elliptic operators so that the only singularity comes from the lack of definiteness, in Section 3 we extend several of the results of Section 2 to the case of more general elliptic operators for $n=1$. Specifically, we first find sharp bounds for the Haupt and Richardson indices (defined below) in the case:

$$(1.\ 3)\qquad -(py')' + qy = \lambda w\, y,$$

$$(1.\ 4)\qquad y(a) = y(b) = 0$$

where, in particular, $p \geqq 0, \dfrac{1}{p} \in L^1(a, b)$.

An estimation of a related, but not equivalent, quantity is treated in [2]. We next discuss the possible existence of simple non-real eigenvalues for (1. 3)–(1. 4). At the end of this section we show that the estimation of the real part of a simple non real eigenvalue may be realized provided one can estimate various quadratic forms, in itself a nontrivial undertaking. If $p(x) > \delta > 0$ for some constant δ and p, q, w are regular then many of the results in Section 3 have much simpler proofs, based on the arguments in Section 2, which we usually indicate.

Finally in Section 4, we abandon elliptically altogether and briefly discuss the highly pathological mixed problem for the $n=1$ case where $p(x)$, in (1. 3), is allowed to change its sign in $\Omega = (a, b)$ and $1/p \in L^1(a, b)$. We exploit the features of an example, first presented in [3], where *every* $\lambda \in \mathbb{C}$ is an eigenvalue of (1. 2)–(1. 3) to show that many of the results herein and parts of the general theory may not be improved upon by relaxing the sign condition on $p(x)$. This example generates an alternate example of a symmetric linear operator in a Krein space (see [4]) whose spectrum is all of $\mathbb{C}$. Apparently the only other such example known is due to H. Langer [4], Chapter 5. The main difference between our example and that of Langer's is that $\mathbb{C}$ consists only of continuous spectrum in the latter whereas in our case $\mathbb{C}$ consists only of eigenvalues.

After a brief excursion into Sturm theory we present an eigenvalue existence and asymptotics result for the *Neumann* problem associated with (1. 3) where $q(x)=0$ and $w(x)>0$. We also give a "positivity result" related to the eigenfunctions of the smallest positive and largest negative eigenvalues.

2. Uniformly elliptic problems

As stated in the Introduction, we consider in a bounded domain $\Omega \subset \mathbb{R}^n$ the elliptic expression:

$$lu = -\Delta u + qu = \lambda w u \tag{2. 1}$$

with $u \in \mathring{W}^{1,2}(\Omega)$, q, w real and in $L^\infty(\Omega)$. We assume l, w indefinite: there exist $\varphi, \eta, \psi \in C_0^\infty(\Omega)$ such that $(w\varphi, \varphi)<0$, $(w\psi, \psi)>0$, $B(\eta, \eta)<0$ where B denotes the form associated with l and $(\,,\,)$ represents the $L^2(\Omega)$ inner product. We also assume that for some sphere $B_1 \subset \Omega$, $\mu(\{x \mid w(x)=0\} \cap B_1)=0$.

Since $l-\xi w$ is bounded below for any fixed constant ξ, we let $L-\xi w$ denote the associated Friedrich's operator. It is our purpose to discuss the spectrum of:

$$Lu = \lambda w u. \tag{2. 2}$$

An eigenvector of (2. 2) is called a *ghost state* iff $(wu, u)=0$. We observe that any nonreal eigenvalue of (2. 2) has only eigenvectors which are ghost states (by the symmetry of l, w).

For any subset S of $\mathring{W}^{1,2}(\Omega)$ we let $|S|$ denote the largest number of linearly independent vectors in S, while $\# S^+$ (resp. $\# S^-$) denotes the largest number of vectors in S such that $(wu_i, u_j)=\delta_{ij}$ (resp. $-\delta_{ij}$). We observe that $\# S^{+,-} \leqq |S|$, with strict inequality possible due to the presence of ghost states. In the sequel, where the situation is clear from the context, we merely write $\# S$ for $\# S^+$ or $\# S^-$.

Finally, if S is any set of scalars, we let $|S|$ denote the number of points of S.

It is useful to also recall the following definitions and results. Let $T: L^2 \to L^2$ be a compact operator with eigenvalue λ. The root subspace S associated with λ is the space $\{v \mid (T-\lambda)^s v=0 \text{ for some } s\}$. The members of S are termed root vectors for λ while $\dim(S)$ is the algebraic multiplicity of λ. We recall also the following continuity result (see [11], p. 212): Let $B \subset \mathbb{C}$ be a given domain bounded by a smooth curve, and assume that the spectrum of T in B consists of a finite number of isolated eigenvalues of total algebraic multiplicity m. Then in the operator norm there exists a ball $B_\varepsilon(T)$ centered at T, with radius $\varepsilon=\varepsilon(T)$, such that if $T_1 \in B_\varepsilon(T)$ then the spectrum of T_1 in B also consists of isolated eigenvalues of total algebraic multiplicity m. In the applications to follow, $Tu=L^{-1}(wu)$, and observe that induction shows:

$$\left(T-\frac{1}{\lambda}\right)^{s+1} v=0 \quad \text{iff} \quad (L-\lambda w)v = w\left(\sum_{i=1}^{s} u_i\right) \quad \text{where} \quad \left(T-\frac{1}{\lambda}\right)^{i} u_i=0$$

$$\text{for} \quad i=1,\dots,s.$$

We begin by stating:

Theorem 2. 0. (a) *Equation* (2. 2) *admits two* (*resp. one*) *positive eigenfunction*(*s*) *iff* $L-\xi w>$ (*resp.* $\geqq$) 0 *for some real* ξ.

(b) *If* (2. 2) *admits a positive eigenfunction u with eigenvalue* λ *then*:

(i) λ *is simple* (*i.e.* dim (*eigenspace*) $=1$);

(ii) *there are no ghost states except, possibly, u itself*;

(iii) *if* μ *is any other eigenvalue with eigenvector* v *then* $(\mu-\lambda)(wv,v)>0$.

(c) *Let* $S=\{u \mid \lambda(wu,u)\leqq 0$, *u eigenfunction of* (2. 2) *with eigenvalue* $\lambda\}$. *Then* $|S|$ *is finite.*

(d) *For any fixed* λ, *let* $S=\{u \mid Lu=\lambda wu\}$. *Then* $|S|$ *is finite.*

(e) *L,w do not have nontrivial common null space.*

(f) *If* L^{-1} *exists and* $\mu,\lambda\neq 0$ *then root vectors u, v corresponding to eigenvalues* λ,μ $(\lambda\neq\bar{\mu})$ *satisfy* $(wu,v)=0$.

(g) *Eigenvectors corresponding to different eigenvalues of* (2. 2) *are linearly independent.*

Proof. Parts (a), (b) are explicitly shown in [1]. Part (c) is proved in [15] or [5]. Part (d) is an immediate consequence of the Sobolev Embedding Theorems. To see part (e) assume $Lv=wv=0$. Then $v\equiv 0$ in a sphere $B_1\subset\Omega$ by our assumption on w. Since $|\Delta v|\leqq K(|v|)$ for some constant K (depending on the coefficients) we find $v\equiv 0$ by the Unique Continuation Theorem given in [21], p. 240. Next, for part (f) we follow the procedure of [10], p. 130 (see also [5]). Specifically, assume $\left(T-\frac{1}{\lambda}\right)^r u=\left(T-\frac{1}{\mu}\right)^s v=0$. If $r=s=1$, u, v are eigenvectors and the result is immediate. Suppose the result is true for any indices with sum $<r+s$. Set: $u'=\left(T-\frac{1}{\lambda}\right)u$, $v'=\left(T-\frac{1}{\mu}\right)v$. We conclude $(wu,v')=(wu',v)=0$ by assumption, whence:

$$\begin{aligned}0&=\lambda(wu',v)=\lambda(wL^{-1}(wu),v)-(wu,v)\\&=\lambda(u,wL^{-1}(wv))-(wu,v)\\&=\frac{\lambda}{\bar{\mu}}\,[\bar{\mu}(wu,v')+(wu,v)]-(wu,v)=\left[\frac{\lambda}{\bar{\mu}}-1\right](wu,v)\end{aligned}$$

and the result follows. Finally, for part (g), assume $(L-\lambda_i w)u_i=0$, for $i=1,\ldots,j$. Since all u_i are eigenvectors, if $\sum_{i=1}^{j} c_i u_i=0$ then $\sum_{i=1}^{j} w\lambda_i c_i u_i=0$ and we conclude

$$\sum_{i=1}^{j-1} w(\lambda_i-\lambda_j)c_i u_i=0 \quad\text{in}\quad B_1\subset\Omega,$$

where B_1 is a sphere in Ω in which $w\neq 0$ a.e. We note, by induction, that $u_1\equiv 0$ in B_1. Unique Continuation then shows $u_1\equiv 0$ in Ω and, by induction, $u_i\equiv 0$ for $i=2,\ldots,j$.

We remark that, in general, eigenvectors corresponding to different eigenvalues need not be independent. Indeed, if $Lv = wv \equiv 0$ then $Lv - \lambda wv = 0$ for any λ but this is not possible here. We also note that results for the cases $\lambda = 0$ or L not invertible, explicitly excluded in some of the proof of Theorem 2. 0, may be always obtained by a shift in λ, or L as shown in arguments introduced below.

We note in passing that the space A spanned by the eigenvectors corresponding to the nonpositive spectrum of L may be used to estimate $|S|$ in part (c) as was done in [15]. We take this opportunity to modify slightly some estimates in [15]. For every complex λ (with $\operatorname{Im}\lambda > 0$), let $n(\lambda)$ denote the dimension of its eigenspace. If λ is real and $(wu, u) \neq 0$ for any of its eigenvectors u, then we let $n(\lambda)$ again denote the dimension of the eigenspace of λ if $\lambda(wu, u) \leqq 0$, while we set $n(\lambda) = 0$ otherwise. Finally, if λ is real and $(wu, u) = 0$ for some eigenvector u, then $n(\lambda)$ will denote the maximal dimension of spaces spanned by linearly independent eigenvectors $\{g_i\}$ such that:

$$\lambda(wg_i, g_i) \leqq 0, \ (wg_i, g_j) = 0, \quad (i \neq j).$$

The classical Spectral Theorem shows that the sum of $n(\lambda)$ over $\{\lambda \mid \operatorname{Im}\lambda \geqq 0\}$ does not exceed $\dim(A)$. The estimates given in [15] are to be modified accordingly, but note that if the eigenspaces are one dimensional then no modification is required. For an alternate explicit estimate illustrating this connection, see also [5], or [12]. We shall not need such an estimate in the sequel, except in obvious special cases.

Theorem 2. 0 indicates that positive eigenfunctions cannot be used to investigate the spectrum of truly indefinite problems. Indeed, such problems may be characterized by the absence of (positive) ground states. This is in sharp contrast to the definite cases (see e.g. [13], [21], [9]) where there are eigenvalue(s) with positive eigenvector(s) at the onset of the spectrum.

Select a constant $k_0 > 0$ such that $L_1 = L + k_0 > 0$. It will be useful to consider in the sequel the problem:

$$L_1 u = \lambda wu + ku \tag{2. 3}$$

with eigenpair (λ, k). For fixed $\lambda \in R$ we let $k_j = k_j(\lambda)$ denote the j^{th} eigenvalue of (2. 3) as given by the Courant min.-max. principle. Observe that in particular, the real eigenvalues of the original problem (2. 2) correspond to the set:

$$\bigcup_j k_j^{-1}(k_0).$$

We first show that if $\mu\{x \mid w(x) = 0\} = 0$ then the set of λ's associated with ghost states must be bounded uniformly for bounded k.

Theorem 2. 1. *Assume $\mu\{x \mid w(x) = 0\} = 0$. If (λ, k) is associated with an eigenvector u such that $\lambda(wu, u) \leqq 0$ and $L_1 u = \lambda wu + ku$ with $|k| < \beta$ then there exists a constant $\gamma = \gamma(\beta) > 0$ such that $|\lambda| < \gamma$.*

Proof. It will suffice to show, by compactness, that if $L_1 u_n - k_n u_n = \lambda_n wu_n$ with $k_n \to k$ and $\lambda_n(wu_n, u_n) \leqq 0$ then, for some constant γ, $|\lambda_n| < \gamma$. To see this, set $(u_n, u_n) = 1$.

It follows that:

$$\|u_n\|_{1,2}^2 \leqq c(L_1 u_n, u_n) \leqq Q$$

for some constant Q, where $\| \ \|_{1,2}$ denotes the $\mathring{W}^{1,2}$ norm. We conclude that u_n has a convergent subsequence (also denoted by u_n) to some u in L^2. Let φ_j denote an eigenfunction of the (regular) positive operator L_1, with eigenvalue μ_j. We conclude

$$\mu_j(u, \varphi_j) = (u, (L_1)\varphi_j) = \lim_{n\to\infty} [((L_1 - k_n)u_n, \varphi_j) + k_n(u_n, \varphi_j)]$$
$$= \lim_{n\to\infty} [\lambda_n(wu_n, \varphi_j) + k_n(u_n, \varphi_j)].$$

If $|\lambda_n| \to \infty$ it follows that $(wu_n, \varphi_j) \to 0$ with n, i.e. $(wu, \varphi_j) = 0$. But $\{\varphi_j\}$ is complete in L^2 and, consequently, $wu \equiv 0$. By our assumption on w this implies $u \equiv 0$, which contradicts $(u, u) = 1$, and the result follows.

We next recall the following results. A brief indication of the proof is given for the reader's convenience.

Theorem 2. 2. *Let $y = k_j(\lambda)$ denote the real eigencurves of (2. 3) as given by the Courant min.-max. principle.*

(a) *$k_j(\lambda)$ is Lipschitz continuous (indeed $|k_j(\lambda) - k_j(\mu)| \leqq |\lambda - \mu| \, \|w\|_{L^\infty}$);*

(b) *$k_j(\lambda) \to +\infty$ as $j \to \infty$, uniformly for λ in compact sets;*

(c) *$y = k_1(\lambda)$ is concave;*

(d) *$k_j(\lambda) \to -\infty$ as $|\lambda| \to \infty$;*

(e) *Problem (2. 2) has infinitely many eigenvalues $\{\lambda_i^\pm\}$, $\lambda_i^\pm \to \pm\infty$.*

Proof. Part (a) is immediate from the Courant min.-max. theorem, while (b) is a direct consequence of the compactness of $(L_1 - \lambda w + \tau)^{-1}$ for τ large and comparison with $(L_1 - [\sup_{\lambda \in K} |\lambda|] \, \|w\|_{L^\infty})$ with K compact. Part (c) follows from (d) and either known results (see, e.g., [23]) or it may be shown directly from the proof of Theorem 2. 0(b). Next, for part (d), consider $Mu = wu$. This is a self-adjoint map, $L^2(\Omega) \to L^2(\Omega)$, with both positive and negative essential spectrum $\varepsilon(T)$. Let $\alpha \in \varepsilon(T) \cap R^+$. By the spectral theorem there exist $\{u_i\}$ orthonormalized in L^2 such that $\|wu_i - \alpha u_i\| \to 0$. Let $\{\varphi_i\} \subset C_0^\infty(\Omega)$, $\|\varphi_i - u_i\| < 1/i$. We observe: $(\varphi_i, \varphi_j) \to \delta_{ij}$, $(w\varphi_i, \varphi_j) \to \alpha\delta_{ij}$ as $i, j \to \infty$. Now let p be given and choose $k > 0$ sufficiently large so that $A \geqq \alpha/2I$ and $B \geqq 1/2I$ where A (resp. B) is the $p \times p$ symmetric matrix with entries: $(w\varphi_{i+k}, \varphi_{j+k})$ (resp. $(\varphi_{i+k}, \varphi_{j+k})$) for $i, j = 1, \ldots, p$. Let $\{\xi_i\}_{i=1}^{p-1}$ be given in L^2 and S the space generated by them. Choose $\{c_i\}_1^p$ such that $\sum_1^p c_i^2 = 1$ and $v = \sum_1^p c_i \varphi_{i+k} \in S^\perp$. It follows that:

$$(L_1 v - \lambda wv, v) \leqq [q_1 - \lambda q_2] \, (v, v)$$

with constants $q_1, q_2 > 0$ independent of λ and the specific ξ_i, whence $k_p(\lambda) \to -\infty$ as $\lambda \to \infty$. The case $\lambda \to -\infty$ is identical with $\beta \in \varepsilon(T) \cap R^-$ replacing α. Finally, for part (e), let $P > 0$ be given. Select I_0 such that if $i > I_0$ then $k_i(\lambda) > k_0$ for $|\lambda| < P$, whence

$\{\lambda \mid \lambda > P\} \cap k_i^{-1}(k_0) \neq \emptyset$ by parts (a)—(d). This shows the existence of $\{\lambda_i^+\}$. The existence of $\{\lambda_i^-\}$ is identical.

We observe the following consequences.

Corollary 2. 3. *Let $k_0 \in R$ be given and assume $0 < \lambda_0 \in k_i^{-1}(k_0)$ for some i. Then:* (a) *for all $j > i$ there exists $\lambda_j \in k_j^{-1}(k_0)$ such that $\lambda_j \geqq \lambda_0$;* (b) *there exists an integer $J_0 = J_0(k_0)$ such that if $j \geqq J_0$ then $(wu_j, u_j) > 0$ for any eigenfunction of* (2. 3) *corresponding to (λ_j, k_0). Analogous results for $\lambda_0 < 0$. If $\mu\{x \mid w(x) = 0\} = 0$ and $|k_0| < \beta$ then $J_0 = J_0(\beta)$.*

Proof. Since $k_j(\lambda) \geqq k_i(\lambda)$ for all λ, the existence of λ_j is immediate from Theorem 2. 2. Next, note that if u_j satisfies $L_1 u_j - k_0 u_j = \lambda_j w u_j$, Theorem 2. 0 (c), (g) give a contradiction if $\lambda_j(wu_j, u_j) \leqq 0$ for infinitely many indices j. The uniformity of J_0 in β is a consequence of Theorem 2. 1.

Corollary 2. 4. *For any $\alpha \in R$ there exists an integer $J_0 = J_0(\alpha)$ such that if $j \geqq J_0$ then $k_j^{-1}(\alpha) = \{\lambda_1, \lambda_2\}$ with $\lambda_1 < 0 < \lambda_2$. If $\mu\{x \mid w(x) = 0\} = 0$ and $|\alpha| < \beta$ then $J_0 = J_0(\beta)$.*

Proof. By Theorem 2. 2(b), (d), there exists J_0' such that if $j \geqq J_0'$ then $\{\lambda_1, \lambda_2\} \subset k_j^{-1}(\alpha)$ with $\lambda_1 < 0 < \lambda_2$. We show that for $J_0 \geqq J_0'$, sufficiently large, $k_j^{-1}(\alpha) = \{\lambda_1, \lambda_2\}$. Assume to the contrary, that there exist $\lambda_2, \lambda_3 \in (0, \infty) \cap k_j^{-1}(\alpha)$ with $0 < \lambda_2 < \lambda_3$. Since the other possible case: $\lambda_1, \lambda_3 \in (-\infty, 0) \cap k_j^{-1}(\alpha)$ is identical, we do not consider it explicitly. By the continuity of k_j, we may assume that either there exists a sequence $\lambda_n \to \lambda_3^-$ such that $k_j(\lambda_n) \leqq k_j(\lambda_2) = k_j(\lambda_3) = \alpha$, or that there exists a sequence $\lambda_n \to \lambda_2^+$ such that $k_j(\lambda_n) \geqq k_j(\lambda_2) = k_j(\lambda_3) = \alpha$. Again we treat only the first possibility, since the same arguments may be used in the second case. Let $\{u_n\}$ be a normalized (in L^2) sequence of eigenfunctions:

$$L_1 u_n - \lambda_n w u_n = k_j(\lambda_n) u_n .$$

Clearly $\{u_n\}$ is bounded in $\mathring{W}^{1,2}(\Omega)$ and, without loss of generality, $\lim u_n \to v$ weakly in $\mathring{W}^{1,2}(\Omega)$ and strongly in $L^2(\Omega)$. Note that v satisfies:

$$L_1 v - \lambda_3 w v = k_j(\lambda_3) v = \alpha v .$$

Hence:

$$\begin{aligned}(u_n, (\alpha + \lambda_3 w) v) &= (L_1 u_n, v)\\ &= ((k_j(\lambda_n) + \lambda_n w) u_n, v)\end{aligned}$$

or:

$$(\lambda_n - \lambda_3)(w u_n, v) = [\alpha - k_j(\lambda_n)]\,(u_n, v) .$$

We conclude $(wv, v) = \lim_n [(wu_n, v)] \leqq 0$. Hence $\lambda_3(wv, v) \leqq 0$ and this contradicts Corollary 2. 3 if it occurs for infinitely many j.

Observe that the arguments of Corollary 2. 4 and Theorem 2. 0(c) also show that for any α, j, the set $k_j^{-1}(\alpha)$ is finite.

Corollary 2. 5. *For any real k there exists a countable set Z such that* $L_1 - k - \tau w$ *does not have ordinary eigenvalue zero for any complex* $\tau \notin Z$.

Proof. Let $Z = \bigcup_i k_i^{-1}(k)$. By the remark after Corollary 2. 4, Z is countable. We adjoin to Z — if need be — a finite number of purely complex eigenvalues and conclude that for $\tau \notin Z$ the equation $L_1 u - ku - \tau wu = 0$ implies $u \equiv 0$.

We shall apply this result in the future by assuming, without loss of generality, that $(L_1 - k)^{-1}$ exists.

We now proceed to estimate the eigenvalues of (2. 2). For this it is convenient to introduce the following notation: Let (λ, k) be an eigenpair for (2. 3). We decompose the eigenspace associated with the pair into the following sets, the last three of which are in particular the positive, neutral, negative parts (see [4], Chapter 1):

$$S_0 = \{v \mid (v, v) = 1\},$$

$$S_1 = \{v \mid (wv, v) > 0\},$$

$$S_2 = \{v \mid (wv, v) = 0, \ v \text{ nontrivial}\},$$

$$S_3 = \{v \mid (wv, v) < 0\}.$$

Observe that if $S_1 \neq \emptyset$ and $S_3 \neq \emptyset$ then $S_2 \neq \emptyset$: see e.g. [4], Chapter 1. Let $n_i = |S_i|$ for $i = 0, 2$, $n_1 = \# S_1^+$, $n_3 = \# S_3^-$. We also observe the following relationships between the n_i. A short proof is given for convenience.

Lemma 2. 6. n_i *is finite for* $i = 0, \ldots, 3$. *Assume* $S_2 = \emptyset$ *then either* $S_1 = \emptyset$ *or* $S_3 = \emptyset$. *Furthermore, in this case, if* S_1 *(resp.* S_3*)* $\neq \emptyset$ *then* n_1 *(resp.* n_3*)* $= n_0$.

Proof. That n_0, n_2 are finite is an immediate consequence of the Sobolev compact embedding theorems. We show n_1, n_3 are finite by showing $n_1 \leqq n_0$ and $n_3 \leqq n_0$. Indeed let $\{u_i\}_1^p$ be given, $(wu_i, u_j) = \delta_{ij}$. We orthonormalize $\{u_i\}$ in L^2 and thus create an orthonormal set $\{v_i\}_1^p$, whence $n_1 \leqq n_0$. The proof that $n_3 \leqq n_0$ is identical. Finally, assume $S_2 = \emptyset$, and without loss of generality that $S_1 \neq \emptyset$, so that $S_3 = \emptyset$. Given $\{v_i\}_1^m$ eigenvectors such that $(v_i, v_j) = \delta_{ij}$ we now orthonormalize with respect to $(w\cdot, \cdot)$ to obtain the set $\{u_i\}_1^m$ such that $(wu_i, u_j) = 0$ if $i \neq j$. Since $S_2 = S_3 = \emptyset$ we conclude $\{u_i\} \subset S_1$ i.e. $(wu_i, u_i) > 0$, whence $n_0 \leqq n_1$.

If $S_2 = \emptyset$ we thus define the multiplicity of (λ, k) to be n_0. If $S_2 \neq \emptyset$ then the multiplicity of (λ, k) is given by the four vector (n_0, n_1, n_2, n_3). Observe that in such a case there are n_2 linearly independent ghost states associated with (λ, k). The same terminology will, in particular, be used for complex λ.

Again motivated by [4], we term (λ, k) a positive (resp. negative) eigenpair iff $S_2 = S_3 = \emptyset$ (resp. $S_1 = S_2 = \emptyset$). An associated eigenvector u will be termed positive (resp. negative). Note that this definition does not imply any actual sign conditions on (λ, k) or the function u.

Fix k and let (λ, k) be a positive eigenpair of multiplicity m. We order the eigenvectors associated with (λ, k) by the procedure of Lemma 2. 6. Specifically, we specify and fix $\{v_1, \dots, v_m\}$, orthonormal in L^2, by the Courant min.-max. theorem. We then construct $\{u_1, \dots, u_m\}$ by the Gram-Schmidt procedure of the Lemma. Assume that, for a fixed k, the positive eigenpairs (μ, k) with $\mu < \lambda$ have been designated as the first j^{th} pairs. Then λ is termed the $i+j^{\text{th}}$ positive eigenvalue and is denoted by λ_{i+j} iff it is associated with u_i. An identical procedure is followed for negative eigenpairs, while the precise ordering of multiple (real or complex) eigenvalues with associated ghost states is left to convenience. Observe that λ_i is well defined since by Corollary 2. 3, (λ, k) is negative (resp. positive) — for fixed k — if $\lambda \ll$ (resp. $\gg$)0. We thus construct for each k two sequences $\{u_i^{\pm}\}_1^{\infty}$ of positive (resp. negative) eigenvectors with associated eigenvalues $\lambda_i^{\pm} \to \pm\infty$ as $i \to \infty$, and such that $(wu_i^{\pm}, u_j^{\pm}) = \pm\delta_{ij}$. These sequences constitute possibly different sets from those given in Theorem 2. 2(e). Finally, observe that in determining that λ is the i^{th} positive eigenvalue, we do not count the negative eigenvalues nor those associated with ghost states.

Henceforth, (λ, k) will be termed the $(i, j)^{\text{th}}$ positive eigenpair iff $(\lambda, k) = (\lambda_i, k_j(\lambda_i))$ where λ_i is the i^{th} positive eigenvalue of (2. 3). In view of the symmetry present in the results, we shall explicitly consider only positive eigenpairs.

We have the following estimate relating i and j:

Corollary 2. 7. *For any k there exists a constant $e = e(k)$ such that if (λ, k) is the positive $(i, j)^{\text{th}}$ eigenpair then $|i - j| < e(k)$.*

Proof. Let k be given and note that there exists $J_0 \in R$ such that: for some i_0 and $\lambda_{i_0} \in k_{J_0}^{-1}(k) \cap (0, \infty)$, (λ_{i_0}, k) is the $(i_0, J_0)^{\text{th}}$ positive eigenpair, and by Corollary 2. 3 and Lemma 2. 6, for $m > 0$, $\lambda_{i_0+m} \in k_{J_0+m}^{-1}(k) \cap (0, \infty)$ implies (λ_{i_0+m}, k) is the $(i_0 + m, J_0 + m)^{\text{th}}$ positive eigenpair. Hence, the $(i, j)^{\text{th}}$ eigenpair — with i, j large — satisfies $|i - j| = |(i_0 + m) - (J_0 + m)| = |i_0 - J_0| = e$, and the bound follows since there is at most a finite number of i, j which do not satisfy this estimate. This completes the proof.

Fix k and let G denote any linear space generated by a linearly independent set of root vectors $\{g_i\}_{i=1}^m$ such that $(wg_i, g_j) = ((L_1 - k)g_i, g_j) = 0$ for all i, j. Note that any such G is finite dimensional, and we let M denote the set of possible G. If no such vectors exist, we set $G = \{0\}$, and note $|G|$ is then 0. Observe that the set of all non real root vectors may not be a member of M. For example if $\{\mu, \bar{\mu}\}$ are a nonreal pair of eigenvalues of (2. 2) with eigenvectors $u, \bar{u}$ then $\operatorname{sp}\{u, \bar{u}\} \in M$ iff $(wu, \bar{u}) = 0$. We also observe that if $g \in G \in M$ and h is a positive eigenvector, then $(wg, h) = 0$ by Theorem 2. 0(f).

We now state our first estimate.

Theorem 2. 8. *Let (λ, k) be the $(i, j)^{\text{th}}$ positive eigenpair. Then*:

$$\sup_{G \in M} \left[\sup_{\substack{V \\ V^\perp \cap G = \{0\}}} \left[\inf_{\substack{u \in V^\perp \\ (wu, u) > 0}} \frac{((L_1 - k)u, u)}{(wu, u)} \right] \right] \leqq \lambda$$

$$(2.4) \qquad \leqq \inf_{G \in M} \left[\sup_{\substack{S \\ S^\perp \cap G = \{0\}}} \left[\inf_{\substack{u \in S^\perp \\ (wu, u) > 0}} \frac{((L_1 - k), u, u)}{(wu, u)} \right] \right],$$

where: S, V *are subspaces in* L^2 *with* $\dim(S)=j-1$, $\dim V=i-1+|G|$; $S^\perp, V^\perp$ *are understood in the* L^2 *sense*; *the members of each subspace or set are allowed to be complex*; $u \in \overset{\circ}{W}{}^{1,2}(\Omega)$.

Proof. Observe that there exist $j-1$ eigenvectors of (2.3) such that $L_1 v_\alpha - \lambda w v_\alpha = k_\alpha v_\alpha$ for $\alpha=1,\dots,j-1$ with $k_\alpha \leqq k$. Let v be the eigenfunction associated with (λ, k) and note $(wv, v)>0$ as (λ, k) is positive. Set $S=\{v_\alpha\}_1^{j-1}$. Then $v \in S^\perp$ and $(L_1 v - kv, v) = \lambda(wv, v)$ while if $\tau \in S^\perp$ and $(w\tau, \tau)>0$ then $(L_1\tau - \lambda w\tau, \tau) \geqq k(\tau, \tau)$ implies

$$\lambda = \inf_{\substack{\tau \in S^\perp \\ (w\tau,\tau)>0}} \frac{(L_1\tau - k\tau, \tau)}{(w\tau, \tau)}.$$

Next, for any $G \in M$ we show $S^\perp \cap G = \{0\}$. Suppose $t = \sum c_\alpha g_\alpha \in S^\perp \cap G$ with $\{g_\alpha\}$ linearly independent root vectors spanning G. Note $(wg_\alpha, g_\beta)=0$ for all α, β and $((L_1-k)(\sum c_\alpha g_\alpha), \sum c_\alpha g_\alpha)=0$ by definition of G or, equivalently,

$$((L_1 - \lambda w)(\sum c_\alpha g_\alpha), (\sum c_\alpha g_\alpha)) = k(\sum c_\alpha g_\alpha, \sum c_\alpha g_\alpha).$$

By the Spectral Theorem, since $t \in S^\perp$, we conclude $t = \sum c_\alpha g_\alpha$ is an eigenvector of $L_1 - \lambda w$ with eigenvalue k: $L_1 t - \lambda w t = kt$. That is: $t = cu$ for some constant c and eigenvector u associated with k. But then $\sum c_\alpha g_\alpha - cu = 0$, whence $c=0$ since root vectors corresponding to eigenvalues which are not complex conjugate are $(w\cdot,\cdot)$ orthogonal, and (λ, k) is a positive eigenpair. Whence $t=0$ and $S^\perp \cap G = \{0\}$. To establish the other inequality: choose any $G \in M$, set $d = i + |G|$ and select any V with $\dim V = d-1$, $V^\perp \cap G = \{0\}$. Let $v = \sum_1^d c_\alpha \eta_\alpha$ with $\{\eta_\alpha\}$ either root vectors associated with ghost states in G or positive eigenfunctions with eigenvalue pair (μ, k), $(\mu \leqq \lambda)$. We observe that $\{\eta_\alpha\}$ is never empty since the eigenvector of (λ, k) is a member. Choose $\{c_\alpha\}$ such that $v \in V^\perp$. Since $V^\perp \cap G = \{0\}$ we observe that expressing v as $v = g + h$ — with $g \in G$ and h a linear combination of positive eigenvectors — yields: $h=0$ implies $v = g \in V^\perp \cap G$ and hence $v=0$, $c_\alpha = 0$ for $\alpha = 1,\dots,d$. We conclude $h \neq 0$. We observe that $(wg, g)=0$ by definition of G, while $(wg, h)=0$ and $(wh, h)>0$ since positive eigenpairs have eigenvectors orthogonal to G in $(w\cdot,\cdot)$ by the arguments preceding this theorem. It follows that $((L_1-k)v, v) \leqq \lambda(wv, v)$, and:

$$\lambda \geqq \inf_{\substack{u \in V^\perp \\ (wu,u)>0}} \frac{((L_1-k)u, u)}{(wu, u)}.$$

We remark that if $j=1$ then $G=\{0\}$ and $L_1 + \tau w > 0$ for some τ by Theorem 2.0. Theorem 2.8 is then contained in the classical min.-max. principle for (left) definite problems.

For completeness, we observe heuristically that it may be possible to increase the dimension of the space V on the left hand side of (2.4). This could, in principle, improve the estimate and would be done by counting not only the i positive eigenfunctions corresponding to (μ, k) with $\mu \leqq \lambda$, but also some eigenfunctions z corresponding to nonpositive pairs (δ, k) — of higher eigenspace multiplicity — and such that $(wz, z)>0$.

If such eigenfunctions satisfy the various orthogonality relations needed in our arguments, they could also be used in the proof of Theorem 2. 8. Possibly, one could also enlarge M along the same lines. While abstract results can be stated along these arguments, estimate (2. 4) suffices for our explicit examples and we continue to work with it.

As an immediate consequence we note the following Corollary which shows that if $j=i$ then no ghost states are present and the min.-max. principle holds.

Corollary 2. 9. *Let k be fixed and (λ, k) be the $(i, j)^{\text{th}}$ positive eigenpair.*

(a) $j \geqq i + \sup\{|G|; G \in M\}$.

(b) *If $j=i$ then there are no ghost states and*

$$\lambda = \sup_S \left[\inf_{\substack{u \in S^\perp \\ (wu, u) > 0}} \frac{((L_1 - k)u, u)}{(wu, u)} \right]. \tag{2. 5}$$

(c) *There exists a constant $C = C(k)$ such that, for any λ,* (2. 4) *holds with* $\dim V = i - 1 + |G|$, $\dim S = i - 1 + |G| + C$.

Proof. (a) This is similar to the proof of Theorem 2. 0(c). Suppose $j < i + |G_0|$ for some G_0. Without loss of generality, we may assume that $L_1 - \lambda w - k$ has j nonpositive ordinary eigenvalues but that the $j+1^{\text{st}}$ is positive, since, if this is not the case, by our numbering scheme we need only replace (i, j) by $(i+m, j+m)$ for some m not exceeding the multiplicity of (λ, k). We also note that there exist i positive eigenvectors $\{u_\tau\}_1^i$ of $(L_1 - k)u_\tau = \lambda_\tau w u_\tau$ with $\lambda_\tau \leqq \lambda$. Consider the vector $t = \sum_{\tau=1}^{i} c_\tau u_\tau + \sum_{\tau=1}^{|G_0|} d_\tau g_\tau$ with $g_\tau \in G_0$. We select c_τ, d_τ so that t is perpendicular to the eigenvectors in the nonpositive spectrum of $L_1 - \lambda w - k$. Since $\{u_\tau\}$, $\{g_\tau\}$ are linearly independent, $t \neq 0$. But $((L_1 - k)t, t) \leqq \lambda(wt, t)$ whence $((L_1 - k - \lambda w)t, t) \leqq 0$. This contradiction shows the result. Next, for part (b), if $j=i$ then $G = \{0\}$ from part (a) and the min.-max. follows from Theorem 2. 8, since the trial spaces S, V in (2. 4) have the same dimension. Finally, for part (c) we merely employ Theorem 2. 8 and Corollary 2. 7.

For future convenience we also state:

Corollary 2. 10. *Let k be fixed and let (λ, k) be the $(i, j)^{\text{th}}$ positive eigenpair with $j = i + |G|$ for some $G \in M$. Then* (2. 5) *holds for λ with* $\dim S = j - 1$, $S^\perp \cap G = \{0\}$.

We now remark on the relation of our results to previously known estimates for the cases of "no negative squares" (see e.g. [20], [25], [5]). An example is known (see [2] and below) where the (1, 2) and (2, 2) positive eigenpairs exist and there are no ghost states. Consequently even if M consists only of $\{0\}$ then we cannot conclude $i=j$ for all i, j. In the same example, it can be shown that setting $\dim S = 0$ in (2. 5) (i.e. $S = \{0\}$) yields $-\infty$, while $\dim S = 1$ gives the (2, 2) eigenpair. The details of these remarks are given at the end of this Section. What this shows is that, even if no ghost states present, we cannot hope to precisely locate all eigenpairs (i, j), for fixed k, by a max.-min. argument of the above type, and the preceding results cannot be strengthened in this direction. Observe that the fact that the (λ, k) in the example are not positive (in the sense $\lambda > 0$) is irrelevant, since these results are independent of shifts in λ.

We proceed to show, however, that under suitable assumptions a variant of the min.-max. is valid for fixed k and large (depending on k) λ regardless of the presence of ghost states. We conjecture this is true in general.

To formulate these results we observe that if $L_1 u - \lambda w u = k u$, with λ complex, then by Corollary 2. 5, $(L_1 - \mu w - k)^{-1}$ exists for some μ. We thus have:

$$\frac{1}{(\lambda - \mu)} u = T u$$

where $T u = (L_1 - \mu w - k)^{-1}(w u)$ is a compact map from $L^2 \to L^2$ by the Sobolev Embedding Theorems. Assume u is the only eigenvector but that the root subspace has dimension 2. We conclude that, for some v, $L_1 v - \lambda w v - k v = w u$, whence $(w u, \bar{u}) = 0$ by the Fredholm Theory. Conversely, also by the Fredholm Theory, $(w u, \bar{u}) = 0$ implies the existence of v such that $L_1 v - \lambda w v - k v = w u$. It follows that a nonreal λ has root subspace of dimension at least 2 iff $(w u, \bar{u}) = 0$ holds, as well as our earlier relation $(w u, u) = 0$. Given any ball B and operator T we denote, in the sequel, by $R(B, T)$ the total root dimension of T in B.

We also assume from now on that $\mu\{x \mid w(x) = 0\} = 0$.

Theorem 2. 11. *Let k_0 be as above and assume that for any $k \leqq k_0$, all eigenvalues of $L_1 u - k u = \lambda w u$ have one dimensional eigenspaces and root spaces of dimension $\leqq 2$. Let $L_1 u - k_0 u = \lambda w u$ with $\lambda > 0$ sufficiently large. Then*:

$$\lambda = \sup_{G \in M} \left[\sup_{\substack{V \\ G \cap V^{\perp} = \{0\}}} \left[\inf_{\substack{u \in V^{\perp} \\ (w u, u) > 0}} \frac{(L_1 u - k_0 u, u)}{(w u, u)} \right] \right] \tag{2. 6}$$

where $\dim(V)$ *depends on* λ. *Conversely for* $\dim V$ *sufficiently large, the right hand side of* (2. 6) *gives an eigenvalue of* $L_1 u - k_0 u = \lambda w u$.

It is useful to first show the following result. We recall that $y = f(x)$ is said to have a strict local max. (resp. min) at $x = x_0$ iff $f(x) > f(x_0)$ (resp. $f(x) < f(x_0)$) for x near x_0, $x \neq x_0$.

Lemma 2. 12. *Under the conditions of Theorem* 2. 11, *let $L_1 v - k v = \lambda_0 w v$ have a real ghost state for some λ_0 with $k \leqq k_0$. Then $k = k_j(\lambda_0)$ for some j is either a strict local max. or a local min. of the curve $y = k_j(\lambda)$.*

Proof of Lemma 2. 12. Since the eigenspaces are one dimensional, the curves $y = k_j(\lambda)$ must be twice continuously differentiable. Suppose λ_0 is associated with a ghost state then $k_j'(\lambda_0) = 0$ but $k_j''(\lambda_0) \neq 0$ since the root space does not exceed two. These remarks follow immediately from the formulas $k_j'(\lambda_0)$ and $k_j''(\lambda_0)$ need satisfy by perturbation theory (see [11], [23]). We conclude that k_j has either a strict maximum or minimum at λ_0.

Proof of Theorem 2. 11. For λ_0 chosen sufficiently large, we associate (λ_0, k_0) with the $(i, j)^{\text{th}}$ eigenpair as before. Consider the curve $y = k_j(\lambda)$ with j chosen so large that all ghost states associated with eigenvalues of the problem for $0 \leqq k \leqq k_0 = k_j(\lambda_0)$ have real part properly inside the values of the two members of $k_j^{-1}(k)$. This is possible for λ_0 (i.e. i) sufficiently large by Theorem 2. 1. Let $0 > k$ be small enough.

By Theorem 2. 0, $L_1 u - ku = \lambda wu$ will have positive solutions and the min.-max. will apply. We conclude

$$\lambda_i = \sup_{\substack{W \\ \dim W = j-1}} \inf_{\substack{u \perp W \\ (wu,u)>0 \\ G \cap W^\perp = \{0\}}} \frac{(L_1 u - ku, u)}{(wu, u)}.$$

Here $\dim W = i-1 = j-1$ (as $G = \{0\}$) and formula (2. 6) holds. Let now $k \uparrow k_0$ and let $i_1 = i_1(k)$ denote the number of positive eigenpairs (μ, k) of $L_1 u = \mu wu + ku$ with μ not exceeding the larger member of $k_j^{-1}(k)$. Note that if k is small, then $i_1(k) = j$, by the positivity of the problem. Given any k we consider the following set $G \in M = M(k)$. Let

$G_1(k) = \{u \mid u$ is a real ghost eigenvector for some $(\lambda, k)\}$,

$G_2(k) = \{u \mid u$ is a complex ghost eigenvector for some (λ, k) with associated root space of dimension 1 and $\operatorname{Im} \lambda > 0\}$,

$G_3(k) = \{u \mid u$ is a complex ghost eigenvector for some (λ, k) with associated root space of dimension 2 and $\operatorname{Im} \lambda > 0\}$.

Set $G(k) = \operatorname{span}\{G_1(k) \cup G_2(k) \cup G_3(k) \cup \bar{G}_3(k)\}$. Then $G \in M(k)$ since, in particular, if $u \in G_3(k)$ then $(wu, \bar{u}) = 0$. We claim that $i_1(k) + |G(k)| \geqq j$ for $k \leqq k_0$ whence it will follow that $i_1(k_0) + \sup\{|G|, G \in M(k_0)\} \geqq j$ and the result. To see this, set $f(k) = i_1(k) + |G(k)|$. Observe that $f(k) = j$ for k sufficiently small as was noted earlier. We use a standard argument to show that f is constant. Let $k_1 \leqq k_0$ be given and set: $k_j^{-1}(k_1) = \{\lambda_1, \lambda_2\}$, $S = \{\lambda \mid (\lambda, k_1)$ is an eigenpair of (2. 3), $\operatorname{Re}(\lambda) \in [\lambda_1, \lambda_2]$ and the eigenvector is positive or in $G\}$. Since S is finite and $S \neq \emptyset$ by choice of j, we construct a family of disjoint spheres $\{S_i\}_{i=1}^p$ such that $S \subset \bigcup_{i=1}^p S_i$ and $|S \cap S_i| = 1$. Without loss of generality we may assume $T_{k_1} = (L - k_1)^{-1} w$ exists, and we apply once again the root subspace continuity argument. Observe that if (μ, k_1) is a positive eigenpair with μ in S_i then S_i must contain a δ with positive eigenpair $(\delta, k_1 + \varepsilon)$ for $\varepsilon > 0$ sufficiently small. Exactly the same argument applies if (μ, k_1) is associated with an eigenvector u with $u \in G_2(k)$. If (μ, k_1) is an eigenpair with $\mu \in G_1(k)$ then by Lemma 2. 12 we find that S_i contains δ with either $(\delta, k_1 + \varepsilon)$ a positive eigenpair or the pair $(\delta, k_1 + \varepsilon)$, $(\bar{\delta}, k_1 + \varepsilon)$ with $\delta \neq \bar{\delta}$, whence the eigenvector for one of the pair $(\delta, \bar{\delta})$ is in $G_2(k_1 + \varepsilon)$. Finally, if $\mu \in S_i$ with associated eigenvector in $G_3(k_1) \cup \bar{G}_3(k_1)$ then by continuity, either $\delta \in S_i$ for some $(\delta, k_1 + \varepsilon)$ with eigenvector also in $G_3(k_1 + \varepsilon) \cup \bar{G}_3(k_1 + \varepsilon)$ or there exist a pair $(\delta_1, k_1 + \varepsilon)$, $(\delta_2, k_1 + \varepsilon)$ with $\delta_1, \delta_2 \in S_i$, δ_1, δ_2 complex and $\delta_1 \neq \delta_2$, but $\operatorname{sign}(\operatorname{Im} \delta_1) = \operatorname{sign}(\operatorname{Im} \delta_2)$. By symmetry, (μ, k_1) with $\mu \in S_i$ implies the existence of $(\bar{\mu}, k_1)$ with $\bar{\mu} \in S_j$ for some j and we conclude that $(\bar{\delta}_1, k_1 + \varepsilon)$, $(\bar{\delta}_2, k_1 + \varepsilon) \in S_j$, whence we may assume $\operatorname{Im}(\delta_1)$, $\operatorname{Im}(\delta_2) > 0$ and conclude that the two associated eigenvectors are in $G_2(k_1 + \varepsilon)$. In either case, we replace a pair of eigenvectors in $G_3(k_1) \cup \bar{G}_3(k_1)$ corresponding to $(\bar{\mu}, \mu)$ by either a pair in $G_3(k_1 + \varepsilon) \cup \bar{G}_3(k_1 + \varepsilon)$ or by a pair in $G_2(k_1 + \varepsilon)$. This shows that $f(k_1 + \varepsilon) \geqq f(k_1)$. Now let $g(k)$ represent the same set as $f(k)$ with $\operatorname{Im}(\lambda) > 0$ replaced by $\operatorname{Im}(\lambda) < 0$, positive eigenpairs in S replaced by negative eigenpairs in S. We conclude, exactly in the same way, that $g(k_1 + \varepsilon) \geqq g(k_1)$. But note that $g(k_1) + f(k_1) = g(k_1 + \varepsilon) + f(k_1 + \varepsilon) =$ total root space dimension in the slab $\operatorname{Re} \lambda \in [\lambda_1, \lambda_2]$. This is a constant in k since by choice of j no eigenvalue associated with a ghost state can have real part in $(-\infty, \lambda_1] \cup [\lambda_2, \infty)$, and all eigenspaces are one dimensional by assumption. We thus conclude if $\varepsilon>0$, sufficiently small, that $f(k_1+\varepsilon)=f(k_1)$. But our arguments are reversible, whence also $f(k_1-\varepsilon)=f(k_1)$.

Since the number of eigenpairs we consider for any k is finite and bounded independent of k (recall: $j \geqq i_1(k_1)+|G(k_1)|$), we conclude that $f(k_1)=$ constant for $k_1 \leqq k_0$, whence $f(k_1)\equiv j$.

We may use these results to give existence and localization criteria for complex eigenvalues. In view of the simplicity requirements, Corollary 2. 10 may be more suitable than Theorem 2. 11 if $n>1$.

Theorem 2. 13. *Let* $\delta=\max\limits_{\lambda\in R} k_1(\lambda)$,

$$m=\inf\{k \mid \lambda\in k_j^{-1}(k) \text{ such that } (\lambda, k) \text{ is associated with a ghost state, } j\geqq 2\}.$$

Then for $k\in(\delta, m)$ *there exist two complex eigenvalues of Problem* (2. 3).

Proof. Let $\max\limits_{\lambda\in R} k_1(\lambda)=\delta$ be achieved at $\lambda=\lambda_0$, and assume $k\in(\delta,\delta+\varepsilon)$, $\varepsilon>0$ sufficiently small. Observe that if $0<k-\delta<\varepsilon$ then the nearest real eigenvalue of L_1-k to λ_0 comes from the graph of $y=k_2(\lambda)$. Since $k_1(\lambda)$ is simple for all λ, we conclude that for ε small, L_1-k has no real eigenvalue near λ_0. Once again, consider the problem: $T_k(u)=(L_1-k)^{-1}[wu]=(\lambda)^{-1}u$. We again apply the perturbation theory argument and conclude T_k has an eigenvalue near $1/\lambda_0$, i.e. there exists u such that $(L_1-k)u=\mu(k)u$ with $\mu(k)$ near λ_0. From our earlier arguments, $\mu(k)$ is complex. We conclude, from the simplicity of $y=k_1(\lambda)$, that $\mu(k)$, $\overline{\mu(k)}$ are the only (complex) eigenvalues near λ_0. This pair cannot recombine until $k=m$, whence the existence follows.

To apply our results we need an estimation of m, δ. For some cases this is easy:

Corollary 2. 14. *Let* $\Omega=(-a,a)\times(0,a)$ *in* $\mathbb{R}^2$ *and assume* $w(x,y)=-w(-x,y)$. *If* $k\in\left(\frac{5}{4}\frac{\pi^2}{a^2}, \frac{2\pi^2}{a^2}\right)$ *then* $Lu=(-\Delta-k)u=\lambda wu$ *has exactly two complex (purely imaginary) eigenvalues. If* $k=\frac{5}{4}\frac{\pi^2}{a^2}$ *or* $k=\frac{2\pi^2}{a^2}$ *then this operator has a real ghost state. If* $k\leqq 2\frac{\pi^2}{a^2}$ *then the min.-max.* (2. 5) *holds unchanged for positive eigenpairs, with smallest trial space S of dimension* $=2$, $S^{\perp}\cap G=\{0\}$.

Proof. We apply the ideas of Theorem 2. 13 and the symmetry of w, Ω to conclude $\delta=\frac{5\pi^2}{4a^2}$. We thus find for $k\in\left(\frac{5}{4}\frac{\pi^2}{a^2}, \frac{2\pi^2}{a^2}\right)$ a complex pair μ, $\bar{\mu}$ which must be purely imaginary or, by the symmetry of the problem, there would be at least four: μ, $\bar{\mu}$, $-\mu$, $-\bar{\mu}$. This would contradict the observation, shown below, that there are at most two. This pair cannot recombine earlier than at $m=2\frac{\pi^2}{a^2}$. Again by our earlier results $(L-k)u=\lambda wu$ has a real ghost state at $\frac{5\pi^2}{4a^2}, \frac{2\pi^2}{a^2}$. Finally, to see that there are exactly two complex eigenvalues, observe that for $k<2\frac{\pi^2}{a^2}$, $L-k$ has only one negative (classical) eigenvalue, whence there can be at most one complex pair $\{\mu, \bar{\mu}\}$ of eigenvalues by the remarks following Theorem 2. 0. Finally, to see that (2. 5) holds, observe that there can be no other ghost states but the eigenvectors of $\{\mu, \bar{\mu}\}$ by our arguments, whence all real eigenpairs (λ, k) (for k in this range) must be positive or negative. We thus have the $(i, i+1)$ positive pairs for $i=1, 2, \ldots$. Since

$$j=i+1=i+\sup\{|G|, G\in M\}$$

the min.-max. holds by Corollary 2. 10.

We conclude with the example mentioned after Corollary 2. 10. In [2] the following problem is considered:

$$-y''-ky=\lambda wy,$$
$$y(\pm 1)=0,$$
$$w(x)=\operatorname{sign} x,$$

and numerical tabulation of eigenvalues λ (as functions of k) is given. In terms of our notation we give some of these results as: consider

$$-y''-\lambda wy=k_2(\lambda)y.$$

Then $k_2(\lambda)$ has a minimum at $\lambda=0$ of π^2. Observe also that, as mentioned earlier, $-y''-k_j(\lambda)y=\lambda wy$ has a root subspace of dimension 2 iff $k_j'=0$ but $k_j''\neq 0$ at λ. The root subspace of $k_2(\lambda)$ at $\lambda=0$ has dimension two (from the two complex eigenvectors being absorbed) whence $k_2''\neq 0$ and for λ near zero, $k_2'(\lambda)=-(wu, u)<(\text{resp.} >)0$ if $\lambda<(\text{resp.} >)0$. We conclude that for fixed $k>\pi^2$, near π^2, there exist λ_1, λ_2 near zero such that $\lambda_i(wu_i, u_i)<0$ for $i=1, 2$. Since for such k, $-y''-ky$ has only two negative eigenvalues, the estimate of Theorem 2. 1(c) shows that there are no other λ such that $\lambda(wu, u)\leqq 0$. In particular, there are no ghost states and if $-y''-\lambda wy=ky$ and $\lambda_i\neq\lambda>0$ then $(wy, y)>0$. We conclude that if $\lambda\in k_j^{-1}(k)$ (with $j>2$) then $k_j'(\lambda)\neq 0$. We must therefore have, for this k, the positive eigenpairs (1, 2), (2, 2), (i, i) for $i=3, \ldots, \infty$ with the λ associated with the (1, 2) positive eigenpair less than zero. We conclude that λ corresponding to the (i, i) positive pair is given by the min.-max. over spaces of dimension $i-1$ for $i=2, \ldots$, while the min.-max. over spaces of dimension 0 (i.e. the min. over $(wu, u)>0$) gives $-\infty$ by the choice:

$$u_\varepsilon=\begin{cases}\sin\left(\dfrac{\pi(x+1-\varepsilon)}{2-\varepsilon}\right) & x>-1+\varepsilon,\\[2ex] 0 & x\leqq -1+\varepsilon\end{cases}$$

and letting $\varepsilon\to 0^+$. This shows, as earlier stated that the (1, 2) positive pair is not given by such a min.-max. formula, but that the others are.

3. The elliptic case for $n=1$

We now consider the extensions and implications of the results of Section 2 to the case of ordinary differential operators which are not uniformly elliptic. We immediately observe from Theorem 2. 4 that if $u_j^{\pm}$ is the eigenfunction of (2. 2) corresponding to $\lambda_j^{\pm}$ as given in Theorem 2. 2(e) then there exists an integer J_0 such that for $j \geqq J_0$ then

$$\#\text{ of zeros of }(u_{j+1}^{\pm}) = \#\text{ of zeros of }(u_j^{\pm})+1,$$

while if we set

$$z(k)=\inf\{\#\text{ of zeros of } u \mid (L_1-k)u=\lambda w u\}$$

then $\lim_{k\to\infty} z=\infty$. This shows the existence for regular problems of the indices introduced below, which was first shown by Richardson [24] and Haupt [8], for the regular case and by Mingarelli [17] for more general coefficients. We also point out that it is inconvenient for what follows to term (λ, k) positive iff $(wu, u)>0$. Instead we shall henceforth term (λ, k) positive iff $\lambda>0$ unless otherwise specified. Furthermore, in this section we shall assume the more general conditions: $1/p$, q, $w\in L(a, b)$, $p(x)>0$ a.e. on (a, b) and w not a.e. zero. Note that here we allow — unlike Section 2 — p to vanish on some sets of measure zero. If the coefficients are regular many of the results follow immediately from Section 2 as we indicate below. The boundary problem under consideration is

$$ly=-(p(x)y')'+q(x)y=\lambda w(x)y, \tag{3. 1}$$

$$y(a)=0,\quad y(b)=0, \tag{3. 2}$$

We recall that we assume that $\lambda=0$ is not an eigenvalue of (3. 1), (3. 2) and that the problem

$$ly=\lambda y, \tag{3. 3}$$

$$y(a)=y(b)=0 \tag{3. 4}$$

has exactly N negative eigenvalues $\{\lambda_i\}_{i=1}^N$ (with $\lambda_i<\lambda_j$ if $j>i$) with $N\geqq 1$.

The Richardson index, n_R^+, of (3. 1), (3. 2), $(\lambda>0)$, is defined as that smallest non-negative integer for which (3. 1)–(3. 2) has no real eigenfunctions with precisely n zeros in (a, b) for $n<n_R^+$ whereas for $n\geqq n_R^+$ there is at least one real eigenfunction having exactly n zeros in (a, b). (For negative eigenvalues the Richardson index, n_R^-, is defined accordingly.)

The Haupt index, n_H^+, of (3. 1)–(3. 2), $(\lambda>2)$ is defined as that smallest non-negative integer for which there is precisely one (independent) real eigenfunction with n zeros in (a, b) for each $n\geqq n_H^+$. (A similar definition applies for the negative eigenvalues.)

For general measurable coefficients their existence is part of the folklore in the area (see [14]). The estimates to be derived below are for the indices n_R^+, n_H^+. Similar discussions apply to n_R^- and n_H^- and so will be omitted. We note that $n_R^+\leqq n_H^+$ in general.

As in the regular case, there exist an infinite sequence of positive and negative eigenvalues of the problem (3. 1)–(3. 2). This is discussed in [14]. If $w(x)$ is a.e. of one sign on (a, b) the definitions of the Haupt and Richardson indices coincide [6] and there is an at most finite negative spectrum.

We denote the first positive eigenvalue of (3. 1)–(3. 2) by λ^*. The number $\mathcal{N}(\lambda^*)$ denotes the number of zeros of a corresponding eigenfunction in (a, b).

Theorem 3. 1. (a) *For* $N \geqq 1$, $\mathcal{N}(\lambda^*)$ *is either equal to* N *or* $N-1$.

(b) *If* $\mathcal{N}(\lambda^*) = N$, $N \geqq 1$, *we have* $0 \leqq n_R^+ \leqq N \leqq n_H^+$.

(c) *If* $\mathcal{N}(\lambda^*) = N-1$, $N \geqq 2$, *we have* $0 \leqq n_R^+ \leqq N-1 \leqq n_H^+$.

All the estimates are precise. (Similar estimates hold if $\lambda^* < 0$.)

Proof. a) Let $y(x, \lambda_N)$ be the eigenfunction of (3. 3)–(3. 4) satisfying $y(a, \lambda_N) = 0$, $(py')(a, \lambda_N) = 1$. By Sturm-Liouville theory $y(x, \lambda_N)$ has precisely $N-1$ zeros in (a, b). Writing $y(x, \lambda)$ for the solution of (3. 3) which satisfies $y(a, \lambda) = 0$, $(py')(a, \lambda) = 1$ it follows that $y(x, 0)$ must have precisely N zeros in (a, b). Thus every non-trivial solution of

$$-(p(x)y')' + q(x)y = 0,$$

$$y(a) = 0, \quad (py')(a) \neq 0$$

must have precisely N zeros in (a, b). We introduce a Prüfer angle in the usual way: Let $\tan\theta(x, \lambda) = y(x, \lambda)/(py')(x, \lambda)$ where y is a solution of (3. 3). Then θ is uniquely defined by the condition $\theta(a, \lambda) = 0$ for each λ. Since θ increases at each zero of y and those λ's for which $\theta(b, \lambda) = n\pi$, $n \in I$ are the eigenvalues, it follows that since $y(b, 0) \neq 0$, by assumption,

$$N\pi < \theta(b, 0) < (N+1)\pi. \tag{3. 7}$$

Now let $\varphi(x, \lambda)$ denote the Prüfer angle for (3. 1) defined uniquely by the requirement that $\varphi(a, \lambda) = 0$ for each λ. Let $z(x, \lambda)$ denote the solution of (3. 1) satisfying $z(a, \lambda) = 0$, $(pz')(a, \lambda) = 1$. Then the Prüfer angle for z must satisfy

$$\varphi(b, 0) = \theta(b, 0)$$

(since $y(x, 0) = z(x, 0)$ for each x). Thus, by (3. 7),

$$N\pi < \varphi(b, 0) < (N+1)\pi. \tag{3. 8}$$

It now follows that either $\varphi(b, \lambda^*) = N\pi$ in which case $\mathcal{N}(\lambda^*) = N-1$ or $\varphi(b, \lambda^*) = (N+1)\pi$, in which case $\mathcal{N}(\lambda^*) = N$. This proves (a). We now prove (b), the proof of (c) being omitted as it is completely analogous to this case. Let $\mathcal{N}(\lambda^*) = N$. We show that $n_R^+ \equiv n_R \leqq N$. For if $n_R > N$ then, by definition, there can be no eigenfunction having N zeros in (a, b). But $z(x, \lambda^*)$ is such a solution. Hence

$$0 \leqq n_R \leqq N.$$

In order to prove that $n_H^+ \geqq N$ we assume, on the contrary, that $n_H \equiv n_H^+ \leqq N-1$. The following argument shows that it is sufficient to prove that $n_H = N-1$ is impossible. If $n_H = N-1$ then for each $n \geqq N-1$ there is precisely one eigenfunction having n zeros in (a, b). In particular, there is precisely one eigenfunction having $N-1$ zeros and precisely one eigenfunction having N zeros in (a, b). Returning to our Prüfer angle φ we see that there are now three cases: (1) For every $\lambda > \lambda^*$, $\varphi(b, \lambda) \geqq (N+1)\pi$. (2) For every $\lambda > \lambda^*$, $\varphi(b, \lambda) \geqq N\pi$ and (3) There exists $\lambda^+ > \lambda^*$ for which $\varphi(b, \lambda^+) < N\pi$. In the first case every eigenfunction has at least N zeros in (a, b) and thus $n_H \geqq N$ which is impossible. In the second case $\varphi(b, \lambda) = (N+1)\pi$ has at least two solutions for λ, and so there are at least two eigenfunctions with N zeros in (a, b), i.e. $n_H \geqq N+1$ which is a contradiction. In the last case there exists $\lambda^* < \lambda < \lambda^+$ such that $\varphi(b, \lambda) = N\pi$ but since $\varphi(b, \lambda) \longrightarrow +\infty$ as $\lambda \longrightarrow +\infty$ (see [3]) it follows that there exists $\mu > \lambda^+$ for which $\varphi(b, \mu) = N\pi$ again, i.e., there are at least two eigenfunctions with $N-1$ zeros in (a, b) so that $n_H \geqq N$. This final contradiction proves that $n_H = N-1$ is impossible. Thus $n_H \geqq N$.

Observe that if the coefficients are regular so that the results of Section 2 apply, we may give a short proof of Theorem 3. 1 which we sketch as follows: since λ^* is the first positive eigenvalue and has $M = \mathcal{N}(\lambda^*)$ zeros, then $k_{M+1}(\lambda^*) = 0$ and $k_{M+2} > 0$ in $[0, \lambda^*]$, $k_M < 0$ in $[0, \lambda^*]$. If $k_{M+1}(0) > 0$ then l has exactly M negative eigenvalues, while if $k_{M+1}(0) < 0$ then l has exactly $M+1$ negative eigenvalues. Since n_R^+, n_H^+ count intersections of the λ axis with the curves $y = k_j(\lambda)$, the estimates follow.

To show the sharpness of Theorem 3. 1 we consider the following examples.

1. (Precision in Theorem 3. 1(b)) The example

$$-y'' - (9\pi^2/16)y = \lambda w(x) y,$$

$$y(0) = y(2) = 0$$

where $w(x) = \operatorname{sgn}(1-x)$ on $(0, 2)$ shows that $n_R^+ = n_H^+ = 1 = N$.

2. That $n_R^+ = 0$ may occur can be found in [17].

3. (Precision in Theorem 3. 1(c).) Here the second example on page 40 of [2] shows that $N = 2$, $\mathcal{N}(\lambda^*) = 1$, while $n_R^+ = 1$ and $n_H^+ = 2$.

We now focus our attention on the existence of simple non-real eigenvalues. We recall that an eigenvalue (real or not) of (3. 1)–(3. 2) is said to be non-simple if it is a double zero of the characteristic equation defining it.

Lemma 3. 2. *Let $\lambda \in \mathbb{C}$ be an eigenvalue of* (3. 1)–(3. 2) *and* $y(x, \lambda)$ *a corresponding eigenfunction. Then λ is non-simple if and only if*

$$(wy, \bar{y}) = 0. \tag{3. 9}$$

Proof. See [16], [19]. Also note the Fredholm arguments given in Section 2 for regular coefficients.

Lemma 3. 3. *Let* $\lambda \in \mathbb{C}$, $\operatorname{Im}\lambda \neq 0$ *be a non-simple eigenvalue of* (3. 1)−(3. 2). *If* $y(x, \lambda) \equiv u + iv$ *is a corresponding eigenfunction then*

$$(wu, v) = (wu, u) = (wv, v) = 0, \tag{3. 10}$$

$$(lu, u) = (lv, v) = 0 \tag{3. 11}$$

and

$$(lu, v) = 0. \tag{3. 12}$$

Proof. From (3. 9) it follows immediately that

$$(wu, v) = 0, \quad (wu, u) = (wv, v). \tag{3. 13}$$

Furthermore, since $\operatorname{Im}\lambda \neq 0$, we know that (see e.g. [24])

$$(wy, y) = 0. \tag{3. 14}$$

Combining (3. 14) with (3. 13) we obtain (3. 10). Now since λ is a (non-simple) eigenvalue

$$(ly, \bar{y}) = \lambda(wy, \bar{y}) = 0.$$

Separating real and imaginary parts yields (3. 12). However, since $\operatorname{Im}\lambda \neq 0$, we also have $(ly, y) = 0$, from which (3. 11) follows.

Theorem 3. 4. *Assume* l *has one negative eigenvalue. Let* $w \in L(a, b)$ *be chosen so that* (3. 1)−(3. 2) *has non-real eigenvalues. Then every non-real eigenvalue is simple.*

Proof. Let ψ be a (real) eigenfunction corresponding to λ_1, the smallest eigenvalue of (3. 3)−(3. 4). Then $\psi(x) > 0$ in (a, b) and by the Courant min.-max. theorem,

$$\lambda_2 = \inf_{f \in \mathscr{A}} \left\{ \frac{(lf, f)}{(f, f)} \right\} \tag{3. 15}$$

where λ_2 is the next largest eigenvalue, (the usual extremal properties of the eigenvalues), and

$$\mathscr{A} = \{ f \in AC[a, b] : pf' \in AC[a, b], f \not\equiv 0, f(a) = f(b) = 0 \text{ and } (f, \psi) = 0 \}.$$

If possible, let $w \in L(a, b)$ be such that the corresponding problem (3. 1)−(3. 2) has a non-simple non-real eigenvalue, λ. Since $N = 1$, there are precisely two such eigenvalues λ, $\bar{\lambda}$ (since they occur in complex conjugate pairs and the simplicity of λ is equivalent to that of $\bar{\lambda}$), see [15]. Let $u + iv$ denote an eigenfunction corresponding to λ. Then we can choose $\alpha, \beta \in \mathbb{R}$ such that $\varphi \equiv \alpha u + \beta v$ is L^2-orthogonal to ψ, i.e., $(\varphi, \psi) = 0$. We see that $\varphi \not\equiv 0$, since $\varphi \equiv 0$ forces $u \equiv -\alpha v$ i.e. the complex eigenfunction can be chosen to be real and this is impossible since $\operatorname{Im}\lambda \neq 0$. Furthermore $\varphi \in \mathscr{A}$. But Lemma 3. 3 then shows $\lambda_2 \leqq 0$ which is contrary to our hypothesis.

Let λ, $\operatorname{Im}\lambda \neq 0$, be a simple non-real eigenvalue. We may then normalize a corresponding eigenfunction by setting $(wy, \bar{y}) = 1$. In this case, if we assume $p = 1$ for simplicity, an estimate for $|\operatorname{Re}\lambda|$ may be obtained provided some qualitative results for the solutions are available. For example one can show that

$$|\operatorname{Re}\lambda| \leqq \left(1 + \varepsilon + \frac{c(\varepsilon)}{\lambda_0}\right) \|y'\|_2^2$$

where $\varepsilon > 0$ is arbitrary, $\lambda_0 > 0$ is the smallest eigenvalue of $-d^2/dx^2$ on (a, b) and

$$c(\varepsilon) = \frac{1}{\varepsilon}\left(\int_a^b |q|\, dx\right)^2.$$

Here $\|y'\|_2$ denotes the usual L^2-norm of the derivative of the eigenfunction y defined above. In order to prove this we use the inequality

$$\left|\int_a^b q f^2\, dx\right| \leqq c(\varepsilon) \int_a^b |f|^2\, dx + \varepsilon \int_a^b |f'|^2\, dx$$

along with Wirtinger's inequality.

4. Mixed problems

While Section 2 dealt with uniformly elliptic problems and Section 3 with possibly degenerate elliptic problems, this section deals with mixed problems for the case $n = 1$, i.e. p need no longer be of fixed sign. We know of no counterpart of such results for the case $n > 1$. As usual we do, however, assume $1/p$, q, $w \in L(a, b)$ and w not a.e. zero.

In the first place we show that most of the results in Section 3 cannot be improved upon by neglecting a sign condition on $p(x)$.

Example. Let $q \equiv 0$ in (3. 1)–(3. 2), $w \equiv p$ and consider the boundary problem

$$-(p(x)y')' = \lambda p(x) y,$$

$$y(-1) = 0 = y(1)$$

where $p(x) = \operatorname{sgn}(x)$. We recall that solutions $y \in AC[-1, 1]$ along with py'.

Now for each $\lambda \in \mathbb{C}\backslash\{0\}$, fixing a determination for the root, the function

$$y(x, \lambda) = \frac{\sin[\sqrt{\lambda}(1 - |x|)]}{\sqrt{\lambda}}, \quad -1 \leqq x \leqq +1$$

satisfies the differential equation and the boundary conditions. Thus y is an eigenfunction for each $\lambda \neq 0$. If $\lambda = 0$ we may define an eigenfunction by

$$y(x, 0) = -\int_{-1}^{x} \frac{ds}{p(s)}.$$

Thus every $\lambda \in \mathbb{C}$ is an eigenvalue of this problem. This example, first reported in passing in [3] in the context of spectral asymptotics and oscillation theory has the following features:

1. There is an interval $I \subset \mathbb{R}$ (e.g. $I=(0, 1)$) corresponding to which each $\lambda \in I$ generates a positive eigenfunction (in (a, b)), in sharp constrast with the results of Section 2 (see also [1]).

2. The operator T defined by

$$(Tf)(x) = -(p(x)f')'$$

on $D(T)=\{f \in L^2(-1, 1): f, pf' \in AC(-1, 1), (pf')' \in L^2(-1, 1) \text{ and } f(-1)=0=f(1)\}$ has infinitely many negative eigenvalues, and this in turn, puts no restriction on the number of pairs of non-real eigenvalues, cf., [15]. When $p(x)>0$ a.e. T, as defined above is semi-bounded from below.

3. The Richardson index $n_R^+ = 0$ while the Haupt index does not even exist! If $\lambda < 0$ *all* the eigenfunctions are non-zero in $(-1, 1)$. Thus n_R^- does not exist!

4. Every real eigenvalue is non-simple, in contrast with the case $p(x)>0$ a.e. [16].

We remark that if on the space $L^2(-1, 1)$ we introduce the generally indefinite inner product [,] by

$$[f, g] = \int_{-1}^{1} f\bar{g}p\,dx$$

where $p(x)=\operatorname{sgn} x$, as in our example, we turn $L^2(-1, 1)$ into a Krein space which we denote by K, (for these notions again see [4]). The operator T defined by

$$(Tf)(x) = -\frac{1}{p(x)}(p(x)f'(x))'(x)$$

on $D(T)=\{f \in AC(-1, 1), pf' \in AC(-1, 1), Tf \in L^2(-1, 1) \text{ and } f(-1)=0=f(1)\}$ satisfies the relation

$$[Tf, g] = [f, Tg], \quad f, g \in D(T).$$

Thus T is a symmetric linear operator on K. Furthermore $\sigma(T)=\mathbb{C}$ and in fact the spectrum is pure point. Thus T is a symmetric linear operator on a Krein space K whose spectrum is pure point only and fills all of $\mathbb{C}$.

H. Langer gave the apparently only other known example of a symmetric linear operator on a Krein space whose spectrum fills all of $\mathbb{C}$. In his case however, the spectrum is purely continuous.

In the sequel we make use of the "reciprocal transformation", see e.g. [22]. That is, let $p, q: [a, b] \longrightarrow \mathbb{R}$, $1/p, q \in L(a, b)$. If y is a solution of the equation

$$(p(x)y')' + q(x)y = 0$$

then $z = py'$ satisfies the "reciprocal equation"

$$\left(\frac{1}{q(x)} z'\right)' + \frac{1}{p(x)} z = 0.$$

Use of this transformation permits the study of (3. 1—2) when the leading term $p(x)$ changes sign in (a, b). For a given solution y, the function $z = py'$ will be termed the "derivative solution".

Observe that if $q(x)$ is a.e. of one sign on (a, b), then for any real nontrivial solution y of

$$(p(x)y')' + q(x)y = 0,$$

the derived solution has only finitely many zeros in $[a, b]$. (This is obtained by applying Sturm theory to the reciprocal equation.) It is possible that y itself has an infinite number of zeros in $[a, b]$, (see [3]). If $p(x), q(x) \in C[a, b]$ and $q(x)$ is of one sign on $[a, b]$ then between any two zeros of $py'(x)$ there is always at least one zero of y.

Let $1/p_i$, $q_i \in L(a, b)$, $i = 1, 2$ and $q_1(x) > \delta > 0$ a.e. on (a, b). If,

$$\frac{1}{p_1(x)} < \frac{1}{p_2(x)} \quad \text{and} \quad q_1(x) < q_2(x) \quad \text{a.e.} \quad \text{on } [a, b]$$

then between any two zeros of the quantity $p_1 y_1'$ where y_1 is a solution of

$$(p_1 y_1')' + q_1 y_1 = 0$$

there is always at least one zero of every quantity of the form $p_2 y_2'$ where y_2 is a solution of

$$(p_2 y_2')' + q_2 y_2 = 0.$$

(This may be shown by applying Sturm's comparison theorem to the reciprocal equation.) Let $1/p$, $q \in L(a, b)$, q a.e. of one sign on (a, b). Then between any two consecutive zeros of a derived solution there is exactly one zero of every other derived solution linearly independent of the first. (This follows from Sturm's separation theorem applied to the reciprocal equation.)

We now focus on weighted-problems with indefinite leading-terms.

Theorem 4. 1. *Let $1/p$, $w \in L(a, b)$, $w(x) > \delta > 0$ a.e. on $[a, b]$ and $p(x)$ changes sign on sets of positive measure. Then the Neumann problem corresponding to the equation*

$$-(p(x)y')' = \lambda w(x) y \quad on \quad [a, b]$$

(i.e., $py'(a) = py'(b) = 0$) has a sequence of eigenvalues $\lambda_n^{\pm} \cup \{0\}$ with

$$\cdots < \lambda_1^- < \lambda_0^- < 0 < \lambda_0^+ < \lambda_1^+ < \cdots$$

and

$$\lambda_n^{\pm} \sim \pm n^2 \pi^2 / \left(\int_a^b \sqrt{(w/p)_+}\, ds \right)^2$$

as $n \to \infty$.

Furthermore the derived solutions corresponding to $\lambda_0^{\pm}$ are strictly positive in (a, b).

Proof. The reciprocal equation here is of the form

$$-\left(\frac{1}{\lambda w(x)} z' \right)' = \frac{1}{p(x)} z \qquad (\lambda \neq 0)$$

and the corresponding boundary problem is of Dirichlet type, i.e.,

$$z(a) = 0, \qquad z(b) = 0.$$

Thus, for $\lambda \neq 0$, we seek $\lambda \in \mathbb{C}$ such that

$$-\left(\frac{1}{w(x)} z' \right)' = \frac{\lambda}{p(x)} z,$$

$$z(a) = 0, \quad z(b) = 0$$

has non-trivial solutions. But it is known that this problem has a doubly infinite sequence of real eigenvalues only whose asymptotic behaviour is as above, [3], and with eigenfunctions $z(x, \lambda_0^{\pm})$ being strictly positive in (a, b). The result now follows via the "inverse" reciprocal transformation.

We note that for reasons alluded to earlier it is unlikely that one may replace "derived solutions" by the actual eigenfunctions y.

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