Werk

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De curvis catenariis sphaericis dissertatio analytico-geometrica.

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(Auct. Dr. *Christoph. Gudermann*, Math. Prof. ord. Monast. Guestph.)

17.

De nexu inter abscissas et areas curvarum homologe coniugatarum catenariarum.

Si formulas (2) et (3) articuli praecedentis aut additione aut subtractione coniungimus eas, in quibus integralia continent parametros, quorum alter est alterius complementum, et si insuper arcus AM et CN homologe coniugatos esse censes, quo $v_1 = v$ fit, oriuntur

$$\tfrac{1}{2}(x-x_1) + \tfrac{1}{2}(f+f_1) = \frac{h(a+c)v}{l(1+c)(1-a)} + {}'S(v, q) + {}'S(v, L-q),$$

$$\tfrac{1}{2}(f+f_1) - \tfrac{1}{2}(x-x_1) = \frac{h(a+c)v}{l(1-c)(1+a)} + {}'S(v, r) + {}'S(v, L'-r),$$

quae praeceptis theoriae functionum modularium adhibitis contrahuntur ad

$$\operatorname{tang}\left(\tfrac{1}{2}(f+f_1) + \tfrac{1}{2}(x-x_1)\right) = \frac{h(a+c)}{l(1+c)(1-a)} \cdot \frac{tn\, v}{dn v} = \frac{h}{(1+c)(1-a)} \cdot \operatorname{tang} s,$$

$$\operatorname{tang}\left(\tfrac{1}{2}(f+f_1) - \tfrac{1}{2}(x-x_1)\right) = \frac{h(a+c)}{l(1-c)(1+a)} \cdot \frac{tn\, v}{dn v} = \frac{h}{(1-c)(1+a)} \cdot \operatorname{tang} s,$$

sive adhibitio $h = \sqrt{((1-a)(1-b)(1+c)(1+d))} = \sqrt{((1+a)(1+b)(1-c)(1-d))}$, ad

$$\operatorname{tang}\left(\tfrac{1}{2}(f+f_1) + \tfrac{1}{2}(x-x_1)\right) = \sqrt{\frac{(1-b)(1+d)}{(1-a)(1+c)}} \cdot \operatorname{tang} s,$$

$$\operatorname{tang}\left(\tfrac{1}{2}(f+f_1) - \tfrac{1}{2}(x-x_1)\right) = \sqrt{\frac{(1+b)(1-d)}{(1+a)(1-c)}} \cdot \operatorname{tang} s.$$

Si utimur iisdem formulis, quibus in artic. 10, et insuper triangulum characteristicum 2Δ adhibemus, determinatum formula $\operatorname{tang}\Delta = \operatorname{tang}\alpha \cdot \operatorname{tang} s$ articuli 2, formulae illae abeunt in

$$1. \quad \begin{cases} \operatorname{tang}\left(\tfrac{1}{2}(f+f_1) + \tfrac{1}{2}(x-x_1)\right) = \dfrac{\operatorname{tang}\alpha \cdot \operatorname{tang} s}{\operatorname{tang}\left(\frac{1}{4}\pi - g\right)} = \dfrac{\operatorname{tang}\Delta}{\operatorname{tang}\left(\frac{1}{4}\pi - g\right)}, \\ \operatorname{tang}\left(\tfrac{1}{2}(f+f_1) - \tfrac{1}{2}(x-x_1)\right) = \dfrac{\operatorname{tang}\alpha \cdot \operatorname{tang} s}{\operatorname{tang}\left(\frac{1}{4}\pi + g\right)} = \dfrac{\operatorname{tang}\Delta}{\operatorname{tang}\left(\frac{1}{4}\pi + g\right)}. \end{cases}$$

Quare si duo triangula sphaerica rectangula construuntur, quorum unum cathetos habeat $f+f_1+(x-x_1)$ et $\frac{1}{2}\pi-2g$, alterum vero cathetos $f+f_1-(x-x'_1)$ et $\frac{1}{2}(\pi+2g)$, duo triangula cum triangulo characteristico 2Δ ad arcum parabolicae catenariae $s=\frac{1}{2}(AM+CN)$, cuius parameter $\alpha=\frac{1}{2}(A+C)$, pertinente eiusdem erunt magnitudinis arealis.

E formulis illis derivantur facillime formulae novae

$$2.\quad \begin{cases} \operatorname{tang}(f+f_1) = \dfrac{\sin 2\alpha . \sin 2s}{\cos 2g(\cos 2\alpha+\cos 2s)} = \dfrac{\operatorname{tang} 2\Delta}{\cos 2g}, \\ \operatorname{tang}(x-x') = \dfrac{\operatorname{tang} 2g . \sin 2\alpha \sin 2s}{1+\cos 2\alpha \cos 2s} = \operatorname{tang} 2g . \sin 2\Delta. \end{cases}$$

E formulis his perspicis, semisummam arearum $\frac{1}{2}(f+f_1)$, quae ad arcus homologe coniugatos AM et CN pertinent, maiorem esse area parabolicae catenariae, cuius arcui s semisumma arcuum AM et CN aequalis est, et abscissam curvae ellipticae arcui AM subtensam longiorem esse abscissa, quae arcui hyperbolico CN subtenditur coniugato, dummodo puncta arcuum extrema M et N non sunt vertices, qui vertices A et C proxime sequantur. Etenim si puncta M et N coïncidunt cum verticibus B et D, vel quod idem est, si sumitur arcus parabolicus $s=\frac{1}{2}\pi$, est $\frac{1}{2}(f+f_1)=\Delta=\frac{1}{2}\pi$, et abscissae x et x_1 eo extensae, ut ad arcus AB et CD pertineant, aequales sunt.

Formulae (2) etiam edocent, nexum inter quatuor quantitatis $2g$, 2Δ, $x-x_1$ et $f+f_1$ exhiberi geometrice adiumento trianguli rectanguli sphaerici; quod si ita construatur, ut cathetus 2Δ cum hypotenusa $f+f_1$ faciat angulum constantem $2g$, cathetus alter angulo $2g$ oppositus erit abscissarum differentia $x-x_1$. Quo in triangulo si catheto 2Δ oppositus angulus 2η appellatur, formulis (2) addendae sunt sequentes octo:

$$3.\quad \begin{cases} \cos(f+f_1)=\cos 2\Delta . \cos(x-x_1), & \operatorname{tang}(x-x')=\operatorname{tang}(f-f_1).\cos 2\eta, \\ \sin(x-x_1)=\sin(f+f_1).\sin 2g, & \operatorname{tang} 2\Delta=\operatorname{tang} 2\eta.\sin(x-x_1), \\ \cos(f+f_1)=\cot 2g.\cot 2\eta, & \cos 2g=\cos(x-x_1).\sin 2\eta, \\ \sin 2\Delta=\sin(f+f_1).\sin 2\eta, & \cos 2\eta=\cos 2\Delta.\sin 2g, \end{cases}$$

ita ut formulae decem nexum inter quantitates quinque $2g$, 2Δ, $x-x_1$, $f-f_1$, 2η omnimode exprimant.

18.

De chorda MN *punctorum homologe coniugatorum, et de quadrigono* AMNB.

E formulis (2) articuli praecedentis facillime derivantur quatuor hae:

$$1.\quad \begin{cases} \sin(f+f_1)=\dfrac{\sin 2\alpha \sin 2s}{2\cos^2\mu \sin 2\eta}; & \cos(f+f_1)=\dfrac{\cos 2g(\cos 2\alpha+\cos 2s)}{2\cos^2\mu \sin 2\eta}, \\ \sin(x-x_1)=\dfrac{\sin 2g \sin 2\alpha \sin 2s}{2\cos^2\mu \sin 2\eta}; & \cos(x-x_1)=\dfrac{\cos 2g}{\sin 2\eta} \text{ (ut antea),} \end{cases}$$

dummodo non negligis, esse

$$\begin{aligned} &\sqrt{(\sin^2 2\alpha \sin^2 2s + \cos^2 2g(\cos 2\alpha + \cos 2s)^2)} \\ &= \sqrt{(\sin^2 2g \sin^2 2\alpha \sin^2 2s + \cos^2 2g(1 + \cos 2\alpha \cos 2s)^2)} \\ &= \sqrt{((1 + \cos 2\alpha \cos 2s)^2 - \sin^2 2g(\cos 2\alpha + \cos 2s)^2)} \\ &= (1 + \cos 2\alpha \cos 2s)\sqrt{(1 - \sin^2 2g . \cos^2 2\Delta)} \\ &= (1 + \cos 2\mu) . \sin 2\eta = 2\cos^2\mu . \sin 2\eta, \end{aligned}$$

si 2μ est hypotenusa trianguli characteristici supra recepti 2Δ, et cuius catheti sunt 2α et $2s$.

Iisdem signis utendo formulas (7, 8, 9) transformamus, ut migrent in

$$2.\quad \cos y . \cos y_1 = \frac{\cos^2\mu}{\cos^2 g} . \sin 2\eta;$$

$$3.\quad 1 - \sin y . y_1 = \frac{\cos^2\mu}{\cos^2 g}; \quad \sin y - \sin y_1 = \frac{\cos^2\mu}{\cos^2 g} \cos 2\eta;$$

$$4.\quad \cos\tfrac{1}{2}(y+y_1) = \frac{\cos\mu}{\cos g} . \sin(\tfrac{1}{4}\pi+\eta);$$

$$5.\quad \sin\tfrac{1}{2}(y-y_1) = \frac{\cos\mu}{\cos g} . \sin(\tfrac{1}{4}\pi-\eta).$$

E formulis (3) divisione oritur

$$\frac{\sin y - \sin y_1}{1 - \sin y \sin y_1} = \cos 2\eta = \sin(\tfrac{1}{2}\pi - 2\eta),$$

sive $\dfrac{\mathfrak{Tang}\,\mathfrak{L}y - \mathfrak{Tang}\,\mathfrak{L}y_1}{1 - \mathfrak{Tang}\,\mathfrak{L}y . \mathfrak{Tang}\,\mathfrak{L}y_1} = \mathfrak{Tang}(\mathfrak{L}y - \mathfrak{L}y_1) = \mathfrak{Tang}\,\mathfrak{L}(\tfrac{1}{2}\pi - 2\eta)$, ideoque est

$$6.\quad \mathfrak{L}y - \mathfrak{L}y_1 = \mathfrak{L}(\tfrac{1}{2}\pi - 2\eta).$$

Si arcubus circulorum maximorum coniungimus tum vertices A et C, tum puncta homologe coniugata M et N, oritur quadrigonum $ACNM$, cuius latus $AC = A + C = 2\alpha$, cuius latera AM et CN sunt arcus coniugati duarum curvarum, quorum semisumma $= s$ et cuius latus quartum MN est chorda inter puncta homologe coniugata, cuius longitudinem nunc eruamus. Est vero MN latus trigoni MON, cuius duo reliqua latera sunt $OM = \frac{1}{2}\pi - y$ et $ON = \frac{1}{2}\pi + y_1$, et angulum $MON = x - x_1$ faciunt. Quia vero cos MN = cos OM cos ON + sin OM sin ON cos MON, sive

$$\cos MN = -\sin y \sin y_1 + \cos y \cos y_1 \cos(x - x_1),$$

primo componimus $\cos y \cos y_1 . \cos(x - x_1) = \frac{\cos^2\mu}{\cos^2 g} \cos 2g$, quare $\cos MN = \frac{\cos^2\mu}{\cos^2 g} + \frac{\cos^2\mu}{\cos^2 g} \cos 2g - 1 = 2\cos^2\mu - 1 = \cos 2\mu$, ideoque

$$MN = 2\mu.$$

Chorda igitur MN *iungens puncta homologe coniugata* M *et* N, *aequalis est hypotenusae trianguli characteristici* 2Δ *pertinentis ad arcum parabolicum* s.

Chorda, cuius longitudinem modo determinavimus, cum applicatis punctorum M et N homologe coniugatorum angulos facit acutos M et N, et trigonum MON, cuius latus $MN = 2\mu$ esse invenimus, continet angulum N ipsum, loco anguli M vero ipsius supplementum. Quare formulae trigonometriae sphaericae praebent relationes

$$\sin \tfrac{1}{2}(M-N) \cos \mu = \sin \tfrac{1}{2}(y-y_1) . \sin \tfrac{1}{2}(x-x_1),$$
$$\cos \tfrac{1}{2}(M-N) \cos \mu = \cos \tfrac{1}{2}(y-y_1) . \cos \tfrac{1}{2}(x-x_1),$$

quae formulis (4) et (5) mutantur in

$$\sin \tfrac{1}{2}(M-N) . \cos g = \sin (\tfrac{1}{4}\pi - \eta) . \sin \tfrac{1}{2}(x-x_1),$$
$$\cos \tfrac{1}{2}(M-N) . \cos g = \cos (\tfrac{1}{4}\pi - \eta) . \cos \tfrac{1}{2}(x-x_1).$$

Si vero easdem trigonometriae sphaericae formulas applicas ad triangulum rectangulum in fine articuli praecedentis inventum, invenis

$$\sin (\tfrac{1}{4}\pi - \eta) . \sin \tfrac{1}{2}(x-x_1) = \sin (\tfrac{1}{2}(f+f_1) - \Delta) . \cos g,$$
$$\cos (\tfrac{1}{4}\pi - \eta) . \cos \tfrac{1}{2}(x-x_1) = \cos (\tfrac{1}{2}(f+f_1) - \Delta) . \cos g;$$

quare prodeunt formulae

$$\cos \tfrac{1}{2}(M-N) = \cos (\tfrac{1}{2}(f+f_1) - \Delta), \text{ et } \sin \tfrac{1}{2}(M-N) = \sin (\tfrac{1}{2}(f+f_1) - \Delta),$$

quae sine ullo dubio monstrant, esse

$$M - N = f + f_1 - 2\Delta.$$

Chorda MN lineam abscissarum secans in V, duo facit triangula rectangula VPM et VQN, si memineris esse y applicatam PM et y_1 applicatam QN, in quibus, si angulus $PVM = QVN = V$ est,

$$\text{area } \mathfrak{M} \text{ trianguli } VPM = V + M - \tfrac{1}{2}\pi, \text{ et}$$
$$\text{area } \mathfrak{N} \text{ trianguli } VQN = V + N - \tfrac{1}{2}\pi,$$
$$\text{ideoque } \mathfrak{M} - \mathfrak{N} = M - N = f + f_1 - 2\Delta.$$

Quia vero ex intuitu ipso patet, esse aream quadrigoni $AMNC = f + f_1 - \mathfrak{M} + \mathfrak{N}$, *invenitur area quadrigoni* AMNC $= 2\Delta$, *sive duplo areae parabolicae, dummodo arcus parabolicus* s *est semisumma arcuum* AM *et* CN, *et curvae parabolicae parameter* α *est semisumma applicatarum* A *et* C, *quae ad vertices* A *et* C *pertinent.*

Si praeter quadrigonum $AMNC$ construitur quadrigonum $amnc$, cuius latera $ac=AC$ et $mn=MN$ sint arcus circulorum maximorum, cuius vero latera am et cn sint arcus aequales parabolici et insuper symmetrice aequales, nec non convexi versus quadrigoni aream, quadrigonum $AMNC$ aequalis erit quadrigono $amnc$, dummodo $am=cn=\frac{1}{2}(AM+CN)$ et anguli a et c sunt recti. Si angulus $m=n=\Theta$ esse ponitur, angulus $NMA=\Theta$ et angulus $MNC=\Theta_1$, est angulus $\lambda=\Theta+M$ et angulus $\lambda_1=\Theta_1-N$, quare $\lambda+\lambda_1=\Theta+\Theta_1+M-N$, ideoque $\lambda+\lambda_1=\Theta+\Theta_1+f+f_1-2\Delta$, et quia $\Delta=\Theta+\sigma-\frac{1}{2}\pi$, sive $2\Delta=2\Theta+2\sigma-\pi$, formula $f+f_1=2\sigma+\lambda+\lambda_1-\pi$ in artic. **15.** inventa migrat in simpliciorem hanc:

$$\theta=\tfrac{1}{2}(\Theta+\Theta_1).$$

Sed paullo post etiam demonstrabitur, esse $\theta=\Theta=\Theta_1$, ideoque quadrigonis duobus $AMNC$ aequalem esse nonnisi perimetrum et aream, sed aequales etiam esse angulos tunc patefiet, quae mira sane est curvarum catenariarum sphaericarum proprietas, et mirum in modum obstrusa.

19.

De nexu inter abscissas et areas curvarum reciprocarum.

Pro abscissarum x' et x'_1, initio punctum sumimus, quod abscissarum x et x_1 initio est e diametro oppositum, quum linea normalis, quae per vertices A et C transit, etiam vertices A' et C' transit, et lineam abscissarum in novo abscissarum initio secat. Quare est

$$x'=\int_{a'}\frac{-(h+ez')\partial z'}{(1-z'^2)\sqrt{Z'}} \quad \text{et} \quad f'=\int_{a'}\frac{-(h+ez')z'\partial z'}{(1-z'^2)\sqrt{Z'}},$$

ideoque

$$x'+f'=\int_{a'}\frac{-(h+ez')\partial z'}{(1-z')\sqrt{Z'}}=-\frac{2ev}{l}+\frac{2(h+e)}{l}\cdot\int_0\frac{\partial v}{1-z'},$$

$$x'-f'=\int_{a'}\frac{-(h+ez')\partial z'}{(1+z')\sqrt{Z'}}=\frac{2ev}{l}+\frac{2(h-e)}{l}\cdot\int_0\frac{\partial v}{1+z'},$$

quibus in formulis argumentum v determinatur iis (6) artic. **11.** Si substituimus

$$z'=\frac{(b'+c')a'-(a'-b')c'\,sn^2v}{b'+c'+(a'-b')sn^2v},$$

invenimus formulas

$$\text{1.}\quad\begin{cases}x'+f'=\dfrac{2(h+ea').v}{l(1-a')}-2.S(v,q')=\text{sectori } A'O'M',\\ x'-f'=\dfrac{-(h+ea').v}{l(1+a)}+2.S(v,r')=\text{sectori } A'OM',\end{cases}$$

si parametri q' et r' determinantur formulis

$$2.\quad \begin{cases} sn'q' = \sqrt{\frac{(1+c')(a'+d')}{(1+d')(a'+c')}}, & sn'r' = \sqrt{\frac{(1-c')(a'+d')}{(1-d')(a'+c')}}, \\ cn'q' = \sqrt{\frac{(1-a')(c'-d')}{(1+d')(a'+c')}}, & cn'r' = \sqrt{\frac{(1+a')(c'+d')}{(1-d')(a'+c')}}, \\ dn'q' = \sqrt{\frac{(1-b')(c'-d')}{(1+d')(b'+c')}}, & dn'r' = \sqrt{\frac{(1+b')(c'-d')}{(1-d')(b'+c')}}, \\ snc'q' = \sqrt{\frac{(1-a')(b'+c')}{(1-b')(a'+c')}}, & snc'r' = \sqrt{\frac{(1+a')(b'+c')}{(1+b')(a'+c')}}, \\ cnc'q' = \sqrt{\frac{(1+c')(a'-b')}{(1-b')(a'+c')}}, & cnc'r' = \sqrt{\frac{(1-c')(a'-b')}{(1+b')(a'+c')}}, \\ dnc'q' = \sqrt{\frac{(1+d')(a'-b')}{(1-b')(a'+d')}}, & dnc'r' = \sqrt{\frac{(1-d')(a'-b')}{(1+b')(a'+d')}}. \end{cases} \quad \text{et}$$

E formulis antecedentibus, quae spectant ad curvam reciprocitatis lege coniunctam cum catenaria elliptica illico fiunt eae, quae spectant ad abscissam x'_1 et aream f'_1 curvae cum catenaria hyperbolica reciprocitatis lege coniunctae; dummodo permutamus x' et x'_1, f' et f'_1, a' et c', b' et d', v' et v_1, e et $-e$, quo, quia q' in $L'-r'$ et r' in $L'-q'$ abit, adipiscimur formulas

$$3.\quad \begin{cases} x'_1 + f'_1 = \frac{2(h-e.c')}{l(1-c')} - 2.S(v_1, L'-r') = \text{sectori } C'ON', \\ x'_1 - f'_1 = \frac{2(h-e.c')}{l(1+c')} + 2.S(v_1, L'-q') = \text{sectori } C'O'N', \end{cases}$$

in quibus argumentum v_1 ex applicata $y'_1 = \arcsin(z'_1)$ computandum est per formulas (**3.** artic. **12**).

Facillime hinc derivabis formulas, quibus exprimuntur areae sectorum $B'OM'$, $B'O'M'$, $C'ON'$, $C'O'N$; quas vero brevitatis causa omittimus.

20.

De nexu inter areas et abscissas, quae ad arcus homologe coniugatos A'M' *et* C'N' *reciprocos pertinent.*

Ponamus iterum, puncta M et N duarum curvarum AM et CN esse homologe coniugata; erunt etiam puncta nunc M' et N' homologe coniugata, ideoque et arcus $A'M'$ et $C'N'$ ipsi, i. e. erit argumentum $v_1 = v$. Quare e formulis artic. praeced. nunc aut addendo aut subtrahendo componimus:

$$\tfrac{1}{2}(f'+f'_1) - \tfrac{1}{2}(x'_1 - x') = \frac{(h-e)(a'+c').v}{l(1-a')(1+c')} - S(v, q') - S(v, L'-q'),$$

$$\tfrac{1}{2}(f'+f'_1) + \tfrac{1}{2}(x'_1 - x') = \frac{(h-e)(a'+c')v}{l(1+a')(1-c')} - S(v, r') - S(v, L-r'),$$

quas facillime reduces ad simpliciores

$$1.\quad \begin{cases} \operatorname{tang}\left(\frac{1}{2}(f'+f'_1)-\frac{1}{2}(x'_1-x')\right)=\dfrac{\operatorname{tang}\sigma}{\operatorname{tang}(\alpha+g)}, \\ \operatorname{tang}\left(\frac{1}{2}(f'+f'_1)+\frac{1}{2}(x'_1-x')\right)=\dfrac{\operatorname{tang}\sigma}{\operatorname{tang}(\alpha-g)} \end{cases}$$

unde perspicis, si arcus reciproci $A'M'$ et $C'N'$ non extendantur usque ad vertices proxime sequentes B' et D', fore abscissam $x'_1 > x'$, i. e. eam, quae ad ellipticam curvam reciprocam pertinet, superari ab ea, quae ad hyperbolicam reciprocam; et insuper esse $\frac{1}{2}(f'+f'_1) < \frac{1}{2}\pi$. Si vero puncta M' et N' collocamus in B' et D', vel quod idem, si $\sigma=\frac{1}{2}\pi$, est $x'_1=x'$ et simul $f'+f'_1=\pi$. E formulis modo erutis componimus duas:

$$2.\quad \begin{cases} \operatorname{tang}(f'+f'_1)=\dfrac{\sin 2\alpha \sin 2\sigma}{\cos 2g\cos 2\sigma-\cos 2\alpha}, \\ \operatorname{tang}(x'_1-x')=\dfrac{\sin 2g.\sin 2\sigma}{\cos 2g-\cos 2\alpha\cos 2\sigma}, \end{cases}$$

e quibus reductionibus facillimis invenies

$$3.\quad \begin{cases} \cos(x'_1-x')=\dfrac{\cos 2g-\cos 2\alpha\cos 2\sigma}{\sqrt{((1-\cos(2\alpha+2g)\cos 2\sigma)(1-\cos(2\alpha-2g)\cos 2\sigma))}}, \\ \sin(x'_1-x')=\dfrac{\sin 2g\sin 2\sigma}{\sqrt{((1-\cos(2\alpha+2g)\cos 2\sigma)(1-\cos(2\alpha-2g)\cos 2\sigma))}}, \\ \cos(f'+f'_1)=\dfrac{\cos 2g\cos 2\sigma-\cos 2\alpha}{\sqrt{((1-\cos(2\alpha+2g)\cos 2\sigma)(1-\cos(2\alpha-2g)\cos 2\sigma))}}, \\ \sin(f'+f'_1)=\dfrac{\sin 2\alpha.\sin 2\sigma}{\sqrt{((1-\cos(2\alpha+2g)\cos 2\sigma)(1-\cos(2\alpha-2g)\cos 2\sigma))}}; \end{cases}$$

quae formulae relationes complures suppeditant, quarum praecipuae sic exhibentur:

$$4.\quad \begin{cases} \sin 2g.\sin(f'+f'_1)=\sin 2\alpha.\sin(x'_1-x'), \\ \cos(f'+f')=\cos 2\sigma\cos(x'_1-x')-\dfrac{\cos 2\alpha}{\sin 2g}\sin 2\sigma\sin(x'_1-x'), \\ \cos(x'_1-x')=\cos 2\sigma\cos(f'_1+f')+\dfrac{\cos 2g}{\sin 2\alpha}\sin 2\sigma\sin(f'_1+f'), \\ \sin 2\alpha\sin 2\sigma\cos(x'_1-x')=(\cos 2g-\cos 2\alpha\cos 2\sigma)\sin(f'_1+f'), \\ \sin 2g\sin 2\sigma\cos(f'_1+f')=(\cos 2g\cos 2\sigma-\cos 2\alpha)\sin(x'_1-x'), \\ (\cos 2g-\cos 2\alpha\cos 2\sigma)\cos(f'_1+f')=\cos 2g\cos 2\sigma-\cos 2\alpha)\cos(x'_1-x'). \end{cases}$$

Si construitur trigonum, cuius latera 2σ, x'_1-x' et f'_1+f' variabilia esse scimus, anguli duo constantes inerunt, scilicet angulus lateri x'_1-x' oppositus erit $=$ arc cos $\left(\frac{\cos 2g}{\cos 2\alpha}\right)$, et angulus lateri f'_1+f' oppositus

$= \pi - \text{arc cos}\left(\frac{\cos 2\alpha}{\sin 2g}\right)$. Angulus vero variabilis tertius, lateri 2σ oppositus, erit $= \text{arc cos}\left(\frac{\sin 2\alpha \sin 2g \cos 2\sigma - \cos 2\alpha \cos 2g}{\sin^2 2g - \cos^2 2\alpha}\right)$.

21.

Interpretatio geometrica rationis, quae intercedit inter arcum catenariae ellipticae AM *et arcum homologe coniugatum* CN *catenariae hyperbolicae.*

Coniungat chorda $M'N'$ puncta coniugata duorum arcuum $A'M'$ et $C'N'$, quo oritur triangulum $M'ON'$, cuius latus $OM' = \frac{1}{2}\pi - y$, latus $ON' = \frac{1}{2}\pi + y'$ et angulus $M'ON' = x'_1 - x'$; quare est

$$\cos M'N' = -\sin y' \sin y'_1 + \cos y' \cos y'_1 \cos(x'_1 - x');$$

at formula (4 artic. 15) adhibita praebet

$$\cos y' . \cos y'_1 . \cos(x'_1 - x') = \frac{\cos 2g - \cos 2\alpha \cos 2\sigma}{1 + \cos 2g}, \quad \text{et quia}$$

$$\sin y' . \sin y'_1 = \frac{\cos 2g (1 + \cos 2\alpha \cos 2\sigma)}{1 + \cos 2g},$$

illico prodit $\cos M'N' = -\cos 2\alpha . \cos 2\sigma = \cos 2\alpha' . \cos 2s$, dummodo $\alpha' = \frac{1}{2}\pi - \alpha$ ponimus. Quia vero ad arcum parabolicum s, cuius parameter α, pertinet arcus reciprocus itidem parabolicus σ, cuius parameter est α': vides $\cos 2\alpha' . \cos 2\sigma$ esse cosinum trianguli characteristici $2\Delta'$, cuius catheti sunt 2α et 2σ, quod pertineat ad arcum parabolicum reciprocum σ; unde perspicis, *chordam* M'N', *aequalem esse hypontenusae huius trianguli* $2\Delta'$, *sive aequalem esse duplici applicatae, quae ad punctum extremum arcus* σ *reciproci parabolici pertineat.*

Quia M est centrum sphaericum circuli maximi tangentis curvam catenariam ellipticam in M, et quia pariter N' est centrum circuli maximi curvam catenariam hyperbolicam in N tangentis, vice versa intersectio horum circulorum tangentium, quae sit T, erit centrum cordae $M'N'$ sphaericum, et angulus MTN, quem duo circuli tangentes secum faciunt, erit supplementum chordae $M'N'$ ipsius. Si angulus iste $= 2v$ ponitur, est $M'N' + 2v = \pi$, ideoque $\cos 2v = -\cos M'N'$, sive

$$\cos 2v = -\cos 2\alpha . \cos 2\sigma,$$

quae formula similitudinem quandam habet cum formula $\cos 2\mu = \cos 2\alpha . \cos 2s$, Quia vero inter arcus rericoprocos σ et s parabolicos intercedit relatio

$$(1 + \cos 2\alpha \cos 2\sigma)(1 - \cos 2\alpha \cos 2s) = \sin^2 2\alpha, \quad \text{est}$$
$$(1 + \cos 2\nu)(1 - \cos 2\mu) = 4\sin^2\alpha \cos^2\alpha, \quad \text{sive}$$
$$\cos\nu . \sin\mu = \sin\alpha \cos\alpha.$$

Triangulum MTN praeter angulum 2ν continet angulos Θ et Θ_1 (artic. 18), et cum e formulis trigonometriae sphaericae nota est relatio

$$\frac{\sin\frac{1}{2}(\Theta+\Theta_1)}{\cos\nu} = \frac{\cos\frac{1}{2}(MT-NT)}{\cos\mu}, \quad \text{quia} \quad \sin\tfrac{1}{2}(\Theta+\Theta_1) = \sin\theta = \frac{\sin 2\alpha}{\sin 2\mu},$$
$$\text{invenitur} \quad \cos\tfrac{1}{2}(MT-NT) = \frac{\sin 2\alpha}{\sin 2\mu}\cdot\frac{\cos\mu}{\cos\nu} = 1, \quad \text{sive}$$
$$MT = NT; \quad \text{ideoque} \quad \Theta = \Theta_1 = \theta.$$

Chorda igitur MN *iungens duo puncta homologe coniugata* M *et* N *in catenaria elliptica et hyperbolica est ea, quae cum duabus curvis angulos internos* Θ *et* Θ_1 *faciat aequales*; quae est interpretatio geometrica rationis, quae locum habet inter puncta homologe coniugata et arcus homologe coniugatos.

Nunc primum patet, quadrigonis $amnc$ et $AMNC$ (art. 18.) aequales esse angulos $a = A = \frac{1}{2}\pi$, $c = C = \frac{1}{2}\pi$, $m = M$ (sive $\theta = \Theta$) et $n = N$ (sive $\theta = \Theta_1$), quorum vero latera sunt ea, ut sit $ac = AC = 2\alpha$, $mn = MN = 2\mu$ et $am = cn = \frac{1}{2}(AM+CN)$.

Nunc etiam perspicies, in quadrigono $AMNC$, cuius latera $A'M'$ et $C'N'$ reciprocitatis lege coniuncta sunt cum arcubus AM et CN duarum catenariarum homologe coniugatis, pariter esse angulum $A' = C' = \frac{1}{2}\pi$ et angulum $M' = N' = \theta' = \pi - MT$.

Aream huius quadrigoni e theoremate de quadraturis sphaericis generalissime facile est computare. Si omnia ipsius latera essent arcus circulorum maximorum, area foret $2.\frac{1}{2}\pi + 2\theta' - 2\pi = 2\theta' - \pi$; quia vero latera $A'M'$ et $C'N'$ sunt arcus curvarum versus quadrigoni aream convexi, addendi sunt arcus reciproci AM et CN; quare ipsius area est $2\theta' - \pi + AM + CN = 2\theta' + 2s - \pi$. Quia vero area catenariae parabolicae reciprocae, quae ad ipsius arcum σ pertinet, est $= \Delta' = \theta' + s - \frac{1}{2}\pi$, ideoque $2\Delta' = 2\theta' + 2s - \pi$, *aream quadrigoni* $A'M'N'C' = 2\Delta'$ esse, *vel areae trianguli characteristici, ad catenariam parabolicam reciprocam pertinenti aequalem esse patet.*

Quia anguli M' et N' in quadrigono $A'M'N'C'$ sunt iidem, quos circuli maximi, qui arcus $A'M'$ et $C'N'$ tangunt in punctis M' et N', faciunt cum chorda $M'N'$ punctorum coniugatorum, circuli hi in puncto quodam T, quod

est centrum chordae MN, ite se intersecabunt, ut fiat triangulum isoscele $M'T'N'$, sive $M'T' = N'T' = \pi - 0$ sit.

Si per puncta M et N homologe coniugata, ducimus curvarum AM et CN lineas normales, etiam hae ob aequales tangentes $MT = NT$ in puncto qnodam U ita se intersecabunt, ut sit $UM = UN$, et ut circulus maximus per punta T' et U ductus, normaliter in partes aequales dividat chordam punctorum M et N coniugatorum. Quare U est radius circuli minoris, qui curvas catenarias AMB et CND simul tangit in punctis coniugatis M et N. Si r est radius huius circuli minoris, facillime invenies:

$$\text{tang } r = \frac{1-\cos 2\alpha \cos 2s}{\sin 2\alpha} = \frac{\sin 2\alpha}{1+\cos 2\alpha \cos 2\sigma} = \frac{\sin^2 \mu}{\sin \alpha \cos \alpha} = \frac{\sin \alpha \cos \alpha}{\cos^2 \nu};$$

quae formula edocet radium hunc r aequalem esse radio curvaturae sive lineae normali, pertinenti ad punctum arcus curvae parabolicae extremum, cuius longitudo $= s$, et parameter $= \alpha$ (confer. artic. 3, ubi $y = \mu$). Idem radius aequalis est normali, quae ad curvam parabolicam pertinet.

Quare si circulus minor, cuius radius aequalis est radio osculi sive lineae normali, quae ad arcum parabolicum s quendam pertinet, ita applicatur inter duas curvas catenarias AMB et CND, ut has curvas simul tangat, puncta contactus M et N erunt homologe coniugata et simul ita posita, ut sit $\frac{1}{2}(AM+CN) = s$. Methodus haec, inveniendi puncta coniugata inter eas, sane referenda est problematum solutiones, quae dicuntur mechanicae, quae vero adhibendae sunt, si alias et quidem geometricas frustra quaerimus. Notio saltem huius circuli est adhibenda, qui simul has curvas tangat, quia ratio quae intercedit inter puncta homologe coniugata, clarissime explicatur, si dicimus, *puncta homologe coniugata esse puncta contactus duarum curvarum cum eodem circulo minore.*

Datis duobus punctis homologe coniugatis M et N, facillime ducuntur lineae, quae curvas catenarias in his punctis tangant. Si linea tangens est ducenda per punctum M, ex altero puncto N describatur circulus, cuius radius est 2α, a puncto M ad circulum duci possunt duae lineae tangentes sphaerico-rectae, quorum una (quam facillime distingues ab altera inutili) quae circulum tangat in R, etiam curvam AM in M tanget. Pari modo invenitur circulus maximus, qui alteram curvam CN tangat in N. Si enim ducimus radium NR, oritur trigonum rectangulum NRM, cuius cathetus $NR = 2\alpha$, et hypotenusa 2μ est, quare triangulum hoc erit characteristicum 2Δ, cathetus alter $MR = 2s = AM + CN$, area aequalis erit areae quadrigoni

$AMNC$, et angulus $NMR = 0 = \Theta$, ideoque MR curvam AM in M tangit. Inventis duabus lineis tangentibus MT et NT, addo, angulum RNT esse complementum arcus 2σ parabolici reciproci. Quia enim $2\Delta = 2\Theta + 2\sigma - \pi$ et triangulum characteristicum $2\Delta = \frac{1}{2}\pi + NMR + MNR - \pi = NMR + MNT - RNT - \frac{1}{2}\pi = 2\Theta - RNT - \frac{1}{2}\pi$, est $2\Theta + 2\sigma - \pi = 2\Theta - RNT - \frac{1}{2}\pi$, sive

angulus $RNT = \frac{1}{2}\pi - 2\sigma$.

Vides ergo, angulum $RNT = 0$ esse, si $\sigma = \frac{1}{4}\pi$, eumque negativum fieri, si $\sigma > \frac{1}{4}\pi$ sumatur. Pari modo duci circulos maximos, qui curvas reciprocas $A'M'$ et $C'N'$ in M' et N' tangant, vix monendum est.

23.

Reductiones formularum, quibus abscissae punctorum homologe coniugatorum exprimuntur.

Ad absolvendas reductiones propositas, omnes quantitates constantes, praesertim eae, quae ad octo vertices quatuor curvarum pertinent, exprimendae sunt ita, ut nonnisi a duabus quantitatibus pro arbitrio sumendis pendeant. Quantitates binae constantes sunt α et tang g. Ponamus brevitatis causa

$$\varrho = \operatorname{tang} g = \sqrt{(1 - \sin 2\beta \sin 2\gamma)}.$$

Quia

$$1.\quad \begin{cases} \cos(\gamma+\beta) = \varrho\cos\alpha, & \text{sit } \varepsilon = \sin(\gamma+\beta) = \sqrt{(1-\varrho^2\cos^2\alpha)}, \\ \sin(\gamma-\beta) = \varrho\sin\alpha, & \text{sit } \varepsilon' = \cos(\gamma-\beta) = \sqrt{(1-\varrho^2\sin^2\alpha)}, \end{cases} \quad \text{et quia}$$

erunt

$$2.\quad e = \varrho\cos 2\alpha, \qquad h = (1-\varrho^2)\sin\alpha\cos\alpha, \qquad \operatorname{tang} 2g = \frac{e}{h}\operatorname{tang} 2\alpha;$$

$$3.\quad \begin{cases} a = \sin A = +\cos(\beta+\gamma-\alpha) = +\varrho\cos^2\alpha + \varepsilon\sin\alpha, \\ b = \sin B = +\cos(\alpha+\gamma-\beta) = -\varrho\sin^2\alpha + \varepsilon'\cos\alpha, \\ c = \sin C = -\cos(\alpha+\beta+\gamma) = -\varrho\cos^2\alpha + \varepsilon\sin\alpha, \\ d = \sin D = +\cos(\alpha+\beta-\gamma) = +\varrho\sin^2\alpha + \varepsilon'\cos\alpha, \\ \text{et} \\ a' = \cos A = \sin(\beta+\gamma-\alpha) = -\varrho\sin\alpha\cos\alpha + \varepsilon\cos\alpha, \\ b' = \cos B = \sin(\alpha+\gamma-\beta) = +\varrho\sin\alpha\cos\alpha + \varepsilon'\sin\alpha, \\ c' = \cos C = \sin(\alpha+\varepsilon+\gamma) = +\varrho\sin\alpha\cos\alpha + \varepsilon\cos\alpha, \\ d' = \cos D = \sin(\alpha+\beta-\gamma) = -\varrho\sin\alpha\cos\alpha + \varepsilon'\sin\alpha. \end{cases}$$

E formulis his invenis, componendo, sequentes:

$$4.\quad \begin{cases} ac = \sin^2\alpha - \varrho^2\cos^2\alpha, & bd = \cos^2\alpha - \varrho^2\sin^2\alpha, \\ a + c = 2\varepsilon\sin\alpha, & b + d = 2\varepsilon'\cos\alpha; \end{cases}$$

5.
$$\begin{cases} \sqrt{((1+a)(1+c))} = \varepsilon+\sin\alpha; & \sqrt{((1+a)(1-c))} = (1+\varrho)\cos\alpha; \\ \sqrt{((1+b)(1+d))} = \varepsilon'+\cos\alpha; & \sqrt{((1-b)(1+d))} = (1+\varrho)\sin\alpha; \\ \sqrt{((1-a)(1-c))} = \varepsilon-\sin\alpha; & \sqrt{((1-a)(1+c))} = (1-\varrho)\cos\alpha; \\ \sqrt{((1-b)(1-d))} = \varepsilon'-\cos\alpha; & \sqrt{((1+b)(1-d))} = (1-\varrho)\sin\alpha; \end{cases}$$

6.
$$\begin{cases} a'c' = \cos^2\alpha - \varrho^2\cos^2\alpha, & b'd' = \sin^2\alpha - \varrho^2\sin^2\alpha, \\ a'+c' = 2\varepsilon\cos\alpha, & b'+d' - 2\varepsilon'\sin\alpha; \end{cases}$$

7.
$$\begin{cases} \sqrt{((1-a')(1-c'))}=\varepsilon-\cos\alpha; & \sqrt{((1-a')(1+c'))}=\sin\alpha+\varrho\cos\alpha; \\ \sqrt{((1-b')(1-d'))}=\varepsilon'-\sin\alpha; & \sqrt{((1+b')(1-d'))}=\cos\alpha+\varrho\sin\alpha; \\ \sqrt{((1+a')(1+c'))}=\varepsilon+\cos\alpha; & \sqrt{((1+a')(1-c'))}=\sin\alpha-\varrho\cos\alpha; \\ \sqrt{((1+b')(1+d'))}=\varepsilon'+\sin\alpha; & \sqrt{((1-b')(1+d'))}=\cos\alpha-\varrho\sin\alpha; \end{cases}$$

8.
$$\begin{cases} l=\sqrt{((a+d)(b+c))}=\sqrt{((a'+d)(b'+c'))}=\sqrt{((\varepsilon'\cos\alpha+\varepsilon\sin\alpha)^2-\varrho^2)} \\ \qquad =\sqrt{(1-\varrho^2-2\varrho^2\sin^2\alpha\cos^2\alpha+\varepsilon\varepsilon'\sin 2\alpha)}, \\ \sqrt{(l^2-l'^2)}=\sqrt{((a+c)(b+d))}=\sqrt{((a'+c')(b'+d'))}=\sqrt{(2\varepsilon\varepsilon\sin\alpha)}, \\ l'=\sqrt{((b-a)(d-c))}=\sqrt{((a'-b')(c'-d'))}=\sqrt{((\varepsilon'\cos\alpha-\varepsilon\sin\alpha)^2-\varrho^2)} \\ \qquad =\sqrt{(1-\varrho^2-2\varrho^2\sin^2\alpha\cos^2\alpha-\varepsilon\varepsilon'\sin 2\alpha)}, \end{cases}$$

quibus quantitatibus exprimuntur moduli

$$\lambda = \frac{l'}{l} \quad \text{et} \quad \lambda' = \frac{\sqrt{(l^2-l'^2)}}{l}.$$

Formulae nunc reducendae sunt in (artic. 16.) inventa (2 et 3), e quibus ita exhibitis:

$$x+f=\frac{2hv}{l(1+d)} + 2.'D(v, q); \qquad x_1+f_1=\frac{2hv}{l(1+b)} + 2.'D(v, L'-r);$$
$$x-f=\frac{2hv}{l(1-d)} - 2.'D(v, r); \qquad x_1-f_1=\frac{2hv}{l(1-b)} - 2.'D(v, L-q),$$

componimus primo has:

$$\tfrac{1}{2}(x+x_1) + \tfrac{1}{2}(f+f_1) = \frac{hv}{l(1+d)} + \frac{hv}{l(1+b)} + 'D(v, q) + 'D(v, L'-r),$$
$$\tfrac{1}{2}(x+x_1) - \tfrac{1}{2}(f+f_1) = \frac{hv}{l(1-d)} + \frac{hv}{l(1-b)} - 'D(v, r) - 'D(v, L'-q),$$

sive, paullum mutatas:

$$\tfrac{1}{2}(x+x_1) + \tfrac{1}{2}(f+f_1) = \frac{hv}{l} + \frac{h(1-bd)v}{l(1+b)(1+d)} + 'D(v, q) + 'D(v, L'-r),$$
$$\tfrac{1}{2}(x+x_1) - \tfrac{1}{2}(f+f_1) = \frac{hv}{l} + \frac{h(1-bd)v}{l(1-b)(1-d)} - 'D(v, r) - 'D(v, L'-q).$$

Adhibenda nunc est formula generalis

$$'D(v,a)+'D(v,b)='D(v,a+b)-(\lambda^2 tn'a\,tn'b\,tn'(a+b)).v$$
$$+ \operatorname{arctang}\left(\frac{\lambda'^2 sn'a\,sn'b\,sn'(a+b)\,tnv\,dnv}{dn^2v-\lambda'^2 cn'a\,cn'b\,cn'(a+b)\,sn^2v}\right),$$

et in additione hac parametrorum non negligendum est, e formulis (art. 16.)

$$tn'q = \sqrt{\frac{(1-b)(a+d)}{(1+d)(b-a)}} \quad \text{et} \quad tn'r = \sqrt{\frac{(1+b)(a+d)}{(1-d)(b-a)}}$$

patere, esse parametrum $q<r$. Hinc sequitur, summam $m=q+(L'-r)=L'-(r-q)$ esse quadrante L' minorem; e contra, summa $r+(L'-q)=L'+r-q=2L'-m$ quadrante L' maior est. Ponamus re vera $m=L'-(r-q)$, et formula prima migrabit in

$$\tfrac{1}{2}(x+x_1)+\tfrac{1}{2}(f+f_1)=\frac{hv}{l}+\left(\frac{h(1-bd)}{l(1+b)(1+d)}-\lambda^2 tn'q\,tnc'r\,tn'm\right)v+{}'D(v,m)+\varphi,$$

si φ computatur e formula

$$\text{tang}\,\varphi=\frac{\lambda'^2\,sn'q\,sn'r.sn'm}{1-cn'q\,cnc'r\,cn'm.cnc^2v}\cdot\frac{tn\,v}{dn\,v}.$$

Ante omnia reliqua eruendae nunc sunt valores functionum modularium novi parametri $m=q+L'-r$. Adhibita formula notissima

$$cn'm=\frac{cn'q\,cnc'r-sn'q\,snc'r\,dn'q\,dnc'r}{1-\lambda'^2 sn'^2q\,sn'^2r}$$

e formulis (1. art. 16) hoc loco substitutis, oritur primo:

$$cn'm=l'.\frac{\sqrt{[(1+b)(1+d)(1-a)(1-c)]}-\sqrt{[(1-b)(1-d)(1+a)(1+c)]}}{(1-a)(1-c)(b+d)-(1-b)(1-d)(a+c)},$$

quae formulis (3 et 5) mutatur in

$$cn'm=\tfrac{1}{2}l'.\frac{(\varepsilon'+\cos\alpha)(\varepsilon-\sin\alpha)-(\varepsilon'-\cos\alpha)(\varepsilon+\sin\alpha)}{\varepsilon'\cos\alpha(\varepsilon-\sin\alpha)^2-\varepsilon\sin\alpha(\varepsilon'-\cos\alpha)^2}.$$

Quia vero numerator reducitur ad $2(\varepsilon\cos\alpha-\varepsilon'\sin\alpha)$, et denominator ad $(\varepsilon\varepsilon'-\sin\alpha\cos\alpha)(\varepsilon\cos\alpha-\varepsilon'\sin\alpha)$, prodit valor ipsius $cn'm$ satis simplex; e quo facillimo negotio derivantur valores functionum reliquarum, scilicet:

9.
$$\begin{cases} sn'm=\dfrac{(1+\varrho^2)\sin\alpha\cos\alpha}{\varepsilon\varepsilon'-\sin\alpha\cos\alpha}=\dfrac{\sin\alpha\cos\alpha}{\cos^2g(\varepsilon\varepsilon'-\sin\alpha\cos\alpha)}; & snc'm=\dfrac{l}{\varepsilon\varepsilon'+\sin\alpha\cos\alpha}, \\ cn'm=\dfrac{l'}{\varepsilon\varepsilon'-\sin\alpha\cos\alpha}, & cnc'm=\dfrac{\sin\alpha\cos\alpha}{\cos^2g(\varepsilon\varepsilon'+\sin\alpha\cos\alpha)}, \\ tn'm=\dfrac{\sin\alpha\cos\alpha}{l'\cos^2g}, & tnc'm=\dfrac{l\cos^2g}{\sin\alpha\cos\alpha}, \\ dn'm=\dfrac{l'(\varepsilon\varepsilon'+\sin\alpha\cos\alpha)}{l(\varepsilon\varepsilon'-\sin\alpha\cos\alpha)}, & dnc'm=\dfrac{\varepsilon\varepsilon'-\sin\alpha\cos\alpha}{\varepsilon\varepsilon'+\sin\alpha\cos\alpha}. \end{cases}$$

Si de reductione formulae alterius agitur, quia $tn'(2L'-m)=-tn'm$, $cn'(2L'-m)=-cn'm$, $sn'(2L'-m)=sn'm$, ${}'D(v,2L'-m)=-{}'D(v,m)$ est, pervenies ad

$$\tfrac{1}{2}(x+x_1)-\tfrac{1}{2}(f+f_1)=\frac{hv}{l}+\left(\frac{h(1-bd)}{l(1-b)(1-d)}-\lambda^2 tnc'q\,tn'r\,tn'm\right).v'+{}'D(v,m)-\varphi_1,$$

si angulum φ' computas e formula

$$\operatorname{tang}\varphi_1 = \frac{\lambda'^2\, snc'q\, sn'r\, sn'm}{1+cnc'q\, cn'r\, cn'm.cnc^2 v}\cdot\frac{tn\, v}{dn\, v}.$$

Quia vero facillime invenies, esse

$$\frac{h(1-bd)}{l(1+b)(1+d)} - \lambda^2 tn'q\, tnc'r\, tn'm = \frac{h(1-bd)}{l(1-b)(1-d)} - \lambda^2 tnc'q\, tn'r\, tn'm = 0,$$

formulae reductae sunt

$$10.\quad \begin{cases} \frac{1}{2}(x+x_1) + \frac{1}{2}(f+f_1) = \frac{hv}{l} + 'D(v, m) + \varphi, \\ \frac{1}{2}(x+x_1) - \frac{1}{2}(f+f_1) = \frac{hv}{l} + 'D(v, m) - \varphi_1. \end{cases}$$

Subtrahendo invenis relationem simplicem

$$11.\quad f+f_1 = \varphi + \varphi',$$

addendo vero hanc:

$$12.\quad \tfrac{1}{2}(x+x_1) = \frac{hv}{l} + 'D(v, m) + \tfrac{1}{2}(\varphi-\varphi_1).$$

E formulis his derivantur aliae, quarum ope singulas abscissas etiam non habita ad punctum coniugatum ratione computare possumus. Quia enim

$$\tfrac{1}{2}(f+f_1) + \tfrac{1}{2}(x-x_1) = \operatorname{arc\,tang}\left(\frac{\operatorname{tang}\alpha.\operatorname{tang}s}{\operatorname{tang}(\frac{1}{4}\pi-g)}\right),$$
$$\tfrac{1}{2}(f+f_1) - \tfrac{1}{2}(x-x_1) = \operatorname{arc\,tang}\left(\frac{\operatorname{tang}\alpha\,\operatorname{tang}s}{\operatorname{tang}(\frac{1}{4}\pi+g)}\right),$$

formulae hae coniunctae cum formulis (10), aut addendo, aut subtrahendo, praebent:

$$13.\quad \begin{cases} x = \frac{hv}{l} + 'D(v, m) + \varphi - \operatorname{arc\,tang}\left(\frac{\operatorname{tang}\alpha\,\operatorname{tang}s}{\operatorname{tang}(\frac{1}{4}\pi+g)}\right), \\ x_1 = \frac{hv}{l} + 'D(v, m) + \varphi - \operatorname{arc\,tang}\left(\frac{\operatorname{tang}\alpha\,\operatorname{tang}s}{\operatorname{tang}(\frac{1}{4}\pi+g)}\right), \end{cases}$$

vel si mavis, has:

$$14.\quad \begin{cases} x = \frac{hv}{l} + 'D(v, m) + \operatorname{arc\,tang}\left(\frac{\operatorname{tang}\alpha\,\operatorname{tang}s}{\operatorname{tang}(\frac{1}{4}\pi-g)}\right) - \varphi_1, \\ x_1 = \frac{hv}{l} + 'D(v, m) + \operatorname{arc\,tang}\left(\frac{\operatorname{tang}\alpha\,\operatorname{tang}s}{\operatorname{tang}(\frac{1}{4}\pi+g)}\right) - \varphi_1. \end{cases}$$

Si re vera non habita ratione ad punctum homologe coniugatum computare vis abscissam x, substitues in his formulis valorem $\operatorname{tang}s = \sqrt{\frac{(z-a)(c+z)}{(b-z)(d+z)}}$; si vero abscissam x_1, substitues valorem $\operatorname{tang}s = \sqrt{\frac{(z_1-c)(a+z_1)}{(d-z_1)(b+z_1)}}$.

Quia componitur

$$\lambda'^2 snc'q\, sn'r\, .\, sn'm = \frac{\varepsilon \sin\alpha \sin 2\alpha(\varepsilon' + \cos\alpha)}{l\cos^2 g(\varepsilon+\sin\alpha)(\varepsilon\varepsilon' - \sin\alpha\cos\alpha)},$$

$$cnc'q\, cn'r\, cn'm = \frac{l'^2(\varepsilon' - \cos\alpha)}{2\varepsilon'\cos\alpha(\varepsilon+\sin\alpha)(\varepsilon\varepsilon' - \sin\alpha\cos\alpha)} \quad \text{est}$$

15. $$\text{tang}\,\varphi_1 = \frac{\varepsilon\varepsilon'\sin^2 2\alpha}{l\cos^2 g} \cdot \frac{\varepsilon' + \cos\alpha}{2\varepsilon'\cos\alpha(\varepsilon+\sin\alpha)(\varepsilon\varepsilon' - \sin\alpha\cos\alpha) + l'^2(\varepsilon' - \cos\alpha)cnc^2 v} \cdot \frac{tn\, v}{dn\, v}.$$

Et quia

$$\lambda'^2 sn'q\, snc'r\, .\, sn'm = \frac{\varepsilon \sin\alpha \sin 2\alpha(\varepsilon' - \cos\alpha)}{l\cos^2 g(\varepsilon-\sin\alpha)(\varepsilon\varepsilon' - \sin\alpha\cos\alpha)},$$

$$cn'q\, cnc'r\, cn'm = \frac{l'^2(\varepsilon' + \cos\alpha)}{2\varepsilon'\cos\alpha(\varepsilon-\sin\alpha)(\varepsilon\varepsilon' - \sin\alpha\cos\alpha)}, \quad \text{est}$$

16. $$\text{tang}\,\varphi = \frac{\varepsilon\varepsilon'\sin^2 2\alpha}{l\cos^2 g} \cdot \frac{\varepsilon' - \cos\alpha}{2\varepsilon'\cos\alpha(\varepsilon-\sin\alpha)(\varepsilon\varepsilon' - \sin\alpha\cos\alpha) - l'^2(\varepsilon' + \cos\alpha)cnc^2 v} \cdot \frac{tn\, v}{dn\, v}.$$

Quia generaliter $\frac{\lambda'^2 sn'a\, sn'b\, sn'(a+b).tn\, v\, dn\, v}{dn^2 v - \lambda'^2 cn'a\, cn'b\, cn'(a+b)\, sn^2 v} = \frac{\lambda'^2 sn'a\, sn'b\, sn'(a+b) tn\, v\, dn\, v}{1 - dn'a\, dn'b\, dn'(a+b)\, sn^2 v}$ est, et producta

$$dnc'q\, dn'r\, dn'm = \frac{l'^2}{l^2} \cdot \frac{(\varepsilon-\sin\alpha)(\varepsilon\varepsilon' + \sin\alpha\cos\alpha)}{(\varepsilon+\sin\alpha)(\varepsilon\varepsilon' - \sin\alpha\cos\alpha)},$$

$$dn'q\, dnc'r\, .\, dn'm = \frac{l'^2}{l^2} \cdot \frac{(\varepsilon+\sin\alpha)(\varepsilon\varepsilon' + \sin\alpha\cos\alpha)}{(\varepsilon-\sin\alpha)(\varepsilon\varepsilon' - \sin\alpha\cos\alpha)},$$

priores formulae etiam sic exhibentur:

17. $$\begin{cases} \text{tang}\,\varphi_1 = \dfrac{l\varepsilon \sin\alpha \sin 2\alpha}{\cos^2 g} \cdot \dfrac{(\varepsilon' + \cos\alpha)\, tn\, v\, dn\, v}{l'^2(\varepsilon-\sin\alpha)(\varepsilon\varepsilon' - \sin\alpha\cos\alpha) - l'^2(\varepsilon-\sin\alpha)\varepsilon\varepsilon' + \sin\alpha\cos\alpha).sn^2 v}, \\ \text{tang}\,\varphi = \dfrac{l\varepsilon \sin\alpha \sin 2\alpha}{\cos^2 g} \cdot \dfrac{(\varepsilon' - \cos\alpha)\, tn\, v\, dn\, v}{l'^2(\varepsilon-\sin\alpha)(\varepsilon\varepsilon' - \sin\alpha\cos\alpha) - l'^2(\varepsilon+\sin\alpha)(\varepsilon\varepsilon' + \sin\alpha\cos\alpha sn^2 v} \end{cases}$$

Si in formulis (10, 11, 12) sumitur $s = \frac{1}{2}\pi$, quo $v = L$ fit et $\varphi = \varphi_1 = \frac{1}{2}\pi$, abscissae eo extensae aequales sunt, et arcubus AB et CD subtensae sunt. Abscissa haec est

$$\frac{h.L}{l} + {}'D(L, m).$$

24.

Reductiones formularum, quibus abscissae punctorum coniugatorum in curvis reciprocis exprimuntur.

Reducendae nunc sunt formulae in (artic. 19.) inventae:

$$x' + f' = \frac{2(h+ea')}{l(1-a')}.v - 2.S(v, q'); \qquad x' - f' = \frac{2(h+ea')}{l(1+a')} + 2.S(v, r'),$$

$$x'_1 + f'_1 = \frac{2(h-ec')}{l(1-c')}.v - 2.S(v, L'-r'); \qquad x'_1 - f'_1 = \frac{2(h-ec')}{l(1+c')} + 2.S(v, L'-q'),$$

e quibus componimus primo

$$\tfrac{1}{2}(x'+x'_1)+\tfrac{1}{2}(f'+f'_1)=\left(\frac{h+ea'}{l(1-a')}+\frac{h-ec'}{l(1-c')}\right)v-S(v,q')-S(v,L'-r'),$$

$$\tfrac{1}{2}(x'+x'_1)-\tfrac{1}{2}(f'+f'_1)=\left(\frac{h+ea'}{l(1+a')}+\frac{h-ec'}{l(1+c')}\right)v+S(v,r')+S(v,L'-q'),$$

vel, si mavis,

$$\tfrac{1}{2}(x'+x'_1)+\tfrac{1}{2}(f'+f'_1)=\frac{hv}{l}+\left(\frac{h(1-a'c')+e(a'-c')}{l(1-a')(1-c')}\right)v-S(v,q')-S(v,L'-r'),$$

$$\tfrac{1}{2}(x'+x'_1)+\tfrac{1}{2}(f'+f'_1)=\frac{hv}{l}+\left(\frac{h(1-a'c')+e(a'-c')}{l(1+a')(1-c')}\right)v+S(v,r')+S(v,L'-q').$$

E formulis (2 artic. 19) patet esse parametrum $q'>r'$, quare parametrorum summa $q'+L'-r'=L'+(q'-r')$ quadrante L' maior, altera vero summa $r'+L'-q'=L'-(q'-r')$ quadrante L' minor. Si igitur ponimus iterum

$$m=r'+L'-q'=L'-(q'-r'),$$

altera summa $q'+L'-r=2L'-m$ quadrante maior erit. Adhibita formula

$$cn'm=\frac{cn'r'cnc'q'-sn'r'snc'q'dn'r'dnc'q'}{1-\lambda'^2sn'^2rsnc'^2q'},$$

componimus vero e substitutis formulis (2 artic. 19):

$$cn'm=\frac{\sqrt{[(1+a')(1+c')(1-b')(1-d')]}-\sqrt{[(1-a')(1-c')(1+b')(1+d')]}}{(1-b')(1-d')(a'+c')-(1-a')(1-c')(b'+d')}.l',$$

quae adhibitis (3 — 7 artic. praec.) reducitur ad $cn'm=\dfrac{l'}{\varepsilon\varepsilon'-\sin\alpha\cos\alpha}$, unde persqicis parametrum m esse eundem ac in articulo praeced., ideoque esse $L'-(r-q)=L'-(q'-r')$, sive

$$1.\quad r-q=q'-r'.$$

Quo invento, hac occasione videamus, summae $r+q$ et $q'+r'$ aequales sint, nec ne. Formularum pro $cn'm$ loco nunc habemus formulas

$$2.\quad cn'(L'-q-r)$$
$$=l'.\frac{\sqrt{[(1+b)(1+d)(1-a)(1-c)]}+\sqrt{[(1-b)(1-d)(1+a)(1+c)]}}{(1-a)(1-c)(b+d)-(1-b)(1-d)(a+c)}=\frac{l'}{\varepsilon\cos\alpha-\varepsilon'\sin\alpha},$$

$$3.\quad cn'(L'-q'-r')$$
$$=l'.\frac{\sqrt{[(1+a')(1+c')(1-b')(1-d')]}+\sqrt{[(1-a')(1-c')(1+b')(1+d')]}}{(1-b')(1-d')(a'+c')-(1-a')(1-c')(b+d')}=\frac{l'}{\varepsilon'\cos\alpha-\varepsilon\sin\alpha},$$

unde perspicis, summas $q-r$ et $q'-r'$ esse diversas.

Quia $S(v,r')+S(v,L'-q')=S(v,m)-(\lambda^2\,tn'r'\,tnc'q'\,tn'm)v$ + arc tang $\dfrac{\lambda^2sn'r'snc'q'sn'm\,snv\,cnv\,dnv}{cn'r'cnc'q'cn'm\,dn^2v+\lambda^2sn^2v}$ qua in formula $\dfrac{\lambda^2sn'r\,snc'q'sn'm\,snv\,cnv\,dnv}{cn'r\,cnc'q\,cn'm\,dn^2v+\lambda^2sn^2v}$ $=\dfrac{\lambda^2\lambda'^2sn'r\,snc'q'sn'm\,snv.sncv}{dn'r\,dnc'q'dn'm-\lambda^2snc^2v}=\psi$, nanciscimur

$$\tfrac{1}{2}(x'+x'_1) - \tfrac{1}{2}(f'+f'_1)$$
$$= \frac{hv}{l} + [(\frac{h(1-a'c')+e'(a'-c')}{l(1+a')(1+c')} - \lambda^2 tn'r'tnc'q'tn'm)]v + S(v, m) + \psi,$$

et pariter

$$\tfrac{1}{2}(x'+x'_1) + \tfrac{1}{2}(f'+f'_1)$$
$$= \frac{hv}{l} + [(\frac{h(1-a'c')+e'(a'-c')}{l(1-a')(1-c')} - \lambda^2 tnc'r tn'q tn'm)]v + S(v, m) + \pi - \psi_1.$$

Et quia

$$\text{tum } \frac{h(1-a'c')}{l(1+a')(1+c')} + \frac{e(a'-c')}{l(1+a')(1+c')} - \lambda^2 tn'r'tnc'q'tn'm = 0,$$
$$\text{tum } \frac{h(1-a'c')}{l(1-a')(1-c')} + \frac{e(a'-c')}{l(1-a')(1-c')} - \lambda^2 tnc'r'tn'q'tn'm = 0$$

invenies, formulae reductae sunt

$$4. \quad \begin{cases} \tfrac{1}{2}(x'+x'_1) - \tfrac{1}{2}(f'+f'_1) = \dfrac{hv}{l} + S(v, m) + \psi, \\ \tfrac{1}{2}(x'+x'_1) + \tfrac{1}{2}(f'+f'_1) = \dfrac{hv}{l} + S(v, m) + \pi - \psi_1, \end{cases}$$

unde concludis, addendo,

$$5. \quad \tfrac{1}{2}(x'+x'_1) = \frac{hv}{l} + S(v, m) + \tfrac{1}{2}\pi - \tfrac{1}{2}(\psi_1 - \psi),$$

et subtrahendo,

$$6. \quad f'+f'_1) = \pi - \psi - \psi_1.$$

Quia insuper est

$$\tfrac{1}{2}(x'-x'_1) - \tfrac{1}{2}(f'+f'_1) = -\operatorname{arc\,tang}\left(\frac{\operatorname{tang}\sigma}{\operatorname{tang}(\alpha+\gamma)}\right), \text{ et}$$
$$\tfrac{1}{2}(x'-x'_1) + \tfrac{1}{2}(f'+f'_1) = +\operatorname{arc\,tang}\left(\frac{\operatorname{tang}\sigma}{\operatorname{tang}(\alpha-\gamma)}\right),$$

e formulis (4) obtines

$$7. \quad \begin{cases} x' = \dfrac{hv}{l} + S(v, m) + \psi + \operatorname{arc\,tang}\left(\dfrac{\operatorname{tang}\sigma}{\operatorname{tang}(\alpha-\gamma)}\right), \\ x'_1 = \dfrac{hv}{l} + S(v, m) + \psi + \operatorname{arc\,tang}\left(\dfrac{\operatorname{tang}\sigma}{\operatorname{tang}(\alpha+\gamma)}\right), \end{cases}$$

nec non has:

$$8. \quad \begin{cases} x' = \dfrac{hv}{l} + S(v, m) + \pi - \psi_1 - \operatorname{arc\,tang}\left(\dfrac{\operatorname{tang}\sigma}{\operatorname{tang}(\alpha+\gamma)}\right), \\ x'_1 = \dfrac{hv}{l} + S(v, m) + \pi - \psi_1 - \operatorname{arc\,tang}\left(\dfrac{\operatorname{tang}\sigma}{\operatorname{tang}(\alpha-\gamma)}\right). \end{cases}$$

Ope harum formularum singulas abscissas, quae ad puncta curvarum reciprocarum pertinent, etiam non habita ratione puncti coniugati computare potes,

dummodo in computanda abscissa x' adhibes valorem $\operatorname{tang}\sigma = \sqrt{\frac{(a'-z')(c'+z')}{(z'-b')(d'+z')}}$, in computanda vero abscissa x' valorem $\operatorname{tang}\sigma = \sqrt{\frac{(c'-z'_1)(a'+z'_1)}{(z'_1-d')(b'+z'_1)}}$.

Ad computandum denique angulum ψ, compones formulam

9. $$\operatorname{tang}\psi = \frac{\varepsilon \sin\alpha \sin 2\alpha}{l.\cos^2 g} \cdot \frac{(\varepsilon - \cos\alpha)\, snv\, snc\, v}{(\varepsilon'+\sin\alpha)(\varepsilon\varepsilon'+\sin\alpha\cos\alpha)-(\varepsilon'-\sin\alpha)(\varepsilon\varepsilon'-\sin\alpha\cos\alpha).snc^2 v},$$

et pariter angulus ψ_1 determinatur formula

10. $$\operatorname{tang}\psi_1 = \frac{\varepsilon \sin\alpha \sin 2\alpha}{l.\cos^2 g} \cdot \frac{(\varepsilon + \cos\alpha)\, snv\, snc\, v}{(\varepsilon'-\sin\alpha)(\varepsilon\varepsilon'+\sin\alpha\cos\alpha)-(\varepsilon'+\sin\alpha)(\varepsilon\varepsilon'-\sin\alpha\cos\alpha).snc^2 v}.$$

Simul perspicis abscissas quatuor arcubus AB, CD, $A'B'$ et $C'D'$ subtensas aequales esse omnes, et valorem hunc communem esse

$$\frac{hL}{l} + \mathbf{S}(L, m) + \tfrac{1}{2}\pi = \frac{hL}{l} + {}'\boldsymbol{D}(L, m).$$

E praeceptis geometriae sphaericae nota est curvarum sphaericarum proprietas generalis ea, *ut subtangentes lineae, quae ad puncta reciproca pertinent, aequales sint*, et *insuper noscitur, differentiam abscissarum arcubus reciprocis subtensarum esse complementum lineae illae subtangentis.* Quum vero ex aequatione differentiali catenariae ellipticae facillime invenitur, esse lineam subtangentem, quae ad punctum M pertineat, aequalem

$$\operatorname{arc\,tang}\left(\frac{hz}{\sqrt{Z}}\right),$$

et pariter subtangentem, quae ad punctum curvae hyperbolicae N pertineat, esse

$$\operatorname{arc\,tang}\left(\frac{hz_1}{\sqrt{Z_1}}\right),$$

pervenimus ad relationes:

$$x' - x = \tfrac{1}{2}\pi - \operatorname{arc\,tang}\left(\frac{hz}{\sqrt{Z}}\right) = \operatorname{are\,tang}\left(\frac{\sqrt{Z}}{hz}\right), \text{ et}$$

$$x'_1 - x' = \tfrac{1}{2}\pi - \operatorname{arc\,tang}\left(\frac{hz_1}{\sqrt{Z_1}}\right) = \operatorname{arc\,tang}\left(\frac{\sqrt{Z_1}}{hz_1}\right),$$

e quibus, substituendo valores quantitatum x, x_1, x'_1 et x' in articulis praecedentibus inventos, oriuntur relationes inter angulos φ, φ_1, ψ et ψ_1, quas etiam ulterius reduces, si memineris, esse $\operatorname{tang}\sigma = \frac{\operatorname{tang} s}{\operatorname{tang}\alpha}$, et e theoria integralium, quae a functionibus modularibus pendent, notam esse formulam generalem

$${}'\boldsymbol{D}(v, m) - \mathbf{S}(v, m) = \operatorname{arctang}\left(\frac{sn'm}{snc'm} \cdot \frac{snv}{sncv}\right).$$

25.

Reductiones formularum, quibus areae coniugatae linearum catenarium et arcus reciproci exprimuntur.

Eruamus primo semidifferentiam arcuum homologe coniugatorum $A'M'$ et $C'N'$, quia est nexus e praeceptis geometriae sphaericae notus, hanc differentiam inter et differentiam arearum, quae ad curvarum catenariarum ipsarum arcus homologe coniugatos AM et CN pertinent. Reducenda igitur est formula ex iis (artic. 11.) composita:

$$\tfrac{1}{2}(A'M'-C'N') = \frac{(a'-c')v}{l} - \mathbf{S}(v, p') + \mathbf{S}(v, L'-p').$$

E formulis

$$tn'p' = \sqrt{\frac{a'+d'}{c'-d'}} = \sqrt{\frac{\varepsilon\cos\alpha+\varepsilon'\sin\alpha-2\varrho\sin\alpha\cos\alpha}{\varepsilon\cos\alpha-\varepsilon'\sin\alpha+2\varrho\sin\alpha\cos\alpha}}, \quad \text{et}$$

$$tnc'p' = \sqrt{\frac{b'+c'}{a'-b'}} = \sqrt{\frac{\varepsilon\cos\alpha+\varepsilon'\sin\alpha+2\varrho\sin\alpha\cos\alpha}{\varepsilon\cos\alpha-\varepsilon'\sin\alpha-2\varrho\sin\alpha\cos\alpha}},$$

patet primum, esse parametrum $L'-p' > p'$, sive $p' < \frac{1}{2}L$, quare integralium differentia reducetur ad unicum, cuius parameter est $L'-2p' = n$, et eruendae ante omnia sunt formulae, quibus exprimantur valores functionum modularium huius argumenti n. Est primo $tnc'n = tn'(2p') = \frac{2tn'p.dn'p}{1-tn'^2p'dn'^2p'}$ $= \frac{2l}{b'+c'-a'-d'} = \frac{l}{\varrho\sin 2\alpha}$ et ideo $tn'n = \frac{\varrho\sin 2\alpha}{l'}$. Quare habemus formulas

1.
$$\begin{cases} sn'n = \dfrac{\operatorname{tang} g\sin 2\alpha}{\varepsilon\cos\alpha-\varepsilon'\sin\alpha}, & snc'n = \dfrac{l}{\varepsilon\cos\alpha+\varepsilon'\sin\alpha}, \\ cn'n = \dfrac{l}{\varepsilon\cos\alpha-\varepsilon'\sin\alpha}, & cnc'n = \dfrac{\operatorname{tang} g\sin 2\alpha}{\varepsilon\cos\alpha+\varepsilon'\sin\alpha}, \\ tn'n = \dfrac{\operatorname{tang} g\sin\alpha}{l'}, & \text{et} \quad tnc'n = \dfrac{l}{\operatorname{tang} g\sin 2\alpha}, \\ dn'n = \dfrac{l'(\varepsilon\cos\alpha+\varepsilon'\sin\alpha)}{l(\varepsilon\cos\alpha-\varepsilon'\sin\alpha)}, & dnc'n = \dfrac{\varepsilon\cos\alpha-\varepsilon'\sin\alpha}{\varepsilon\cos\alpha+\varepsilon'\sin\alpha}, \end{cases}$$

Quia in articulo praeced. invenimus $cn'(L'-q-r) = \frac{l'}{\varepsilon\cos\alpha-\varepsilon'\sin\alpha} = cn'n$ patet, novum parametrum esse $n = \pm(L'-q-r) = L'-2p'$, ideoque esse

2. aut $p' = \frac{1}{2}(q+r)$ aut $L'-p' = \frac{1}{2}(q+r)$.

Valere formulam secundam, ideoque primam esse falsam, in art. seq. monstrabitur.

Quia $\mathbf{S}(v, L'-p') - \mathbf{S}(v, p') = \mathbf{S}(v, n) + v(\lambda^2\, inc'p'\, tn'p\, tn'n)$ $-$ arc tang $\left(\frac{\lambda^2\lambda'^2\, snc'p'\, sn'p'\, sn'n\, snv\, sncv}{dnc'p'\, dn'p'\, dn'n-\lambda^2\, snc^2v}\right)$, est $\frac{1}{2}(A'M' - C'N') = (\frac{a'-c'}{l}$ $+ \lambda^2 tnc'p' tn'p tn'n)v + \mathbf{S}(v, n) - \chi$, ponendo

$$\text{tang}\,\chi = \frac{\lambda^2\lambda'^2 snc'p'sn'p'sn'n\,snc\,snc\,v}{dnc'p\,dn'p'\,dn'n - \lambda^2 snc^2 v}.$$

Quia vero $tn'p\,tnc'p = \frac{1}{\lambda}$, est $\lambda^2 tnc'p'\,tn'p'\,tn'n = \lambda\,tn'n = \frac{\text{tang}\,g\sin 2\alpha}{l}$, et quum $\frac{a'-c'}{l} = -\frac{\text{tang}\,g\sin 2\alpha}{l}$, inveniinus

$$\tfrac{1}{2}(A'M'-C'N') = \mathbf{S}(v,n) - \text{arc tang}\left(\frac{2\varepsilon'\sin\alpha}{l}\cdot\frac{\text{tang}\,g\sin 2\alpha\,snv\,sncv}{\varepsilon\cos\alpha+\varepsilon'\sin\alpha-(\varepsilon\cos\alpha-\varepsilon'\sin\alpha)snc^2v}\right).$$

Quia insuper $\frac{1}{2}(A'M'+C'N') = \sigma$, arcus reciproci ipsi exprimuntur formulis

$$3.\quad \begin{cases} A'M' = \sigma + \mathbf{S}(v,n) \\ \quad - \text{arc tang}\left(\frac{2\varepsilon\sin\alpha}{l}\cdot\frac{\text{tang}\,g\sin 2\alpha.snv\,sncv}{\varepsilon\cos\alpha+\varepsilon'\sin\alpha-(\varepsilon\cos\alpha-\varepsilon'\sin\alpha)snc^2v}\right), \\ C'N' = \sigma - \mathbf{S}(v,n) \\ \quad + \text{arc tang}\left(\frac{2\varepsilon'\sin\alpha}{l}\cdot\frac{\text{tang}\,g\sin 2\alpha.snv\,sncv}{\varepsilon\cos\alpha+\varepsilon'\sin\alpha-(\varepsilon\cos\alpha-\varepsilon'\sin\alpha)snc^2v}\right). \end{cases}$$

Si arcus $A'M'$ et $C'N'$ eo extendimus, ut fiunt $A'B'$ et $C'D'$, est $\sigma = \frac{1}{2}\pi$ et $v = L$; quare est

$$4.\quad \begin{cases} \text{arcus}\,A'B' = \frac{1}{2}\pi + \mathbf{S}(L,n), \\ \text{arcus}\,C'D' = \frac{1}{2}\pi - \mathbf{S}(L,n). \end{cases}$$

In formulis (3) est $= \frac{2\varepsilon\cos\alpha}{l}\cdot\frac{tn\,v}{dnv}$. Si vero arcus $A'M'$ et $C'N'$ computare vis, e formulis mode erutis, ratione ad arcum coniugatum non habita, substituas $\sigma = \text{arc tang}\left(\sqrt{\frac{(c'-z')(c'+z')}{(z'-d')(d'+z')}}\right)$ in computando $A'M'$, at $\sigma = \text{arc tang}\left(\sqrt{\frac{(c'-z'_1)(a'+z'_1)}{(z'_1-d')(b'+z'_1)}}\right)$ in computando $C'N'$.

Areae f et f_1 etiam inveniuntur e formulis his, dummodo memineris esse

$$5.\quad \begin{cases} f = A'M' - \text{arc cos}\left(\frac{h}{\cos y(\sin y-e)}\right), \\ f_1 = C'N' - \text{arc cos}\left(\frac{h}{\cos y_1(\sin y_1+e)}\right), \end{cases}$$

et quia cosinus in formulis his obvii unitati aequales fiunt, si areae f et f_1 usque ad applicatas B et D extenduntur, areae, quae nunc sint F et F_1, exprimuntur formulis satis simplicibus

$$6.\quad \begin{cases} F = A'B' = \frac{1}{2}\pi + \mathbf{S}(L,n), \\ F_1 = C'D' = \frac{1}{2}\pi - \mathbf{S}(L,n). \end{cases}$$

Eodem perveniemus, si formulas, quibus summae et differentiae abscissarum et arearum exprimuntur in (artic. **23.**) probatas alio modo coniungimus. Oriuntur primo:

$$\tfrac{1}{2}(f-f_1)+\tfrac{1}{2}(x-x_1)=\frac{hv}{l(1+d)}-\frac{hv}{l(1+b)}+{}'D(v,q)-{}'D(v,L'-r),$$

$$\tfrac{1}{2}(f-f_1)-\tfrac{1}{2}(x-x_1)=\frac{hv}{l(1-b)}-\frac{hv}{l(1-d)}+{}'D(v,r)-{}'D(v,L'-q).$$

Ante omnia nunc eruendum est, parametrorum differentiae an sint positivae nec ne. Invenitur autem ope formulae $tn'(a+b)=\frac{tn'a\,dn'b+dn'a\,tn'b}{1-tn'a\,dn'b\,tn'b\,dn'a}$, generalis esse $tn'(q+r)=-tn'2p'$, unde concludimus esse $L-(q+r)$ negativum. Quare vides esse $q>L'-r$ et pariter $r>L'-q$, ideoque esse $L'-p'=\frac{1}{2}(q+r)$, sive $q+r-L'=n$.

Quia est ${}'D(v,q)-{}'D(v,L'-r)={}'D(v,n)+(\lambda^2 tn'q\,tnc'r.tn'n).v$ ponendo

$$\text{tang}\,\chi=\frac{\lambda'^2 sn'q\,snc'r\,sn'n.tnv.dnv}{1-dn'q\,dnc'r\,dn'n\,sn^2v},\quad\text{invenimus primo}$$

$$\tfrac{1}{2}(f-f_1)+\tfrac{1}{2}(x-x_1)=\left(-\frac{h(d-b)}{l(1+d)(1+b)}+\lambda^2 tn'q\,tnc'r\,tn'n\right)v+{}'D(v,n)-\chi,$$

et pariter

$$\tfrac{1}{2}(f-f_1)-\tfrac{1}{2}(x-x_1)=\left(\frac{-h(d-b)}{l(1-d)(1+b)}+\lambda^2 tnc'q\,tn'r\,tn'n\right)v+{}'D(v,n)-\chi_1,$$

$$\text{ponendo}\quad\text{tang}\,\chi_1=\frac{\lambda'^2 snc'q\,sn'r\,sn'n\,tnv\,dnv}{1-dnc'q\,dn'r\,dn'n\,sn^2v},$$

quae vero, cum facillime tibi persuadeas, esse

$$\frac{-h(d-b)}{l(1+d)(1+b)}+\lambda^2 tn'q\,tnc'r\,tn'v=\frac{-h(d-b)}{l(1-d)(1+b)}+\lambda^2 tnc'q\,tn'r\,tn'v=0,$$

contrahuntur ad simplices:

$$7.\quad\begin{cases}\tfrac{1}{2}(f-f_1)+\tfrac{1}{2}(x-x_1)={}'D(v,n)-\chi,\\ \tfrac{1}{2}(f-f_1)-\tfrac{1}{2}(x-x_1)={}'D(v,n)-\chi_1,\end{cases}$$

e quarum combinatione obtines

$$8.\quad x-x_1=\chi_1-\chi,$$

$$9.\quad \tfrac{1}{2}(f-f_1)={}'D(v,n)-\tfrac{1}{2}(\chi+\chi_1).$$

Fomulae, quas ad computandos angulos χ et χ_1 applicabis, satis reductae, sunt:

$$10.\quad\text{tang}\,\chi=\frac{2l\varepsilon\sin\alpha.\text{tang}\,g\sin 2\alpha.(\varepsilon-\cos\alpha).tnv.dnv}{l^2(\varepsilon\cos\alpha-\varepsilon'\sin\alpha)(\varepsilon-\sin\alpha)-l'^2(\varepsilon\cos\alpha+\varepsilon'\sin\alpha)(\varepsilon+\sin\alpha)sn^2v}$$

et

$$11.\quad\text{tang}\,\chi_1=\frac{2l\varepsilon\sin\alpha\,\text{tang}\,g\sin 2\alpha(\varepsilon+\cos\alpha).tnv.dnv}{l^2(\varepsilon\cos\alpha-\varepsilon'\sin\alpha)(\varepsilon+\sin\alpha)-l'^2(\varepsilon\cos\alpha+\varepsilon'\sin\alpha)(\varepsilon-\sin\alpha)sn^2v}.$$

Si formulas

$$\tfrac{1}{2}(f+f_1)+\tfrac{1}{2}(x-x_1)=\text{arc tang}\left(\frac{\text{tang}\,\alpha\,\text{tang}\,s}{\text{tang}\,(\frac{1}{4}(\pi-g))}\right),$$

$$\tfrac{1}{2}(f+f_1)-\tfrac{1}{2}(x-x_1)=\text{arc tang}\left(\frac{\text{tang}\,\alpha\,\text{tang}\,s}{\text{tang}\,(\frac{1}{4}(\pi+g))}\right),$$

in (art. 17.) inventas, coniungis cum formulis modo erutis (7.), prodeunt novae:

$$12.\quad \begin{cases} f = +\,'D(v, n) - \chi + \text{arc tang}\left(\dfrac{\text{tang}\,\alpha.\text{tang}\,s}{\text{tang}\,(\frac{1}{4}\pi+g)}\right), \\ f_1 = -\,'D(v, n) + \chi + \text{arc tang}\left(\dfrac{\text{tang}\,\alpha.\text{tang}\,s}{\text{tang}\,(\frac{1}{4}\pi-g)}\right), \end{cases}$$

et pariter

$$13.\quad \begin{cases} f = +\,'D(v, n) - \chi_1 + \text{arc tang}\left(\dfrac{\text{tang}\,\alpha\,\text{tang}\,s}{\text{tang}\,(\frac{1}{4}\pi-g)}\right), \\ f = -\,'D(v, n) + \chi_1 + \text{are tang}\left(\dfrac{\text{tang}\,\alpha\,\text{tang}\,s}{\text{tang}\,(\frac{1}{4}\pi+g)}\right), \end{cases}$$

quarum ope singulas areas f et f_1, non habita ratione ratione alterius coniugati, duplici modo computare potes. Areae usque ad applicatas extensae, quae ad vertices proximos superiores pertinent, e formulis his inveniuntur ponendo $s = \frac{1}{2}\pi$ et $v = L$, quo obtines

$$F = 'D(L, n) \quad \text{et} \quad F_1 = \pi - 'D(L, n),$$

quae cum supra inventis formulis (6.) congruunt.

26.

Reductiones formularum, quibus arcus coniugati linearum catenariarum et areae reciprocae exprimuntur.

Reducenda est formula ex iis articuli (8 et 9.) composita haec

$$\tfrac{1}{2}(AM - CN) = -\frac{(d-b+2e)v}{l} + 'D(v, p) - 'D(v, L'-p),$$

in quo non negligenda est ratio inter parametros p et $L' - p$. Quia est $tn'p = \sqrt{\frac{a+d}{b-a}} = \sqrt{\frac{\varepsilon'\cos\alpha+\varepsilon\sin\alpha+\varrho}{\varepsilon'\cos\alpha-\varepsilon\sin\alpha-\varrho}}$ et $tnc'p = \sqrt{\frac{b+c}{d-c}} = \sqrt{\frac{\varepsilon'\cos\alpha+\varepsilon\sin\alpha-\varrho}{\varepsilon'\cos\alpha-\varepsilon\sin\alpha+\varrho}}$, vides esse

$$p > L' - p, \quad \text{sive} \quad p > \tfrac{1}{2}L'.$$

Quare ponamus $u = 2p - L'$, quam differentiam positivam esse modo invenimus, et adhibeamus formulam

$$'D(v, p) - 'D(v, L'-p) = 'D(v, u) + (\lambda^2 tn'p\, tnc'p\, tn'u)v - M,$$

$$\text{ponendo} \quad \text{tang}\, M = \frac{\lambda'^2 sn'p\, snc'p\, sn'u.tnv\, dnv}{1 - dn'p\, dnc'p\, dn'u\, sn^2 v}, \quad \text{quo fit}$$

$$\tfrac{1}{2}(AM - CN) = (\lambda^2 tn'p\, tnc'p\, tn'u - \frac{d-b+2e}{l})v + 'D(v, u) - M.$$

Ante omnia eruendi sunt valores functionum modularium novi parametri u. Quia vero $tnc'u = tn'(2L'-2p) = -\,tn'2p = -\,\frac{2tn'p\, dn'p}{1 - tn'^2 p\, dn'^2 p} = \frac{l}{\varrho}$, ideoque $tn'u = \frac{\varrho}{l'}$, invenimus formulas:

$$1.\quad \begin{cases} sn'u = \dfrac{\varrho}{\varepsilon'\cos\alpha-\varepsilon\sin\alpha}, & snc'u = \dfrac{l}{\varepsilon'\cos\alpha+\varepsilon\sin\alpha}, \\ cn'u = \dfrac{l'}{\varepsilon'\cos\alpha-\varepsilon\sin\alpha}, & cnc'u = \dfrac{\varrho}{\varepsilon'\cos\alpha+\varepsilon\sin\alpha}, \\ tn'u = \dfrac{\varrho}{l}, & tnc'u = \dfrac{l}{\varrho}, \\ dn'u = \dfrac{l'(\varepsilon'\cos\alpha+\varepsilon\sin\alpha)}{l(\varepsilon'\cos\alpha-\varepsilon\sin\alpha)}, & dnc'u = \dfrac{\varepsilon'\cos\alpha-\varepsilon\sin\alpha}{\varepsilon'\cos\alpha+\varepsilon\sin\alpha}. \end{cases} \quad \text{et}$$

Nunc vero est $\lambda^2 tn'p\,tnc'p\,tn'u = \lambda\,tn'u = \frac{\varrho}{l}$, et quia $\frac{d-b+2e}{l} = \frac{2\varrho\sin^2\alpha+2e}{l} = \frac{\varrho+e}{l}$, invenimus formulam

$$2.\quad \tfrac{1}{2}(AM-CN) = -\frac{ev}{l} + {}'D(v,u) - M,$$

in qua angulum M computabis ope formulae

$$3.\quad \text{tang}\, M = \frac{2\varrho l\varepsilon\sin\alpha.tnv.dnv}{l^2(\varepsilon'\cos\alpha-\varepsilon\sin\alpha)-l'^2(\varepsilon'\cos\alpha+\varepsilon\sin\alpha)sn^2v}.$$

Quia insuper est $\frac{1}{2}(AM+CN) = s$, oriuntur

$$4.\quad \begin{cases} AM = s - \dfrac{ev}{l} + {}'D(v,u) - M, \\ CN = s + \dfrac{ev}{l} - {}'D(v,u) + M, \end{cases}$$

quarum ope et singulos arcus non habita ratione alterius coniugati computare possumus. Quia in (artic. 24.) invenimus $cn'(L'-q'-r') = \frac{l'}{\varepsilon'\cos\alpha-\varepsilon\sin\alpha}$, videmus esse $cn'(L'-q'-r') = cn'u$, unde concludimus esse

$$u = \pm(L'-q'-r') = 2p' - L', \quad \text{quare}$$

$$\text{aut}\quad p = \tfrac{1}{2}(q'+r'), \quad \text{aut}\quad p = L' - \tfrac{1}{2}(q'+r') \quad \text{erit.}$$

Quia vero $tn'(q'+r') = \frac{tn'q\,dn'r+tn'r\,dn'q}{1-tn'q\,tn'r.dn'r\,dn'q} = +\,tnc\,u$, est $p = L' - \frac{1}{2}(q'+r')$ et $u = L'-(q'+r')$, et quum e formulis (artic. 19.) componi possunt formulae

$$\tfrac{1}{2}(f'-f_1') + \tfrac{1}{2}(x'-x'_1) = \left(\frac{h+ea'}{l(1-a')} - \frac{h-ec'}{l(1-c')}\right)v - S(v,q) + S(v,L'-r'),$$

$$\tfrac{1}{2}(f'-f_1') - \tfrac{1}{2}(x'-x'_1) = \left(\frac{h-ec'}{l(1+c')} - \frac{h+ea'}{l(1+a')}\right)v + S(v,L'-q') - S(v,r),$$

vel, si mavis,

$$\tfrac{1}{2}(f'-f_1') + \tfrac{1}{2}(x'-x'_1) = -\frac{ev}{l} + \left(\frac{e(1-a'c')-h(c'-a')}{l(1-a')(1-c')}\right)v + S(v,L'-r') - S(v,q'),$$

$$\tfrac{1}{2}(f'-f_1') - \tfrac{1}{2}(x'-x'_1) = -\frac{ev}{l} + \left(\frac{e(1-a'c')-h(c'-a')}{l(1+a')(1+c')}\right)v + S(v,L'-q') - S(v,r'),$$

ambae integralium differentiae reducuntur ad idem integrale $S(v, u)$, cuius parameter $u = L' - r' - q'$, adhibitis formulis

$$S(v, L'-r') - S(v, q') = S(v, u) + (\lambda^2 tnc' r' tn' q' tn' u)v - \Omega, \quad \text{si ponitur}$$

$$\text{tang } \Omega = \frac{\lambda^2 l'^2 snc' r' sn' q sn' u . snv sncv}{dnc' r' dn' q dn' u - \lambda^2 sn^2 v} \quad \text{et}$$

$$S(v, L'-q') - S(v, r') = S(v, u) + (\lambda^2 tnc' q' tn' r' tn' u)v - \Omega_1, \quad \text{posito}$$

$$\text{tang } \Omega_1 = \frac{\lambda^2 \lambda'^2 snc' q' sn' r' sn' u . snv sncv}{dnc' q' dn' r' dn' u - \lambda^2 sn^2 v}.$$

Quia est $\lambda^2 tnc' r' tn' q' tn' u = \frac{\varrho}{l} \cdot \frac{\varepsilon + \cos\alpha}{\varepsilon - \cos\alpha} = \frac{\varrho \sin^2\alpha - \varrho^3 \cos^2\alpha}{l(\varepsilon - \cos\alpha)^2}$ et $\frac{e(1-a'c') - h(c'-a')}{l(1-a')(1-c')}$ $= \frac{-\varrho \sin^2\alpha + \varrho^3 \cos^2\alpha}{l(\varepsilon - \cos\alpha)^2}$, formula prima reducitur ad

$$5. \quad \begin{cases} \frac{1}{2}(f'-f_1') + \frac{1}{2}(x'-x'_1) = -\frac{ev}{l} + S(v, u) - \Omega, \\ \text{et pariter altera ad} \\ \frac{1}{2}(f'-f_1') + \frac{1}{2}(x'-x'_1) = -\frac{ev}{l} + S(v, u) - \Omega_1; \end{cases}$$

quare, tum addendo tum subtrahendo, invenis

$$6. \quad \frac{1}{2}(f'-f_1') = -\frac{ev}{l} + S(v, u) - \frac{1}{2}(\Omega + \Omega_1)$$

et

$$7. \quad x' - x'_1 = \Omega - \Omega_1.$$

Formulae ad computandos angulos auxiliares aptissimae nunc vero sunt

$$8. \quad \begin{cases} \text{tang } \Omega = \frac{2\varrho l \varepsilon' \sin\alpha . (\varepsilon + \cos\alpha) . snv sncv}{(\varepsilon' - \sin\alpha)(\varepsilon' \cos\alpha + \varepsilon \sin\alpha) - (\varepsilon' + \sin\alpha)(\varepsilon' \cos\alpha - \varepsilon \sin\alpha) sn^2 v}, \\ \text{tang } \Omega_1 = \frac{2\varrho l \varepsilon' \sin\alpha . (\varepsilon - \cos\alpha) . snv sncv}{(\varepsilon' + \sin\alpha)(\varepsilon' \cos\alpha + \varepsilon \sin\alpha) - (\varepsilon' - \sin\alpha)(\varepsilon' \cos\alpha - \varepsilon \sin\alpha) sn^2 v}. \end{cases}$$

Si cum formulis (5) coniungimus inventas in (artic. 20.) has:

$$\frac{1}{2}(f'+f_1') - \frac{1}{2}(x'-x'_1) = \text{arc tang}\left(\frac{\text{tang}\,\sigma}{\text{tang}\,(\alpha+g)}\right),$$

$$\frac{1}{2}(f'+f_1') + \frac{1}{2}(x'-x'_1) = \text{arc tang}\left(\frac{\text{tang}\,\sigma}{\text{tang}\,(\alpha-g)}\right);$$

tum addendo, tum subtrahendo, oriuntur:

$$9. \quad \begin{cases} f' = -\frac{ev}{l} + S(v, u) - \Omega + \text{arc tang}\left(\frac{\text{tang}\,\sigma}{\text{tang}\,(\alpha+g)}\right), \\ f_1' = \frac{ev}{l} - S(v, u) + \Omega + \text{arc tang}\left(\frac{\text{tang}\,\sigma}{\text{tang}\,(\alpha-g)}\right), \end{cases}$$

vel etiam hae

$$10.\quad \begin{cases} f' = -\frac{ev}{l} + \mathbf{S}(v, u) - \Omega_1 + \text{arc tang}\left(\frac{\text{tang}\,\sigma}{\text{tang}\,(\alpha - g)}\right), \\ f_1' = +\frac{ev}{l} - \mathbf{S}(v, u) + \Omega_1 + \text{arc tang}\left(\frac{\text{tang}\,\sigma}{\text{tang}\,(\alpha + g)}\right). \end{cases}$$

Si arcus AM et CN extendimus usque ad vertices proximos B et D, formulae (4) praebent

$$11.\quad AB = {}'D(\boldsymbol{L}, u) - \frac{e.L}{l} \quad \text{et} \quad CD = \pi - {}'D(\boldsymbol{L}, u) + \frac{e.L}{l}.$$

Areae denique f' et f_1' extensae usquae ad applicatas, quae ad vertices B' et D' pertinent, si sunt F' et F_1', valent formulae

$$12.\quad \begin{cases} F' = -\frac{e.L}{l} + S(\boldsymbol{L}, u) + \frac{1}{2}\pi, \\ F_1' = +\frac{e.L}{l} + S(\boldsymbol{L}, u) + \frac{1}{2}\pi. \end{cases}$$

Si inventa in praecedentibus inde ab artic. (23) perlustramus, videmus pendere abscissas x et x_1 ab integrali ${}'\boldsymbol{D}(v,m)$ et abscissas x' et x'_1 ab integrali $S(v,m)$; arcus $A'M'$ et $C'N'$ pendere ab integrali $S(v,n)$ et areas f et f_1 ab integrali ${}'\boldsymbol{D}(v, n)$; arcus AM et CN ab integrali ${}'\boldsymbol{D}(v, u)$, arcus vero f' et f_1' ab integrali $\mathbf{S}(v, u)$. Quia autem integralia ${}'\boldsymbol{D}(v, m)$, ${}'\boldsymbol{D}(v, n)$, ${}'\boldsymbol{D}(v, u)$ facillime reducuntur ad integralia $\mathbf{S}(v, m)$, $\mathbf{S}(v, n)$, $\mathbf{S}(v, u)$, sive haec ad illa, ope formulae generalis ${}'\boldsymbol{D}(v,m) - \mathbf{S}(v,m) = \text{arc tang}\left(\frac{sn'm}{snc'm} \cdot \frac{snv}{sncv}\right)$, patet, *computum duodecim quantitatum* x, x_1, x', x_1', AM, CN, A'M', C'N', f, f_1, f', f_1' *pendere nonnisi a computo trium integralium.* Haec de curvis coniugatis sufficiant. Eaedem curvae etiam non homologe coniugatae spectari possunt; tunc vertices A et B coniugati sunt cum verticibus D et C, et crescente applicata y, decrescit applicata y_1. Si disquisitionem hanc novam suscipis, invenies curvarum proprietates geometricas pares et formulas inventis similes. Quam rem attentione dignam addigitasse hoc loco sufficiat, et ad reliquas curvarum formas considerandas iam transeamus.

27.

De catenaria hyperbolica ea, cuius coniugata elliptica ita degenerat, ut fiat circulus.

Invenimus in articulo (5), curvam ellipticam ita degenerare posse, ut fiat circulus, quo vero in casu curva catenaria hyperbolica non est circulus. Hoc evenit, si sumitur $v = \gamma$, quo fit $\delta = 0$, ideoque $A = B = \frac{1}{2}\pi - \gamma$. Si e centro lineae abscissarum describis circulum minorem radio sphaerico $= \gamma$,

is est catenaria elliptica degenerata, cuius aequatio est $y = \frac{1}{2}\pi - \gamma$. Formula artic. (5.) $\cos\delta' = \frac{\sin\nu}{\cos\gamma}$ nunc fit $\cos\delta' = \operatorname{tang} g$, ideoque $\sin\delta' = \sqrt{(1-\operatorname{tang}^2\gamma)} = \frac{\sqrt{(\cos 2\gamma)}}{\cos\gamma}$. Quia in artic. (6.) $\delta = \beta - \alpha$, fit $\alpha = \beta$, $\delta' = \frac{1}{2}\pi - 2\alpha$, aequatio $\sin(\alpha+\beta) = \frac{\sin\gamma}{\cos\nu}$ abit in $\sin 2\alpha = \operatorname{tang}\gamma$. Fiunt $e = \frac{\cos 2\gamma}{\cos\gamma} = \cos\gamma(1-\operatorname{tang}^2\gamma)$ $= \cos 2\alpha\sqrt{\cos 2\gamma}$ et $h = \frac{1}{2}(\sin^2 2\alpha . \sin 2\gamma) = \frac{\sin^3\gamma}{\cos\gamma} = \sin^2\gamma . \operatorname{tang}\gamma$

1.
$$\begin{cases} a = \cos\gamma = \dfrac{1}{\sqrt{(1+\varrho^2)}}, \\ b = \cos\gamma = \dfrac{1}{\sqrt{(1+\varrho^2)}}, \\ c = \dfrac{\sin^2\gamma - \cos\gamma\sqrt{\cos 2\gamma}}{\cos\gamma} = \sin\gamma\operatorname{tang}\gamma - \cos\gamma\sqrt{(1-\operatorname{tang}^2\gamma)} = \dfrac{\varrho^2 - \sqrt{(1-\varrho^2)}}{\sqrt{(1+\varrho^2)}}, \\ d = \dfrac{\sin^2\gamma + \cos\gamma\sqrt{\cos 2\gamma}}{\cos\gamma} = \sin\gamma\operatorname{tang}\gamma + \cos\gamma\sqrt{(1-\operatorname{tang}^2\gamma)} = \dfrac{\varrho^2 + \sqrt{(1-\varrho^2)}}{\sqrt{(1+\varrho^2)}}, \end{cases}$$

si ponitur $\varrho = \operatorname{tang}\gamma$.

Ne applicata $C = \arcsin(c)$ fiat negativa, $\operatorname{tang}^2\gamma > \sqrt{(1-\operatorname{tang}^2\gamma)}$ sive $\operatorname{tang}\gamma > \sqrt{\left(\frac{\sqrt{5}+1}{2}\right)}$ esse debet.

2.
$$\begin{cases} a' = \sin\gamma = \dfrac{\varrho}{\sqrt{(1+\varrho^2)}}, \\ b' = \sin\gamma = \dfrac{\varrho}{\sqrt{(1+\varrho^2)}}, \\ c' = \sin\gamma + \operatorname{tang}\gamma\sqrt{(\cos 2\gamma)} = \sin\gamma(1+\sqrt{(1-\operatorname{tang}^2\gamma)}) = \dfrac{\varrho(1+\sqrt{(1-\varrho^2)})}{\sqrt{(1+\varrho^2)}}, \\ d' = \sin\gamma - \operatorname{tang}\gamma\sqrt{(\cos 2\gamma)} = \sin\gamma(1-\sqrt{(1-\operatorname{tang}^2\gamma)}) = \dfrac{\varrho(1-\sqrt{(1-\varrho^2)})}{\sqrt{(1+\varrho^2)}}, \end{cases}$$

$$e = \frac{1-\varrho^2)}{\sqrt{(1+\varrho^2)}}, \qquad h = \frac{\varrho^3}{1+\varrho^3}.$$

Aequatio differentialis curvae hyperbolicae est

3. $$\partial x = \frac{h\partial y}{\cos y\sqrt{(\cos^2 y(\sin y + e)^2 - h^2)}} = \frac{h\partial z}{(1-z^2)(z+a)\sqrt{((z-c)(d-z))}}.$$

Sed ante omnia eruenda est relatio inter h et e conditionalis adimplenda, quo linea catenaria elliptica fiat circulus.

Ex aequationibus $(1+\varrho^2)e^2 = (1-\varrho)^2$ et $h^2(1+\varrho^2)^2 = \varrho^6$ per regulas Algebrae facillime invenis:

$$\varrho^2 = \tfrac{1}{2}(e^2+2-e\sqrt{(e^2+8)}), \text{ sive } 1+\varrho^2 = (\tfrac{1}{2}(\sqrt{(e^2+8)}-e))^2$$

$$\text{et } \varrho^2 = \frac{(1-e^2)^2+(4-e^2)h^2}{2-e^2-e^4+e^2h^2}.$$

Si valorem hunc substituis in aequatione $(1+\varrho^2)e^2 = (1-\varrho^2)^2$, invenis

$$(1+e^2-2e^4+(2e^2-4)h^2)^2 = (3e^2-3e^4+4e^2h^2)(2-e^2-e^4+e^2h^2),$$

quae, evoluta et reducta, est haec:

$$(1-e^2)^4 - (8+12e^2-21e^4+e^6)h^2 + 16(1-e^2)h^4 = 0,$$

ideoque, reiecto factore $1-e^2$ tribus terminis communi, migrat in satis simplicem

$$4. \quad 16h^4 - (8+20e^2-e^4)h^2 + (1-e^2)^3 = 0.$$

Adimpleta aequatione hac conditionali, alterutra saltem curva est circulus minor, quia e^2 non vero quantitas c ipsa aequationi inest.

Sit iterum area curvae $= f$, ideoque $\partial f = z.\partial x = \frac{hz.\partial z}{(1-z^2)(z+a)\sqrt{(z-c)(d-z)}}$, et erit

$$\partial x + \partial f = \frac{h\partial z}{(1-z)(z+a)\sqrt{((z-c)(d-z))}} \quad \text{et} \quad \partial x - \partial f = \frac{h\partial z}{(1+z)(z+a)\sqrt{((z-c)(d-z))}},$$

quas formalas ut integremus, ponamus

$$t = \sqrt{\frac{z-c}{d-z}}, \quad \text{et vice versa} \quad z = \frac{c+dt^2}{1+t^2},$$

quo fit $\frac{\partial z}{\sqrt{((z-c)(d-z))}} = \frac{2(d-c).\partial t}{1+t^2}$, ideoque

$$\partial x + \partial f = \frac{2h(d-c)(1+t^2).\partial t}{[(1-c)+(1-d)t^2][a+c+(a+d)t^2]}$$

$$= \frac{2h(d-c)}{1+a} \cdot \frac{\partial t}{1-c+(1-d)t^2} + \frac{2h(d-c)}{1+a} \cdot \frac{\partial t}{a+c+(a+d)t^2},$$

$$\partial x - \partial f = \frac{2h(d-c)(1+t^2).\partial t}{[(1+c)+(1+d)t^2][a+c+(a+d)t^2]}$$

$$= \frac{2h(d-c)}{1-a} \cdot \frac{\partial t}{(a+c)+(a+d)t^2} - \frac{2h(d-c)}{1-a} \cdot \frac{\partial t}{1+c+(1+d)t^2}.$$

Si igitur sumuntur tres anguli sive arcus circulorum maximorum tales, ut sit

$$\text{tang } P = t.\sqrt{\frac{1-d}{1-c}}; \quad \text{tang } Q = t.\sqrt{\frac{1+d}{1+c}}; \quad \text{tang } R = t.\sqrt{\frac{a+d}{a+c}},$$

integrando inveniuntur formulae

$$x+f = \frac{2h(d-c).R}{(1+a)\sqrt{((a+c)(a+d))}} + \frac{2h(d-c).P}{(1+a)\sqrt{((1-c)(1-d))}},$$

$$x-f = \frac{2h(d-c).R}{(1-a)\sqrt{((a+c)(a+d))}} - \frac{2h(d-c).Q}{(1-a)\sqrt{((1+c)(1+d))}}.$$

Quia ob aequationem $a = b$ valet formula $\sqrt{((1-a)(1-b)(1+c)(1+d))} = (1-a)\sqrt{((1+c)(1+d))} = h$, et $d-c = 2\cos\gamma\sqrt{(1-\text{tang}^2\gamma)} = 2\sqrt{\cos 2\gamma}$, $(a+c)(a+d) = (\frac{1}{\cos\gamma} - \sqrt{\cos 2\gamma})(\frac{1}{\cos\gamma} + \sqrt{\cos 2\gamma}) = \text{tang}^2\gamma(2+\cos 2\gamma)$, est

$$x + f = 4(1-\cos\gamma)\sqrt{\frac{\cos 2\gamma}{2+\cos 2\gamma}}.R + 4\sqrt{\cos 2\gamma}.P,$$

$$x - f = 4(1+\cos\gamma)\sqrt{\frac{\cos 2\gamma}{2+\cos 2\gamma}}.R - 4\sqrt{\cos 2\gamma}.Q,$$

unde addendo et subtrahendo concludimus

$$x = 4\sqrt{\left(\frac{\cos 2\gamma}{2+\cos 2\gamma}\right)}.R + 2\sqrt{\cos 2\gamma}.(P-Q),$$

$$f = 2\sqrt{\cos 2\gamma}.(P+Q) - 4\cos\gamma.\sqrt{\frac{\cos 2\gamma}{2+\cos 2\gamma}}.R.$$

Si ponitur $t = \text{tang}\,\varphi$, est

$$z = c\cos\cos^2\varphi + d\sin^2\varphi, \quad \text{sive}$$

$$5. \qquad \sin y = z = \frac{\varrho^2 - \cos 2\varphi.\sqrt{(1-\varrho^2)}}{\sqrt{(1+\varrho^2)}}.$$

Hinc sequitur $\text{tang}\,R = \text{tang}\,\varphi.\sqrt{\frac{a+d}{a+c}}$, ideoque

$$\text{tang}\,2R = \frac{2\sin\varphi\cos\varphi\sqrt{((a+d)(a+c))}}{(a+c)\cos^2\varphi-(a+d)\sin^2\varphi}, \quad \text{sive}$$

$$\text{tang}\,2R = \frac{\sin\gamma\sqrt{(2+\cos 2\gamma)}.\sin 2\varphi}{\cos 2\varphi - \cos\gamma\sqrt{\cos 2\gamma}}.$$

Quia autem e formula generali

$$\sqrt{((1-\sin C)(1+\sin D))} = \cos\tfrac{1}{2}(D+C) + \sin\tfrac{1}{2}(D-C), \quad \text{et}$$
$$\sqrt{((1+\sin C)(1-\sin D))} = \cos\tfrac{1}{2}(D+C) - \sin\tfrac{1}{2}(D-C)$$

sequitur

$$\sqrt{((1-c)(1-d)} = \cos\gamma + \sin\delta' = \cos\gamma + \sqrt{(1-\text{tang}^2\gamma)} = \frac{\cos^2\gamma+\sqrt{\cos 2\gamma}}{\cos\gamma}, \quad \text{et}$$

$$\sqrt{((1+c)(1-d))} = \cos\gamma - \sin\delta' = \cos\gamma - \sqrt{(1-\text{tang}^2\gamma)} = \frac{\cos^2\gamma-\sqrt{\cos 2\gamma}}{\cos\gamma},$$

facillime componimus

$$\text{tang}\,(Q-P) = \frac{\sqrt{\cos 2\gamma}}{\cos\gamma}\cdot\frac{\sin 2\varphi}{c'\cos^2\varphi+d'\sin^2\varphi} \quad \text{et} \quad \text{tang}\,(P+Q) = \frac{\cos\gamma.\sin 2\varphi}{c'\cos^2\varphi-d'\sin^2\varphi}, \quad \text{sive}$$

$$\text{tang}\,(Q-P) = \frac{\sqrt{(1-\text{tang}^2\gamma)}}{\sin\gamma}\cdot\frac{\sin 2\varphi}{1-\cos 2\varphi\sqrt{(1-\text{tang}^2\gamma)}},$$

$$\text{tang}\,(Q+P) = \cot\gamma\cdot\frac{\sin 2\varphi}{\sqrt{(1-\text{tang}^2\gamma)}-\cos 2\varphi}.$$

Si valores hos substituis, habes formulas

$$6. \qquad x = 2\sqrt{\frac{\cos 2\gamma}{2+\cos 2\gamma}}.\text{arctang}\left(\frac{\sin\gamma\sqrt{(2+\cos 2\gamma)}\sin 2\varphi}{\cos 2\varphi-\cos\gamma.\sqrt{\cos 2\gamma}}\right)$$
$$- 2\sqrt{(\cos 2\gamma)}.\text{arc tang}\left(\frac{\sqrt{(1-\text{tang}^2\varphi)}}{\sin\gamma}\cdot\frac{\sin 2\varphi}{1-\cos 2\varphi\sqrt{(1-\text{tang}^2\gamma)}}\right),$$

$$7.\quad f = 2\sqrt{\cos\gamma}\,\operatorname{arc\,tang}\left(\frac{\cot\gamma\sin 2\varphi}{\sqrt{(1-\tan g^2\gamma)-\cos 2\varphi}}\right)$$
$$-\,2\cos\gamma\sqrt{\frac{\cos\gamma}{2+\cos 2\gamma}}\,.\operatorname{arc\,tang}\left(\frac{\sin\gamma\sqrt{(2+\cos 2\gamma)}.\sin 2\varphi}{\cos 2\varphi-\cos\gamma\sqrt{\cos 2\gamma}}\right).$$

Si arcus curvae $CN=s$ ponitur, est $\partial s = \frac{(z+e)\partial z}{(z+a)\sqrt{((z-c)(d-z))}} = \frac{2(d-c)(z+e).\partial t}{(z+a)(1+t^2)}$
$= 2(d-c).\frac{c+e+(d+e)t^2}{(a+c+(d+a)t^2)(1+t^2)} = 2(d-c).\frac{\partial t}{1+t^2} - \frac{2(d-c)(a-e)\partial t}{a+c+(a+d)t^2}$. Quare, integrando, invenis formulam

$$s = 2(d-c).\varphi - \frac{2(d-c)(a-e)}{\sqrt{((a+c)(a+d))}}.R,\quad \text{sive}$$

$$8.\quad s = 4\sqrt{\cos 2\gamma}.\varphi - 4\sin\gamma\sqrt{\frac{\cos 2\gamma}{2+\cos 2\gamma}}.R$$

in qua anguli φ et R sunt iidem ac in formulis praecedentibus. Si ponimus $z=d$, fit $\varphi = P = Q = R = \frac{1}{2}\pi$, quare est

$$\text{Abscissa arcui } CD \text{ subtensa} = 2\pi.\sqrt{\frac{\cos 2\gamma}{2+\cos 2\gamma}},$$
$$\text{Arcus } CD \quad \text{ipse} \quad = 2\pi(\sqrt{\cos 2\gamma} - \sin\gamma\sqrt{\frac{\cos 2\gamma}{2+\cos 2\gamma}}),$$
$$\text{et area ad arcum } CD \text{ pertinens} = 2\pi(\sqrt{\cos 2\gamma} - \cos\gamma\sqrt{\frac{\cos 2\gamma}{2+\cos 2\gamma}}).$$

Curvae reciprocitatis lege coniunctae cum curva CD aequatio differentialis est

$$\partial x' = \frac{-(h-ez').\partial z'}{(1-z'^2)(z'+a')\sqrt{((z'-d')(c'-z'))}}.$$

Integrationibus etiam nunc inveniuntur formulae, quarum ope abscissas, areas et arcus computare potes, et quidem formulae ita constructae, ut functiones in ipsis contentae transcendentes sint arcus cyclici; quare in his rebus diutius commorari superfluum esse videtur.

28.

De catenaria hyperbolica ea, cuius coniugata elliptica est imaginaria.

Vidimus in casu III. (articuli 5.) curvam ellipticam esse imaginariam, at curvam hyperbolicam esse realem, neque esse circulum, si ν sumatur talis, ut sit contentus inter fines γ et $\frac{1}{2}\pi - \gamma$.

Sit iterum

$$e = \frac{\cos 2\gamma}{\cos\nu}\quad \text{et}\quad 2h = \sin 2\gamma.\tan g^2\nu.$$

Si insuper ponitur $\sin y = z$ et $Z = \cos^2 y(\sin y + e)^2 - h^2$, curvae hyperbolicae aequatio differentialis erit

$$1. \quad \partial x = \frac{h.\partial z}{(1-z^2).\sqrt{Z}},$$

Quia nunc

$$A = \tfrac{1}{2}\pi - \gamma - \delta, \qquad C = \gamma - \delta',$$
$$B = \tfrac{1}{2}\pi - \gamma + \delta, \quad \text{et} \quad D = \gamma + \delta',$$

in quibus $\cos\delta = \frac{\cos\nu}{\cos\gamma}$ et $\cos\delta' = \frac{\sin\gamma}{\cos\nu}$, ponamus iterum $\frac{1}{\cos\nu} = \varrho$, quo fit

$$\cos\delta = \varrho\cos\gamma \quad \text{et} \quad \cos\delta' = \varrho\sin\gamma;$$

vel, si δi ponitur loco δ, est

$$A = \tfrac{1}{2}\pi - \gamma - \delta i, \qquad C = \gamma - \delta',$$
$$B = \tfrac{1}{2}\pi - \gamma + \delta i, \qquad D = \gamma + \delta';$$

in quibus quantitates δ et δ' determinantur formulis his:

$$2. \quad \mathfrak{Cos}\,\delta = \varrho\cos\gamma \quad \text{et} \quad \cos\delta' = \varrho\sin\gamma.$$

Sit insuper

$$3. \quad \mathfrak{Sin}\,\delta = \varepsilon = \sqrt{(\varrho^2\cos^2\gamma - 1)} \quad \text{et} \quad \varepsilon' = \sin\delta' = \sqrt{(1-\varrho^2\sin^2\gamma)}.$$

Quia nunc

$$a = \sin A = \cos\gamma\,\mathfrak{Cos}\,\delta - i\sin\gamma\,\mathfrak{Sin}\,\delta, \qquad c = \sin\gamma\cos\delta' - \cos\gamma\sin\delta' = \sin C,$$
$$b = \sin B = \cos\gamma\,\mathfrak{Cos}\,\delta + i\sin\gamma\,\mathfrak{Sin}\,\delta, \qquad d = \sin\gamma\cos\delta' + \cos\gamma\sin\delta' = \sin D,$$
$$a' = \cos A = \sin\gamma\,\mathfrak{Cos}\,\delta + i\cos\gamma\,\mathfrak{Sin}\,\delta, \qquad c' = \cos\gamma\cos\delta' + \sin\gamma\sin\delta' = \cos C,$$
$$b' = \cos B = \sin\gamma\,\mathfrak{Cos}\,\delta - i\cos\gamma\,\mathfrak{Sin}\,\delta, \qquad d' = \cos\gamma\cos\delta' - \sin\gamma\sin\delta' = \cos D,$$

oriuntur formulae satis simplices:

$$4. \quad \begin{cases} a = \varrho\cos^2\gamma - i\varepsilon\sin\gamma, \\ b = \varrho\cos^2\gamma + i\varepsilon\sin\gamma, \\ c = \varrho\sin^2\gamma - \varepsilon'\cos\gamma, \\ d = \varrho\sin^2\gamma + \varepsilon'\cos\gamma, \end{cases} \quad \text{et} \quad \begin{cases} a' = \varrho\sin\gamma\cos\gamma + i\varepsilon\cos\gamma, \\ b' = \varrho\sin\gamma\cos\gamma - i\varepsilon\cos\gamma, \\ c' = \varrho\sin\gamma\cos\gamma + \varepsilon'\sin\gamma, \\ d' = \varrho\sin\gamma\cos\gamma - \varepsilon'\sin\gamma; \end{cases}$$

$$5. \quad e = \varrho\cos 2\gamma, \qquad 2h = (\varrho^2 - 1)\sin 2\gamma;$$

$$Z = (z-c)(d-z)(z^2 + (a+b)z + ab).$$

Adhibeamus formulas in (§. 233.) theoriae functionum modularium demonstratas. Sit

$$n^2 = c^2 + (a+b)c + ab = (c+a)(c+b) \quad \text{et}$$
$$n'^2 = d^2 + (a+b)d + ab = (d+a)(d+b),$$

sive

$$n^2 = (\varrho - \varepsilon'\cos\gamma)^2 + \varepsilon^2\sin^2\gamma = \varrho^2 + \cos 2\gamma - 2\varrho\varepsilon'\cos\varrho,$$
$$n'^2 = (\varrho + \varepsilon'\cos\gamma)^2 + \varepsilon^2\sin^2\gamma = \varrho^2 + \cos 2\gamma + 2\varrho\varepsilon'\cos\gamma,$$

$$6. \quad n^2 n'^2 = (\varrho^2-1)^2 + (\varrho^4-1)\sin^2 2\gamma = (\varrho^2-1)^2\left(1 + \frac{\varrho^2+1}{\varrho^2-1}\sin^2 2\gamma\right).$$

Quia

7.
$$\begin{cases} ab = \varrho^2\cos^2\gamma - \sin^2\gamma, & cd = \varrho^2\sin^2\gamma - \cos^2\gamma, \\ a+b = 2\varrho\cos^2\gamma, & c+d = 2\varrho\sin^2\gamma, \\ 1+ab = (1+\varrho^2)\cos^2\gamma, & 1+cd = (1+\varrho^2)\sin^2\gamma, \\ \text{inveniuntur} & \\ \sqrt{((1+a)(1+b))} = (\varrho+1)\cos\gamma, & \sqrt{((1+c)(1+d))} = (\varrho+1)\sin\gamma, \\ \sqrt{((1-a)(1-b))} = (\varrho-1)\cos\gamma, & \sqrt{((1-c)(1-d))} = (\varrho-1)\sin\gamma. \end{cases}$$

Est insuper

$$8. \quad m = \sqrt{(ab - \tfrac{1}{2}(a+b)^2)} = \sqrt{(-\tfrac{1}{2}(a-b)^2)} = \varepsilon\sin\gamma$$

et

$$cd + \tfrac{1}{2}(c+d)(a+b) + ab = \varrho^2\cos^2\gamma - \sin^2\gamma + 2\varrho\sin^2\gamma.\varrho\cos^2\gamma + \varrho^2\sin^2\gamma - \cos^2\gamma,$$

sive

$$9. \quad cd + \tfrac{1}{2}(c+d)(a+b) + ab = \varrho^2 - 1 + \tfrac{1}{2}\varrho^2\sin^2 2\gamma.$$

Quare, ponendo $cd + \frac{1}{2}(c+d)(a+b) + ab = nn'.\mu$, fit

10.
$$\begin{cases} \mu = \cos 2\theta = \dfrac{\varrho^2 - 1 + \frac{1}{2}\varrho^2\sin^2 2\gamma}{(\varrho^2-1)\sqrt{(1+\frac{\varrho^2+1}{\varrho^2-1}\sin^2 2\gamma)}} = \dfrac{1+\frac{\varrho^2}{2\varrho^2-2}\sin^2 2\gamma}{\sqrt{(1+\frac{\varrho^2+1}{\varrho^2-1}\sin^2 2\gamma)}}, \\ \sqrt{(1-\mu^2)} = \sin 2\theta = \dfrac{\varepsilon\varepsilon'\sin 2\gamma}{(\varrho^2-1)\sqrt{(1-\frac{\varrho^2+1}{\varrho^2-1}\sin^2 2\gamma)}}, \\ \operatorname{tang} 2\theta = \dfrac{\varepsilon\varepsilon'\sin 2\gamma}{\varrho^2 - 1 + \frac{1}{2}\varrho^2\sin^2 2\gamma}. \end{cases}$$

Si adhibito modulo

11. $\lambda = \sqrt{(\frac{1}{2}(1-\mu))} = \sin\theta$ et coniugato $\lambda' = \sqrt{(\frac{1}{2}(1+\mu))} = \cos\theta$, ponimus

$$12. \quad \operatorname{tang}\tfrac{1}{2}am\,v = \sqrt{\frac{n'(z-c)}{n(d-z)}},$$

et vice versa

$$z = \frac{nd(1-cnv) + n'c(1+cnv)}{n(1-cnv) + n'(1+cnv)} = \frac{nd + n'c + (n'c - nd)\,cnv}{n' + n + (n'-n)\,cnv},$$

erit $\frac{\partial z}{\sqrt{Z}} = \frac{\partial u}{\sqrt{(nn')}}$. Formulae $\lambda = \sqrt{(\frac{1}{2}(1-\mu))}$ et $\lambda' = \sqrt{(\frac{1}{2}(1+\mu))}$ etiam sic exhiberi possunt:

$$\lambda = \sqrt{\frac{2\sqrt{((c+a)(c+b)(d+a)(d+b))} - 2ab - 2cd - (c+d)(a+b)}{4\sqrt{[(c+a)(c+b)(d+a)(d+b)]}}},$$

$$\lambda' = \sqrt{\frac{2\sqrt{((c+a)(c+b)(d+a)(d+b))} + 2ab + 2cd + (c+d)(a+b)}{4\sqrt{[(c+a)(c+b)(d+a)(d+b)]}}},$$

et quia $2ab + 2cd + (c+d)(a+b) = (d+a)(c+b) + (c+a)(d+b)$, invenis

$$13.\quad \left\{\begin{aligned} \lambda &= \frac{\sqrt{((d+a)(c+b))}-\sqrt{((c+a)(d+a))}}{2i\sqrt{(nn')}} \\ &= \frac{\sqrt{(\varrho^2-1+\frac{1}{2}\varrho^2\sin^2 2\gamma+i\varepsilon\varepsilon'\sin 2\gamma)}-\sqrt{((\varrho^2-1+\frac{1}{2}\varrho^2\sin^2 2\gamma-i\varepsilon\varepsilon'\sin 2\gamma)}}{2\sqrt{(nn')}}, \\ \lambda' &= \frac{\sqrt{((d+a)(c+b))}+\sqrt{((c+a)(d+b))}}{2\sqrt{(nn')}} \\ &= \frac{\sqrt{(\varrho^2-1+\frac{1}{2}\varrho^2\sin^2 2\gamma+i\varepsilon\varepsilon'\sin 2\gamma)}+\sqrt{(\varrho^2-1+\frac{1}{2}(\varrho^2\sin^2 2\gamma-i\varepsilon\varepsilon'\sin 2\gamma)}}{2\sqrt{(nn')}}, \end{aligned}\right.$$

unde perspicis, modulus λ et λ' et aliis modis computari posse.

Sit iterum f area curvae ad arcum CN pertinens, sive $\partial f = \frac{z\partial z}{(1-z^2)\sqrt{Z}} = \frac{hz\partial u}{(1-z^2)\sqrt{(nn')}}$, ideoque

$$\partial x + \partial f = \frac{h}{\sqrt{(nn')}}\cdot\frac{\partial u}{1-z} \quad \text{et} \quad \partial x - \partial f = \frac{h}{\sqrt{(na')}}\cdot\frac{\partial u}{1+z}.$$

Quia

$$1+z = \frac{n(1+d)+n'(1+c)+(n'(1+c)-n(1+d))cnv}{n'+n+(n'-n)cnv},$$

$$1-z = \frac{n(1-d)+n'(1-c)+(n'(1-c)-n(1-d))cnv}{n+n'+(n'-n)cnv},$$

oriuntur formulae integrandae:

$$\partial x + \partial f = \frac{h}{\sqrt{(nn')}}\cdot\frac{n+n'+(n'-n)cnv}{n(1-d)+n'(1-c)+(n'(1-c)-n(1-d))cnv}\cdot\partial v,$$

$$\partial x - \partial f = \frac{h}{\sqrt{(nn')}}\cdot\frac{n+n'+(n'-n)cnv}{n(1+d)+n'(1+c)+(n'(1+c)-n(1+d))cnv}\cdot\partial v.$$

Quia fractio formae $\frac{A+Bcnv}{C+Dcnv} = \frac{B}{D} + \frac{AD-BC}{CD}\cdot\frac{1}{1+\frac{D}{C}cnv}$, formula prima reducitur ad

$$x+f = \frac{h(n'-n)v}{(n'(1-c)-n(1-d)\sqrt{(nn')}} + \frac{2h(d-c).\sqrt{(nn')}}{n'^2(1-c)^2-n^2(1-d)^2}\cdot\int_0\frac{\partial v}{1+g\cdot cnv},$$

$$\text{si } g = \frac{n'(1-c)-n(1-d)}{n'(1-c)+n(1-d)} \text{ ponitur, et altera ad}$$

$$x-f = \frac{h(n'-n)v}{(n'(1+c)-n(1+d)\sqrt{(nn')}} - \frac{2h(d-c)\sqrt{(nn')}}{n'^2(1+c)^2-n^2(1+d)^2}\cdot\int_0\frac{\partial v}{1+g'\cdot cnv},$$

$$\text{si } g' = \frac{n'(1+c)-n(1-d)}{n'(1+c)+n(1+d} \text{ ponimus.}$$

29.

His praeparatis integrationes absolvimus. Comparemus integrale $\int_0\frac{\partial v}{1+g.cnv}$ cum formula (2.) in (§. 212.) theoriae functionum modularium demonstrata:

$$\int_0 \frac{tnc'p\,dnc'p\,.\partial v}{1-cnc'p\,.cnv} = \frac{v}{sn'p\,snc'p} - \text{arc tang}\left(\frac{\lambda\, cnc\, v}{\lambda' cn'p}\right) - \mathbf{S}(c, p).$$

Quem in finem ponamus $cnc'p = g$, sive

$$cnc'p = \frac{n'(1-c)-n(1-d)}{n'(1-c)+n(1-d)}.$$

Hac ex formula deducimus facillime $\frac{1-cnc'p}{1+cnc'p} = \frac{n(1-d)}{n'(1-c)}$, sive

$$1. \qquad \text{tang}\,\tfrac{1}{2}amc'p = \sqrt{\frac{n(1-d)}{n'(1-c)}}.$$

Porro est

$$snc'p = \frac{2\sqrt{(nn')}.\sqrt{((1-c)(1-d))}}{n'(1-c)+n(1-d)};\ \lambda' snc'p = \frac{\sqrt{((d+a)(c+b))}+\sqrt{((c+a)(d+b))}}{n'(1-c)+n(1-d)}\sqrt{((1-c)(1-d))},$$

ideoque $dnc'p = \sqrt{(1-\lambda'^2 snc'^2 p)} = \frac{(d-c)\sqrt{((1+a)(1+b))}}{n'(1-c)+n(1-d)}$,

$$tnc'p = \frac{2\sqrt{(nn)}.\sqrt{((1-c)(1-d))}}{n'(1-c)-n(1-d)}.$$

E formulis his componis $tnc'p.dnc'p = \frac{2(d-c)\sqrt{(nn')}.\sqrt{((1+a)(1+b)(1+c)(1+d))}}{n'^2(1-c)^2-n^2(1-d)^2}$, sive

$$tnc'p\,.dnc'p = \frac{2h(d-c)\sqrt{(nn')}}{n'^2(1-c)^2-n^2(1-d)^2}, \text{ quare est}$$

$$x+f = \frac{h(n'-n)v}{(n'(1-c)-n(1-d))\sqrt{(nn')}} + \frac{v}{sn'p\,snc'p} - \text{arc tang}\left(\frac{\lambda\, cnc\, v}{\lambda' cn'p}\right) - \mathbf{S}(c, p).$$

Quia est $sn'p = \frac{cnc'p}{dnc'p}$, illico obtines

$$sn'p = \frac{n'(1-c)-n(1-d)}{(d-c)\sqrt{((1+a)(1+b))}}, \text{ ideoque}$$

$$\frac{1}{snc'p\,snc'p} = \frac{d-c}{2\sqrt{(nn')}}\cdot\frac{n'(1-c)+n(1-d)}{n'(1-c)-n(1-d)}\cdot\frac{\sqrt{((1+a)(1+b))}}{\sqrt{((1-c)(1-d))}}$$
$$= \frac{h(d-c)}{2(1-c)(1-d)\sqrt{(nn')}}\cdot\frac{n'(1-c)+n(1-d)}{n'(1-c)-n(1-d)};$$

quo valore substituto formulam contrahes ad satis simplicem hanc:

$$x+f = \frac{h(1-\frac{1}{2}(c+d))v}{(1-c)(1-d).\sqrt{(nn')}} - \text{arc tang}\left(\frac{\lambda\, cnc\, v}{\lambda' cn'p}\right) - \mathbf{S}(c, p),$$

in qua valores $1-\varrho\sin^2\gamma$ pro $1-\frac{1}{2}(c+d)$ et $(\varrho-1)^2.\sin^2\gamma$ pro $(1-c)(1-d)$ substituendi sunt.

Si formulam differentialem, qua $\partial x - \partial f$ exprimitur, comparas cum ea, quae summam $\partial x+\partial f$ perhibet, videbis, permutanda esse $-a, -b, -c, -d$ cum a, b, c, d; quo si parameter p abit in q, valent formulae

$$\text{tang } \tfrac{1}{2} amc' q = \sqrt{\frac{n(1+d)}{n'(1+c)}},$$

$$cnc' q = \frac{n'(1+c)-n(1+d)}{n'(1+c)+n(1+d)},$$

$$snc' q = \frac{2\sqrt{(nn')}.\sqrt{((1+a)(1+d))}}{n'(1+c)+n(1+d)},$$

$$dnc' q = \frac{(d-c)\sqrt{((1+a)(1+b))}}{n'(1+c)+n(1+d)},$$

$$tnc' q = \frac{2\sqrt{(nn')}.\sqrt{((1+c)(1+d))}}{n'(1+c)-n(1+d)},$$

$$sn' q = \frac{n'(1+c)-n(1+d)}{(d-c)\sqrt{((1-a)(1-b))}},$$

et

$$x - f = \frac{h(1+\frac{1}{2}(c+d))v}{(1+c)(1+d)\sqrt{(nn')}} + \text{arc tang}\left(\frac{\lambda\, cnc\, v}{\lambda' cn' q}\right) + \mathbf{S}(v, q).$$

Si vis abscissam x et aream f eo extendere, ut pertineant ad arcum CD inter vertices proximos contentum, est $z = d$, ideoque $v = 2L$, quare nunc est $cnc\, v = 0$, et

$$X + F = \frac{h(2-c-d)L}{(1-c)(1-d)\sqrt{(nn')}} - \mathbf{S}(2L, p),$$

$$X - F = \frac{h(2+c+d)L}{(1+c)(1+d)\sqrt{(nn')}} + \mathbf{S}(2L, q).$$

30.

Derivatu facilior est formula, quae longitudinem arcus hyperbolicis $CN = s$ exprimit. Quia est $\partial s = \frac{(z+c)\partial z}{\sqrt{Z}} = \frac{(z+e)\partial v}{\sqrt{(nn')}}$, est

$$s = \frac{ev}{\sqrt{(nn')}} + \frac{1}{\sqrt{(nn')}} \int_0 \frac{nd+n'c+(n'c-nd)\,cnv}{n'+n+(n'-n)\,cnv}.\partial v,$$

quae, adhibita formula $\frac{A+B\,cnu}{C+D\,cnu} = \frac{B}{D} + \frac{AD-BC}{CD} \cdot \frac{1}{1+\frac{D}{C}cnu}$, mutatur in

$$s = \frac{ev}{\sqrt{(nn')}} + \frac{(n'c-nd)}{n'-n} \cdot \frac{v}{\sqrt{(nn')}} + \frac{2\sqrt{(nn')}.(d-c)}{n'^2-n^2} \cdot \frac{\partial v}{1+\frac{n'-n}{n'+n}cnv},$$

quare integratio contingit adhibita eadem formula generali, qua in articulo praeced. usi sumus. Eligamus parametri signum r et ponamus nunc

$$cnc'r = \frac{n'-n}{n'+n}, \quad \text{quo fiunt}$$

$$snc'r = \frac{2\sqrt{(nn')}}{n'+n},$$

$$\operatorname{tang} \tfrac{1}{2} amc'r = \sqrt{\frac{n}{n'}},$$

$$tnc'r = \frac{2\sqrt{(nn')}}{n'-n},$$

$$\lambda' snc\, r = \frac{\sqrt{((d+a)(c+b))}+\sqrt{((c+a)(d+b))}}{n'+n},$$

$$dnc'r = \frac{d-c}{n'+n},$$

$$sn'r = \frac{n'-n}{d-c},$$

unde compones $tnc'r\, dnc'r = \frac{2(d-c)\sqrt{(nn')}}{n'^2-n^2}$ et $\frac{1}{sn'r\, snc'r} = \frac{d-c}{2\sqrt{(nn')}} \cdot \frac{n'+n}{n'-n}$, quare est

$$s = \frac{ev}{\sqrt{(nn')}} + \frac{n'c-nd}{n'-n} \cdot \frac{v}{\sqrt{(nn')}} + \frac{d-c}{2\sqrt{(nn')}} \cdot \frac{n'+n}{n'-n} \cdot v - \operatorname{arc\,tang}\left(\frac{\lambda cncv}{\lambda' cn'r}\right) - \mathbf{S}(v, r),$$

sive, magis reducta,

$$s = \frac{(e+\frac{1}{2}(c+d))v}{\sqrt{(nn')}} - \operatorname{arc\,tang}\left(\frac{\lambda cncv}{\lambda' cn'r}\right) - \mathbf{S}(v, r).$$

Si arcus s extenditur a puncto sive vertice C usque ud verticem proximum D, arcus exprimitur formula

$$CD = \frac{(2e+c+d)L}{\sqrt{(nn')}} - \mathbf{S}(2L, r).$$

Formulae ab inventis parum diversae inveniuntur, quibus tum abscissa x', tum area f' curvae reciprocae et arcus reciprocus $C'N'$ ipse exprimuntur; sed has res difficultatibus obnoxias nullis mittamus nunc omnes, cum relationes inter curvas reciprocas intercedentes sint notissimae.

Monasterii Guestph. d. 8. Martii 1845.