Werk

Titel: Continued fractions and density results for Dedekind sums.

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Continued fractions and density results for Dedekind sums

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For integers h and k with $k \geqq 1$ and $(h, k) = 1$, let $s(h, k)$ be the Dedekind sums, as defined and discussed in [2]. According to [2], p. 28, it is unknown whether the points $(h/k, s(h, k))$ are dense in the plane. It is the object of this paper to show that they are. In the process, a formula will be given for $s(h, k)$ in terms of the simple continued fraction (s.c.f.) expansion of h/k.

As in [2], it is convenient to consider $s(h, k)$ as a function of a single rational argument, by defining $s(h/k) = s(h, k)$. That this function is well defined is clear from the conditions on h and k. Note that s is periodic with period 1.

The reader is referred to [1], ch. 7, for the elementary properties of continued fractions which will be used here.

Theorem 1. *Let $\langle a_0, \ldots, a_r \rangle$ be a finite simple continued fraction. Then*

$$s(\langle a_0, \ldots, a_r \rangle) = \frac{-1+(-1)^r}{8} + \frac{1}{12}(\langle 0, a_1, \ldots, a_r \rangle + (-1)^{r+1}\langle 0, a_r, \ldots, a_1 \rangle + a_1 - a_2 + \cdots + (-1)^{r+1} a_r).$$

Proof. The proof is by induction on r. For $r = 0$, both sides of the equation are 0, so the theorem holds in this case.

Let $r \geqq 1$ and assume the result for $r-1$. By the periodicity of s, we have $s(\langle a_0, \ldots, a_r \rangle) = s(\langle 0, a_1, \ldots, a_r \rangle)$. The reciprocity theorem for Dedekind sums [2], p. 4 implies that

$$s(h/k) = -\frac{1}{4} + \frac{1}{12}\left(\frac{h}{k} + \frac{k}{h} + \frac{1}{hk}\right) - s(k/h)$$

for positive relatively prime integers h and k. Applying this with $\dfrac{h}{k} = \langle 0, a_1, \ldots, a_r \rangle$ and using the induction hypothesis for $s(\langle a_1, \ldots, a_r \rangle)$ gives

$$s(\langle a_0, \ldots, a_r \rangle) = -\frac{1}{4} + \frac{1}{12}\left(\langle 0, a_1, \ldots, a_r \rangle + \langle a_1, \ldots, a_r \rangle + \frac{1}{hk}\right) - s(\langle a_1, \ldots, a_r \rangle)$$

$$= -\frac{1}{4} + \frac{1}{12}\left(\langle 0, a_1, \ldots, a_r\rangle + \langle a_1, \ldots, a_r\rangle + \frac{1}{hk}\right) - \frac{-1+(-1)^{r-1}}{8}$$

$$-\frac{1}{12}(\langle 0, a_2, \ldots, a_r\rangle + (-1)^r\langle 0, a_r, \ldots, a_2\rangle + a_2 - a_3 + \cdots + (-1)^r a_r)$$

$$= \frac{-1+(-1)^r}{8} + \frac{1}{12}\left(\langle 0, a_1, \ldots, a_r\rangle + \frac{1}{hk} + (-1)^{r+1}\langle 0, a_r, \ldots, a_2\rangle\right.$$

$$\left. + a_1 - a_2 + \cdots + (-1)^{r+1} a_r\right),$$

since $\langle a_1, \ldots, a_r\rangle - \langle 0, a_2, \ldots, a_r\rangle = a_1$. Hence it suffices to show that

$$\langle 0, a_r, \ldots, a_1\rangle = \langle 0, a_r, \ldots, a_2\rangle + \frac{(-1)^{r+1}}{hk}.$$

Letting $b_i = a_{r+1-i}$, we must show that

$$\langle 0, b_1, \ldots, b_r\rangle - \langle 0, b_1, \ldots, b_{r-1}\rangle = \frac{(-1)^{r+1}}{hk},$$

where $\frac{h}{k} = \langle 0, b_r, \ldots, b_1\rangle$. Denote the convergents to $\langle 0, b_1, \ldots, b_r\rangle$ by $\frac{h_i}{k_i}$, so that

$$\langle 0, b_1, \ldots, b_r\rangle = \frac{h_r}{k_r}$$

and

$$\langle 0, b_1, \ldots, b_{r-1}\rangle = \frac{h_{r-1}}{k_{r-1}}.$$

It is known (see [1], sec. 7. 3, prob. 4) that

$$\frac{k_r}{k_{r-1}} = \langle b_r, \ldots, b_1\rangle;$$

but this also equals $\langle a_1, a_2, \ldots, a_r\rangle = \frac{k}{h}$. Hence, as both fractions are irreducible, $k = k_r$ and $h = k_{r-1}$ and the equality to be proved becomes $\frac{h_r}{k_r} - \frac{h_{r-1}}{k_{r-1}} = \frac{(-1)^{r+1}}{k_{r-1}k_r}$. This, however, is a well-known identity (see, e.g. [1], Theorem 7. 5), so that the proof is complete.

Note that every rational number has a unique s.c.f. expansion $\langle a_0, \ldots, a_r\rangle$ for which r is even. For this expansion, Theorem 1 reduces to

$$s(\langle a_0, \ldots, a_r\rangle) = \frac{1}{12}(\langle 0, a_1, \ldots, a_r\rangle - \langle 0, a_r, \ldots, a_1\rangle + a_1 - a_2 + \cdots - a_r).$$

Theorem 2. *The set of points $(h/k, s(h/k))$ is dense in the plane.*

Proof. Let (x, y) be a point in the plane and ε a positive real number. It suffices to exhibit a rational number h/k such that $\left|x - \frac{h}{k}\right| < \varepsilon$ and $|y - s(h/k)| < \varepsilon$.

Since the points (α, β), with α irrational and β rational, are dense in the plane, we may assume, without loss of generality, that x is irrational and y is rational. Hence x has an infinite s.c.f. expansion $x = \langle b_0, b_1, \cdots \rangle$. Denote the convergents to x by $\frac{h_i}{k_i} = \langle b_0, \ldots, b_i \rangle$.

Note that for any integer $s \geqq 0$ and any real number $\alpha > 0$, we have

$$\begin{aligned} |x - \langle b_0, \ldots, b_{s-1}, \alpha \rangle| &\leqq \left| x - \frac{h_{s-1}}{k_{s-1}} \right| + \left| \frac{h_{s-1}}{k_{s-1}} - \langle b_0, \ldots, b_{s-1}, \alpha \rangle \right| \\ &= \left| x - \frac{h_{s-1}}{k_{x-1}} \right| + \left| \frac{h_{s-1}}{k_{s-1}} - \frac{\alpha h_{s-1} + h_{s-2}}{\alpha k_{s-1} + k_{s-2}} \right| \\ &= \left| x - \frac{h_{s-1}}{k_{s-1}} \right| + \frac{1}{k_{s-1}(\alpha k_{s-1} + k_{s-2})} \leqq \left| x - \frac{h_{s-1}}{k_{s-1}} \right| + \frac{1}{k_{s-1} k_{s-2}}. \end{aligned}$$

Since, as $s \to \infty$, $\frac{h_{s-1}}{k_{s-1}} \to x$, $k_{s-1} \to \infty$, and $k_{s-2} \to \infty$, we may choose s so large that, for all $\alpha > 0$, $|x - \langle b_0, \ldots, b_{s-1}, \alpha \rangle| < \varepsilon$.

Note that $x - 12y$ is irrational so that it has an infinite s.c.f. expansion, say, $x - 12y = \langle d_0, d_1, \ldots \rangle$. As above, we may choose t so large that, for all $\alpha > 0$, $|x - 12y - \langle d_0, \ldots, d_{t-1}, \alpha \rangle| < \varepsilon$. Moreover, this choice can be made in such a way that $s + t$ is even.

We now choose positive integers m and n such that

$$b_1 - b_2 + \cdots + (-1)^s b_{s-1} + (-1)^{s+1} m + (-1)^{s+2} n + (-1)^{t-1} d_{t-1} + \cdots - d_1 = b_0 - d_0.$$

Such a choice is clearly possible; indeed, one of m and n may be chosen to be 1.

Finally, let

$$\frac{h}{k} = \langle b_0, \ldots, b_{s-1}, m, n, d_{t-1}, \ldots, d_1 \rangle.$$

By the choice of s, $\left| x - \frac{h}{k} \right| < \varepsilon$. Also, since $s + t$ is even,

$$\begin{aligned} s(h/k) &= \frac{1}{12} (\langle 0, b_1, \ldots, b_{s-1}, m, n, d_{t-1}, \ldots, d_1 \rangle - \langle 0, d_1, \ldots, d_{t-1}, n, m, b_{s-1}, \ldots, b_1 \rangle \\ &\quad + b_1 - b_2 + \cdots + (-1)^s b_{s-1} + (-1)^{s+1} m + (-1)^{s+2} n + (-1)^{t-1} d_{t-1} + \cdots - d_1) \\ &= \frac{1}{12} ((x - b_0 + \delta_1) - (x - 12y - d_0 + \delta_2) + b_0 - d_0), \end{aligned}$$

say, where the choices of s and t imply that $|\delta_1| < \varepsilon$ and $|\delta_2| < \varepsilon$. Thus

$$s(h/k) = y + \frac{1}{12} (\delta_1 - \delta_2)$$

and

$$|y - s(h/k)| < \frac{1}{12}(\varepsilon + \varepsilon) < \varepsilon,$$

which completes the proof.

The following corollary of Theorem 2 answers another question raised in [2], p. 28.

Corollary. *The values of $s(h/k)$ are dense on the real axis.*

References

[1] *Ivan Niven* and *Herbert Zuckerman*, An Introduction to the Theory of Numbers, 3rd ed., New York 1972.
[2] *Hans Rademacher* and *Emil Grosswald*, Dedekind Sums, Carus Mathematical Monograph **16** (1972).

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