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Characteristic and order of conically differentiable points

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1. Introduction

1. 1. In [5], P. Scherk described a theory of order characteristics for the interior differentiable points of arcs in projective n-space. The characteristic of a differentiable point p was a certain n-tuple of integers, which were defined by the intersection and support properties of all the hyperplanes through p and it was shown that the sum of these numbers was associated with the order of the arc in the neighbourhood of p. Later, Lane and Scherk developed an analogous theory of characteristics for differentiable points in the conformal plane [1]. Characteristics of parabolically differentiable points were dealt with by Lane and Singh [4]. In the present paper, we define the characteristic of a conically differentiable point of an arc in the projective plane [2], and prove two main results.

Theorem 1. *The conical order of a conically differentiable interior point of an arc is at least as great as the sum of the digits of its characteristic.*

Theorem 2. *Let p be a conically differentiable interior point of an arc which is the union of two arcs of conical order five. Then the conical order of p is equal to the sum of the digits of the characteristic of p.*

1. 2. Prerequisites I. The definitions and notations used in this paper are the same as in [2] and [3]. For the convenience of the reader, a summary of the background, which will be needed here, is given below.

The letters $P, Q, \ldots$ denote points in the real projective plane; the German letters, $\mathfrak{T}, \mathfrak{L}, \ldots$ denote lines. $\mathfrak{L}(P, Q)$ is the line through P and Q. γ denotes a conic, which may degenerate into a pair of lines, a double line, a double segment, or a point. The interior [exterior] of a non-degenerate conic γ is denoted by γ_* [γ^*]. The closure of a family α of conics will be denoted by $\bar{\alpha}$.

The three-parameter family of non-degenerate conics which touch a line $\mathfrak{T}$ at a point P is denoted by τ. If no three of the points P, Q, R, S are collinear, and Q, R, S do not lie on $\mathfrak{T}$, the conic of τ through Q, R, S is denoted by $\gamma(\tau; Q, R, S)$.

If $\gamma \in \tau$, $\varphi = \varphi(\gamma)$ denotes the two parametric subfamily of τ which consists of those conics of τ which have at least three-point contact with γ at P; φ_R is the subfamily of φ through $R \notin \mathfrak{T}$. Let $\varphi_P(\gamma)$ denote the subfamily of $\varphi(\gamma)$, each of whose members has at least four-point contact with γ at P.

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An *arc* A in the real projective plane is the continuous image of a real interval. The images of distinct points of the parameter interval are to be considered as different points of A even though they may coincide in the plane. The letters $p, s, q, \ldots$ denote points of arcs. The parameters s and p are supposed to be distinct and s will usually be "sufficiently close" to p. We shall restrict our attention to arcs A with the property that each point p of A has a neighbourhood N on the parameter interval, such that for every $s \in N$, $s \neq p$, the image points s and p in the projective plane are distinct. This condition is automatically satisfied if p has finite linear order. It also holds true if A is *differentiable* at p, i. e., if A satisfies the following condition.

Condition I. *If the parameter s is sufficiently close to p, $s \neq p$, the line ps is uniquely determined. It converges as s tends to p* ([2], 4. 2).

The limit straight line $\mathfrak{T}$ is the ordinary tangent of A at p.

Condition I implies:

(i) If A is differentiable at an interior point p, then the non-degenerate, non-tangent conics through p all intersect A at p or all of them support ([2] Theorem 2).

The arc A is called *twice conically differentiable* at p if it satisfies the following.

Condition II. *Let A satisfy Condition* I *at p and let Q and R be any fixed points, $Q \notin \mathfrak{T}$, $R \notin \mathfrak{T}$; p, Q, R not collinear. If s is close to p, $s \neq p$, the unique tangent conic $\gamma(\tau; s, Q, R)$ converges as s tends to p* ([2], 5. 1).

The limit conic of Condition II is called the *osculating conic* of A at p through Q and R and it is denoted by $\gamma(\sigma; Q, R)$. The family of all the osculating conics of A at p is denoted by σ.

(ii) If Condition II holds for two points Q and R such that p, Q, R are not collinear and $Q \notin \mathfrak{T}$, $R \notin \mathfrak{T}$, then it holds for every such pair of points ([2] Theorem 3).

Condition II implies the following statements.

(iii) If the arc A intersects $\mathfrak{T}$ at p, then the conics of σ are degenerate ([2] Theorem 4).

(iv) If A satisfies Condition II at p, then σ is one of the subfamilies φ of τ described above (*Type* 1); or σ consists of the pairs of distinct lines through p, both of them different from $\mathfrak{T}$ (*Type* 2); or σ consists of the pairs of lines one of which is $\mathfrak{T}$, while the other does not pass through p (*Type* 3) ([2] Theorem 5). We note that if p is of Type 1, or Type 2, and s is sufficiently close to p, then $s \notin \mathfrak{T}$.

(v) The conics of $\tau - \sigma$ all support A at an interior point p, except when p is of Type 2 and A intersects $\mathfrak{T}$ at p, in which case they all intersect A at p ([2] Theorem 7).

The arc A is called *three times conically differentiable* at p if it satisfies the following.

Condition III. *Let A be twice conically differentiable at p. If p is of Type* 1 *and $Q \notin \mathfrak{T}$, then $\gamma(\sigma; s, Q)$ converges as s tends to p* ([2], 6. 1).

The limit conic is called the *superosculating conic* of A at p through Q and is denoted by $\gamma(\varrho; Q)$. The family of all the superosculating conics is denoted by ϱ.

(vi) If Condition III holds for a single point $Q \notin \mathfrak{T}$, then it holds for all such points ([2] Theorem 9).

Condition III implies the next three statements.

(vii) ϱ is either one of the subfamilies φ_p of φ described earlier (*Type* 1a); or ϱ consists of the pairs of (distinct) lines through p one of which is $\mathfrak{T}$ (*Type* 1b) ([2] Theorem 10).

(viii) The conics of $\sigma - \varrho$ all support A at an interior point p or all of them intersect. If p is of Type 1a, they all support or all intersect according as A has or has not a cusp at p ([2] Theorem 11).

(ix) If p is of Type 1b and $\gamma \in \sigma$, $\gamma' \in \sigma$, then $s \notin (\gamma_* \cap \gamma'^*) \cup (\gamma^* \cap \gamma'_*)$, if s is sufficiently close to p ([2], 6. 5 Corollary).

A is called *conically differentiable at* p if it satisfies the following condition.

Condition IV. *Let A be three times conically differentiable at p. If p is of Type* 1a *then $\gamma(\varrho; s)$ converges as s tends to p* ([2], 7. 1).

The limit *ultraosculating conic* is denoted by $\gamma(p)$. It is non-degenerate (*Type* 1a(i)), or the point conic p (*Type* 1a (ii)), or the double line on $\mathfrak{T}$ (*Type* 1a (iii)).

(x) Condition IV implies that the conics of $\varrho - \gamma(p)$ all support A at an interior point p ([2] Theorem 12).

Remarks. (1) If p is of Type 2 or Type 3, Conditions III and IV are trivially satisfied. The superosculating conic through Q is then defined to be

$$\lim_{s \to p} \gamma(\sigma; s, Q) = \mathfrak{T} \cup pQ.$$

If p is of Type 1b, Condition IV automatically holds. If p is not of Type 1a define $\gamma(p)$ to be

$$\lim_{s \to p} (\mathfrak{T} \cup ps) = \text{double line on } \mathfrak{T}.$$

(2) We shall adopt the convention that the double line on $\mathfrak{T}$ supports A at p, even if A crosses $\mathfrak{T}$ at p.

(3) If the conic γ consists of a pair of distinct lines through p, the arc A will be said to support [intersect] γ at p, if there exist one-sided open neighbourhoods of p on A which lie in the same region [in different regions] with respect to γ.

1. 3. Prerequisites II. An arc A has the *conical order* n if no conic meets A more than n times and at least one conic meets A exactly n times. A_5 denotes an arc of conical order five; it is automatically convex. The *conical order of a point* p of A is the minimum of the conical orders of all the neighbourhoods of p on A; thus it is not less than five.

Let s, t, u, v, w be mutually distinct points on A sufficiently close to p. We call A *strongly differentiable* at p if the line uv converges as u and v tend to p. A is called *twice strongly conically differentiable* at p if A satisfies the above condition at p and if for every pair of distinct points Q and R which are not collinear with p, and do not lie on $\mathfrak{T}$ the conic $\gamma(t, u, v, Q, R)$ is uniquely defined and converges as t, u, v converge to p. We say that A is *three times strongly conically differentiable* at p if, in addition, for every point R, $R \notin \mathfrak{T}$, $\gamma(t, u, v, w, R)$ is uniquely defined and converges as t, u, v, w converge to p. Finally, A is called *four times strongly conically differentiable* at p if the previous condition holds and if $\gamma(s, t, u, v, w)$ is uniquely defined and converges as s, t, u, v, w converge on A to p; cf. ([3], 3).

(xi) An end-point p of Type 1 or Type 3 [Type 2] of A_5 is four times [three times] strongly conically differentiable ([3] Theorem 6). If p is of Type 2, then the limit conics of $\gamma(s, t, u, v, w)$ as s, t, u, v, w tend to p on A_5 are degenerate conics on $\mathfrak{T}$.

(xii) An interior point p of A_5 is three times strongly conically differentiable ([3] Theorem 7). It is of Type 1a (i) with respect to each of the two arcs into which A_5 is decomposed by p ([3] Theorem 5).

(xiii) If $A = A_5 \cup p \cup A_5'$ is convex and conically differentiable at the point p of Type 1 a (i), and if $\gamma(p)$ intersects A at p, then p has the conical order five ([3] Theorem 10); cf. 3. 1.

2. Characteristics

2. 1. We associate with each interior conically differentiable point p of Type k of an arc A, a (*conical*) *characteristic*, $(a_0a_1a_2a_3a_4;\ k)$, $k = 1\,\text{a(i)}, 1\,\text{a(ii)}, 1\,\text{a(iii)}, 1\,\text{b}, 2$, or 3. The numbers a_i are equal to 1 or 2; $i = 0, 1, 2, 3, 4$. They are determined as follows:

$$a_0 \equiv 1\,[\equiv 0] \pmod 2,$$

if the non-tangent conics through p all intersect [support] A at p; cf. 1. 2 (i).

$$a_0 + a_1 \equiv 1\,[\equiv 0] \pmod 2,$$

if the non-osculating tangent conics of A at p all intersect [support] A at p; cf. 1. 2 (v).

$$a_0 + a_1 + a_2 \equiv 1\,[\equiv 0] \pmod 2,$$

if the non-superosculating osculating conics of A at p all intersect [support] A at p; cf. 1. 2 (viii).

$$a_0 + a_1 + a_2 + a_3 \equiv 1\,[\equiv 0] \pmod 2,$$

if the superosculating conics $\pm\gamma(p)$ all intersect [support] A at p; cf. 1. 2 (x).

$$a_0 + a_1 + a_2 + a_3 + a_4 \equiv 1\,[\equiv 0] \pmod 2,$$

if $\gamma(p)$ intersects [supports] A at p.

2. 2. *Let p be of Type* 1 a. Thus A does not cross its tangent at p; cf. 1. 2 (iii). The definitions in 2. 1 and the results 1. 2 (v), (viii), (x) imply the following relations.

If A has no cusp [has a cusp] at p, then

$$a_0 = a_1 = a_2 = a_3 = 1\,[= 2].$$

$$a_4 = \begin{cases} 1 \text{ or } 2 & k = 1\,\text{a(i)} \\ & \text{if} \\ 2 & k = 1\,\text{a(ii) or } 1\,\text{a(iii)}. \end{cases}$$

Thus, if p is of Type 1 a, $\sum_{i=0}^{4} a_i$ is even except when $k = 1\,\text{a(i)}$ and $\gamma(p)$ intersects A at p.

Let p be of Type 1 b. Again A does not cross its tangent at p. *If A has no cusp [has a cusp] at p, then*

$a_0 = a_1 = 1\,[= 2]$.

a_2 *can be either* 1 *or* 2.

$a_0 + a_1 + a_2 + a_3 \equiv 1\,[\equiv 0] \pmod 2$; cf. 1. 2 Remark (3).

$a_0 + a_1 + a_2 + a_3 + a_4 \equiv 0 \pmod 2$; *thus* $a_4 = 1\,[= 2]$.

Let p be of Type 2 *or Type* 3. The statement (v) and the Remarks in 1. 2 imply the following.

Case 1. *Let A have no cusp at p.*

If A supports $\mathfrak{T}$ at p, then for both $k = 2$ and $k = 3$,

$$a_0 = a_1 = a_3 = a_4 = 1;\ a_2 = 2.$$

If A intersects $\mathfrak{T}$ at p, then

$$a_0 = a_2 = 1,\ a_1 = a_3 = a_4 = 2 \text{ if } k = 2;$$
$$a_0 = a_1 = a_2 = a_3 = 1,\ a_4 = 2 \text{ if } k = 3.$$

Case 2. *Let A have a cusp at p.*

If A supports $\mathfrak{T}$ at p (cusp of the second kind), *then for both $k = 2$ and $k = 3$,*

$$a_0 = a_1 = a_2 = a_3 = a_4 = 2.$$

If A intersects $\mathfrak{T}$ at p (cusp of the first kind), *then*

$$a_0 = 2,\ a_1 = a_2 = a_3 = a_4 = 1 \text{ if } k = 2;$$
$$a_0 = a_1 = a_3 = 2,\ a_2 = a_4 = 1 \text{ if } k = 3.$$

2. 3. We summarize the preceding discussion in the following table.

Points having no cusp				*Cusp points*	
(11111; 1a (i))	$\gamma(p)$ intersects A at p			(22221; 1a (i))	cusps of the second kind
(11112; 1a (i)$_*$)	a small nbd. of p on $A - p$ lies	inside	$\gamma(p)$	(22222; 1a (i)$_*$)	
(11112; 1a (i)*)		outside		(22222; 1a (i)*)	
(11112; 1a (ii))	$\gamma(p) = p$			(22222; 1a (ii))	
(11112; 1a (iii))	$\gamma(p)$ = double line on $\mathfrak{T}$			(22222; 1a (iii))	
(11121; 1b)	conics of σ intersect A at p			(22112; 1b)	
(11211; 1b$_*$)	a small nbd. of p on $A - p$ lies	inside	a conic of σ	(22222; 1b$_*$)	
(11211; 1b*)		outside		(22222; 1b*)	
(11211; 2)	A does not cross $\mathfrak{T}$ at p			(22222; 2)	
(11211; 3)				(22222; 3)	
(12122; 2)	A crosses $\mathfrak{T}$ at p			(21111; 2)	cusps of the first kind
(11112; 3)				(22121; 3)	

2. 4. We give examples of interior points p of arcs with the characteristics listed in Section 2. 3. In all these examples the arc is decomposed by the point p into two subarcs of the minimal conical order five; cf. ([2], 7. 3) and ([3], 4. 5, 7. 3).

(11111; 1a(i)) Any interior conically differentiable point p of an arc A_5.

(11112; 1a(i)$_*$) [(11112; 1a(i)*)] Take an A_5 with end-point p of Type 1a(i) and $A_5 - p < \gamma(p)_*$ [$A_5 - p < \gamma(p)^*$]. The image A_5' of A_5 under a harmonic homology which maps $\gamma(p)$ into itself and preserves p, will lie in $\gamma(p)_*$ [$\gamma(p)^*$]. The point p will be conically differentiable on $A_5 \cup p \cup A_5'$. It has the indicated characteristic.

In the following examples, we take the point p given by $s = 0$ on the indicated arcs. The number $\delta > 0$ is sufficiently small. Take the union of the arcs in each of the first four examples.

$$\left.\begin{array}{ll} (11112;\ 1a(ii)) & x = \pm s^2,\ y = s^4 + s^7; \\ (11112;\ 1a(iii)) & x = \pm s^2,\ y = s^4 - s^7; \\ (11121;\ 1b) & x = s^2,\ y = s^4 + s^5 \\ & x = -s^2,\ y = s^4 - s^5; \\ (11211;\ 1b^*) & x = \pm s^2,\ y = s^4 - s^5; \end{array}\right\} \quad 0 \leqq s \leqq \delta.$$

$$\left.\begin{array}{ll} (11211;\ 1b_*) & x = s^3,\ y = s^6 + s^8; \\ (11211;\ 2) & x = s^3,\ y = s^4; \\ (11211;\ 3) & x = s,\ y = s^4; \\ (12122;\ 2) & x = s^3,\ y = s^5; \\ (11112;\ 3) & x = s,\ y = s^3. \end{array}\right\} \quad -\infty < s < \infty.$$

Examples of the cusp points can be constructed in a similar way.

2. 5. For the sake of completeness, we also define characteristics for those interior conically differentiable points which have infinite conical order.

If every neighbourhood of p on A has points in common with a line $\mathfrak{L}$, then $\mathfrak{L} = \mathfrak{T}$ and p is of Type 3; thus the conics of $\bar{\sigma}$ neither intersect nor support A at p; cf. ([2], 5. 1, 5. 6). We then define the characteristic of p to be $(1, 1, \infty, \infty, \infty; 3)$ if p is not a cusp point, and $(2, 2, \infty, \infty, \infty; 3)$ if A has a cusp at p. (In both cases A can cross $\mathfrak{T}$ at p.)

If every neighbourhood of p on A has points in common with a non-degenerate conic γ, then p is of Type 1a(i) and $\gamma = \gamma(p)$; cf. ([2] 5. 10 and 7. 2). We assign to p the characteristic $(2, 2, 2, 2, \infty;$ 1a) or $(1, 1, 1, 1, \infty;$ 1a) according as A has or has not a cusp at p; cf. 1. 2 (viii). (By 1. 2 (iii), A does not cross $\mathfrak{T}$ at p.)

From now on we exclude these cases.

3. The conical order of an arc

This section is concerned with the proof of Theorem 1.

3. 1. Let p be a conically differentiable interior point of the arc A. We may assume that p has finite conical order. Let $(a_0 a_1 a_2 a_3 a_4; k)$ be the characteristic of p. We have to show that the conical order of p is not less than $a_0 + a_1 + a_2 + a_3 + a_4$. We shall actually prove that *for every neighbourhood M of p on A, there exists a conic arbitrarily close to $\gamma(p)$ which intersects M at not less than $\sum_{i=0}^{4} a_i$ points.*

This result is trivial when p has the characteristic $(11111;$ 1a(i)).

3. 2. It will be convenient to introduce certain pencils of conics λ^i, $i = 0, 1, 2, 3, 4$.

Choose any non-degenerate conic γ^1 through p, $\gamma^1 \notin \tau$. Let p, Q_1, Q_2, Q_3, Q_4 be mutually distinct points on γ^1;

$$Q_i \notin \mathfrak{T};\ Q_1 \notin \gamma(p);\ Q_2 \notin \gamma(\varrho;\ Q_1);\ Q_3 \notin \gamma(\sigma;\ Q_1, Q_2).$$

Let λ^0 be the pencil of the conics through Q_1, Q_2, Q_3, Q_4. Let λ^1 be the pencil of conics through p, Q_1, Q_2, Q_3. Thus

$$\gamma(\lambda^1;\ s) = \gamma(p,\ s,\ Q_1,\ Q_2,\ Q_3) \text{ and } \gamma(\lambda^0;\ p) = \gamma(\lambda^1;\ Q_4) = \gamma^1.$$

Denote the pencil of the conics in τ through Q_1 and Q_2 by λ^2. Thus

$$\gamma(\lambda^2;\ s) = \gamma(\tau;\ s,\ Q_1,\ Q_2) \text{ and } \lim_{s \to p} \gamma(\lambda^1;\ s) = \gamma(\lambda^2;\ Q_3).$$

The pencil of the conics in σ through Q_1 shall be denoted by λ^3. Thus

$$\gamma(\lambda^3;\ s) = \gamma(\sigma;\ s,\ Q_1) \text{ and } \lim_{s\to p} \gamma(\lambda^2;\ s) = \gamma(\lambda^3;\ Q_2).$$

Finally, write $\lambda^4 = \varrho$. Hence

$$\gamma(\lambda^4;\ s) = \gamma(\varrho;\ s);\ \lim_{s\to p} \gamma(\lambda^3;\ s) = \gamma(\lambda^4;\ Q_1),\ \text{and } \lim_{s\to p} \gamma(\lambda^4;\ s) = \gamma(p).$$

It will be convenient to write

$$\lim_{s\to p} \gamma(\lambda^i;\ s) = \gamma^{i+1},.$$

With one exception noted further on, the conics of the pencil λ^i are in $1-1$ correspondence with the points of a (projective) line if λ^i is completed by including γ^{i+1}.

We observe that if p is not of Type 1a, then $\gamma^4 = \mathfrak{T} \cup pQ_1$, and if p is not of Type 1a(i) or 1a(ii), then $\gamma^5 = \gamma(p)$ is the double line on $\mathfrak{T}$; (cf. 1. 2, Remark (1)).

3. 3. Lemma 1. *Let M be any neighbourhood of p on A. Then λ^i contains conics γ, arbitrarily close to but different from γ^{i+1} which intersect M outside p in not less than a_i points*; $i = 0, 1, 2, 3, 4$.

Proof. If $i \leqq 3$ or if $i = 4$ and p is not of Type 1a(ii) or 1a(iii), we consider an open interval $\mu^i < \lambda^i$ such that $\gamma^{i+1} \in \mu^i$. Then γ^{i+1} decomposes this interval into two open subintervals μ^i_L and μ^i_R. Let $\gamma^i_L[\gamma^i_R]$ be the other end-conic of $\mu^i_L[\mu^i_R]$. We then define the open regions,

$$E^i_L = \{\gamma^i_{L^*} \cap \gamma^{i+1^*}\} \cup \{\gamma^{i^*}_L \cap \gamma^{i+1}_*\},$$

and

$$E^i_R = \{\gamma^i_{R^*} \cap \gamma^{i+1^*}\} \cup \{\gamma^{i^*}_R \cap \gamma^{i+1}_*\},$$

except in the two cases where $i = 3$ and p is of Type 1b, and $i = 2$ and p is of Type 3. In these cases we define γ^{i+1}_* to be the limit of the interior of a conic of λ^i which tends to γ^{i+1} through μ^i_L. We then define E^i_L as above, but we take

$$E^i_R = \{\gamma^i_{R^*} \cap \gamma^{i+1}_*\} \cup \{\gamma^{i^*}_R \cap \gamma^{i+1^*}\}.$$

If $i = 4$, and p is of Type 1a(ii) or 1a(iii), then λ^4 is in $1-1$ correspondence with the points of an open interval, γ^5 corresponding to an end-point. We then consider a half-open interval $\mu^4_L < \lambda^4$ with one end-conic equal to γ^5. Let γ^4_L be the other end-conic of μ^4_L. Then only the region E^4_L is defined. It reduces to $E^4_L = \gamma^4_{L^*}$ when $\gamma(p) = p$, and to $E^4_L = \gamma^{4^*}_L$ when $\gamma(p)$ is the double line on $\mathfrak{T}$.

In all cases the conics of μ^i_L pass through the region E^i_L.

We can choose our neighbourhood $M = N \cup p \cup N'$ so small that γ^{i+1}, γ^i_L, γ^i_R do not meet N or N'. Let $s \in N$, $s' \in N'$. Since $\gamma(\lambda^i;\ s)$ is close to γ^{i+1} when s is close to p, it follows that either all the conics $\gamma(\lambda^i;\ s)$ lie in μ^i_L or they all lie in μ^i_R. Suppose that $\gamma(\lambda^i;\ s) \in \mu^i_L$. Thus $N < E^i_L$.

Let $e \in N$. Then $\gamma(\lambda^i;\ e)$ is the end-conic of a one-sided neighbourhood ν of γ^{i+1} in λ^i. If s moves from e to p, then $\gamma(\lambda^i;\ s)$ moves in λ^i from $\gamma(\lambda^i;\ e)$ continuously to γ^{i+1}. Hence the conics $\gamma(\lambda^i;\ s)$ omit none of the conics of ν, i. e., every conic of ν meets N.

Let $\gamma \in \nu$. Thus γ lies between $\gamma(\lambda^i;\ e)$ and γ^{i+1} in ν. If s is sufficiently close to p, then $s \notin \gamma$, and γ will also lie between $\gamma(\lambda^i;\ e)$ and $\gamma(\lambda^i;\ s)$. Since $e \notin \gamma$ and since the points s and e lie in E^i_L, they will also be separated by γ.

Since p has finite order, we may assume that M also has finite order. In particular γ will meet N in a finite number of points only and at least one of them will be a point of intersection.

Similarly, there exists a one-sided neighbourhood ν' of γ^{i+1} in λ^i such that each of its conics meet N'. If N' is sufficiently small, then ν' can be chosen such that each conic of ν' intersects N'.

If $a_i = 1$, then one of the conics γ^{i+1} and γ^i_L intersects while the other supports M at p. Hence $N' < E^i_R$. If $a_i = 2$, then γ^{i+1} and γ^i_L either both intersect or both support, hence $N' < E^i_L$. Thus the conics $\gamma(\lambda^i; s')$ belong to μ^i_R[to μ^i_L] if $a_i = 1$[$= 2$]. Since $\nu < \mu^i_L$, it follows that ν and ν' lie on opposite sides of γ^{i+1} or on the same side depending on whether $a_i = 1$ or $= 2$. This implies Lemma 1.

3. 4. *We can now readily verify Theorem* 1. We first approximate $\gamma(p)$ by a super-osculating conic γ^4; then γ^4 by an osculating conic $\gamma^3 \notin \varrho$; then γ^3 by a non-osculating tangent conic γ^2, which is in turn approximated by a non-tangent conic γ^1 through p; and finally, γ^1 by one, γ^0, which does not pass through p.

Let M_4 be a neighbourhood of p on A. By Lemma 1, there exists a conic $\gamma^4 \in \lambda^4 = \varrho$ close to but different from $\gamma(p)$, which intersects M_4 in not less than a_4 points s_4 outside p.

In the following, λ^3 is the pencil of conics of σ through a point Q of γ^4, $Q \notin \mathfrak{T}$; λ^2 is the pencil of τ through two distinct points Q, R of γ^3 outside p; p, Q, R not collinear; $Q, R \notin \mathfrak{T}$; λ^1 is the pencil through p and three other distinct points of γ^2; and finally, γ^0 is the pencil through four distinct points of γ^1 outside p. In the following, we put $i = 4, 3, 2$, and 1, in turn.

In M_i we construct mutually disjoint neighbourhoods B_i of the points s_i and M_{i-1} of p. By Lemma 1, there exists a conic $\gamma^{i-1} \in \lambda^{i-1}$ close to but different from γ^i, which intersects M_{i-1} in not less than a_{i-1} points s_{i-1} outside p, and which also intersects each of the $\sum_{n=i}^{4} a_n$ arcs $B_i, \ldots, B_4$.

Altogether, γ^0 is close to $\gamma(p) = \gamma^5$ and intersects M_4 in not less than

$$a_0 + a_1 + a_2 + a_3 + a_4$$

points, all of which are different from p.

4. Preparatory remarks

4. 1. Let r, s, t, u, v lie on A_5 between its end-points p and e. Let p be of Type 1 b* or 2 [1 b_* or 3]. (Thus a small neighbourhood N of p on A_5 lies outside [inside] a conic of σ for Type 1 b* [1 b_*]; cf. 1. 2 (ix).) We note that *for both Types* 1 b* *and* 2 [1 b_* *and* 3], *a small* N *lies inside* [*outside*] *a given* $\gamma(r, s, t, u, v)$ *or* $\gamma(\tau; t, u, v)$ *but outside* [*inside*] *a* $\gamma(p, s, t, u, v)$. Also, e *lies outside* [*inside*] $\gamma(r, s, t, u, v)$, *and, in particular, outside* [*inside*] $\gamma(p, s, t, u, v)$ *and* $\gamma(\tau; t, u, v)$. Thus *as* r, s, t, u, v *tend to* p, $\gamma(r, s, t, u, v)$, $\gamma(p, s, t, u, v)$, *and* $\gamma(\tau; t, u, v)$ *tend to the double line on* $\mathfrak{T}$ (if p is of Type 2, this limit can also be a double segment on $\mathfrak{T}$ or the point p) *in such a way that the limit of the interior* [*exterior*] *of these conics is void*; cf. 1. 3 (xi).

In these cases, it is convenient to orient the pair of lines $\gamma(\varrho; e)$ such that A_5 lies inside [outside] $\gamma(\varrho; e)$. Then $\gamma(s, t, u, v, e)_*$ tends to $\gamma(\varrho; e)_*$ as s, t, u, v tend to p.

4. 2. Let p and e be the end-points of A_5. Let p be of Type 1 a(i)* (i. e., $A_5 < \gamma(p)^*$), 1 a(ii), 1 b*, or 2. Define $\gamma = pe \cup \mathfrak{T}$, $A_5 < \gamma_*$. If p, s, t, u, v and b, c, d, e lie on A_5 in that order, with s, t, u preceding b, c, d respectively, then

$$\gamma(p, b, c, d, e)^* \cap \gamma^* < \gamma(p, s, c, d, e)^* \cap \gamma^* < \gamma(p, s, t, d, e)^* \cap \gamma^*$$
$$< \gamma(p, s, t, u, e)^* \cap \gamma^* < \gamma(p, s, t, u, v)^* \cap \gamma^* < \gamma(p, s, t, u, v)^*.$$

In particular, when e is of Type 1 and $b = c = d = e$, we obtain

$$\gamma(p;\ \varrho_e)^* \cap \gamma^* < \gamma(p, s, t, u.\ v)^*,$$

and when $p = s = b$ and $c = d = e$,

$$\gamma(\tau;\ \sigma_e)^* \cap \gamma^* < \gamma(\tau;\ t, u, v)^*.$$

If p is of Type 1, and we put $p = s = t = b = c$ and $d = e$, we obtain

$$\gamma(\sigma;\ \tau_e)^* \cap \gamma^* < \gamma(\sigma;\ u, v)^*.$$

Symmetrically, if p is of Type 1a(i)$_*$, 1a(iii), 1b$_*$, or 3, it is convenient to put $\gamma = pe \cup \mathfrak{T}$ with $A_5 < \gamma^*$ and we obtain

$$\gamma(p, b, c, d, e)_* \cap \gamma_* < \gamma(p, s, t, u, v)_*,$$

and if p is of Type 1,

$$\gamma(\sigma;\ \tau_e)_* \cap \gamma_* < \gamma(\sigma;\ u, v)_*.$$

4. 3. Let p, t, u, v lie on a convex arc A. Let $\gamma' = pu \cup pv$, $\gamma'' = uv \cup \mathfrak{T}$, $t \in \gamma'^* \cap \gamma''_*$. Then

$$A \cup \gamma(\tau;\ t, u, v) < (\gamma'^* \cap \gamma''_*) \cup (\gamma'_* \cap \gamma''^*) \cup \{p, u, v\}.$$

This implies that $\gamma(\tau;\ t, u, v)$ *and* A *touch* $\mathfrak{T}$ *from the same side.* Thus, a sufficiently small plane neighbourhood (depending on t, u, and v) of p is decomposed by $\mathfrak{T}$ into two disjoint regions, such that A and $\gamma(\tau;\ t, u, v)$ have no points in common with one of these regions; cf. ([2], 3. 3).

4. 4. Let A have either a cusp of the first kind or an inflection point at p. Let $A = e \cup B \cup p \cup B' \cup e'$, where B and B' are convex, and have the common tangent $\mathfrak{T}$, $A \cap \mathfrak{T} = p$. Choose a line $\mathfrak{L}$, $P \notin \mathfrak{L}$. By making A smaller we may assume $\mathfrak{L} \cap \bar{A} = \emptyset$. Thus e and e' lie in different regions of $\mathfrak{T} \cup \mathfrak{L}$. By ([3], 2. 8), there exists a neighbourhood N of p on B such that if $u, v \in N$, then $\mathfrak{L}(u, v) \cup \mathfrak{L}$ is so close to $\mathfrak{T} \cup \mathfrak{L}$ that e and e' also lie in different regions of $\mathfrak{L}(u, v) \cup \mathfrak{L}$. Thus $\mathfrak{L}(u, v)$ meets A with an odd multiplicity, and hence three times. This yields the following result.

If A has a cusp of the first kind or a point of inflection at p, then there exists a neighbourhood $M = N \cup p \cup B'$ of p on A which has linear order three.

Remark. This is a slightly strengthened version of the observation that an interior differentiable point of a plane elementary arc has the linear order three, if it has the characteristic (1, 2) or (2, 1); cf. [5].

4. 5. Suppose that no three of the points P, Q, R, S are collinear. Then the degenerate members, $PQ \cup RS$, $PR \cup QS$, and $PS \cup QR$, of the family of all the conics through P, Q, R, S decompose the set of its non-degenerate members into three mutually disjoint subfamilies. In either of the topologies of the space of all the conics considered in ([2], 2), each of these subfamilies is homeomorphic to an interval. Suppose $\gamma, \gamma', \gamma''$ belong to the same subfamily and γ lies between γ' and γ''. Then $\gamma_* > \gamma'_* \cap \gamma''_*$.

4. 6. Let p and e be the end-points of a convex arc A. Let N be a small neighbourhood of p on A. We may assume that N does not meet a suitable line $\mathfrak{L}_\infty$. Let α be the family of all the conics through four fixed points s, t, u, v of N. Then α is decomposed by $\gamma_1 = \mathfrak{L}(s, t) \cup \mathfrak{L}(u, v)$, $\gamma_2 = \mathfrak{L}(s, v) \cup \mathfrak{L}(t, u)$, and $\gamma_3 = \mathfrak{L}(s, u) \cup \mathfrak{L}(t, v)$ into three subfamilies α_1, α_2, and α_3, where α_1, say, is bounded by γ_1 and γ_2, α_2 by γ_1 and γ_3, and α_3 by γ_3 and γ_2. In the affine plane obtained by removing $\mathfrak{L}_\infty$, each conic of $\alpha_2[\alpha_3]$ is a hyperbola with s and t [s and v] on one branch and u and v [t and u] on the other.

We orient $\mathfrak{T}$ such that $N < \mathfrak{T}_*$. The orientation of a line $\mathfrak{L}(s, t)$ is induced by that of $\mathfrak{T}$. Thus $p \in \mathfrak{L}(s, t)_*$. Then one branch of each conic of α_2 lies in the set

$$R_2 = \{\mathfrak{L}(s, u)_* \cap \mathfrak{L}(s, t)^*\} \cup \{\mathfrak{L}(s, t)_* \cap \mathfrak{L}(s, u)^* \cap \mathfrak{L}(t, v)_*\}$$
$$\cup \{\mathfrak{L}(t, v)^* \cap \mathfrak{L}(u, v)_* \cap \mathfrak{L}(s, t)^*\} \cup \{s, t\},$$

and the other branch lies in a similar set. Similarly, one branch of a conic of α_3 lies in

$$R_3 = \{[\mathfrak{L}(t, v)^* \cup \mathfrak{L}(s, u)^*] \cap \mathfrak{L}(s, v)_* \cap \mathfrak{L}(t, u)_*\}$$
$$\cup \{\mathfrak{L}(s, u)_* \cap \mathfrak{L}(t, v)_* \cap \mathfrak{L}(s, v)^*\} \cup \{s, v\},$$

for example. Thus *the conics of α_2 and α_3 do not meet N outside s, t, u, v*. The conics of α_1 lie in

$$\{\mathfrak{L}(s, t)^* \cap \mathfrak{L}(s, v)^* \cap \mathfrak{L}(t, u)_*\} \cup \{\mathfrak{L}(s, t)_* \cap \mathfrak{L}(t, u)^* \cap \mathfrak{L}(u, v)_*\}$$
$$\cup \{\mathfrak{L}(t, u)_* \cap \mathfrak{L}(u, v)^* \cap \mathfrak{L}(s, v)^*\} \cup \{\mathfrak{L}(s, v)_* \cap \mathfrak{L}(s, t)_* \cap \mathfrak{L}(u, v)_*\}$$
$$\cup \{\mathfrak{L}(s, t)^* \cap \mathfrak{L}(u, v)^*\} \cup \{s, t, u, v\}.$$

The points s, t, u, v lie on each conic $\gamma \in \alpha_1$ in the indicated order and *if γ is a hyperbola, then they all lie on the same branch.*

From the above we obtain that *no hyperbola through four points of N can have three of these points on one branch and the fourth point on the other branch.*

4. 7. In the next two sections, we introduce a local Euclidean metric into the projective plane with p the origin, $\mathfrak{T}$ the x-axis, and A_5 in the first quadrant.

Let p be an end-point of A_5 of Type 3, and let $N_4 < N_3 < N_2 < N_1$ be suitable small neighbourhoods of p on A_5.

Let $f(\gamma)$ be the slope of the conjugate axis of a conic γ (a hyperbola) through five points r, s, t, u, v of N_1. We consider the behaviour of $f(\gamma)$ as these points move on N_1.

When the five points are all at p, the conic γ is the double line on $\mathfrak{T}$, and $f(\gamma)$ can be taken to be 0. If we keep four of the points fixed at p, and move the fifth point v on N_1 away from p, then $f\{\gamma(\varrho; v)\}$ increases monotonically.

Next, we hold three of the points fixed at p, and keep v fixed on N_1 while the fifth point u is moved away from p on N_1. Then $f\{\gamma(\sigma; u, v)\}$ also increases monotonically.

Next, we hold two of the points fixed at p, and keep u and v fixed on N_2, while t moves monotonically away from p on N_1. Then $f\{\gamma(\tau; t, u, v)\}$ varies monotonically, and, by comparing the end-cases $t = p$ and $t = e$, where e is the end-point of N_1, we obtain that $f\{\gamma(\tau; t, u, v)\}$ increases as t moves from p to e; cf. 1. 3 (xi).

Next, we hold one point fixed at p, and keep t, u, v fixed on N_3 and let s move monotonically on N_2 from p to its end-point. Then $f\{\gamma(p, s, t, u, v)\}$ varies monotonically, and we obtain, as above, that it increases monotonically.

Finally, in the same fashion, we conclude that $f\{\gamma(r, s, t, u, v)\}$ *is positive for all choices of r, s, t, u, v on N_4.*

4. 8. *Let p be an end-point of A_5 of Type* 2, *and let N be a small neighbourhood of p on A_5.* We shall prove that *if the conic through five points of A_5 is an ellipse* [a hyperbola], *then its centre lies to the right* [*left*] *of the y-axis.*

Case 1: We first consider the family α of tangent conics of A_5 at p through $u, v \in N$. α is decomposed by $\gamma_1 = \mathfrak{T} \cup \mathfrak{L}(u, v)$ and $\gamma_2 = \mathfrak{L}(p, u) \cup \mathfrak{L}(p, v)$ into two subfamilies α_1

and α_2. The conics of α_2 are hyperbolas and they do not meet A_5 outside u and v; cf. 4. 3.

Let $F(\gamma)$ denote the abscissa of the centre of the conic $\gamma \in \alpha$. For any real a, there are at most two conics $\gamma \in \alpha$ with $F(\gamma) = a$. We consider the variation in $F(\gamma)$ as γ runs through α, starting with γ_2, where $F(\gamma_2) = 0$, and proceeding through α_2, γ_1, and α_1.

As the hyperbola γ runs continuously and monotonically through α_2, γ_1, and α_1 as far as $\gamma(\tau; u, v, e)$, $F(\gamma)$ increases to a local maximum, then decreases to $F\{\gamma(\tau; u, v, e)\}$. From this point on, γ meets A_5 five times. $F(\gamma)$ continues to decrease towards $-\infty$. There is a discontinuity of $F(\gamma)$ when γ is a parabola, and then γ becomes an ellipse with $F(\gamma)$ decreasing from $+\infty$ to a local minimum $> F(\gamma_1) > 0$. $F(\gamma)$ then increases once more to $+\infty$ and $F(\gamma)$ has another discontinuity. Finally γ becomes and remains a hyperbola with $F(\gamma)$ increasing from $-\infty$ to o.

γ is a conic of τ through three points of A_5 if and only if γ lies between $\gamma(\tau; u, v, e)$ and γ_2 in α_1. In this interval, $F(\gamma)$ is negative when γ is a hyperbola, but positive when γ is an ellipse.

Case 2: α is the family of conics through $p, t, u, v \in N$. As in 4. 6, with $s = p$, α is decomposed by γ_1, γ_2 and γ_3, into three subfamilies α_1, α_2, and α_3. The discussion is similar to that of Case 1 when $F(\gamma_2) < 0$, but when $F(\gamma_2) > 0$, γ runs from $+\infty$ to $-\infty$ as an ellipse and then from $+\infty$ to $-\infty$ as a hyperbola. However, in the interval of α in which γ meets A_5 five times, i. e., the interval of α_1 bounded by $\gamma(\tau; t, u, v)$ and $\gamma(p, t, u, v, e)$, $F(\gamma)$ remains positive when γ is an ellipse, but $F(\gamma)$ is negative when γ is a hyperbola.

Case 3: α is the family of conics through $s, t, u, v \in N$. The discussion is similar to that of Case 2. We omit the details.

4. 9. We continue the discussion of 4. 4 and 4. 6. Let $M = N \cup p \cup N'$, where p is a cusp point or an inflection point. Let e and e' be the end-points of M. We may assume that M does not meet a suitable line $\mathfrak{L}_\infty$ and that a line through two points of N or of N' meets M exactly three times. Let $\mathfrak{L}(s, t)$, $\mathfrak{L}(s, u)$, $\mathfrak{L}(s, v)$, $\mathfrak{L}(t, u)$, $\mathfrak{L}(t, v)$, and $\mathfrak{L}(u, v)$, meet N' at, say, a', b', c', f', g', h', respectively. These points lie on N' either in the order indicated, or in the order a', b', f', c', g', h'.

It can be verified that *every conic of* α_3 *meets* N' *exactly twice.* One of these points lies between b' and c' and the other lies between f' and g'. We indicate a proof. The set R_3, defined in 4. 6, decomposes the rest of the plane into two disjoint connected subsets. Furthermore, R_3 contains one branch of each hyperbola γ of α_3. Since p and c' are separated by R_3, and hence by the branch of γ in R_3, γ meets N' with an odd multiplicity between b' and c'.

Similarly the other branch of γ meets N' with an odd multiplicity between f' and g'. Finally, γ cannot meet N' elsewhere. By 4. 6, applied to N', each branch of γ must meet N' exactly once.

We next consider the conics γ of α_2. One branch, say γ_L, of γ lies in the set R_2, defined in 4. 6. Since $h' \in \gamma_L^*$ and $g' \in \gamma_{L^*}$, γ_L meets N' between g' and h' with an odd multiplicity. Similarly, since $a' \in \gamma_L^*$ and $b' \in \gamma_{L^*}$, γ_L meets N' between a' and b' with an odd multiplicity. γ_L cannot meet N' outside the above intervals. The other branch of γ lies in $\mathfrak{T}_*$, and hence it cannot meet N'.

Using the same methods, we can verify that a conic γ of α_1 can meet N' between p and a', between c', f', and between h' and e', but nowhere else.

5. Conically elementary points

5. 1. An interior point p of A is *conically elementary* if there exists a neighbourhood of p on A which is decomposed by p into two one-sided neighbourhoods of conical order five. We can restate Theorem 2 as follows:

Let p be an interior, conically elementary, conically differentiable point of an arc A. Let $(a_0a_1a_2a_3a_4; k)$ be the characteristic of p. Then p has the conical order $\sum_{i=0}^{4} a_i$.

Theorem 2 remains valid if p is counted $\sum_{i=0}^{n} a_i$ times on a conic γ, where $n = 0, 1, 2, 3, 4$ according as γ is a non-tangent conic through p, $\gamma \in \tau - \sigma$, $\gamma \in \sigma - \varrho$; $\gamma \in \varrho - \gamma(p)$, and $\gamma = \gamma(p)$, respectively.

The remainder of this paper is concerned with the proof of Theorem 2.

5. 2. We may assume that A itself is decomposed by p into two open arcs A_5 and A_5' of conical order five. Thus the order of A, and therefore that of p, is not greater than ten. This fact, combined with Theorem 1, implies that *Theorem 2 holds trivially when the characteristic of p is of the form* (22222; k).

5. 3. Lemma 2. *There exists a neighbourhood $M = N \cup p \cup N'$ of p on A such that a conic γ through five points of $N \cup p$ or of $N' \cup p$ will meet A with an even multiplicity, except when p is of Type* (11111; 1a(i)) *or* (22221; 1a(i)). *In the exceptional cases, γ meets A with an odd multiplicity. Thus γ meets A with a multiplicity $\equiv \sum_{0}^{4} a_i \pmod 2$.*

Proof. Suppose that p is [is not] of Type (11111; 1a(i)) or (22221; 1a(i)). Thus $\gamma(p)$ intersects [supports] A at p and since $\gamma(p)$ does not meet A outside p, the end-points of A lie on opposite sides [on the same side] of $\gamma(p)$. By 1. 3 (xi), we can choose M so small that γ is so close to $\gamma(p)$ (to a degenerate conic on $\mathfrak{T}$ if p is of Type 2) that the end-points of A lie on opposite sides [on the same side] of γ also. This implies our statement.

Corollary 1. *If A has no cusp [has a cusp] at p, then every non-tangent conic through p and four points of N or of N' will meet A outside p with an odd [even] multiplicity, except when p is of Type* 1a(i) *and $\gamma(p)$ intersects A at p.*

Corollary 2. *A conic of τ through three points of N or of N' will meet A outside p with an even multiplicity, except when p is of Type* 1a(i) *and $\gamma(p)$ intersects A at p, or when p is of Type* 2 *and A crosses $\mathfrak{T}$ at p. In the exceptional cases, such a conic will meet $A - p$ with an odd multiplicity.*

Proof. A conic of $\tau - \sigma$ supports A at p except when p is of Type 2 and A crosses $\mathfrak{T}$ at p.

5. 4. Let p, Q_1, Q_2, Q_3, Q_4 be mutually distinct points on a non-tangent conic; $Q_j \notin \mathfrak{T}$; $Q_4 \notin \gamma(p)$; $Q_3 \notin \gamma(\varrho; Q_4)$; $Q_2 \notin \gamma(\sigma; Q_3, Q_4)$.

Lemma 3. *There exists a neighbourhood $M = N \cup p \cup N'$ of p on A such that no conic through $Q_i, \ldots, Q_4$ meets $N \cup p$ or $N' \cup p$ at more than i points, $i = 1, 2, 3, 4$.*

We illustrate the proof by taking $i = 2$, for example. By 1. 2 (xi), we can choose M so small that a conic γ through Q_3, Q_4 and three points of $N \cup p$ or of $N' \cup p$ is so close to $\gamma(\sigma; Q_3, Q_4)$ that $Q_2 \notin \gamma$.

The next result is similar to Lemma 2. Suppose that A does not meet $\gamma(\varrho; Q_4)$, $\gamma(\sigma; Q_3, Q_4)$, $\gamma(\tau; Q_2, Q_3, Q_4)$, $\gamma(p, Q_1, Q_2, Q_3, Q_4)$ outside p.

Lemma 4. *There exists a neighbourhood $M = N \cup p \cup N'$ of p on A such that a conic γ through $Q_i, \ldots, Q_4$ and i points of $N \cup p$ or of $N' \cup p$ meets A with a multiplicity* $\equiv \sum_{j=0}^{i-1} a_j \pmod 2$.

We give the proof for $i = 4$, for example. By 1. 2 (xi), we can choose M so small that the end-points of A will lie on opposite sides [on the same side] of the conic γ if they lie on opposite sides [on the same side] of $\gamma(\varrho; Q_4)$. The end-points of A, however, will lie on opposite sides [on the same side] of $\gamma(\varrho; Q)$ only if $a_0 + a_1 + a_2 + a_3$ is odd [even].

5. 5. *Suppose that p has the characteristic* (22221; 1a(i)). By 5. 3, we can choose a one-sided neighbourhood N of p so small that every conic γ through five points of $N \cup p$ will meet A with an odd multiplicity. Hence γ will meet A_5' with an even multiplicity $\leqq 4$. Hence no conic meets $M = N \cup p \cup A_5'$ more than nine times.

5. 6. *When the characteristic of p is* (11111; 1a(i)), the conditions given in 1. 3 (xiii), for the union of two arcs of conical order five to be of order five, will be satisfied by a small neighbourhood of p on A. Thus p will have conical order five, and Theorem 2 will hold true in this case also.

From now on we consider only those cases where $5 < \sum_{0}^{4} a_i < 9$.

5. 7. Lemma 5. *Let $A = A_5 \cup p \cup A_5'$. There exists a neighbourhood N_1 of p on A_5, such that no conic of ϱ meets $M_1 = N_1 \cup p \cup A_5'$ more than a_4 times outside p.*

Proof. A conic of ϱ can meet A_5 or A_5' at most once each. Thus it meets $A_5 \cup A_5'$ not more than twice. A conic will meet A with a multiplicity $\equiv \sum_{i=0}^{4} a_i \pmod 2$ if it is sufficiently close to $\gamma(p)$. Hence, we can choose $N_1 < A_5$ such that if $s \in N_1$, then $\gamma(\varrho; s)$ will meet $A_5 \cup A_5'$ with a multiplicity $\equiv a_4 \pmod 2$, and hence with the exact multiplicity a_4; cf. 2. 1. This proves Lemma 5. (If $a_4 = 2$, we can even choose $M_1 = A$.)

Remark. In the following sections

$$M_r = N_r \cup p \cup N_r'$$

will denote neighbourhoods of p on A such that

$$N_r < N_{r-1} \text{ and } N_r' < N_{r-1}', \quad r = 2, 3, 4, 5.$$

6. Conics of σ

In this section we prove

Lemma 6. *There exists a neighbourhood $M_2 < M_1$ which is met by every conic of σ at most $a_3 + a_4$ times outside p.*

The proof is given in the next four subsections. Note that Lemma 6 holds trivially if $a_3 + a_4 = 4$.

6. 1. On account of Lemma 5, we need consider only the conics of $\sigma - \varrho$. Hence it suffices to construct one-sided neighbourhoods $N_2 < N_1$, $N_2' < N_1'$ of p such that any conic of σ which meets N_2 or N_2' twice will meet M_1 outside p at most $a_3 + a_4$ times.

By 5. 3, N_2 [N_2'] can be chosen so small that a conic $\gamma \in \sigma$ through two points of N_2 [N_2'] meets M_1 outside p with a multiplicity $\equiv a_3 + a_4 \pmod 2$. Since γ does not meet N_1 or N_1' more than twice each, it can meet M_1 at most four times outside p. This proves Lemma 6 when $a_3 + a_4 = 3$.

We note that Lemma 6 also follows from 4. 4 when p is of Type 3 and $a_3 + a_4 = 3$.

We now consider the cases of Lemma 6 in which $a_3 + a_4 = 2$.

6. 2. *Let p have the characteristic* (21111; 2) *or* (11211; 2). From Lemma 5, the neighbourhoods N_1 and N_1' are such that if $s \in N_1$, then the line ps does not meet N_1'. Since p is of Type 2, every conic of σ is a pair of distinct lines through p neither of which in $\mathfrak{T}$. Hence *no conic of σ meets M_1 more than $a_3 + a_4 = 2$ times outside p*. Thus we may choose our neighbourhood M_2 in Lemma 6 to be equal to M_1 in these two cases.

6. 3. *Type* (11211; 3). We choose $M_2 < M_1$ such that M_2 is convex, i. e., no line meets M_2 more than twice. Since in this case every conic of σ is a pair of lines one of which is $\mathfrak{T}$, *no conic of σ can meet M_2 more than $a_3 + a_4 = 2$ times outside p*.

6. 4. *Types* (11211; $1b_*$) *and* (11211; 1b*). Let e be the end-point of N_1. Choose $N_2' < N_1'$ such that N_2' does not meet $\gamma(\sigma;\ \tau_e)$. Since $\gamma(\sigma;\ \tau_e)$ supports A at p, $N_2' \cup N_1$ lies in $\gamma(\sigma;\ \tau_e)_*$ in the former case, and in $\gamma(\sigma;\ \tau_e)^*$ in the latter.

Suppose, for example, that $N_2' \cup N_1 < \gamma(\sigma;\ \tau_e)^*$. We may assume that $N_1 < \gamma(\varrho;\ e)_*$. Thus

$$N_1 < \gamma(\sigma;\ \tau_e)^* \cap \gamma(\varrho;\ e)_* \text{ and } N_2' < \gamma(\sigma;\ \tau_e)^* \cap \gamma(\varrho;\ e)^*.$$

Let $u, v \in N_1$. Then 4. 2 yields

$$N_2' < \gamma(\sigma;\ \tau_e)^* \cap \gamma(\varrho;\ e)^* < \gamma(\sigma;\ u,\ v)^*.$$

The case where $N_1 \cup N_2' < \gamma(\sigma;\ \tau_e)_*$ can be dealt with similarly. Thus no conic of σ throught two points of N_1 will meet N_2'.

Constructing $N_2 < N_1$ symmetrically, *we obtain a neighbourhood M_2 which no conic of σ will meet more than twice outside p*. This completes the proof of Lemma 6.

7. Conics of τ

7. 1. Lemma 7. *There exists a neighbourhood $M_3 < M_2$ which is met by every conic of τ at most $a_2 + a_3 + a_4$ times outside p.*

The proof of Lemma 7 is given in this section. On account of Lemma 6, we need consider only the conics of $\tau - \sigma$.

Notation: e and e' will denote the end-points of N_2 and N_2'; $t, u, v \in N_2$, $t', u', v' \in N_2'$.

7. 2. *Type* (12122; 2) (inflection point). By Lemma 2, Corollary 2, we can choose $M_3 < M_2$ such that any conic γ of τ through three points of N_3 or of N_3' meets M_2 outside p with an odd multiplicity. Since γ cannot meet N_2 or N_2' more than three times each, it will meet M_2 at most six times outside p. Hence *no conic of τ meets M_3 more than $a_2 + a_3 + a_4 = 5$ times outside p*.

7. 3. In the next five subsections we deal with all the cases for which $a_2 + a_3 + a_4 = 4$. In each case, we prove that *no conic of τ meets some $M \leqq M_2$ more than five times outside p.*

Then by Lemma 2, Corollary 2, $M_3 < M$ can be chosen such that a conic of τ through three points of N_3 or N_3' meets M outside p with an even multiplicity. Thus *no conic of τ meets M_3 outside p at more than $a_2 + a_3 + a_4 = 4$ points.*

7. 4. *Types* (11112; 3) *and* (22121; 3). (A crosses $\mathfrak{T}$ at p.) By 4. 3, applied to N_2 and N_2', $\gamma(\tau;\ t,\ u,\ v)$ and $\gamma(\tau;\ t',\ u',\ v')$ touch $\mathfrak{T}$ from opposite sides. Hence $\gamma(\tau;\ t,\ u,\ v) \neq \gamma(\tau;\ t', u', v')$.

7. 5. *Type* (22112; 1b) (cusp of the second kind) *and Type* (11121; 1b). Here, and in the next subsection also, the conics of $\sigma - \varrho$ all intersect A at p, while

those of $\tau - \sigma$ all support. Thus we may assume that p is of Type $1b_*$ [1b*] with respect to N_2 [N_2']. By 4. 1, a small neighbourhood of p on $M_2 - p$ lies in $\gamma(\tau; t, u, v)^* \cap \gamma(\tau; t', u', v')_*$. Hence $\gamma(\tau; t, u, v) \neq \gamma(\tau; t', u', v')$.

Remark. The case $\gamma(\tau; t, u, v) = \gamma(\tau; t', u', v')$ can also be eliminated by considering the positions of $\gamma(\tau; t, u, v)$, $\gamma(\sigma; u, v)$, and $\gamma(\sigma; u', v')$ with respect to one another and to M_2. This method is illustrated in the next subsection.

7. 6. *Types* (11112; k), $k = 1a(i)_*$, 1a(iii), 1a(i)*, *or* 1a(ii). Consider the last two cases; thus

$$e \in \gamma(\tau; t, u, v)^* \cap \gamma(\sigma; u, v)^* \text{ and } e' \in \gamma(\sigma; u', v')^*.$$

Let $\gamma_0 = \mathfrak{L}(p, t) \cup \mathfrak{T}$, $e \in \gamma_0^*$.

Suppose that $\gamma = \gamma(\tau; t, u, v) = \gamma(\tau; t', u', v')$. Now, $\gamma(\sigma; t', u')$ intersects $\mathfrak{L}(p, t) \cap \gamma_*$ at a point in $\gamma(\sigma; t, u)_*$, while $\gamma(\sigma; t', u')$ meets N_2 between p and t at a point in $\gamma(\sigma; t, u)^*$. Hence $\gamma(\sigma; t', u')$ intersects $\gamma(\sigma; t, u)$ in $\gamma_{0*} \cap \gamma_*$. Symmetrically, they intersect at another point $\neq p$. Thus $\gamma(\sigma; t, u) = \gamma(\sigma; t', u')$. This contradicts Lemma 6.

The proof for the first two cases is similar to the above.

Remark. An alternative proof can be based on a continuity argument. Assume that $\gamma(\tau; t, u, v)$ meets N_2' at t'. Then use $\gamma(\sigma; t, t')$ and $\mathfrak{L}(p, t') \cup \mathfrak{T}$ to separate $\gamma(\tau; t, u, v)$ from the arc pt' of N_2'.

7. 7. *Types* (11211; k); $k = 1b_*$, 3, 1b* *or* 2. Let $\mathfrak{L}$ be any line not through p. We may assume that $M_2 \cap \mathfrak{L} = \emptyset$. Let $\gamma_0 = \mathfrak{L} \cup \mathfrak{T}$ with $M_2 < \bar{\gamma}_{0*}$.

Suppose that $k = 1b_*$ or 3. By 1. 3 (xi) and 4. 1, we can choose $M < M_2$ such that if $t, u, v \in N$, and $t', u', v' \in N'$, then

$$e' \in \gamma(\tau; u, v, e)_*,\ e \in \gamma(\tau; u', v', e')_* \tag{1}$$

and $\{e, e'\} \in \gamma(\tau; t, u, v)_*$. Thus $\gamma(\tau; u, v, e)$ and $\gamma(\tau; u', v', e')$ meet three times in $\bar{\gamma}_{0*}$, and they meet once in γ_0^*.

Suppose that $\gamma(\tau; t, u, v) = \gamma(\tau; t', u', v') = \gamma$. Now γ does not meet $\gamma(\tau; u, v, e)$ outside p, u, v. Hence $v' \in \gamma(\tau; u, v, e)^*$. Since $e' \in \gamma(\tau; u, v, e)_*$, $\gamma(\tau; u', v', e')$ meets $\gamma(\tau; u, v, e)$ between v' and e'. Symmetrically, they also meet between v and e. Altogether, they meet at least five times and hence they coincide. This contradicts (1).

The cases when $k = 1(b)^*$ or 2 are dealt with similarly.

7. 8. *Type* (21111; 2) (cusp of the first kind). Choose N' so small that

$$N' < \gamma(\tau; \sigma_e)^* \cap \gamma(\varrho; e)^*.$$

By 4. 2,

$$\gamma(\tau; \sigma_e)^* \cap \gamma(\varrho; e)^* < \gamma(\tau; t, u, v)^*.$$

Hence $\gamma(\tau; t, u, v)$ does not meet N'. Symmetrically, choose $N < N_2$ so small that if $t', u', v' \in N_2'$, then $\gamma(\tau; t', u', v')$ does not meet N.

It remains to show that *no conic of* τ *meets* N_3 *and* N_3' *exactly twice each, if* M_3 *is sufficiently small.* Such a conic would certainly separate two small one-sided neighbourhoods of p.

Choose $N_3 < N$ so small that if $s, t \in N_3$, then $\mathfrak{L}(s, t)$ meets N_2'. Let $\gamma_0 = ps \cup pt$ and $\gamma_1 = st \cup \mathfrak{T}$. We may assume that $e' \in \gamma_0^* \cap \gamma_1^*$.

Let γ be any conic of τ through s and t. We may assume that a small neighbourhood of p on N' lies in γ_*; thus γ and N touch $\mathfrak{T}$ from opposite sides, and

$$\gamma_* < \{\gamma_{1*} \cap \gamma_0^*\} \cup \{\gamma_1^* \cap \gamma_{0*}\} \cup \{p, s, t\},$$

cf. 4. 3. Thus $e' \in \gamma^*$. This implies that γ meets N_2' with an odd multiplicity. By the above discussion, γ meets N_2' exactly once.

Choosing $N_3' < N'$ symmetrically, we obtain that *no conic of* τ *meets* M_3 *more than* $a_2 + a_3 + a_4 = 3$ *times outside* p.

This completes the proof of Lemma 7.

8. Conics through p

8. 1. The goal of this section is

Lemma 8. *There exists a neighbourhood* M_4 *of* p *contained in* M_3 *which is met by every conic through* p *at most* $a_1 + a_2 + a_3 + a_4$ *times outside* p.

On account of Lemma 7, we consider only the non-tangent conics through p. e and e' are the end-points of N_3 and N_3'; $s, t, u, v \in N_3$, $s', t', u', v' \in N_3'$.

8. 2. *Type* (12122; 2). By Lemma 2, Corollary 1, $M_4 < M_3$ can be chosen such that any conic γ through p and four points of N_4 or of N_4' meets M_3 outside p with an odd multiplicity. Since γ can meet M_3 at most eight times outside p, it meets M_3, and in particular M_4, at most $a_1 + a_2 + a_3 + a_4 = 7$ times outside p.

8. 3. In the next two subsections we deal with the two cases where $a_1 + a_2 + a_3 + a_4 = 6$; thus $a_0 = 2$. In each case, we prove that *no conic through* p *meets some* $M \leqq M_3$ *more than seven times outside* p.

Then, by Lemma 2, Corollary 1, we can choose $M_4 < M$ such that any conic through p and four points of N_4 or of N_4' meets M_3 outside p with an even multiplicity. Thus *no conic through* p *meets* M_4 *more than six times outside* p.

8. 4. *Type* (22112; 1b) (cusp of the second kind). We assume that p is of Type $1b_*$ [1b*] with respect to N_3 [N_3']; cf. 7. 5. By 4. 1, the arc of A between p and s [p and s'] lies in $\gamma(p, s, t, u, v)_*$ [$\gamma(p, s', t', u', v')^*$]. Since the non-tangent conics through p all support A at p, this, however, implies that $\gamma(p, s, t, u, v) \neq \gamma(p, s', t', u', v')$.

8. 5. *Type* (22121; 3) (cusp of the first kind). By 4. 1, $\gamma(p, s, t, u, v)$ converges to the double line on $\mathfrak{T}$, with e and e' in $\gamma(p, s, t, u, v)_*$ as s, t, u, v converge to p on N_3. Hence we can choose $N < N_3$ [$N' < N_3'$] such that if s, t, u, v [s', t', u', v'] lie on N [N'], then the tangent of $\gamma(p, s, t, u, v)$ [$\gamma(p, s', t', u', v')$] at p is so close to $\mathfrak{T}$ that it lies in $\gamma(\varrho; e')^* \cup p$ [$\gamma(\varrho; e)^* \cup p$]. Since the regions $\gamma(\varrho; e)^*$ and $\gamma(\varrho; e')^*$ are disjoint, this implies that $\gamma(p, s, t, u, v) \neq \gamma(p, s', t', u', v')$.

Remark. This case can also be dealt with by applying 4. 7 to N_3 and to N_3'.

8. 6. *Type* (11121; 1b); *Types* (11112; k); $k =$ 1a(i)$_*$, 1a(iii), 3, 1a(i)*, *or* 1a(ii); *and Types* (11211; k); $k =$ 1b*, 2, 1b$_*$, *or* 3. In Type (11121; 1b), we may assume that a small neighbourhood of p on N_3 lies inside a conic of σ. The proof is given below for the first four types and the last two. It also applies to the other types if 'inside' and 'outside' are interchanged appropriately.

Let c, d be fixed points on N_3. Each of the following assertions after the first one can be proved from the preceding one and Lemma 7.

Lemma I [II; III]. *There exists an* $M < M_3$ *such that no conic through* p, c, d *and two points of* N [p, d *and three points of* N; p *and four points of* N] *meets* N' *more than once.*

Proof of I. By Lemma 3, with $i = 2$, choose $N' < N_3'$ such that no conic through p, c, d, e meets N' more than once and $\gamma(\tau; c, d, e)$ does not meet N'. Choose $N < N_3$ such that $c, d, e \notin N$. Let $M = N \cup p \cup N'$. Let $s, t \in N$ and let $\gamma = \gamma(p, s, t, c, d)$ meet N' at s'. Define $\gamma' = \gamma(\tau; s', c, d)$. Since $e \in \gamma(\sigma; c, d)_*$ and $e \notin \gamma'$ for any choice of $s' \in N'$, we have $e \in \gamma_*'$ for all these s'. Now the arc of γ between p and c which contains s lies in γ'^*. Hence $s \in \gamma'^*$. Since $e \in \gamma_*'$, γ' must meet N_3 three times between s and e. By Lemma 7, γ' does not meet M_3 elsewhere.

Let $\gamma_0 = ps' \cup \mathfrak{T}$, $\gamma_1 = \mathfrak{T}_{s'} \cup \mathfrak{T}$, $e \in \gamma_{0*} \cap \gamma_{1*}$. By the above, the arc of N' between p and s' lies in $\gamma'^* \cap \gamma_0^* \cap \gamma_{1*}$. Since γ intersects each of $\mathfrak{T}$, $\mathfrak{L}(p, s')$, γ', at p, γ cannot meet this region. Hence γ does not meet N' elsewhere.

Proof of II. Let $\gamma = \gamma(p, s, t, u, d)$ meet N' at s'. We separate γ from the arc of N' between p and s', as before, by using

$$\gamma' = \begin{cases} \gamma(\tau; s', u, d) & e \in \overline{\gamma(\tau; s', u, d)_*} \\ \gamma(s', p, u, d, e) & \text{if} \quad e \in \gamma(\tau; s', u, d)^*. \end{cases}$$

In the first case, $e \in \overline{\gamma_*'}$, apply the method of I to show that γ' meets N_3 exactly three times and N_3' just once.

In the second case, γ' lies between $\gamma(\tau; s', u, d)$ and $\gamma(p, s, t, u, d)$ and we obtain again $s \in \gamma'^*$. Since a small neighbourhood of p on N lies in γ_*', γ' intersects N between p and s. By Lemma I, γ' does not meet N' outside s'.

The proof of III is similar to that of II; d is replaced by v. From now on we may assume that M is so small that *no conic through p and four points of N [of N'] meets N' [N] more than once.*

It remains to show that *if $M_4 < M$ is sufficiently small, then no conic through p meets N_4 and N_4' three times each.*

Let a and a' be the end-points of M. Choose $N_4 < N$ such that the points a and a' lie inside any conic through p and four points of N_4. Let γ be a conic through p and exactly three points s, t, u of N, all lying in N_4, such that $a \in \gamma_*$. Let $v \in N_4$. Since $a' \in \gamma(p, s, t, u, v)_*$, we readily verify that $a' \in \gamma_*$. Thus γ meets N' with an even multiplicity $\leqq 4$ and hence $\leqq 2$. We construct N_4' symmetrically.

Now, either a or a' would lie inside a conic through p and exactly three points each of N_4 and of N_4'. This is excluded by the above. Thus *no conic through p meets M_4 more than $a_1 + a_2 + a_3 + a_4 = 5$ times outside p.*

Remark. In the (11112; 3) case, Lemma III can also be proved by using 4. 6. If $N < N_3$ is sufficiently small, and $s, t, u, v \in N$, then $\gamma = \gamma(p, s, t, u, v)$ is a hyperbola close to the double line on $\mathfrak{T}$, with p, s, t, u, v lying on the same branch. The other branch of γ meets N_3' with an odd multiplicity. By 4. 6, applied to N_3', γ must meet N_3' exactly once.

8. 7. *Type* (21111; 2) (cusp of the first kind). Let $\gamma = pe \cup \mathfrak{T}$, $N_3 < \gamma_*$; thus $N_3' < \gamma^*$. By 4. 1, $N_3 < \gamma(p, \varrho_e)^*$. Since $\gamma(p, \varrho_e)$ supports M_3 at p we can choose $N' < N_3'$ so small that $N' < \gamma(p; \varrho_e)^* \cap \gamma^*$. If s, t, u, v lie on N_3, then by 4. 2,

$$\gamma(p; \varrho_e)^* \cap \gamma^* < \gamma(p, s, t, u, v)^*.$$

Hence $\gamma(p, s, t, u, v)$ does not meet N', i. e., no conic through p and four points of N_3 meets N'. Symmetrically, there exists a neighbourhood $N < N_3$ such that no conic through p and four points of N_3' meets N. In particular, *no conic through p and four points of N or of N' meets $M = N \cup p \cup N'$ elsewhere.*

It remains to show that *there is an* $M_4 < M$ *such that no conic through* p, *three points of* N_4 $[N_4']$ *and a point of* N_4' $[N_4]$ *meets* M_4 *elsewhere.*

Using 4. 4, choose $M_4 < M$ such that M_4 has linear order three. Shorten M_4 still further so that the line through any two points of N_4 or of N_4' meets M_4 exactly three times. Let $t, u, v \in N_4$, $t' \in N_4'$, no three of these points collinear. Thus tu, tv, and uv meet N_4' at, say, a', b', and c', respectively, and do not meet M_4 elsewhere.

Consider the conic $\gamma = \gamma(t', p, t, u, v)$. There are four cases depending on the position of t' relative to a', b', and c' on N_4'; cf. 4. 9. Suppose, for example, that t' lies between p and a' on N'. We have $e \in \gamma(\sigma; u, v)^*$; cf. 4. 1. By Lemma 7, $e \notin \gamma(\tau; t', u, v)$. Thus $e \in \gamma(\tau; t', u, v)^*$ for all our t'. Since $\gamma(\tau; t', u, v)$ intersects M at p, the arc of N_4' between p and t' lies in $\gamma(\tau; t', u, v)_*$. Thus if $\gamma_0 = pt' \cup \mathfrak{T}$, this arc lies in $\gamma(\tau; t', u, v)_* \cap \gamma_0^*$, say. Since $\gamma(t', p, s, u, v)$ does not meet that region if s is sufficiently close to p, and since this conic depends continuously on s, γ cannot meet this region either. Hence γ does not meet the arc of N_4' between p and a' outside t'. The γ's with t' between p and a' are separated from those with t' beyond c' by $\gamma(\tau; t, u, v)$. If t' lies between a' and b' [between b' and c'] γ belongs to a different subfamily of the set of conics through p, t, u, v; cf. 4. 6, 4. 9.

Similarly $\gamma(t, p, t', u', v')$ does not meet N_4 outside t if $t', u', v' \in N_4'$.

Altogether, *no conic through* p *meets* M_4 *more than* $a_1 + a_2 + a_3 + a_4 = 4$ *times outside* p.

This completes the proof of Lemma 8.

9. The general conic

We complete the proof of Theorem 2 by proving

Lemma 9. *There exists a neighbourhood of* p *which is met by every conic not passing through* p *at most* $\sum_{i=0}^{4} a_i$ *times.*

Let e and e' denote the end-points of M_4.

9. 1. In the next three subsections, we deal with the three cases for which $\sum_{0}^{4} a_i = 8$. In each case we prove that *no conic meets some* $M \leqq M_4$ *ten times.*

Then, by 5. 3, we can choose $M_5 < M$ such that a conic through five points of N_5 or of N_5' meets M with an even multiplicity. Thus *no conic meets* M_5 *more than eight times.*

9. 2. *Type* (22112; 1b) (cusp of the second kind). Using the notation of 8. 4, we obtain from 4. 1 that the arc of M_4 between p and r [p and r'], in particular p, lies in $\gamma(r, s, t, u, v)^*$ [$\gamma(r', s', t', u', v')_*$]. Hence $\gamma(r, s, t, u, v) \neq \gamma(r', s', t', u', v')$.

9. 3. *Type* (22121; 3) (cusp of the first kind). By applying 4. 7 to N_4 and to N_4' we can choose $M < M_4$ such that the conjugate axis of a conic (a hyperbola) through five points of N [of N'] has a positive [negative] slope. Thus *no conic meets* M *ten times.*

9. 4. *Type* (12122; 2) (inflection point). By applying 4. 8 to N_4 and to N_4', we can choose $M < M_4$ such that the abscissa of the centre of a conic through five points of N [N'] is positive [negative] when the conic is an ellipse, but negative [positive] when the conic is a hyperbola. Thus *no conic meets* M *ten times.*

9. 5. *Type* (11121; 1b); *Types* (11112; k), $k =$ 1a(i)$_*$, 1a(iii), 1a(i)*, *or* 1a(ii); *and Types* (11211; k), $k =$ 1b*, 2, 1b$_*$, *or* 3. In the first case, we assume again that a small neighbourhood of p on N_4 lies inside a conic of σ. We use the method of 8. 6, and give a proof for the first three cases and the last two. The other cases are proved similarly.

Let b, c, d, e be fixed points on N_4. Each of the following results, after the first, can be obtained from the preceding one and Lemma 8.

Lemma I [II; III; IV]. *There exists an $M < M_4$ such that no conic through b, c, d and two points of N [c, d and three points of N; d and four points of N; five points of N] meets N' more than once.*

Proof of I. By Lemma 3, with $i = 1$, choose $N' < N_4'$ such that no conic through b, c, d, e meets $N' \cup p$ more than once. Choose $N < N_4$ such that b, c, $d \notin N$. Let $M = N \cup p \cup N'$. Let $s, t \in N$ and let $\gamma = \gamma(s, t, b, c, d)$ meet N' at s'. Define $\gamma' = \gamma(s', p, b, c, d)$. Since $e \in (\tau; b, c, d)_*$, and $e \notin \gamma'$ for any choice of $s' \in N'$, we have $e \in \gamma_*'$ for all these s'. Now the arc of γ between s' and b which contains s lies in γ_*'. Hence $s \in \gamma_*'$. Since $e \in \gamma_*'$, γ' must meet N_4 four times between s and e. By Lemma 8, γ' does not meet M_4 elsewhere.

We can then use γ' to separate the arc of N' between p and s' from γ as in 8. 6. This implies that γ does not meet N' outside s'.

Proof of II. Let $\gamma = \gamma(s, t, u, c, d)$ meet N' at s'. We separate γ from the arc of N' between p and s', as before by using

$$\gamma' = \begin{cases} \gamma(s', p, u, c, d) \\ \gamma(s', u, c, d, e) \end{cases} \text{ if } \begin{matrix} e \in \overline{\gamma(s', p, u, c, d)_*} \\ e \in \gamma(s', p, u, c, d)^*. \end{matrix}$$

In the first case, $e \in \overline{\gamma_*'}$, apply the method of I to show that γ' meets N_4 four times and N_4' just once.

In the second case, γ' lies between γ and $\gamma(s', p, u, c, d)$, and we obtain $s \in \gamma_*'$, $p \in \gamma'^*$. Hence γ' intersects N between p and s. By Lemma I, γ' does not meet N' outside s'.

Similar discussions yield III and IV. From now on we may assume that *no conic through five points of N [of N'] meets N' [N] more than once.*

It remains to show that *there exists an $M_5 < M$ such that no conic through four points of N_5 or of N_5' meets M_5 more than six times.*

We use the same methods as above, and prove successively the following.

Lemma V [VI; VII]. *There exists an $M_5 < M$ such that no conic through c, $d \in M$ and two points of N_5 [through d and three points of N_5; through four points of N_5] meets N_5' more than twice.*

To prove V, for instance, we separate the conic through the six points t', s', s, t, c, d from the arc of N_5' between s' and t' by using

$$\gamma' = \begin{cases} \gamma(t', s', p, c, d) \\ \gamma(t', s', c, d, e) \end{cases} \text{ if } \begin{matrix} e \in \overline{\gamma(t', s', p, c, d)_*} \\ e \in \gamma(t', s', p, c, d)^*. \end{matrix}$$

Altogether, we may assume that *no conic meets M_5 more than six times.*

9. 6. *Type* (21111; 2) (cusp of the first kind). We use a variation of the method of 9. 5. By using Lemmas 3 and 4, we can construct a neighbourhood $M < M_4$ such that no conic through b, c, d and two points of N meets N' more than once. Then Lemmas II, III, IV of 9. 5 can be proved, in turn.

Proof of II. Let $s, t, u \in N$. Let $\gamma = \gamma(s, t, u, c, d)$ meet N' at s'. Let $\gamma_L = \gamma(s', t, v, c, d)$. Then the arc of γ between s' and t which contains s lies in γ_L^*. Hence $s \in \gamma_L^*$. We can also verify that $p \in \gamma_{L^*}$. Therefore γ_L intersects N between p and s.

Hence γ_L does not meet N' outside s'. By Lemma 8, $\gamma_R = \gamma(p, s', t, c, d)$ does not meet M_4 again. Since

$$\gamma < (\gamma_{L*} \cap \gamma_R^*) \cup (\gamma_L^* \cap \gamma_{R*}) \cup \{s', t, c, d\},$$

while

$$N' < (\gamma_{L*} \cap \gamma_{R*}) \cup (\gamma_L^* \cap \gamma_R^*) \cup s',$$

we obtain that γ does not meet N' outside s'.

Similar discussions yield III and IV. From now on, we may assume that *no conic through five points of* N [of N'] *meets* N' [N] *more than once.*

Let $M_5 \in M$ be sufficiently small. Let e' be an end-point of N'.

Let α be the family of conics through four points $s, t, u, v \in N_5$; $\alpha_1, \alpha_2, \alpha_3$ are the subfamilies of α, as in 4. 9. We shall prove that *no conic* $\gamma \in \alpha$ *meets* N' *more than twice.*

Case 1: $\gamma \in \alpha_1$. If γ meets N' at s' between p and a', then the triangle pss' containing the arc of N' between p and s' lies entirely in $\gamma_* \cup \{p, s, s'\}$. Hence γ meets N' exactly once between p and a'.

We can choose N_5 so small that $\gamma' = \gamma(e', s, t, u, v)$ is so close to $\gamma(\varrho, e') = \mathfrak{L}(p, e') \cup \mathfrak{T}$ that γ' meets M_5 with an odd multiplicity; cf. Lemma 4. Hence γ' meets N_5' once or three times and γ' does not meet N' elsewhere. The branch γ_L' of γ' containing s, t, u, v meets N', and the other branch γ_R' contains e'. By 4. 6, γ_L' meets N' exactly once on N_5' and γ_R' meets N' only at e'.

Similarly, a conic γ which lies between γ' and $\mathfrak{L}(s, t) \cup \mathfrak{L}(u, v)$ in α_1 meets N' exactly twice.

Finally, a conic $\gamma \in \alpha_1$ which meets N' between c' and f' can be dealt with as in the following case.

Case 2: $\gamma \in \alpha_2$. Let γ meet N' at s' and t'. Then we can separate γ from the arc of N' between s' and t' by using $\mathfrak{L}(s', t')$ and $\gamma(p, s', t', u, v)$. Hence γ does not meet N' outside s' and t'.

Case 3: $\gamma \in \alpha_3$. By 4. 9, every conic of α_3 meets N_5' exactly twice.

Altogether, we may assume that *no conic meets* M_5 *more than six times.*

9. 7. *Type* (11112; 3) (inflection point). Using 5. 3, we may assume that a conic through five points of N or of N' meets M_4 with an even multiplicity. We use the affine notation of 4. 6. Let $r, s, t, u, v \in N$. Then $\gamma = \gamma(r, s, t, u, v)$ will meet N_4' at a point which lies between h' and e' on N_4'. Since $\gamma(p, s, t, u, v)$ separates γ from the arc of N' between p and h', γ does not meet this arc. Now $p \in \mathfrak{L}(s', t')^*$ for all $s', t' \in N'$. On the other hand, a secant $\mathfrak{L}$ of the branch of γ in $\mathfrak{L}(s, t)^* \cap \mathfrak{L}(u, v)^*$ through two points of γ close to p will be close to $\mathfrak{T}$, and p will lie in $\mathfrak{L}_*$. Hence $\mathfrak{L} \neq \mathfrak{L}(s', t')$.

From now on, we may assume that *a conic through five points of* N *or of* N' *meets* M *exactly six times.*

It remains to prove that *there is an* $M_5 < M$ *such that no conic through four points of* N_5 *or of* N_5' *meets* M_5 *more than six times.*

Let γ be a conic of α_1 through exactly four points s, t, u, v of N. Let e and e' be end-points of M.

Case 1: $p \in \gamma_*$. Here, γ will meet N_5' between p and a' with an odd multiplicity, and between h' and e' with an odd multiplicity. By using 4. 6, applied to N', we obtain that γ meets N' exactly once between p and a' and exactly once between h' and e'.

Case 2: $p \in \gamma^*$. Here $e \in \gamma^*$. We choose $N_5' < N'$ so small that $N_5' < \gamma(\varrho; e)^*$ and choose N_5 so small that if $s, t, u, v \in N_5$, then $\gamma(e, s, t, u, v)$ is so close to $\gamma(\varrho; e)$ that it does not meet N_5'. Then γ does not meet N_5' either, except possibly between c' and f'.

Let $\gamma \in \alpha_1$, meet N_5' between c' and f' at s' and t'. We may assume that $e \in \mathfrak{L}(s', t')_* \frown \mathfrak{L}(u, v)_*$. Let $\gamma' = \gamma(s', t', p, u, v)$. Then $e \in \gamma'^*$ and $t \in \gamma'_*$. Thus γ' meets N' between e and t with an odd multiplicity, and hence three times. By 8. 6, γ' does not meet N' outside s' and t'. Then we can use $\mathfrak{L}(s', t')$ and γ' to separate γ from the arc of N' between s' and t'. Hence no conic of α_1 meets N_5' more than twice.

Symmetrically, we can assume that no conic of the family α_1' through four points of N_5' meets N_5 more than twice.

A conic of α_2 through four points of N_5' would belong to α_1' with respect to these points and this case has already been excluded.

By 4. 9, no conic of α_3 meets N_5' more than twice.

Altogether, we can assume that *no conic meets* M_5 *more than six times.*

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