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Holomorphic flows and complex structures on products of odd dimensional spheres

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0 Introduction

The aim of this paper is the construction and the study of a class $\mathscr{C}$ of complex structures on a product $S^{m,n} = S^{2m-1} \times S^{2n-1}$ of odd dimensional spheres. Any complex manifold in the class $\mathscr{C}$ is a subset of the orbit space of a certain holomorphic flow in $\mathbb{C}^{m+n}$.

Historically the first result concerning the construction of complex structures on $S^{m,n}$ is due to B. Riemann who gave the complete description of the elliptic curves. For $m = 1$, $n > 1$ the first examples were provided by H. Hopf. It is known (cf. [Ha]) that any structure of complex manifold on $S^{1,n}$ whose universal covering is $\mathbb{C}^{n+1} - \{0\}$ is obtained as the quotient of $\mathbb{C}^{n+1} - \{0\}$ by a contracting holomorphic automorphism of $\mathbb{C}^{n+1}$ which fixes the origin. Such a manifold is called a Hopf manifold. In the simply-connected case a class of complex structures was constructed by E. Calabi and B. Eckmann [C-E]. These manifolds are the total space of a holomorphic principal bundle over $\mathbb{P}^{m-1} \times \mathbb{P}^{n-1}$ with fiber an elliptic curve.

The class $\mathscr{C}$ contains all the above examples. For $m = 1$ it consists exactly of the elliptic curves and the Hopf manifolds but for $m, n > 1$ the construction of the class $\mathscr{C}$ increases largely the number of examples. Furthermore the method of construction unifies the Hopf case ($m = 1$ or $n = 1$) with the non Hopf case ($m, n > 1$) and allows us to give a quite precise description of the geometric properties of the complex manifolds so obtained.

Our approach is greatly inspired in Haefliger's paper [Ha] where he describes the versal deformation of the transversely holomorphic foliation on S^{2N-1} induced by a holomorphic flow in $\mathbb{C}^N$ having the origin as an attracting fixed point and meeting transversely the unit sphere. In an Appendix to that

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paper Haefliger shows that the deformations of a Hopf manifold can be studied in a similar manner by substituting the flow by an automorphism of $\mathbb{C}^N$.

We begin also like Haefliger with the contracting flow induced by a vector field ξ in the Poincaré domain written in normal form. This means that

$$\xi = \xi_0 + \xi_1 + \xi_2$$

where

$$\xi_0 = \sum_{k=1}^{N} \lambda_k \zeta_k \frac{\partial}{\partial \zeta_k},$$

with $\operatorname{Im} \lambda_k > 0$, and where $\xi_0 + \xi_1$ is the linear part of ξ in Jordan normal form and $\xi_1 + \xi_2$ commutes with ξ_0. But we add a weak hyperbolicity hypothesis which is essential for our purposes. We assume that the arguments of the eigenvalues λ_k are not all the same. That is, there is $m \in \{1, \ldots, N-1\}$ with $\arg \lambda_i < \arg \lambda_j$ for $1 \leqq i \leqq m$ and $m+1 \leqq j \leqq N$. In this situation $V^{m,n} = \mathbb{C}^N - (\mathbb{C}^m \times \{0\}) \cup (\{0\} \times \mathbb{C}^n)$, where $n = N - m$, is saturated by the flow of ξ. In fact the restriction of the flow to $V^{m,n}$ is a fibration with base space identified to the product of spheres $S^{m,n}$ and this product inherits a structure of complex manifold denoted by $S_\xi^{m,n}$. The class $\mathscr{C}$ is formed by the complex manifolds constructed in this manner.

The paper is organized as follows. Section 1 is devoted to the construction of the complex structures on $S^{m,n}$. The description of the elements in the class $\mathscr{C}$ is given by Theorem 1 in Sect. 1.3. The proof of this result is based in the fact, already used in [Ha], that under the assumptions made the vector field ξ can be triangulated and that consequently its associated flow can be written in terms of exponential polynomials. In Sect. 1.2 we give a characterization of the Lie group of the holomorphic automorphisms of $\mathbb{C}^N$ which keep fixed the diagonal part ξ_0 of ξ. This group will play an important role along the paper. The section ends with the study of certain separability properties of the orbit space of the flow associated to ξ.

In Sect. 2 we study the cohomology and the deformations of the complex manifolds in the class $\mathscr{C}$. In particular we give in Sect. 2.3 a complete description of their Dolbeault's cohomology. This result generalizes at the same time the computations made by D. Mall for Hopf manifolds (cf. [Ma]) and by A. Borel for Calabi-Eckmann manifolds (cf. [Bo]). The proof makes use of several exact sequences of sheaves which are generalizations of sequences already introduced by A. Douady and A. Haefliger (cf. [Do] and [Ha]). We also consider the cohomology of the manifolds $S_\xi^{m,n}$ with values in the sheaf of germs of holomorphic vector fields. This computation is carried out in a complete way only for a generic case. Nevertheless we are still able, also following the ideas in [Ha], to use this result in order to give an explicit description of the versal space of deformations of the complex manifolds $S_\xi^{m,n}$ in almost all the cases. More precisely for any manifold when $m, n > 2$ and in a generic situation when $m = 2$ or $n = 2$ (Theorem 3). The result obtained here is comparable with that of Haefliger in [Ha] and states in particular that, except for some exceptional cases when the underlying manifold is $S^3 \times S^{2n-1}$, any

(small) deformation of $S_\xi^{m,n}$ is still in the class $\mathscr{C}$, i.e. associated to another vector field ξ'.

Section 3 is concerned with the holomorphic line bundles and the analytic subsets of a manifold in the class $\mathscr{C}$. We make use of the cohomology computations in the previous section. In particular, the fact that the Hodge number $h^{0,1} = \dim H^1(S_\xi^{m,n}, \mathcal{O})$ is equal to 1 for any $S_\xi^{m,n}$, is the key in the description of the holomorphic $\mathbb{C}$-principal bundles and the holomorphic line bundles over $S_\xi^{m,n}$ (Propositions 12 and 13). As a consequence we obtain precise information about the meromorphic functions on such a manifold (Corollary 4). In Theorem 5 we compute, in a generic case, the cohomology of $S_\xi^{m,n}$ with values in a line bundle. This generalizes partially some results obtained by D. Mall for Hopf manifolds.

The analytic subvarieties of Hopf manifolds were studied by M. Kato in [Ka2]. We study the analytic subsets of $S_\xi^{m,n}$ when $m,n > 1$ in Sect 3.2 obtaining a complete picture for subvarieties Y of dimension bigger or equal to $\max(m,n)$. Under this assumption on the dimension the pull-back to $V^{m,n}$ of Y extends to a subvariety $\tilde{Y}$ of $\mathbb{C}^N$ by virtue of Remmert's extension theorem. Then the invariance of $\tilde{Y}$ by the flow of ξ enables us to describe it quite precisely showing in particular that it must be algebraic (Theorem 6). The study of the analytic subsets of small dimension remains open. In this section we also show that all the manifolds $S_\xi^{m,n}$, including Hopf manifolds, are endowed with a certain holomorphic vector field without zeros. The flow of this vector field saturates all the subvarieties of positive dimension in the case of Hopf manifolds and any subvariety of dimension greater or equal to $\max(m,n)$ when $m,n > 1$.

In the last section we obtain a classification of the complex manifolds in the class $\mathscr{C}$ in terms of the vector fields ξ (Theorem 7). The statement specifies the conditions that two vector fields ξ and ξ' must satisfy in order to define isomorphic complex manifolds $S_\xi^{m,n}$ and $S_{\xi'}^{m,n}$. For instance if $m,n > 1$ and the vector fields ξ and ξ' are linear then the corresponding manifolds are isomorphic if and only if the vector fields are linearly conjugated. The proof of this result uses in an essential way the description of the holomorphic $\mathbb{C}$-principal bundles over the manifolds $S_\xi^{m,n}$ given in the previous section.

1 Constructing complex structures on $S^{2m-1} \times S^{2n-1}$

1.1 Diagonal vector fields

Let $\lambda = (\lambda_1, \ldots, \lambda_N)$ be a sequence of complex numbers all of them different from zero. Associated to λ there is the diagonal vector field

$$\xi_0 = \lambda_1 \zeta_1 \frac{\partial}{\partial \zeta_1} + \ldots + \lambda_N \zeta_N \frac{\partial}{\partial \zeta_N}, \tag{1}$$

where $\zeta_1, \ldots, \zeta_N$ denote the linear coordinates of $\mathbb{C}^N$. The vector field ξ_0 has an isolated singularity at the origin and induces a (non-singular) foliation $\widehat{\mathscr{F}}_{\xi_0}$

on $\mathbb{C}^N - \{0\}$. The sequence λ (and by extension the vector field ξ_0) is said to belong to the *Poincaré domain* if the convex hull of the points $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$ does not contain $0 \in \mathbb{C}$. In this case the foliation $\widehat{\mathscr{F}}_{\xi_0}$ is transverse to the unit sphere of $\mathbb{C}^N$.

Vector fields in the Poincaré domain enjoy remarkable properties that will be recalled in Sect. 1.2. Nevertheless our interest in Poincaré vector fields is mainly motivated by the transversality property stated in Corollary 1 below.

Let us introduce some notation. The sequence λ is called *hyperbolic* if $\lambda_k \notin \mathbb{R} \cdot \lambda_l$ for $k \neq l$. We will say that λ is *weakly hyperbolic* if there are $k \neq l$ with $\lambda_k \notin \mathbb{R} \cdot \lambda_l$. Let $m, n \in \mathbb{N}^*$ with $m + n = N$ be given. We denote a point ζ in $\mathbb{C}^N$ by a couple (z, w) where $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$ and $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$. Set $L_1 = \{(z,w) | w = 0\}$, $L_2 = \{(z,w) | z = 0\}$ and $V^{m,n} = \mathbb{C}^N - (L_1 \cup L_2) = (\mathbb{C}^m - \{0\}) \times (\mathbb{C}^n - \{0\})$. Clearly ξ_0 is tangent to L_1 and L_2 and therefore $V^{m,n}$ is saturated by the orbits of the corresponding flow. We denote by $\mathscr{F}_{\xi_0}$ the restriction of $\widehat{\mathscr{F}}_{\xi_0}$ to $V^{m,n}$. Let $S^{m,n} = S^{2m-1} \times S^{2n-1}$ be the product of the unit spheres of $\mathbb{C}^m$ and $\mathbb{C}^n$. We will also consider the cones in $\mathbb{C}$ defined by

$$C_1 = \left\{ \rho \cdot \sum_{i=1}^{m} \alpha_i \lambda_i \mid \alpha_i \geqq 0,\ \sum_{i=1}^{m} \alpha_i = 1,\ \rho \in \mathbb{R} \right\}$$

$$C_2 = \left\{ \rho \cdot \sum_{j=1}^{n} \beta_j \lambda_{m+j} \mid \beta_j \geqq 0,\ \sum_{j=1}^{n} \beta_j = 1,\ \rho \in \mathbb{R} \right\}.$$

For the sake of clarity we will sometimes denote $\mu_i = \lambda_i$ for $i \in \{1, \ldots, m\}$ and $\nu_j = \lambda_{m+j}$ for $j \in \{1, \ldots, n\}$.

Proposition 1. *The natural embedding $\iota : S^{m,n} \to \mathbb{C}^N$ is transverse to $\mathscr{F}_{\xi_0}$ if and only if $\lambda_k \notin \mathbb{R}^- \lambda_l$ for any $k, l \in \{1, \ldots, N\}$ and $C_1 \cap C_2 = \{0\}$. In this case λ belongs to the Poincaré domain and each leaf of $\mathscr{F}_{\xi_0}$ meets the product of spheres $S^{m,n}$ at most at one point.*

Corollary 1. *There is a couple of positive integers m, n with $N = m + n$ and a permutation Ψ of the linear coordinates of $\mathbb{C}^N$ such that $\Psi \circ \iota : S^{m,n} \to \mathbb{C}^N$ is transverse to $\mathscr{F}_{\xi_0}$ if and only if λ belongs to the Poincaré domain and is weakly hyperbolic.*

Proof of the Proposition. Set $v = \sum_{i=1}^{m} p_i \partial/\partial z_i + \sum_{j=1}^{n} q_j \partial/\partial w_j$. The real vector $v + \bar{v}$ is tangent to $S^{m,n}$ at a point (z, w) if and only if

$$\mathrm{Re}\left(\sum_{i=1}^{m} \bar{z}_i p_i \right) = \mathrm{Re}\left(\sum_{j=1}^{n} \bar{w}_j q_j \right) = 0 .$$

Therefore $\mathscr{F}_{\xi_0}$ is not transverse to $S^{m,n}$ at a point (z, w) if and only if there is $a \in \mathbb{C}^*$ such that

$$\sum_{i=1}^{m} \mathrm{Re}(a\mu_i) |z_i|^2 = \sum_{j=1}^{n} \mathrm{Re}(a\nu_j) |w_j|^2 = 0 .$$

Since $\sum |z_i|^2 = \sum |w_j|^2 = 1$ the non-transversality of $\mathscr{F}_{\xi_0}$ to $S^{m,n}$ (at some point) is equivalent to the existence of a complex number $a \in \mathbb{C}^*$ and non-negative real numbers $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$ with $\sum \alpha_i = \sum \beta_j = 1$ and fulfilling

$$\sum_{i=1}^{m} \mathrm{Re}(a\mu_i)\alpha_i = \sum_{j=1}^{n} \mathrm{Re}(av_j)\beta_j = 0 . \tag{2}$$

If we set $\gamma_1 = \sum_{i=1}^{m} \mu_i \alpha_i$ and $\gamma_2 = \sum_{j=1}^{n} v_j \beta_j$ then condition (2) is equivalent to the $\mathbb{R}$-linear dependence of the complex numbers γ_1 and γ_2. If γ_1 and γ_2 are both different from zero then the linear dependence means $C_1 \cap C_2 \neq \{0\}$. The linear dependence is also verified if γ_1 or γ_2 are zero. Suppose for instance $\gamma_1 = 0$. Then at least two of the coefficients α_i are different from 0, for example α_1 and α_2. If $\alpha_i = 0$ for $i \geqq 2$ then $\lambda_1 \in \mathbb{R}^- \lambda_2$. If not then $C_1 = \mathbb{C}$ and $C_1 \cap C_2 \neq \{0\}$.

Let us assume now that $\mathscr{F}_{\xi_0}$ is transverse to $S^{m,n}$. The leaf of $\mathscr{F}_{\xi_0}$ through $(z, w) \in S^{m,n}$ is the set of points of the form

$$(z_1 e^{\mu_1 t}, \ldots, z_m e^{\mu_m t}, w_1 e^{\nu_1 t}, \ldots, w_n e^{\nu_n t})$$

for $t \in \mathbb{C}$. If that leaf meets $S^{m,n}$ at another point then there is $t \neq 0$ such that

$$\sum_{i=1}^{m} |z_i|^2 e^{2\mathrm{Re}(\mu_i t)} = \sum_{j=1}^{n} |w_j|^2 e^{2\mathrm{Re}(\nu_j t)} = 1 .$$

These identities are fulfilled for some $(z, w) \in S^{m,n}$ if and only if there are $i_1, i_2 \in \{1, \ldots, m\}$ and $j_1, j_2 \in \{1, \ldots, n\}$ such that $\mathrm{Re}(\mu_{i_1} t), \mathrm{Re}(\nu_{j_1} t) \leqq 0$ and $\mathrm{Re}(\mu_{i_2} t), \mathrm{Re}(\nu_{j_2} t) \geqq 0$. This is equivalent to the existence of $t \in \mathbb{C}^*$ and non-negative real numbers r_1, r_2, s_1, s_2 with $r_1 + r_2 > 0$ and $s_1 + s_2 > 0$ such that

$$\mathrm{Re}(r_1 \mu_{i_1} t + r_2 \mu_{i_2} t) = \mathrm{Re}(s_1 \nu_{j_1} t + s_2 \nu_{j_2} t) = 0 .$$

But this equality implies (2) contradicting the transversality condition.

It remains to prove that λ belongs to the Poincaré domain. If there were a relation $\sum_{k=1}^{N} c_k \lambda_k = 0$ with $c_k \geqq 0$ and $\sum c_k = 1$ then we would have $\sum_{i=1}^{m} c_i \mu_i = -\sum_{j=1}^{n} c_{m+j} \nu_j$. This means that $C_1 \cap C_2 \neq \{0\}$ or that the two terms vanish. Since at least one of the coefficients c_k is different from zero the second possibility also contradicts the hypothesis of transversality. □

1.2 Resonant vector fields of type λ

A λ-resonant monomial vector field on $\mathbb{C}^N$ is a vector field of the form

$$a \cdot \zeta_1^{p_1} \cdot \ldots \cdot \zeta_N^{p_N} \frac{\partial}{\partial \zeta_s}$$

where a is a complex number and p_k are non-negative integers fulfilling the *additive resonance relation*

$$\lambda_s = (p, \lambda) := \sum_{k=1}^{N} p_k \lambda_k .$$

A λ-resonant monomial vector field is called *diagonal* if it is associated to a trivial relation $\lambda_s = \lambda_s$, i.e. it is of the form $a \cdot \zeta_s \partial/\partial\zeta_s$. A λ-resonant vector field on $\mathbb{C}^N$ is a sum of λ-resonant monomial vector fields. Notice that the vector field ξ_0 defined in (1) is λ-resonant. A holomorphic vector field on $\mathbb{C}^N$ is λ-resonant if and only if it commutes with ξ_0. Therefore the vector space $\mathfrak{g}_\lambda$ of λ-resonant vector fields is a Lie algebra.

From now on we will assume that λ belongs to the Poincaré domain. In this case there is only a finite number of possible resonance relations and $\mathfrak{g}_\lambda$ has finite dimension. We recall (cf. [Ha]) that the equation of the flow associated to a vector field η in $\mathfrak{g}_\lambda$ can be triangulated, that is written as

$$\begin{aligned}
&\dot{\zeta}_1 = a_1\zeta_1 \\
&\cdots \\
&\dot{\zeta}_k = a_k\zeta_k + P_k(\zeta_1, \ldots, \zeta_{k-1}) \\
&\cdots
\end{aligned}$$

by a linear change of the coordinates. It follows that η generates a flow globally defined in $\mathbb{C}^N$. Let G_λ denote the group of the holomorphic automorphisms f of $\mathbb{C}^N$ such that $f_*\xi_0 = \xi_0$. This group admits the following characterizations.

Proposition 2. *The group G_λ can be characterized in the following ways:*
(i) *G_λ is the group of polynomial transformations of $\mathbb{C}^N$ of the form*

$$\zeta \mapsto \sum_{(s,p)} a_p^s \zeta^p \vec{e}_s \,, \tag{3}$$

where the sum runs over the sequences (s, p) such that $\lambda_s = \sum p_k\lambda_k$ and $\vec{e}_1, \ldots, \vec{e}_N$ is the canonical basis of $\mathbb{C}^N$, and
(ii) *G_λ is the group generated by the flows associated to the vector fields in $\mathfrak{g}_\lambda$.*

In particular G_λ is a connected Lie group with Lie algebra $\mathfrak{g}_\lambda$.

Remark 1. It follows from the above description that G_λ is an algebraic group.

Proof. Let us prove (i). For a given $f \in G_\lambda$ set $f(\zeta) = \sum_{p\in\mathbb{N}^N} \vec{a}_p\zeta^p$ where $\vec{a}_p = \sum_s a_p^s\vec{e}_s$. One has $(f_*\xi_0)_\zeta = \sum \vec{a}_p(p, \lambda)\zeta^p$ and if we denote by D the diagonal matrix associated to ξ_0 then

$$(\xi_0)_{f(\zeta)} = \sum D \cdot \vec{a}_p\zeta^p \,.$$

The identity $f_*(\xi_0) = \xi_0$ is then equivalent to $D \cdot \vec{a}_p = (p, \lambda)\vec{a}_p$. This gives the form (3) of f. For such a polynomial map invertibility is equivalent to the invertibility of its linear part. This is a consequence of the commutativity of f with the flow ψ of ξ and the contracting property of ψ.

This description shows in particular that G_λ is biholomorphic to the cartesian product of an Euclidean space $\mathbb{C}^r$ by linear groups $\mathrm{GL}(k_i, \mathbb{C})$ where k_i are the multiplicities of the eigenvalues of ξ_0. It follows that G_λ is a connected Lie group.

Finally, it is clear that the group generated by the flows associated to the elements of $\mathfrak{g}_\lambda$ is contained in G_λ. Since these two groups have the same dimension and G_λ is connected they must coincide. In particular $\mathfrak{g}_\lambda$ is the Lie algebra of G_λ. □

Let ξ' be a holomorphic vector field in a neighbourhood of zero in $\mathbb{C}^N$ and having an isolated singularity at the origin. We say that the vector field ξ' is of *type* λ if its linear part has eigenvalues $\lambda_1, \ldots, \lambda_N$. Recall that λ is assumed to belong to the Poincaré domain. In this situation the Poincaré-Dulac theorem (cf. [Ar]) states that there is a biholomorphic map h defined in a neighbourhood of the origin such that $\xi = h_*(\xi')$ is a λ-resonant vector field in *normal form*, i.e. ξ is written as a sum

$$\xi = \xi_0 + \xi_1 + \xi_2 = \xi_0 + \sum_{s,r} R_r^s \frac{\partial}{\partial \zeta_s} \tag{4}$$

where ξ_0 is the diagonal vector field given by (1), $\xi_0 + \xi_1$ is the linear part of ξ in lower triangular Jordan form and ξ_2 is the sum of the resonant monomials of higher degree.

It has been shown by Haefliger (cf. [Ha] p. 242) that for a given $\delta > 0$ there is a diagonal linear change of coordinates k^δ such that

$$k_*^\delta(\xi) = \xi_0 + \sum_{s,r} \delta^{l_{s,r}} R_r^s \frac{\partial}{\partial \zeta_s}$$

with $l_{s,r} \in \mathbb{N}^*$. Therefore the coefficients of the non-diagonal resonant monomials can be made as small as we want. Furthermore up to a change in the coordinate order and multiplication by a non zero constant we can also assume without restriction that the eigenvalues $\lambda_1, \ldots, \lambda_N$ fulfill the following normalization conditions

$$0 \leqq \arg \lambda_1 \leqq \arg \lambda_2 \leqq \ldots \leqq \arg \lambda_N < \pi \tag{5.1}$$

and

$$|\lambda_k| \leqq |\lambda_{k+1}| \quad \text{if } \arg \lambda_k = \arg \lambda_{k+1} . \tag{5.2}$$

We also recall (cf. [Ha]) that if ξ is a vector field of type λ in the normal form (4) then its associated global flow $\psi(t; \zeta) = (\zeta_1(t), \ldots, \zeta_N(t))$ has the form

$$\zeta_s(t) = e^{\lambda_s t} \left(\zeta_s(0) + \sum_{(s,p),r} b_{p,r}^s t^r \zeta^p(0) \right) \tag{6}$$

where the sum runs over the sequences (s, p) such that $\lambda_s = \sum p_i \lambda_i$.

1.3 Products of odd dimensional spheres transverse to a holomorphic flow in $\mathbb{C}^N$

Let ξ be a given holomorphic vector field of type λ defined in a neighbourhood U of 0 in $\mathbb{C}^N$. As above we assume that $\lambda = (\lambda_1, \ldots, \lambda_N)$ belongs to the Poincaré domain and fulfills the normalization condition (5.1). Such a vector field will be called *weakly hyperbolic of type* (m,n), where m,n are positive integers with $m+n=N$, if λ fulfills

$$0 \leqq \arg \lambda_i < \arg \lambda_j < \pi \quad \text{for } 1 \leqq i \leqq m \text{ and } m+1 \leqq j \leqq m+n = N .$$

In this paragraph we show that under the assumption of weakly hyperbolicity the leaf space of the foliation $\widehat{\mathscr{F}}_\xi$ on $U - \{0\}$ induced by ξ contains a compact complex manifold diffeomorphic to a product of odd dimensional spheres which inherits a complex structure in a natural way.

Theorem 1. *Let ξ be a holomorphic vector field defined in a neighbourhood of 0 in $\mathbb{C}^N$ of type λ with λ in the Poincaré domain and fulfilling the normalization condition* (5.1). *Assume that ξ is weakly hyperbolic of type (m,n). Then there are two complex submanifolds L_1' and L_2' in a neighbourhood U of the origin of respective dimensions m and n which are saturated by the foliation $\widehat{\mathscr{F}}_\xi$ on $\mathbb{C}^N$ induced by ξ and there is an embedding τ of $S^{m,n}$ in $\mathbb{C}^N$ transverse to $\widehat{\mathscr{F}}_\xi$ which meets each leaf of the restriction $\mathscr{F}_\xi$ of $\widehat{\mathscr{F}}_\xi$ to $U - (L_1' \cup L_2')$ at exactly one point. If ξ is λ-resonant in the normal form* (4) *and the coefficients of the non-diagonal monomial terms of ξ are small enough then one can take $U = \mathbb{C}^N$, $L_1' = \mathbb{C}^m \times \{0\}$, $L_2' = \{0\} \times \mathbb{C}^n$ and τ equal to the canonical embedding ι of $S^{m,n}$ in $\mathbb{C}^N$.*

Corollary 2. *Let ξ be as in the above theorem. The embedding $\tau : S^{m,n} \to \mathbb{C}^N$ induces a complex structure on $S^{m,n}$. This complex manifold, that we denote by $S^{m,n}_\xi$, is naturally identified with the leaf space of $\mathscr{F}_\xi$. Moreover the vector field ξ defines on $V^{m,n}$ a structure of holomorphic $\mathbb{C}$-principal bundle over $S^{m,n}_\xi$.*

Proof. We can suppose without restriction that ξ is λ-resonant in the normal form (4) and that the eigenvalues of its diagonal part ξ_0 are ordered as in (5). Furthermore, we will also assume that $0 < \arg \lambda_1$ and that the coefficients R_r^s in (4) are small enough to assure that ξ is still transverse to $S^{m,n}$. Given $\varepsilon \in [0,1]$ we set

$$\xi_\varepsilon = \xi_0 + \varepsilon \cdot \sum_{s,r} R_r^s \frac{\partial}{\partial \zeta_s} .$$

We will prove the theorem for all the elements of the one parameter family of vector fields ξ_ε. A direct computation shows that with the above normalizations all the λ-resonant monomial vector fields are tangent to $L_1 = \mathbb{C}^m \times \{0\}$ and $L_2 = \{0\} \times \mathbb{C}^n$. Therefore these two linear subspaces are saturated by the orbits of ξ_ε. Hence it is sufficient to prove that each leaf of the foliation $\mathscr{F}_{\xi_\varepsilon}$ induced by ξ_ε on $V^{m,n}$ meets $S^{m,n}$ exactly at one point. This will be carried out in two steps.

Step 1. *Each leaf of $\mathscr{F}_{\xi_\varepsilon}$ meets $S^{m,n}$.*

The flow ψ_ε associated to ξ_ε is of the form

$$\psi_\varepsilon(t;z,w) = (e^{\mu_1 t}P_1^\varepsilon, \ldots, e^{\mu_m t}P_m^\varepsilon, e^{\nu_1 t}Q_1^\varepsilon, \ldots, e^{\nu_n t}Q_n^\varepsilon) \tag{7}$$

with $P_i^\varepsilon = P_i^\varepsilon(t,z,w)$ and $Q_j^\varepsilon = Q_j^\varepsilon(t,z,w)$ holomorphic polynomials depending analytically on ε. The compact set $H = \{(z,w) \mid \max(\|z\|, \|w\|) = 1\}$, where $\|\cdot\|$ denotes the Euclidean norm, is homeomorphic to the sphere S^{2N-1} and separates $\mathbb{C}^m \times \mathbb{C}^n$ in two connected domains, namely $H^- = \{(z,w) \mid \max(\|z\|, \|w\|) < 1\}$ and $H^+ = \{(z,w) \mid \max(\|z\|, \|w\|) > 1\}$. In this situation we have

Lemma 1. *The intersection of a leaf F of $\mathscr{F}_{\xi_\varepsilon}$ with H is path-connected.*

Proof of Lemma 1. Given $\zeta_1, \zeta_2 \in F \cap H$ let γ be a continuous path in F joining the two points. We define $\beta(u) = \psi_\varepsilon(i \cdot s_0; \gamma(u))$ where

$$s_0 = \begin{cases} \max\{s < 0 \mid \psi_\varepsilon(i \cdot s; \gamma(u)) \in H\} & \text{if } \gamma(u) \in H^- \\ 0 & \text{if } \gamma(u) \in H \\ \min\{s > 0 \mid \psi_\varepsilon(i \cdot s; \gamma(u)) \in H\} & \text{if } \gamma(u) \in H^+ . \end{cases}$$

It follows from the exponential behaviour of ψ_ε and the condition $\operatorname{Im} \lambda_k > 0$ that β is well defined. Clearly β is a continuous path in $F \cap H$ joining the points ζ_1 and ζ_2. □

Let us choose $\alpha \in \mathbb{C}^*$ with $\arg \lambda_m < \arg \alpha < \arg \lambda_{m+1}$. Given $x + iy \in \mathbb{C}$ set $X = x$ and $Y = \operatorname{Re}(\alpha(x+iy))$. Let $x + iy = \eta(X, Y)$ be the inverse transformation. Notice that for $p, q \in \mathbb{R}$ we have

$$\operatorname{Re}((p + q\alpha)(x + iy)) = pX + qY . \tag{8}$$

We set $\mu_i = p_i + \alpha q_i$ and $\nu_j = p'_j + \alpha q'_j$. Then $p_i, q_i, q'_j > 0$ and $p'_j < 0$. Let a point $\zeta_0 \in V^{m,n}$ be given and define the curve C_ε by

$$C_\varepsilon(u) = (z(u), w(u)) = \begin{cases} \psi_\varepsilon(\eta(0,u); \zeta_0) & \text{if } u \leqq 0 \\ \psi_\varepsilon(\eta(-u,0); \zeta_0) & \text{if } u \geqq 0 . \end{cases}$$

By using (7) one easily sees that $\lim_{u \to -\infty} C_\varepsilon(u) = 0$, $\lim_{u \to \infty} z(u) = 0$ and $\lim_{u \to \infty} \|w(u)\| = \infty$. Let A_ε denote the adherence of C_ε, i.e. the union of the curve with the origin, and let A_ε^u be the image of A_ε by $\psi_\varepsilon(\eta(0,u); \cdot)$. Then $\lim_{u \to -\infty} \max\{\|z\| \mid (z,w) \in A_\varepsilon^u\} = 0$. Furthermore A_u is connected, contains zero and is not bounded. So $A_\varepsilon^u \cap H \neq \emptyset$. Thus by taking $u \in (-\infty, 0)$ with absolute value big enough we can find a point $(z,w) \in F$ with $\|z\| \leqq 1$ and $\|w\| = 1$. In an analogous way one can show the existence of another point $(z', w') \in F$ with $\|z'\| = 1$ and $\|w'\| \leqq 1$. Now Lemma 1 implies that $F \cap S^{m,n}$ is non empty.

Step 2. *Each leaf F of $\mathscr{F}_{\xi_\varepsilon}$ meets $S^{m,n}$ at most once.*

We use induction on N. If $N = 2$ then $\xi_\varepsilon = \xi_0$ and for $\mathscr{F}_{\xi_0}$ the assertion has already been shown in Proposition 1. Let us consider the general case. Let k be the index for which $\arg\lambda_{k-1} < \arg\lambda_k = \arg\lambda_N$ and set $j_0 = k - m$. It follows from the assumptions made on the order of the eigenvalues of ξ that the only resonant monomials of type $R\partial/\partial z_1$ and $R'\partial/\partial w_{j_0}$ appearing in (4) are the diagonal ones. Let $\zeta^0 = (z^0, w^0) \in S^{m,n}$ be a fixed point and set $z^0 = (z_1^0, \ldots, z_m^0)$, $w^0 = (w_1^0, \ldots, w_n^0)$. The above remark implies that if we set $(z_t^\varepsilon, w_t^\varepsilon) = \psi_t^\varepsilon(\zeta^0)$ then

$$z_t^\varepsilon = (e^{\mu_1 t} z_1^0, e^{\mu_2 t} P_2^\varepsilon(t, z^0, w^0), \ldots, e^{\mu_m t} P_m^\varepsilon(t, z^0, w^0))$$

and

$$w_t^\varepsilon = (e^{\nu_1 t} Q_1^\varepsilon(t, z^0, w^0), \ldots, e^{\nu_{j_0} t} w_{j_0}^0, \ldots, e^{\nu_n t} Q_n^\varepsilon(t, z^0, w^0))$$

If $z_1^0 = 0$ then the leaf through ζ^0 is contained in the complex hyperplane $\{z_1^0 = 0\}$ and the induction hypothesis concludes the proof. By the same reason the statement is true if $w_{j_0}^0 = 0$. We can thus assume $z_1^0 \cdot w_{j_0}^0 \neq 0$.

Lemma 2. *There is a constant $M > 0$ such that $\psi_\varepsilon(t; \zeta^0) \notin S^{m,n}$ for $|t| > M$ and $\varepsilon \in [0, 1]$.*

Proof of Lemma 2. Let X, Y be the coordinates introduced above and put $t = \eta(X, Y)$. It follows from (8) that

$$|e^{\mu_i t}| = e^{p_i X + q_i Y} \quad \text{and} \quad |e^{\nu_j t}| = e^{p_i' X + q_i' Y}.$$

One has

(a) There is a constant $Y_0 > 0$ such that $\max\{\|z_t^\varepsilon\|, \|w_t^\varepsilon\|\} > 1$ for $Y \geqq Y_0$. This follows from the inequalities

$$\begin{aligned} \max\{\|z_t^\varepsilon\|, \|w_t^\varepsilon\|\} &\geqq \max\{e^{p_1 X + q_1 Y}|z_1^0|, e^{p_{j_0}' X + q_{j_0}' Y}|w_{j_0}^0|\} \\ &\geqq \min\{e^{q_1 Y}|z_1^0|, e^{q_{j_0}' Y}|w_{j_0}^0|\} \end{aligned}$$

and the positivity of q_1 and q_{j_0}'.

(b) There is constant $Y_1 < 0$ such that $\min\{\|z_t^\varepsilon\|, \|w_t^\varepsilon\|\} < 1$ for $Y \leqq Y_1$. By expanding $P_i^\varepsilon(t, z^0, w^0) = P_i^\varepsilon(\eta(X, Y), z^0, w^0)$ with respect to X, Y and ε one can write $|e^{\mu_i t} P_i^\varepsilon(t, z^0, w^0)|^2$ as a finite sum of exponential monomials of the form $R = a\varepsilon^l X^\alpha Y^\beta e^{p_i X} e^{q_i Y}$. Since $p_i > 0$ there is a constant C such that $\|a\varepsilon^l X^\alpha e^{p_i X}\| \leqq C$ for $X \leqq 0$ and then $|R|$ can be made as small as we want by taking Y small enough. For $X \geqq 0$ we can reason in an analogous way with the terms $e^{\nu_j t} Q_j^\varepsilon(t, z^0, w^0)$. This shows the existence of Y_1.

(c) There is $X_0 > 0$ such that $\min\{\|z_t^\varepsilon\|, \|w_t^\varepsilon\|\} < 1$ for $Y_1 \leqq Y \leqq Y_0$ and $|X| \geqq X_0$. This can be shown as assertion (b).

Now the Lemma follows directly from (a), (b) and (c). □

Denote $\varphi(\varepsilon, t) = \psi_\varepsilon(t; \zeta^0)$ and let us suppose that there is $\varepsilon \in [0, 1]$ such that $\varphi(\varepsilon, t) \in S^{m,n}$ for a certain $t \neq 0$. It follows from Lemma 2 that $|t| \leqq M$.

Define

$$\varepsilon_0 = \inf\{\varepsilon \in [0,1] \mid \exists t \neq 0 \text{ with } \varphi(\varepsilon, t) \in S^{m,n}\} .$$

By compacity there are a sequence (t_n), with $t_n \neq 0$ and $t_n \to t_0$, and a decreasing sequence (ε_n), with $\varepsilon_n \to \varepsilon_0$, such that $\varphi(\varepsilon_n, t_n) \in S^{m,n}$. By continuity $\varphi(\varepsilon_0, t_0) \in S^{m,n}$. Let us consider first the case $t_0 = 0$. Then $\varphi(\varepsilon_0, t_0) = \zeta^0$ and it follows from Taylor's formula that

$$\lim_{n\to\infty} \frac{\varphi(\varepsilon_n, t_n) - \varphi(\varepsilon_n, 0)}{t_n} = \frac{\partial \varphi}{\partial t}(\varepsilon_0, 0) = (\xi_{\varepsilon_0})_{\zeta^0} \neq 0 .$$

Thus the above limit would define a non zero complex vector which is at the same time tangent to $\mathscr{F}_{\xi_{\varepsilon_0}}$ and to $S^{m,n}$ getting a contradiction. Suppose now $t_0 \neq 0$. Proposition 1 implies then that $\varepsilon_0 > 0$. The transversality of the foliations $\mathscr{F}_{\xi_r}$ with the product of spheres $S^{m,n}$ would imply in this case that one can still find $0 < \varepsilon' < \varepsilon_0$ and $t' \in \mathbb{C}^*$ with $\varphi(\varepsilon', t') \in S^{m,n}$ but this is not possible because of the choice of ε_0.

Therefore the assumption of the existence of $\varepsilon \in [0,1]$ and $t \neq 0$ with $\psi_\varepsilon(t; \zeta^0) \in S^{m,n}$ leads in any case to a contradiction. This completes the proof of the theorem. □

1.4 Some examples

Corollary 2 provides a general method of constructing complex structures on a product of odd dimensional spheres. In this paragraph we show that any of the classical examples, namely elliptic curves, Hopf manifolds and Calabi-Eckmann manifolds, can also be obtained, as particular cases, by this procedure.

1.4.1 Elliptic curves. Let $m = n = 1$ and take $v \in \mathbb{C}$ with $\operatorname{Im} v > 0$. The $\mathbb{C}$-action on $V^{1,1} = \mathbb{C}^* \times \mathbb{C}^*$ induced by the vector field $\xi = z\dfrac{\partial}{\partial z} + vw\dfrac{\partial}{\partial w}$ is given by $(t, z, w) \mapsto (e^t z, e^{vt} w)$. The intersection with $V^{1,1}$ of the straight line $\{1\} \times \mathbb{C}$ is a copy of $\mathbb{C}^*$ which meets each orbit of the action. The orbit space is then identified to the quotient of $\mathbb{C}^*$ by the discrete subgroup generated by $e^{2i\pi v}$. Any elliptic curve is obtained in this way.

1.4.2 Hopf manifolds. We see here that any Hopf manifold can also be seen, by means of a suspension, as the orbit space of a suitable holomorphic flow. Recall that for a given sequence of complex numbers $\varrho = (\varrho_1, \ldots, \varrho_n)$ a *multiplicative resonance relation* is an identity of the type $\varrho_s = \varrho_1^{p_1} \cdot \ldots \cdot \varrho_n^{p_n}$ where p_j are non-negative integers. A (multiplicative) ϱ-resonant monomial associated to the above relation is a polynomial map $w \mapsto a w_1^{p_1} \cdot \ldots \cdot w_n^{p_n} \cdot \vec{e}_s$.

A Hopf manifold is a compact complex manifold W_f which is the quotient of $\mathbb{C}^n - \{0\}$, where $n \geqq 2$, by an infinite cyclic group acting holomorphically and properly discontinuously on $\mathbb{C}^n - \{0\}$. It is proved in [Ha] that as a generator of this group one can take a polynomial automorphism f of $\mathbb{C}^n$ fixing the origin and whose derivative $f'(0)$ at 0 has eigenvalues $\varrho = (\varrho_1, \ldots, \varrho_n)$ with $0 < |\varrho_j| < 1$ and such that f is ϱ-resonant, i.e. up to a linear change of the

coordinates f is a polynomial map with linear part in lower triangular Jordan form and written as

$$f(w) = \sum_{j=1}^{n} \varrho_j w_j \vec{e}_j + \sum_{(s,p)} a_p^s w^p \vec{e}_s \tag{9}$$

where the terms $a_p^s w^p \vec{e}_s$ are non-diagonal resonant monomials.

Let ξ be a vector field on $\mathbb{C}^{n+1}$ which is of type λ and in the normal form (4). Assume further that $\lambda = (1, \nu_1, \ldots, \nu_n)$ with $\operatorname{Im} \nu_j > 0$. The flow associated to ξ is of the form $\psi(t; z, w) = (e^t z, A(t; z, w))$. As in the case of elliptic curves the intersection of the hyperplane $\{1\} \times \mathbb{C}^n$ with $V^{1,n}$ is a copy of $\mathbb{C}^n - \{0\}$ which meets each orbit of ξ restricted to $V^{1,n}$. Then the complex manifold $S_\xi^{1,n}$ obtained from ξ as in Corollary 2 is biholomorphic to the Hopf manifold obtained as the quotient of $\mathbb{C}^n - \{0\}$ by the action of the group generated by $h(w) = A(2i\pi; 1, w)$. Notice that h is a (multiplicative) ϱ-resonant polynomial map written in normal form, in the sense stated above, where $\varrho_j = e^{2i\pi\nu_j}$.

Let us see that, in a converse way, any Hopf manifold W_f can be obtained by the above procedure. Assume that f is given as in (9). Let ν_j be complex numbers with $\varrho_j = e^{2i\pi\nu_j}$. Since $|\varrho_j| < 1$ one has $\operatorname{Im} \nu_j > 0$. For any resonant monomial $a_p^s w^p \vec{e}_s$ (diagonal or not) appearing in (9) there is an integer q such that

$$\nu_s = q + (p, \nu) .$$

The complex numbers ν_j can be chosen in such a way that all the integers q are non negative (for instance by taking $-1 < \operatorname{Re} \nu_j \leqq 0$). This means that to any monomial in (9) corresponds an (additive) resonance relation associated to $\lambda = (1, \nu_1, \ldots, \nu_n)$. Therefore the complex automorphism F of $\mathbb{C}^{n+1}$ defined by $F(z, w) = (z, f(w))$ is an element of the group G_λ. To conclude we must associate to F a vector field ξ in $\mathfrak{g}_\lambda$, of type λ and written in the normal form (4) such that $F = \exp(2i\pi\xi)$. For this purpose assume that the complex numbers ν_j are ordered in such a way that for $i < j$ one has $\operatorname{Im} \nu_i \leqq \operatorname{Im} \nu_j$ and $\operatorname{Re} \nu_i \leqq \operatorname{Re} \nu_j$ in the case $\operatorname{Im} \nu_i = \operatorname{Im} \nu_j$. Let $\mathfrak{n}$ denote the vector subspace of $\mathfrak{g}_\lambda$ generated by the monomial vector fields in $\mathfrak{g}_\lambda$ of the following two types: (i) of degree bigger than 1 and (ii) of the form $w_r \dfrac{\partial}{\partial w_s}$ with $s > r$. In this situation one has

Lemma 3. *The vector space* $\mathfrak{n}$ *is a nilpotent subalgebra of* $\mathfrak{g}_\lambda$.

Proof. Let us introduce in the set of monomials in $\mathfrak{n}$ the following order. We state that $z \dfrac{\partial}{\partial z}$ is smaller than any other monomial and we say that $w^p \dfrac{\partial}{\partial w_s} \prec w^{p'} \dfrac{\partial}{\partial w_{s'}}$ if $s < s'$ or if $s = s'$ and p' is bigger than p in the lexicographic order. One can easily check that if η_1 and η_2 are monomials in $\mathfrak{n}$ then $\eta_3 = [\eta_1, \eta_2]$ is also a monomial in $\mathfrak{n}$ with $\eta_1 \prec \eta_3$ and $\eta_2 \prec \eta_3$. This proves the lemma. □

By taking exponential of the monomial elements of $\mathfrak{n}$ one can see that the connected subgroup N of G_λ corresponding to the subalgebra $\mathfrak{n}$ is formed by those elements having an unipotent linear part in lower triangular form. Set

$$\xi_0 = z\frac{\partial}{\partial z} + \sum_{j=1}^{n} v_j w_j \frac{\partial}{\partial w_j} .$$

Then one sees by direct computation that $F \circ \exp(-2i\pi\xi_0)$ is in N. Since N is nilpotent the exponential map is onto and there is $\eta \in \mathfrak{n}$ such that

$$\exp(2i\pi\eta) = F \circ \exp(-2i\pi\xi_0) .$$

But ξ_0 is in the center of $\mathfrak{g}_\lambda$. This implies that $F = \exp(2i\pi\xi)$ where $\xi = \xi_0 + \eta$.

Remark 2. Let f be a given contractive automorphism of $\mathbb{C}^n$ in the normal form (9). If f is diagonal one can write it as $\exp(2i\pi\chi)$ with χ a λ-resonant vector field in normal form on $\mathbb{C}^n$. For a general f there is $k \in \mathbb{N}$ such that $f^k = \exp(2i\pi\chi)$. This is proved by M. Kato in [Ka2]. Nevertheless in general the mapping f itself can not be written in this way. An example is given by

$$f(w_1, w_2, w_3) = (\varrho w_1, -\varrho w_2, \varrho^2 w_3 + a w_1^2 + b w_2^2)$$

where a, b and ρ are complex numbers such that $a \cdot b \neq 0$ and $0 < |\varrho| < 1$.

One should remark that different vector fields can give rise, by means of the above construction, to the same Hopf manifold. For a given $j \in \{1, \ldots, n\}$ and $k \in \mathbb{Z}$ set

$$\Phi_{j,k}(z, w) = (z, w_1, \ldots, z^k w_j, \ldots, w_n) . \tag{10}$$

The map $\Phi_{j,k}$ is a holomorphic automorphism of $V^{1,n}$ and the vector fields $(\Phi_{j,k})_* \xi_0$ and $(\Phi_{j,k})_* \xi$ extend to vector fields on $\mathbb{C}^{n+1}$ that we denote by ξ_0' and ξ' respectively. Clearly ξ_0' is a diagonal vector field with eigenvalues $\lambda' = (1, \nu, \ldots, \nu_j + k, \ldots, \nu_n)$. The vector field ξ' commutes with ξ_0' and so it is λ'-resonant. In fact a direct computation shows that ξ' is of type λ' and in the normal form (4). It follows from this construction that

Proposition 3. *The complex manifolds $S_\xi^{1,n}$ and $S_{\xi'}^{1,n}$, where ξ' is the extension to $\mathbb{C}^{n+1}$ of $(\Phi_{j,k})_* \xi$, are holomorphically isomorphic.*

It follows that for a given Hopf manifold $S_\xi^{1,n}$ it is always possible to find a vector field ξ' with $S_{\xi'}^{1,n} \cong S_\xi^{1,n}$ such that ξ' is λ'-resonant where $\lambda' = (1, \nu_1', \ldots, \nu_n')$ does not fulfill relations of the type

$$\nu_s = q + (p, \nu') \quad \text{with } q, p_j \in \mathbb{Z},\ q < 0,\ p_j \geqq 0 .$$

This fact will play an important role in all the discussions concerning the case of Hopf manifolds.

Remark 3. Any complex manifold $S^{n,1}_{\xi}$ is identified with a manifold of the type $S^{1,n}_{\xi'}$ where ξ' is the extension to $\mathbb{C}^{n+1}$ of $\Psi_*^{-1}\xi$ and Ψ is the holomorphic isomorphism from $V^{1,n}$ onto $V^{n,1}$ given by $\Psi(z, w_1, \ldots, w_n) = (w_1, \ldots, w_n, z^{-1})$.

1.4.3 Calabi-Eckmann manifolds. The Calabi-Eckmann manifolds were introduced by these authors in [C-E] as a class of compact, simply connected, non Kähler manifolds. They have the product $S^{2m-1} \times S^{2n-1}$, with $m, n > 1$, as underlying differentiable manifold and are the total space of a principal bundle over $\mathbb{P}^{m-1} \times \mathbb{P}^{n-1}$ with fiber an elliptic curve.

Any such a manifold can be viewed as the orbit space of the $\mathbb{C}$-action on $V^{m,n}$ induced by the vector field $\xi = \sum_i z_i \dfrac{\partial}{\partial z_i} + \nu\left(\sum_j w_j \dfrac{\partial}{\partial w_j}\right)$, where ν is a complex parameter with $\operatorname{Im} \nu \neq 0$. The vector field $\eta = \sum_i z_i \dfrac{\partial}{\partial z_i}$ commutes with ξ inducing a non vanishing vector field $\hat{\eta}$ on $S^{m,n}$. The action associated to $\hat{\eta}$ defines on $S^{m,n}$ the structure of a principal fibration over $\mathbb{P}^{m-1} \times \mathbb{P}^{n-1}$ with fiber the elliptic curve $\mathbb{C}^*/e^{2i\pi\nu}$.

1.5 The leaf space of $\widehat{\mathscr{F}}_\xi$

Let ξ be a vector field of type λ written in the normal form (4) and in the hypothesis of Theorem 1, i.e. weakly hyperbolic of type (m, n). Let $\mathfrak{E}$ denote the leaf space of the foliation $\widehat{\mathscr{F}}_\xi$ defined by ξ on $\mathbb{C}^N - \{0\}$ and let $\pi : \mathbb{C}^N - \{0\} \to \mathfrak{E}$ be the natural projection. In this situation we have

Proposition 4. *The set $\pi(V^{m,n})$ is a maximal separated open subset of $\mathfrak{E}$.*

Proof. The subset $\pi(V^{m,n})$ of $\mathfrak{E}$ is open and separated. Let us assume that V is a separated subset of $\mathfrak{E}$ containing $\pi(V^{m,n})$. Since $\pi(V^{m,n})$ is compact it must be closed in V. But $\pi(V^{m,n})$ is dense in $\mathfrak{E}$ and thus $\pi(V^{m,n}) = V$. □

Proposition 5. *Suppose ξ is hyperbolic (i.e. $\arg \lambda_i \neq \arg \lambda_j$ for $i \neq j$). Then the maximal open separated subsets of the leaf space $\mathfrak{E}$ of $\widehat{\mathscr{F}}_\xi$ are $\pi(V^{m,N-m})$ for $m = 1, \ldots, N-1$.*

Proof. We can assume that the coefficients of the non diagonal terms of ξ are as small as we want. In this situation Guckenheimer's theorem [Gu] states that the foliation $\widehat{\mathscr{F}}_\xi$ is topologically conjugated to $\widehat{\mathscr{F}}_{\xi_0}$ in a neighbourhood of the origin in $\mathbb{C}^N$. Therefore we can reduce to the diagonal case $\xi = \xi_0$.

Let U be an open and saturated subset of $\mathbb{C}^N - \{0\}$ such that $\pi(U)$ is separated. Notice that $\{(z_1, 0, \ldots, 0) | z_1 \in \mathbb{C}^*\}$ is an orbit of ξ which is in the adherence of the leaf through any point of the form $(\zeta_1, \ldots, \zeta_N)$ with $\zeta_1 \neq 0$. So there exist $m \geqq 1$ such that: (i) any point of the form $(z_1, \ldots, z_m, 0, \ldots, 0)$ does not belong to U and (ii) there is a point $(\zeta_1, \ldots, \zeta_{m+1}, 0, \ldots, 0)$ belonging to U (in particular $\zeta_{m+1} \neq 0$).

We want to show that $U \subset V^{m,N-m}$. If this was not the case there would be a point in U of the form $(0, \ldots, 0, w_{m+1}, \ldots, w_N)$. Since U is open we can assume $w_{m+1} \neq 0$. We will see that the points $(\zeta_1, \ldots, \zeta_{m+1}, 0, \ldots, 0)$ and

$(0,\dots,0,w_{m+1},\dots,w_N)$ cannot be separated getting a contradiction. For this purpose it is sufficient to see that for a given $\varepsilon > 0$ one can find complex numbers $\alpha_{m+2},\dots,\alpha_N$, $\beta_1,\dots,\beta_m$ and t with $|\alpha_j| < \varepsilon$, $|\beta_i| < \varepsilon$ and such that

$$\psi(t;\zeta_1,\dots,\zeta_m,\zeta_{m+1},\alpha_{m+2},\dots,\alpha_N) = (\beta_1,\dots,\beta_m,w_{m+1},w_{m+2},\dots,w_N),$$

where ψ is the flow associated to ξ, that is fulfilling the system of equations

$$\begin{aligned} \beta_i &= \zeta_i e^{\lambda_i t} && \text{for } 1 \leqq i \leqq m \\ w_{m+1} &= \zeta_{m+1} e^{\lambda_{m+1} t} && \\ w_j &= \alpha_j e^{\lambda_j t} && \text{for } m+2 \leqq j \leqq N . \end{aligned}$$

These equations are satisfied if one takes $t \in \mathbb{C}$ of the form $t = t_0 + 2ik\pi/\lambda_{m+1}$ where t_0 fulfills $e^{\lambda_{m+1} t_0} = w_{m+1}/\zeta_{m+1}$ and k is a small enough negative integer. □

In the general case the above proposition is no longer true. This is shown by the following examples.

Examples. (i) Let $\xi = \mu\zeta_1\frac{\partial}{\partial\zeta_1} + \zeta_2\frac{\partial}{\partial\zeta_2} + \zeta_3\frac{\partial}{\partial\zeta_3} + \nu\zeta_4\frac{\partial}{\partial\zeta_4}$ with $\operatorname{Im}\mu < 0 < \operatorname{Im}\nu$. This vector field is weakly hyperbolic of types $(1,3)$ and $(3,1)$. Let a small real number $\varepsilon > 0$ be given. The set

$$U = \{((t+\alpha)e^{\mu u},(t+\beta)e^{u},(1-t+\gamma)e^{u},(1-t+\delta)e^{\nu u}) \mid 0 \leqq t \leqq 1;$$
$$|\alpha|,|\beta|,|\gamma|,|\delta| < \varepsilon; u \in \mathbb{C}\}$$

is open, connected and saturated. Its projection $\pi(U)$ in $\mathfrak{E}$ is separated and contained in $\pi(V^{1,3}) \cup \pi(V^{3,1})$. Nevertheless since $(1,1,0,0),(0,0,1,1) \in U$, $\pi(U)$ is not contained in any of the sets $\pi(V^{1,3})$, $\pi(V^{3,1})$.

(ii) Let $\xi = \zeta_1\frac{\partial}{\partial\zeta_1} + (\zeta_2+\zeta_1)\frac{\partial}{\partial\zeta_2} + i\zeta_3\frac{\partial}{\partial\zeta_3}$. This vector field is weakly hyperbolic of type $(2,1)$. The only set $V^{m,n}$ for which ξ is in the hypothesis of Theorem 1 is $V^{2,1}$. The set $U = \{(\zeta_1,\zeta_2,\zeta_3) \mid \zeta_1 \neq 0\}$ is open and saturated. Its projection $\pi(U)$ in $\mathfrak{E}$ is isomorphic to $\mathbb{C} \times \mathbb{C}^*$ and thus separated but it is not contained in $\pi(V^{2,1})$.

2 Dolbeault's cohomology and deformations of $S_\xi^{m,n}$

Throughout this section ξ will denote a vector field of type λ in the normal form (4) and weakly hyperbolic of type (m,n). For a given complex manifold we will denote respectively by $\mathcal{O}$, Θ and Ω^p the sheaves of germs of holomorphic functions, holomorphic vector fields and holomorphic p-forms on the manifold. The aim of this section is to compute the cohomology of the complex manifold $S_\xi^{m,n}$ with values in the above sheaves. This is completely carried out for the sheaves $\mathcal{O}$ and Ω^p. The cohomology with values in Θ is determined

for ξ fulfilling generic conditions. We follow closely the method developed by A. Haefliger in [Ha] where he studies the deformations of the transversely holomorphic foliation on S^{2N-1} induced by the vector field ξ. As applications we show that there are no holomorphic global p-forms on $S_\xi^{m,n}$ and we give a description, in almost all the cases, of the versal space of deformations of the complex manifold $S_\xi^{m,n}$.

2.1 Cohomology with values in $\mathcal{O}$

It is shown in [Fr] that, for $q \geqq 2$, $H^i(\mathbb{C}^q - \{0\}, \mathcal{O}) = 0$ if $i \neq 0, q-1$, that $H^0(\mathbb{C}^q - \{0\}, \mathcal{O})$ is the space $H^0(\mathbb{C}^q, \mathcal{O})$ of entire functions on $\mathbb{C}^q$ and that $H^{q-1}(\mathbb{C}^q - \{0\}, \mathcal{O})$ is a separated Frechet space identified to the space of convergent Laurent series $\sum_{\alpha<0} a_\alpha z^\alpha$ where $\alpha = (\alpha_1, \ldots, \alpha_q)$ is a multiindex of integers, z^α stands for $z^{\alpha_1} \cdot \ldots \cdot z^{\alpha_q}$ and $\alpha < 0$ means $\alpha_i < 0$ for any $0 \leqq i \leqq q$. It is also well known that $H^i(\mathbb{C}^*, \mathcal{O}) = 0$ for $i \geqq 1$ because $\mathbb{C}^*$ is a Stein manifold and that $H^0(\mathbb{C}^*, \mathcal{O})$ is the space of convergent Laurent series on $\mathbb{C}^*$. From these facts and by means of Künneth's formula for complex manifolds with separated cohomology (cf. [De]) one can compute the cohomology groups $H^k(V^{m,n}, \mathcal{O})$ obtaining the results collected in Propositions 6 to 9 below. In order to simplify the notation we will use the following convention: a space whose elements are, for instance, all the convergent Laurent series of the form

$$\sum_{\substack{\alpha \in \mathbb{Z}^m \\ \beta \geqq 0}} a_{\alpha,\beta} z^\alpha w^\beta$$

where $\alpha = (\alpha_1, \ldots, \alpha_m)$ and $\beta = (\beta_1, \ldots, \beta_n)$, will be simply denoted by $\mathfrak{L}^{m,n}_{\alpha;\beta\geqq 0}$.

Proposition 6. (i) $H^0(V^{1,1}, \mathcal{O}) = H^0(\mathbb{C}^* \times \mathbb{C}^*, \mathcal{O})$.
(ii) $H^0(V^{1,n}, \mathcal{O}) = H^0(\mathbb{C}^* \times \mathbb{C}^n, \mathcal{O})$ *for* $n > 1$.
(iii) $H^0(V^{m,n}, \mathcal{O}) = H^0(\mathbb{C}^N, \mathcal{O})$ *for* $m > 1$.

Proposition 7. *Assume* $m = 1$. *For* $k \geqq 1$ *one has*
(i) $H^{n-1}(V^{1,n}, \mathcal{O}) \cong \mathfrak{L}^{1,n}_{\alpha;\beta<0}$.
(ii) $H^k(V^{1,n}, \mathcal{O}) = 0$ *if* $k \neq n-1$.

Proposition 8. *Assume* $1 < m, n$ *and* $m \neq n$. *For* $k \geqq 1$ *one has*
(i) $H^{m-1}(V^{m,n}, \mathcal{O}) \cong \mathfrak{L}^{m,n}_{\alpha<0;\beta\geqq 0}$.
(ii) $H^{n-1}(V^{m,n}, \mathcal{O}) \cong \mathfrak{L}^{m,n}_{\alpha\geqq 0;\beta<0}$.
(iii) $H^{m+n-2}(V^{m,n}, \mathcal{O}) \cong \mathfrak{L}^{m,n}_{\alpha<0;\beta<0}$.
(iv) $H^k(V^{m,n}, \mathcal{O}) = 0$ *otherwise.*

Proposition 9. *Assume* $1 < m = n$. *For* $k \geqq 1$ *one has*
(i) $H^{m-1}(V^{m,m}, \mathcal{O}) \cong \mathfrak{L}^{m,m}_{\alpha<0;\beta\geqq 0} \oplus \mathfrak{L}^{m,m}_{\alpha\geqq 0;\beta<0}$.
(ii) $H^{2m-2}(V^{m,m}, \mathcal{O}) \cong \mathfrak{L}^{m,m}_{\alpha<0;\beta<0}$.
(iii) $H^k(V^{m,m}, \mathcal{O}) = 0$ *otherwise.*

Remark 4. The above isomorphisms can be specified more precisely by means of Cech cohomology using the Stein covering $\mathfrak{W} = \{U_i \times V_j\}_{(i,j)}$ of $V^{m,n}$ where $U_i = \{z \in \mathbb{C}^m \mid z_i \neq 0\}$ and $V_j = \{w \in \mathbb{C}^n \mid w_j \neq 0\}$. For instance the isomorphism in Proposition 8(i) identifies a series $s \in \mathfrak{L}^{m,n}_{\alpha<0;\beta\geqq 0}$ with the cohomology class in $H^{m-1}(\mathfrak{W}, \mathcal{O})$ represented by the cocycle $g = \{g_{(i_1,j_1)\ldots(i_m,j_m)}\}$ determined by the conditions

$$\begin{cases} g_{(i_1,j_1)\ldots(i_m,j_m)} = 0 & \text{if there are } k \neq l \text{ with } i_k = i_l \\ g_{(1,j_1)\ldots(m,j_m)} = s\,. \end{cases}$$

In a similar way the isomorphism in Proposition 8(iii) identifies $r \in \mathfrak{L}^{m,n}_{\alpha<0;\beta<0}$ with the cohomology class in $H^{m+n-2}(\mathfrak{W}, \mathcal{O})$ represented by the cocycle h such that $h_{(i_1,j_1)\ldots(i_{m+n-1},j_{m+n-1})} = r$ if $\{i_1,\ldots,i_m\} = \{1,\ldots,m\}$ and $\{j_m,\ldots,j_{m+n-1}\} = \{j_1,\ldots,j_n\}$ and such that $h_{(i_1,j_1)\ldots(i_{m+n-1},j_{m+n-1})} = 0$ otherwise.

Since the holomorphic tangent bundle of $V^{m,n}$ is trivial, the cohomology group $H^k(V^{m,n}, \Theta)$ (resp. $H^k(V^{m,n}, \Omega^p)$) is identified to the space of expressions of the form $\sum_i a_i \partial/\partial z_i + \sum_j b_j \partial/\partial w_j$ (resp. $\sum_{I,J} a_{I,J} dz^I \wedge dw^J$ with I, J multiindices fulfilling $|I| + |J| = p$) where a_i, b_j (resp. $a_{I,J}$) are elements of $H^k(V^{m,n}, \mathcal{O})$.

The Lie derivative of functions, vector fields and differential forms with respect to ξ induces an action, denoted by L_ξ, on the cohomology groups $H^*(V^{m,n}, \mathcal{O})$, $H^*(V^{m,n}, \Theta)$ and $H^*(V^{m,n}, \Omega^p)$. In the diagonal case $\xi = \xi_0$ and if we identify the cohomology groups to the spaces of power series given above then this action is just the usual Lie derivative.

Let $\pi_\xi : V^{m,n} \to S^{m,n}_\xi$ be the natural projection of $V^{m,n}$ onto the leaf space $S^{m,n}_\xi$ of the foliation $\mathcal{F}_\xi$ induced by ξ. Denote $\mathcal{O}^{tr}_{\mathcal{F}} = \pi^*_\xi \mathcal{O}$. As it is noticed in [Ha] the retraction by deformation of $V^{m,n}$ onto $S^{m,n}_\xi$ along the orbits of ξ induces a natural identification

$$H^*(S^{m,n}_\xi, \mathcal{O}) \cong H^*(V^{m,n}, \mathcal{O}^{tr}_{\mathcal{F}})\,. \tag{11}$$

Therefore $H^*(V^{m,n}, \mathcal{O}^{tr}_{\mathcal{F}})$ inherits all the finiteness and semicontinuity properties of the cohomology of a compact complex manifold with values in a holomorphic vector bundle. In order to compute the cohomology of $S^{m,n}_\xi$ we will use the following exact sequence of sheaves over $V^{m,n}$ considered in [Ha]

$$0 \longrightarrow \mathcal{O}^{tr}_{\mathcal{F}} \longrightarrow \mathcal{O} \xrightarrow{L_\xi} \mathcal{O} \longrightarrow 0\,. \tag{12}$$

Theorem 2. *$H^0(S^{m,n}_\xi, \mathcal{O}) \cong H^1(S^{m,n}_\xi, \mathcal{O}) \cong \mathbb{C}$ and $H^k(S^{m,n}_\xi, \mathcal{O}) = 0$ for $k \geqq 2$.*

Proof. We proceed as Haefliger showing directly the statement for the diagonal vector field and then extending it to the general case by a semicontinuity argument. A straightforward computation, using the description of the cohomology groups $H^k(V^{m,n}, \mathcal{O})$ given at the beginning of this paragraph, shows that for $\xi = \xi_0$ the maps $L_{\xi_0} : H^k(V^{m,n}, \mathcal{O}) \to H^k(V^{m,n}, \mathcal{O})$, appearing in the cohomology exact sequence associated to (12), are isomorphisms for $k \geqq 1$

and that for $k = 0$ the kernel and the cokernel are isomorphic to $\mathbb{C}$. Thus the statement is true for ξ_0. It is known (cf. [K-S]) that the dimension of the cohomology groups of a compact complex manifold X_t with values in a holomorphic bundle B_t which depend smoothly on a parameter t is an upper semi-continuous function of t and that the index $\sum(-1)^k \dim H^k(X_t, B_t)$ is constant. Since $H^0(V^{m,n}, \mathcal{O}^{tr}_{\mathcal{F}})$ contains the constant functions the general case follows. □

2.2 Deformations of $S^{m,n}_{\xi}$

Let Θ_ξ be the sheaf of germs of holomorphic vector fields on $V^{m,n}$ commuting with ξ and set $\Theta^{tr}_{\mathcal{F}} = \Theta_\xi/(\mathcal{O}^{tr}_{\mathcal{F}} \cdot \xi)$. As in [Ha] one has the following exact sequences of sheaves over $V^{m,n}$

$$0 \longrightarrow \Theta_\xi \longrightarrow \Theta \xrightarrow{L_\xi} \Theta \longrightarrow 0\,, \tag{13}$$

$$0 \longrightarrow \mathcal{O}^{tr}_{\mathcal{F}} \xrightarrow{m(\xi)} \Theta_\xi \xrightarrow{p} \Theta^{tr}_{\mathcal{F}} \longrightarrow 0\,. \tag{14}$$

where $m(\xi)$ is multiplication by ξ and p is the natural projection. Multiplication by ξ also sends the sequence (12) into the sequence (13) inducing a commutative diagram. Notice that $\Theta^{tr}_{\mathcal{F}}$ is just the pullback $\pi^*_{\mathcal{F}}\Theta$. Furthermore there is a sheaf $\widehat{\Theta}_\xi$ over $S^{m,n}_{\xi}$ such that $\Theta_\xi = \pi^*_{\mathcal{F}}\widehat{\Theta}_\xi$. In fact $\widehat{\Theta}_\xi$ is the sheaf of germs of holomorphic sections of a certain holomorphic vector bundle over $S^{m,n}_{\xi}$ of rank $m+n$. As in the case of the sheaf of holomorphic functions one has identifications

$$H^*(S^{m,n}, \Theta) \cong H^*(V^{m,n}, \Theta^{tr}_{\mathcal{F}})\,, \tag{15}$$

$$H^*(S^{m,n}, \widehat{\Theta}_\xi) \cong H^*(V^{m,n}, \Theta_\xi)\,. \tag{16}$$

Let K be a small neighbourhood of zero in a vector subspace E of $H^0(V^{m,n}, \Theta)$. Then the holomorphic foliations $\mathcal{F}_{\xi+u}$ on $V^{m,n}$ defined by the vector fields $\xi + u$ for $u \in K$ are still transverse to $S^{m,n}$ and induce a family of complex manifolds $S^{m,n}_{\xi+u}$. One can show as in [Ha] Prop. 2.2 that the Kodaira-Spencer map $T_0K \to H^1(S^{m,n}_{\xi}, \Theta)$ which measures the infinitesimal deformations of the family $S^{m,n}_{\xi+u}$ at $u = u_0$ is just the restriction to $E \equiv T_0K$ of the composition $i \circ p \circ \delta$

$$H^0(V^{m,n}, \Theta) \xrightarrow{\delta} H^1(V^{m,n}, \Theta_\xi) \xrightarrow{p} H^1(V^{m,n}, \Theta^{tr}_{\mathcal{F}}) \xrightarrow{i} H^1(S^{m,n}_{\xi}, \Theta)$$

where δ is the connecting morphism induced by sequence (13) and i is the identification (15) above.

It is a well known fact (cf. for instance [G-H-S]) that if a family of complex structures on a compact manifold parameterized by a smooth space has a Kodaira-Spencer map which is an isomorphism then this family is already versal. We will see in Theorem 3 below that in almost all the cases one can find an appropriate subspace E of $H^0(V^{m,n}, \Theta)$ such that the composition $\iota \circ p \circ \delta$

restricted to E is an isomorphism. This will provide in these cases the desired versal family of deformations of $S_\xi^{m,n}$.

The computation of the cohomology $H^*(S_\xi^{m,n}, \Theta)$, needed in the study of the deformations of $S_\xi^{m,n}$, can be carried out by the method used by Haefliger in [Ha] except in the case $\lambda = (\mu_1, \ldots, \mu_m, \nu_1, \ldots, \nu_n)$ fulfills relations of some of the following two types

$$\mu_s = \sum_i \alpha_i \mu_i + \sum_j \beta_j \nu_j \quad \alpha_i, \beta_j \in \mathbb{Z}, \alpha_i \geqq 0, \beta_j < 0 \tag{17}$$

$$\nu_s = \sum_i \alpha_i \mu_i + \sum_j \beta_j \nu_j \quad \alpha_i, \beta_j \in \mathbb{Z}, \alpha_i < 0, \beta_j \geqq 0\,. \tag{18}$$

Although the difficulty created by the existence of this kind of relations we are still able to describe the versal space of $S_\xi^{m,n}$ if $m \neq 2$ and $n \neq 2$. More precisely we have

Theorem 3. *Let K be a small neighbourhood of zero in a vector subspace of $\mathfrak{g}_\lambda$ complementary to the vector subspace generated by ξ and $[\xi, \mathfrak{g}_\lambda]$. Assume that we are in one of the following cases:*
(i) *$m \neq 2$ and $n \neq 2$,*
(ii) *$m = 2$, $n \neq 2$ and there are no relations of type* (18),
(iii) *$m \neq 2$, $n = 2$ and there are no relations of type* (17),
(iv) *$m = n = 2$ and there are no relations of the types* (17) *or* (18).
Then the family $S_{\xi+u}^{m,n}$ where $u \in K$ is a versal deformation of $S_\xi^{m,n}$.

If there are no relations of types (17) *and* (18) *then $H^0(S_\xi^{m,n}, \Theta) \cong H^1(S_\xi^{m,n}, \Theta)$ and $H^k(S_\xi^{m,n}, \Theta) = 0$ for $k \geqq 2$.*

Remark 5. It has been shown in paragraph 1.4.2 that a manifold $S_\xi^{m,n}$ with $m = 1$ or $n = 1$, i.e. a Hopf manifold, is isomorphic to a manifold $S_{\xi'}^{1,n}$ where ξ' is a vector field of λ'-type and λ' does not fulfill relations of the above types. Therefore we can suppose without restriction that in the case of a Hopf manifold that the vector field ξ has been chosen with this property. Notice that with this assumption $\mathfrak{g}_\lambda$ coincides with the space of holomorphic vector fields on $V^{1,n}$ commuting with ξ_0.

Proof. Let $\mathfrak{g}_\lambda^\perp$ denote the vector subspace of $H^0(V^{m,n}, \Theta)$ of holomorphic vector fields which are sum of non λ-resonant monomials. Notice that in the case of a Hopf manifold this space does not coincide with that considered by Haefliger in [Ha]. If there are no relations of types (17) and (18) one can argue as Haefliger and show that L_ξ maps $\mathfrak{g}_\lambda^\perp$ isomorphically onto itself, that $L_\xi : H^k(V^{m,n}, \Theta) \to H^k(V^{m,n}, \Theta)$ are isomorphisms for $k \geqq 1$ and therefore that $H^k(V^{m,n}, \Theta_\xi) = 0$ for $k \geqq 2$ and $\dim H^0(V^{m,n}, \Theta_\xi) = \dim H^1(V^{m,n}, \Theta_\xi)$. As before this is checked directly for $\xi = \xi_0$ and then extended to the general case by a semicontinuity argument. Remark that if $m, n > 1$ it is proved in [Ha] that $L_\xi : \mathfrak{g}_\lambda^\perp \to \mathfrak{g}_\lambda^\perp$ is an isomorphism without any assumption.

In order to prove the versality of the family parameterized by K it is sufficient to show that the composition $p \circ \delta$ is bijective when restricted to

T_0K. One can see, by means of the commutative diagram of cohomology exact sequences induced by the multiplication by ξ mapping (12) into (13), that the kernel of $p \circ \delta$ is generated by ξ and $[\xi, H^0(V^{m,n}, \Theta)]$ and that $p \circ \delta$ is surjective if and only if δ is surjective. Since $H^2(V^{m,n}, \mathcal{O}_{\mathcal{F}}^{tr}) = 0$ it is sufficient to prove the surjectivity of the connecting morphism δ or equivalently the injectivity of $L_\xi : H^1(V^{m,n}, \Theta) \to H^1(V^{m,n}, \Theta)$. In the case (iv) this follows from the above discussion and in the case (i) one has $H^1(V^{m,n}, \mathcal{O}) = 0$. The case (iii) being similar let us suppose that we are in the situation (ii). Let us assume first that $\xi = \xi_0$. A direct computation shows the injectivity of $L_{\xi_0} : H^1(V^{m,n}, \Theta) \to H^1(V^{m,n}, \Theta)$. Thus the family parameterized by the space K corresponding to ξ_0 is versal at the origin and, by the general theory of deformations, is complete at the points close enough to zero. This implies that, for a general ξ close to ξ_0, any deformation of $S_\xi^{m,n}$ comes from a deformation of the vector field ξ. Therefore a neighbourhood of the origin in $H^0(V^{m,n}, \Theta)$ parameterizes a complete family of deformations of $S_\xi^{m,n}$ and its corresponding Kodaira-Spencer morphism, which is nothing but the composition $i \circ p \circ \delta$, is surjective. This completes the proof. □

Remarks 6. (1) If ξ is in the hypothesis of Theorem 3 then any deformation of $S_\xi^{m,n}$ comes from a deformation of the vector field ξ. This is no longer true in the exceptional cases excluded in the above statement. For these cases one can represent the versal space as a suitable linear subspace of $H^1(V^{m,n}, \Theta_\xi)$ and, at least for $\xi = \xi_0$, the connecting map $\delta : H^0(V^{m,n}, \Theta) \to H^1(V^{m,n}, \Theta_\xi)$ is not surjective. This fact suggests the existence of complex structures on $S^3 \times S^{2n-1}$ which are not obtained from a vector field by the above procedure.

(2) The space K can be identified with the versal space of deformations of the transversely holomorphic foliation induced by ξ on the unit sphere of $\mathbb{C}^N$ (cf. [Ha]).

(3) In the case λ does not have resonance relations other than the trivial ones then $\dim K = \dim \mathfrak{g}_\lambda - 1 = N - 1$.

(4) If $S_\xi^{m,n}$ is a Calabi-Eckmann manifold (cf. 1.4.3) then its versal space has dimension $m^2 + n^2 - 1$.

2.3 Dolbeault's cohomology of $S_\xi^{m,n}$

Let Ω_ξ^p and $\Omega_{b,\xi}^p$ respectively denote the sheaves over $V^{m,n}$ of holomorphic differential p-forms invariant by ξ (i.e. forms ω such that $L_\xi \omega = 09$ and basic p-forms (i.e. forms ω such that $L_\xi \omega = i_\xi \omega = 0$). Notice that $\Omega_{b,\xi}^p = \pi_\xi^* \Omega^p$ where $\pi_\xi : V^{m,n} \to S_\xi^{m,n}$ is the natural projection. One can also see that Ω_ξ^p is the pull-back by π_ξ of the sheaf $\widehat{\Omega}_x i^p$ of holomorphic sections of a certain vector bundle over $S_\xi^{m,n}$. As above there are identifications

$$H^*(S^{m,n}, \Omega^p) \cong H^*(V^{m,n}, \Omega_{b,\xi}^p), \tag{19}$$

$$H^*(S^{m,n}, \widehat{\Omega}_\xi^p) \cong H^*(V^{m,n}, \Omega_\xi^p). \tag{20}$$

In order to compute the Dolbeault cohomology of $S_\xi^{m,n}$ we use the following exact sequences of sheaves over $V^{m,n}$

$$0 \longrightarrow \Omega_\xi^p \longrightarrow \Omega^p \xrightarrow{L_\xi} \Omega^p \longrightarrow 0\,, \tag{21}$$

$$0 \longrightarrow \Omega_{b,\xi}^p \longrightarrow \Omega_\xi^p \xrightarrow{i_\xi} \Omega_{b,\xi}^{p-1} \longrightarrow 0\,. \tag{22}$$

where $1 \leqq p \leqq m+n-1 = S_\xi^{m,n}$. Notice that sequence (21) still makes sense for $p = 0$ and that $\Omega_{b,\xi}^0 = \Omega_\xi^0 = \mathcal{O}_{\mathcal{F}}^{tr}$. We shall also use the Stein open covering $\mathfrak{W} = \{U_i \times V_j\}_{(i,j)}$ of $V^{m,n}$, where $U_i = \{z \in \mathbb{C}^m \mid z_i \neq 0\}$ and $V_j = \{w \in \mathbb{C}^n \mid w_j \neq 0\}$, already considered in Remark 4. Notice that $\mathfrak{W}$ is not acyclic with respect to the sheaves Ω_ξ^p and $\Omega_{b,\xi}^p$.

Proposition 10. *Assume* $1 \leqq m,n$, $m \neq n$ *and let* $0 \leqq p \leqq m+n$. *Then*
(i) $H^k(V^{m,n}, \Omega_\xi^0) \cong \mathbb{C}$ *if* $k = 0, 1$,
(ii) *for* $\xi = \xi_0$ *one has* $H^k(V^{m,n}, \Omega_{\xi_0}^p) \cong \mathbb{C}$ *if* $p = m$ *and* $k = m-1, m$, *if* $p = n$ *and* $k = n-1, n$ *or if* $p = m+n$ *and* $k = m+n-2, m+n-1$,
(iii) *or a general* ξ *one has* $H^k(V^{m,n}, \Omega_\xi^p) = 0$ *in all other cases.*

Proof. The statement for $p = 0$ is just Theorem 2. Assume $\xi = \xi_0$. A direct calculation using the identifications given in Propositions 6 to 9 shows that $L_{\xi_0} : H^k(V^{m,n}, \Omega^p) \to H^k(V^{m,n}, \Omega^p)$ is an isomorphism except if $p = m$ and $k = m-1$, if $p = n$ and $k = n-1$ or if $p = m+n$ and $k = m+n-2$. It follows, using the cohomology exact sequence associated to (21), that for $p \neq 0, m, n, m+n$ the cohomology with values in $\Omega_{\xi_0}^p$ vanishes. If $p = m$ the cohomology groups $H^k(V^{m,n}, \Omega_{\xi_0}^m)$ also vanishes for $k \neq m-1, m$ and $H^{m-1}(V^{m,n}, \Omega_{\xi_0}^m)$, $H^m(V^{m,n}, \Omega_{\xi_0}^m)$ are respectively isomorphic to the kernel and cokernel of $L_{\xi_0} : H^{m-1}(V^{m,n}, \Omega^m) \to H^{m-1}(V^{m,n}, \Omega^m)$ which are both of dimension one. The statement for $p = n, m+n$ can be proved in an analogous way. Finally part (iii), for a general ξ, follows from the diagonal case using the semicontinuity argument. □

In a similar manner one obtains

Proposition 11. *Assume* $1 < m = n$ *and let* $0 \leqq p \leqq 2m$. *Then*
(i) $H^k(V^{m,n}, \Omega_\xi^0) \cong \mathbb{C}$ *if* $k = 0, 1$,
(ii) *for* $\xi = \xi_0$ *one has* $H^{m-1}(V^{m,n}, \Omega_{\xi_0}^m) \cong H^m(V^{m,n}, \Omega_{\xi_0}^m) \cong \mathbb{C}$ *and* $H^{2m-2}(V^{m,n}, \Omega_{\xi_0}^{2m}) \cong H^{2m-1}(V^{m,n}, \Omega_{\xi_0}^{2m}) \cong \mathbb{C}$,
(iii) *for a general* ξ *one has* $H^k(V^{m,n}, \Omega_\xi^p) = 0$ *in all other cases.*

As usual let $h^{p,q} = \dim H^q(S_\xi^{m,n}, \Omega^p)$ denote the Hodge numbers of $S_\xi^{m,n}$.

Theorem 4. (i) *Assume* $1 \leqq m,n$ *and* $m \neq n$. *Set* $u = \min\{m,n\}$ *and* $v = \max\{m,n\}$. *Then* $h^{q,q} = h^{q,q+1} = 1$ *if* $0 \leqq q \leqq u-1$, $h^{q,q} = h^{q,q-1} = 1$ *if* $v \leqq q \leqq m+n-1$ *and* $h^{k,l} = 0$ *otherwise.*
(ii) *Assume* $1 < m = n$. *Then* $h^{q,q} = h^{q,q+1} = 1$ *if* $0 \leqq q \leqq m-1$, $h^{q,q} = h^{q,q-1} = 1$ *if* $m \leqq q \leqq 2m-1$ *and* $h^{k,l} = 0$ *otherwise.*

Proof. (i) We shall assume $m < n$. For any $k \in \{0, \ldots, m-1\}$ one has $H^l(V^{m,n}, \Omega^k_{b,\xi}) = 0$ if $l \neq k, k+1$ and $H^l(V^{m,n}, \Omega^k_{b,\xi}) \cong \mathbb{C}$ if $l = k, k+1$. This is already known for $k = 0$ since $\Omega^0_{b,\xi} = \mathcal{O}^{tr}_{\mathcal{F}}$. For $0 < k \leqq m-1$ it is shown by induction on k. In fact Proposition 10(iii) implies that the connecting morphisms $D^{k-1} : H^{k-1}(V^{m,n}, \Omega^{k-1}_{b,\xi}) \to H^k(V^{m,n}, \Omega^k_{b,\xi})$ and $D^k : H^k(V^{m,n}, \Omega^{k-1}_{b,\xi}) \to H^{k+1}(V^{m,n}, \Omega^k_{b,\xi})$ induced by (22) are in fact isomorphisms. Moreover one can also see by induction and using these connecting morphisms that in the diagonal case $\xi = \xi 0$ the k-cocycle $a^k =$ on the open covering $\mathfrak{W}$ given by

$$a^k_{(i_1,j_1)\ldots(i_{k+1},j_{k+1})} = \sum_{l=1}^{k+1} (-1)^{l+1} \frac{dz_{i_1}}{\mu_{i_1} z_{i_1}} \wedge \ldots \wedge \widehat{\frac{dz_{i_l}}{\mu_{i_l} z_{i_l}}} \wedge \ldots \wedge \frac{dz_{i_{k+1}}}{\mu_{i_{k+1}} z_{i_{k+1}}}$$

defines a non zero cohomology class which generates $H^k(V^{m,n}, \Omega^k_{b,\xi_0}) \cong \mathbb{C}$.

In a similar way, if $n \leqq k \leqq m+n-1$ one has $H^l(V^{m,n}, \Omega^k_{b,\xi}) = 0$ if $l \neq k-1, k$ and $H^l(V^{m,n}, \Omega^k_{b,\xi}) \cong \mathbb{C}$ if $l = k-1, k$. Since $\Omega^{m+n}_{b,\xi} = 0$ the statement for $k = m+n-1$ follows from the cohomology exact sequence induced by (22) where we take $p = m+n$. Then the remaining cases also follow by a descending induction on k. In this case one can also see by induction that in the diagonal case $\xi = \xi_0$ and for $1 \leqq k \leqq m$ the cocycle $b^{m+n-k-1}$ on $\mathfrak{W}$ given by

$$b^{m+n-k-1}_{(i_1,j_1)\ldots(i_{m+n-k},j_{m+n-k})}$$

$$= i_{\xi_0}\left(\sum_{l=0}^{m+n-k} \frac{dz_{i_1}}{\mu_{i_1} z_{i_1}} \wedge \ldots \wedge \frac{dz_{i_l}}{\mu_{i_l} z_{i_l}} \wedge \frac{dw_{j_l}}{\nu_{j_l} w_{j_l}} \wedge \ldots \wedge \frac{dw_{j_{m+n-k}}}{\nu_{j_{m+n-k}} w_{j_{m+n-k}}}\right)$$

defines a non zero cohomology class which generates the cohomology group $H^{m+n-k-1}(V^{m,n}, \Omega^{m+n-k}_{b,\xi_0}) \cong \mathbb{C}$.

The long exact sequence associated to (22) for $p = m$ vanishes except at the following piece

$$0 \to H^{m-1}(\Omega^m_{b,\xi}) \to H^{m-1}(\Omega^m_\xi) \cong \mathbb{C} \xrightarrow{i^{m-1}_\xi} H^{m-1}(\Omega^{m-1}_{b,\xi}) \cong \mathbb{C} \to$$

$$\to H^m(\Omega^m_{b,\xi}) \to H^m(\Omega^m_\xi) \cong \mathbb{C} \to H^m(\Omega^{m-1}_{b,\xi}) \cong \mathbb{C} \to H^{m+1}(\Omega^m_{b,\xi}) \to 0$$

where we have omitted the base space $V^{m,n}$ and the indicated isomorphisms are given by the above discussion and Proposition 10(ii). In an analogous way taking $p = n$ in (22) gives an exact sequence which vanishes except at

$$0 \to H^{n-2}(\Omega^{n-1}_{b,\xi}) \to H^{n-1}(\Omega^n_{b,\xi}) \cong \mathbb{C} \xrightarrow{j^{n-1}} H^{n-1}(\Omega^n_\xi) \cong \mathbb{C} \to$$

$$\to H^{n-1}(\Omega^{n-1}_{b,\xi}) \to H^n(\Omega^n_{b,\xi}) \cong \mathbb{C} \to H^n(\Omega^n_\xi) \cong \mathbb{C} \to H^n(\Omega^{n-1}_{b,\xi}) \to 0\,.$$

We have to prove that $b^{m,q} = 0$ for $q = m-1, m, m+1$ and that $b^{n-1,q} = 0$ for $q = n-2, n-1, n$. Notice that Serre's duality implies $b^{m,m-1} = b^{n-1,n}$,

$b^{m,m} = b^{n-1,n-1}$ and $b^{m,m+1} = b^{n-1,n-2}$. Therefore it is sufficient to prove that the morphisms i_ξ^{m-1} and j^{n-1} in the above exact sequences are in fact isomorphisms. We shall prove this for $\xi = \xi_0$. This result will imply, also by induction, that $H^*(V^{m,n}, \Omega^k_{b,\xi}) \equiv 0$ for $m \leqq k \leqq n-1$ in the diagonal case. The general case will then follow from the semicontinuity argument.

Using Proposition 10 and Remark 4 one can see that the cohomology group $H^{m-1}(V^{m,n}, \Omega^m_{\xi_0}) \cong \mathbb{C}$ is generated by the cohomology class of the $(m-1)$-cocycle c on $\mathfrak{W}$ defined by

$$c_{(i_1,j_1)\ldots(i_m,j_m)} = \frac{dz_{i_1}}{\mu_{i_1} z_{i_1}} \wedge \ldots \wedge \frac{dz_{i_m}}{\mu_{i_m} z_{i_m}} .$$

Since $i_{\xi_0}^{m-1}(a^{m-1}) = c$ it follows that $i_{\xi_0}^{m-1}$ is an isomorphism.

Still using Proposition 10 and Remark 4 one can also see that the cohomology group $H^{n-1}(V^{m,n}, \Omega^n_{\xi_0}) \cong \mathbb{C}$ is generated by the class of the $(n-1)$-cocycle d defined by

$$d_{(i_1,j_1)\ldots(i_n,j_n)} = \frac{dw_{j_1}}{v_{j_1} w_{j_1}} \wedge \ldots \wedge \frac{dw_{j_n}}{v_{j_n} w_{j_n}} .$$

We have $b^{n-1} = d + t$ where t is a sum of terms all of them containing at least one $\dfrac{dz_i}{\mu_i z_i}$. Clearly d is also a cocycle with values in $\Omega^n_{\xi_0}$ and therefore t is so. Since t is a coboundary when thought as a cochain with values in Ω^n and the map $H^{n-1}(V^{m,n}, \Omega^n_{\xi_0}) \to H^{n-1}(V^{m,n}, \Omega^n)$ is injective it follows that d and b^{n-1} define the same cohomology class in $H^{n-1}(V^{m,n}, \Omega^n_{\xi_0})$. This shows that j^{n-1} is also an isomorphism.

(ii) The case $m = n$ is less involved. One computes $h^{p,q}$ by induction on p for $0 \leqq q \leqq m-1$ and any q. The remaining Hodge numbers are then determined by Serre's duality. □

Corollary 3. *There is no global holomorphic differential p-forms on $S_\xi^{m,n}$ for $p > 0$. In particular the canonical line bundle over $S_\xi^{m,n}$ is non trivial.*

Remark 7. The Hodge numbers of a Hopf manifold has been computed by D. Mall in [Ma]. For a Calabi-Eckmann manifold A. Borel has shown in [Bo] that the Dolbeault's cohomology bigraded algebra is isomorphic to $\mathbb{C}[x_{1,1}]/(x_{1,1}^u) \oplus \Lambda(x_{v,v-1}, x_{0,1})$ where $u = \min\{m,n\}$ and $v = \max\{m,n\}$. By means of the representation of the Dolbeault's cohomology in terms of cocycles given through the proof of the above theorem one can easily check that this description remains valid for any manifold $S_\xi^{m,n}$ with $\xi = \xi_0$.

3 Holomorphic line bundles and subvarieties

3.1 Classification of holomorphic $\mathbb{C}$-principal bundles and line bundles over $S_\xi^{m,n}$

The abelian group of isomorphisms classes of holomorphic $\mathbb{C}$-principal bundles over $S_\xi^{m,n}$ is naturally identified to $H^1(S_\xi^{m,n}, \mathcal{O})$. It was proved in Theorem 2

that this cohomology group is isomorphic to $\mathbb{C}$ for any ξ. A more detailed description of these bundles is given in the following proposition.

Proposition 12. *The isomorphisms classes of non trivial holomorphic $\mathbb{C}$-principal bundles over $S_\xi^{m,n}$ are in a one-to-one correspondence with the bundles E_τ, with $\tau \in \mathbb{C}^*$, where E_τ denotes the space $V^{m,n}$ endowed with the $\mathbb{C}$-action defined by the flow associated to $\frac{1}{\tau}\xi$.*

Proof. Let $\{g_{ij}\}$ be a cocycle on a Stein open covering of $S_\xi^{m,n}$ and defining the bundle E_1. A direct computation shows that, for any $\tau \in \mathbb{C}^*$, $\left\{\frac{1}{\tau} g_{ij}\right\}$ is a cocycle defining the bundle E_τ. The cocycle $\{g_{ij}\}$ can not be a coboundary. Otherwise $S_\xi^{m,n}$ could be holomorphically embedded in $\mathbb{C}^N$. Therefore two cocycles $\left\{\frac{1}{\tau} g_{ij}\right\}$ and $\left\{\frac{1}{\tau'} g_{ij}\right\}$ are cohomologous if and only if $\tau = \tau'$. This completes the proof. □

Let $\mathcal{O}^*$ denote the sheaf, over a given complex manifold, of germs of non vanishing holomorphic functions. The group of holomorphic line bundles over $S_\xi^{m,n}$ is naturally identified to $H^1(S_\xi^{m,n}, \mathcal{O}^*)$. The cohomology exact sequence associated to

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{\exp(2i\pi\,\cdot\,)} \mathcal{O}^* \longrightarrow 0 \tag{19}$$

and Theorem 2 imply that $H^1(S_\xi^{m,n}, \mathcal{O}^*) \cong H^1(S_\xi^{m,n}, \mathcal{O}) \cong \mathbb{C}$ if $m,n > 1$ and that $H^1(S_\xi^{m,n}, \mathcal{O}^*) \cong H^1(S_\xi^{m,n}, \mathcal{O})/\mathbb{Z} \cong \mathbb{C}^*$ if $m = 1$ or $n = 1$ but $m + n > 2$. Therefore to each $\mathbb{C}$-principal bundle E_τ over $S_\xi^{m,n}$ there corresponds, via the morphism $\exp(2i\pi\,\cdot\,)_*$, a line bundle F_τ. More precisely if ρ denotes the $\mathbb{C}$-action on $E_\tau \equiv V^{m,n}$ defined by $\frac{1}{\tau}\xi$ then F_τ is the quotient space of $E_\tau \times \mathbb{C}$ by the equivalence relation

$$(x,t) \sim (\rho(u)\cdot x, e^{2i\pi u} t) \quad \text{for } u \in \mathbb{C}\,.$$

In particular one can see that the holomorphic sections of F_τ are naturally identified to the holomorphic functions f in $E_\tau \equiv V^{m,n}$ which satisfy

$$f(\rho(u)\cdot x) = e^{2i\pi u} f(x) \quad \text{for } u \in \mathbb{C}$$

or, equivalently,

$$L_\xi f = 2i\pi\tau \cdot f\,. \tag{20}$$

Let F_0 denote the trivial line bundle over $S_\xi^{m,n}$. We have

Proposition 13. *Every complex line bundle over $S_\xi^{m,n}$ is isomorphic to a line bundle F_τ for a suitable $\tau \in \mathbb{C}$. Moreover*

(i) *Let $m,n > 1$. Then $F_\tau \cong F_{\tau'}$ if and only if $\tau = \tau'$ and the holomorphic sections of F_τ are identified to the holomorphic functions on $\mathbb{C}^N$ fulfilling* (20).

(ii) *Let* $1 = m < n$. *Then* $F_\tau \cong F_{\tau'}$ *if and only if* $2i\pi(\tau - \tau') \in \mathbb{Z}$ *and the holomorphic sections of* F_τ *are identified to the holomorphic functions on* $\mathbb{C}^* \times \mathbb{C}^n$ *fulfilling* (20).

Proof. It only remains to prove that in the case of a Hopf manifold, i.e. if $1 = m < n$, the trivial bundles are exactly those of the form $F_{k/2i\pi}$ for $k \in \mathbb{Z}$. The bundle $F_{1/2i\pi}$ is trivial since the function $f(z,w) \equiv z$ defines a section of it which does not vanish. Assume now that there is an integer q bigger than one for which $F_{1/2i\pi q}$ is trivial. Let a holomorphic function f on $\mathbb{C}^* \times \mathbb{C}^n$ define a section of the bundle, that is $L_\xi f = \frac{1}{q} f$. In particular $h(z) := f(z,0)$ is a holomorphic function on $\mathbb{C}^*$ fulfilling $z \cdot h' = \frac{1}{q} h$. But this is only possible if $h \equiv 0$. In particular $f(1,0) = 0$ and, since $n > 1$, there should exist $w \neq 0$ with $f(1,w) = 0$. Therefore any section of $F_{1/2i\pi q}$ vanishes somewhere contradicting the triviality of the bundle. □

For a given complex manifold M every meromorphic function on M can be written as the quotient s_1/s_2 of two holomorphic sections s_1 and s_2, where s_2 is not identically zero, of a suitable line bundle over M. From the previous proposition we obtain

Corollary 4. *A meromorphic function on* $S_\xi^{m,n}$ *is given by a quotient* u/v, *where* u *and* v *are entire functions on* $\mathbb{C}^N$ *if* $1 < m,n$ *and holomorphic functions on* $\mathbb{C}^* \times \mathbb{C}^n$ *if* $1 = m < n$, *fulfilling a relation of the form*

$$L_\xi u = a \cdot u \quad \text{and} \quad L_\xi v = a \cdot v$$

for some $a \in \mathbb{C}$. *In the diagonal case* $\xi = \xi_0$ *the existence of non constant meromorphic functions over* $S_{\xi_0}^{m,n}$ *is equivalent to the linear dependence over* $\mathbb{Q}$ *of the eigenvalues* $\lambda_1, \ldots, \lambda_N$ *of* ξ_0.

Let $\mathcal{O}_\tau$ denote the sheaf over $S_\xi^{m,n}$ of germs of holomorphic sections of the line bundle F_τ. In view of Proposition 13 the pull-back of $\mathcal{O}_\tau$ by the projection $\pi : V^{m,n} \to S_\xi^{m,n}$ coincides with the kernel of $L_\xi - \tau : \mathcal{O} \to \mathcal{O}$ that we denote by $\mathcal{O}^{tr}_{\mathscr{F},\tau}$. In fact the sequence of sheaves over $V^{m,n}$

$$0 \longrightarrow \mathcal{O}^{tr}_{\mathscr{F},\tau} \longrightarrow \mathcal{O} \xrightarrow{L_\xi - 2i\pi\tau} \mathcal{O} \longrightarrow 0 \tag{21}$$

is exact as it is shown by a calculus in local coordinates. Furthermore there is a natural identification

$$H^*(S_\xi^{m,n}, \mathcal{O}_\tau) \cong H^*(V^{m,n}, \mathcal{O}^{tr}_{\mathscr{F},\tau}) . \tag{22}$$

Therefore the calculus of $H^*(S_\xi^{m,n}, \mathcal{O}_\tau)$ can be made by means of the exact sequence (21) in a way similar to the computation of the cohomology of $S_\xi^{m,n}$ with values in $\mathcal{O}$. In a contrast with that case the cohomology groups $H^q(S_\xi^{m,n}, \mathcal{O}_\tau)$ depend not only on τ but also on ξ. In fact $\dim H^q(S_\xi^{m,n}, \mathcal{O}_\tau)$

is smaller but in general not equal to $\dim H^q(S^{m,n}_{\xi_0}, \mathcal{O}_\tau)$. This can already be seen for the space of global sections of F_ξ using the description given in Proposition 13.

Reasoning as in the proof of Theorem 2 one obtains the results collected in the proposition below. One sees in particular that the cohomology vanishes for those $\tau \in \mathbb{C}$ which do not fulfill relations of type

$$2i\pi\tau = (\alpha, \mu) + (\beta, \nu) \quad \text{with } \alpha \in \mathbb{Z}^m \text{ and } \beta \in \mathbb{Z}^n \,. \tag{23}$$

Let $A_{(+,+)}$ (resp. $A_{(+,-)}$, resp. $A_{(-,+)}$, resp. $A_{(-,-)}$) denote the set of complex numbers τ fulfilling a relation of type (23) with $\alpha, \beta \geqq 0$ (resp. $\alpha \geqq 0$ and $\beta < 0$, resp. $\alpha < 0$ and $\beta \geqq 0$, resp. $\alpha, \beta < 0$) and let A be the disjoint union $A_{(+,+)} \cup A_{(+,-)} \cup A_{(-,+)} \cup A_{(-,-)}$. For a given $\tau \in A_{(+,+)}$ set

$$d(\tau) = \#\{(\alpha, \beta) \in \mathbb{Z}^m \times \mathbb{Z}^n \mid 2i\pi\tau = (\alpha, \mu) + (\beta, \nu) \text{ and } \alpha \geqq 0, \beta \geqq 0\} \,.$$

We define $d(\tau)$ in an analogous way in case τ belongs to $A_{(+,-)}$, $A_{(-,+)}$ or $A_{(-,-)}$. With this notation we have

Theorem 5. *All the cohomology groups $H^q(S^{m,n}_\xi, \mathcal{O}_\tau)$ vanish if $\tau \notin A$. Assume that $\xi = \xi_0$ and that $\tau \in A$, then*

$$H^q(S^{m,n}_\xi, \mathcal{O}_\tau) \cong \mathbb{C}^{d(\tau)} \text{ if } \begin{cases} \tau \in A_{(+,+)} & \text{and } q = 0, 1, \text{ or} \\ \tau \in A_{(-,+)} & \text{and } q = m-1, m, \text{ or} \\ \tau \in A_{(+,-)} & \text{and } q = n-1, n, \text{ or} \\ \tau \in A_{(-,-)} & \text{and } q = m+n-2,\ m+n-1 \,. \end{cases}$$

$H^q(S^{m,n}_\xi, \mathcal{O}_\tau) = 0$ *otherwise.*

Remark 8. In a more general way one can consider the cohomology of $S^{m,n}_\xi$ with values in the sheaf Ω^p_τ of germs of holomorphic p-forms on $S^{m,n}_\xi$ with values in the line bundle F_τ. The pull-back of Ω^p_τ by the projection $\pi_\xi : V^{m,n} \to S^{m,n}_\xi$ is naturally identified to the sheaf $\Omega^p_{b,\xi,\tau}$ of germs of holomorphic p-forms ω fulfilling $L_\xi \omega = 2i\pi\tau \cdot \omega$ and $i_\xi \omega = 0$. Denote by $\Omega^p_{\xi,\tau}$ the kernel of the sheaf morphism $L_\xi - 2i\pi\tau : \Omega^p \to \Omega^p$. Then the following sequences of sheaves over $V^{m,n}$ are exact.

$$0 \longrightarrow \Omega^p_{\xi,\tau} \longrightarrow \Omega^p \xrightarrow{L_\xi - 2i\pi\tau} \Omega^p \longrightarrow 0 \,, \tag{24}$$

$$0 \longrightarrow \Omega^p_{b,\xi,\tau} \longrightarrow \Omega^p_{\xi,\tau} \xrightarrow{i_\xi} \Omega^{p-1}_{b,\xi,\tau} \longrightarrow 0 \,. \tag{25}$$

Since there is an identification $H^*(S^{m,n}_\xi, \Omega^p_\tau) = H^*(V^{m,n}, \Omega^p_{b,\xi,\tau})$ one could compute Dolbeault's cohomology of $S^{m,n}_\xi$ with values in F_τ following the lines of the proof of Theorem 4 and using the cohomology exact sequences associated to (24), (25) and the above proposition. For instance one can easily see that all the cohomology groups $H^q(S^{m,n}_\xi, \Omega^p_\tau)$ vanish in the case τ does not fulfill any relation of type (23). In the case of a Hopf manifold $X = S^{1,n}_\xi \cong \mathbb{C}^n - \{0\}/\langle f \rangle$,

where f is a contraction of $\mathbb{C}^n$ fixing the origin (cf. example 1.4.2) the cohomology groups $H^q(X, \Omega_t^p)$ were calculated by D. Mall in [Ma] under a generic condition on f if $n > 2$ and for all the cases if $n = 2$.

3.2 Subvarieties

The purpose of this paragraph is to describe the analytic subsets of $S_\xi^{m,n}$ whose dimension is bigger or equal to $\max(m,n)$ in the case $m,n > 1$ and of any dimension in the Hopf case. To this end we begin by proving a more precise version of a theorem of M. Kato about the algebraicity of the analytic subsets of $\mathbb{C}^N$ invariant by a contraction. The proof we present here is simpler and more elementary than that of Kato in [Ka1], who uses Rudin's characterization of the algebraic subsets of $\mathbb{C}^N$.

Let ϕ denote a contracting holomorphic automorphism of $\mathbb{C}^N$ fixing the origin and in the normal form (9). The diagonal part ϕ_0 of ϕ is of the form

$$\phi_0(\zeta_1, \ldots, \zeta_N) = (\varrho_1 \zeta_1, \ldots, \varrho_N \zeta_N)$$

with $|\varrho_k| < 1$ for any $1 \leqq k \leqq N$. Let S denote the subset of $\mathbb{C}$ whose elements s are of the form

$$s = \varrho_1^{p_1} \cdot \ldots \cdot \varrho_N^{p_N}$$

with $p_1, \ldots, p_N \in \mathbb{N}$. For a given $s \in S$ let O_s denote the space of s-eigenfunctions of ϕ_0, that is holomorphic functions f on $\mathbb{C}^N$ fulfilling $f \circ \phi_0 = s \cdot f$. The space O_s is finite dimensional and its elements are polynomials. The space of all the entire functions on $\mathbb{C}^N$ is the topological direct sum of the spaces O_s. With this notation we have the following theorem.

Theorem 6. *A ϕ-invariant analytic subset X of $\mathbb{C}^N$ is the vanishing set of a certain number of polynomials belonging to the spaces O_s. In particular X is an algebraic subset of $\mathbb{C}^N$.*

Proof. The automorphism ϕ commutes with ϕ_0 and, for $n \in N$ and $\zeta \in \mathbb{C}^N$, one has

$$\phi_0^{-n} \phi^n \zeta = (\phi_0^{-1} \phi)^n \zeta = P(n, \zeta) \tag{26}$$

where P is a polynomial in n and ζ. In the sequel ζ will be a fixed point in $\mathbb{C}^N$. Let $s \in S$ and $g \in O_s$ be given, then

$$\begin{aligned} g(\phi^n \zeta) = g(\phi_0^n \phi_0^{-n} \phi^n \zeta) &= s^n g(P(n, \zeta)) \\ &= s^n Q(n) \end{aligned} \tag{27}$$

where Q is a polynomial.

Let f be an entire function vanishing on X. If $\zeta \in X$ then $f(\phi^n \zeta) = 0$ for any $n \in \mathbb{N}$. Therefore if we expand f as a series of eigenfunctions f_s in O_s one has

$$\sum_{s \in S} f_s(\phi^n \zeta) = 0 .$$

It follows from (27) that

$$\sum_{s\in S} s^n Q_s(n) = 0\,, \tag{28}$$

where Q_s denotes the polynomial $f_s(P(n,\zeta))$. In order to prove the theorem it suffices to show that the polynomials Q_s vanish identically since one has $Q_s(0) = f_s(\zeta)$. We will use the relation

$$\begin{aligned} f_s(\phi^n\zeta) &= f_s(\phi_0^{n-[n/2]}\phi_0^{[n/2]}\phi_0^{-n}\phi^n\zeta) \\ &= f_s(\phi_0^{n-[n/2]}\phi_0^{[n/2]}P(n,\zeta)) \\ &= s^{n-[n/2]} f_s(\phi_0^{[n/2]}P(n,\zeta))\,. \end{aligned}$$

Let $K>0$ be fixed and set $S_K = \{s\in S \mid K|s|^{1/2} < 1/2\}$. The complementary of S_K in S is finite. Using the normal convergence of $\sum f_s$ in a neighbourhood of zero and the fact that $\lim_{n\to\infty}\phi_0^{[n/2]}P(n,\zeta) = 0$ one deduces that

$$\lim_{n\to\infty}\sum_{s\in S_K} K^n s^{n-[n/2]} f_s(\phi_0^{[n/2]}P(n,\zeta)) = 0$$

which implies $\lim_{n\to\infty}\sum_{s\in S_K} K^n s^n Q_s(n) = 0$ and using (28) that

$$\lim_{n\to\infty}\sum_{s\in S-S_K} K^n s^n Q_s(n) = 0\,.$$

This implies that for an arbitrary $K>0$ one has

$$\lim_{n\to\infty}\sum_{\substack{s\in S-S_K \\ |Ks|\geqq 1}} (Ks)^n Q_s(n) = 0\,.$$

The result is now a consequence of the following elementary lemma:

Lemma 4. *Let $\beta_1,\ldots,\beta_q$ be complex numbers such that $|\beta| \geqq 1$ and $\beta_i \neq \beta_j$ for $i\neq j$. Let $Q_1,\ldots,Q_q$ be polynomials and set $E(n) = \sum_{i=1}^q \beta_i^n Q_i(n)$. Assume that $\lim_{n\to\infty} E(n) = 0$. Then $Q_i \equiv 0$ for any $1 \leqq i \leqq q$.*

Proof. We can suppose without loss of generality that

$$\begin{aligned} &r \equiv |\beta_1| = \ldots = |\beta_k| > |\beta_i| && \text{for } k < i \leqq q \\ \text{and}\quad &\ell \equiv \deg Q_1 = \ldots = \deg Q_h > \deg Q_j && \text{for } h < j \leqq k\,. \end{aligned}$$

For $1 \leqq j \leqq h$ let c_j denote the coefficient of n^ℓ in Q_j. One has

$$E(n) = n^\ell r^n\left(\sum_{j=1}^h c_j u_j^n + o(n)\right)$$

where $u_j = \beta_j/r$. The complex numbers u_j are of modulus 1 and $u_i \neq u_j$ for $i\neq j$. Therefore it is sufficient to prove that for a sequence $v_n = \sum_{j=1}^h c_j u_j^n$,

where $c_j \in \mathbb{C}$ and u_j are as before, then

$$\lim_n v_n = 0 \Rightarrow c_j = 0 \quad \text{for any } 1 \leqq j \leqq 0 .$$

This can be seen by induction on h. For $h = 1$ the result is clear. In the general case, suppose that $\lim_{n\to\infty} v_n = 0$. Then the sequence

$$w_n = u_1^{-n} v_n - u_1^{-(n+1)} v_{n+1} = \sum_{j=2}^{h} c_j \left(1 - \frac{u_j}{u_1}\right)\left(\frac{u_j}{u_1}\right)^n$$

tends also to zero. By the induction hypothesis one has $c_j = 0$ for $j \geqq 2$ and then also $c_1 = 0$. □

As a consequence of the above theorem we obtain the following result.

Proposition 14. (i) *Let X be an analytic subset of $\mathbb{C}^N$ which is invariant by ξ. Then X is also invariant by ξ_0 and in fact $X = \bigcap_{i=1}^{k} f_i^{-1}(0)$ where f_i are polynomials such that $L_{\xi_0} f_i = b_i f_i$ with $b_i \in \mathbb{C}$.*
(ii) *Assume $m = 1$ and let X be a ξ-invariant analytic subset of $\mathbb{C}^* \times \mathbb{C}^n$. Then X is also invariant by ξ_0 and in fact $X = \bigcap_{i=1}^{k} f_i^{-1}(0)$ where f_i are holomorphic functions on $\mathbb{C}^* \times \mathbb{C}^n$ such that $L_{\xi_0} f_i = b_i f_i$ with $b_i \in \mathbb{C}$.*

Proof. Let us begin by proving (ii). We suppose that $\lambda = (1, \nu_1, \ldots, \nu_n)$. Then the flow $\psi(t; z, w) = \exp t\xi$ associated to ξ can be written

$$\psi(t; z, w) = (e^t z, \ldots, e^{\nu_k t} P_k(t, z, w), \ldots) \tag{29}$$

where P_k is a polynomial of the form $P_k(t, z, w) = w_k + t \cdot P_k'(t; z, w)$. For a fixed z we set

$$\exp 2i\pi\xi(z, w) = (z, \phi(w))$$

$$\exp 2i\pi\xi_0(z, w) = (z, \phi_0(w)) .$$

Then ϕ is a contracting automorphism of $\mathbb{C}^n$ and ϕ_0 is its linear part. For a given monomial w^α, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_j \geqq 0$, we have

$$w^\alpha \circ \phi_0 = e^{2i\pi(\alpha, \nu)} w^\alpha .$$

On the other hand the c-eigenfunctions of ξ_0 are given by the convergent Laurent series $\sum_{\alpha \geqq 0, k \in \mathbb{Z}} z^k w^\alpha$ where $(\alpha, \lambda) + k = c$. Notice that, for a given c there is only a finite number of couples $\{\alpha, k\}$ fulfilling the above relation. Any holomorphic function f on $\mathbb{C}^* \times \mathbb{C}^n$ is a sum of functions of the form

$$f_s(z, w) = \sum_{(\alpha, \lambda) \equiv s \bmod \mathbb{Z}} a_{k,\alpha} z^k w^\alpha .$$

For a fixed z the functions f_s so defined give the decomposition of f in terms of eigenfunctions of ϕ_0.

Assume that f vanishes identically on X. Applying Theorem 6 to the automorphism ϕ we see that the functions f_s also vanish on X. Let us write

$$f_s(z,w) = \sum_{j\in I} g_{s+k_j}(w)h_j(z) \tag{30}$$

$$= \sum_{l\in\mathbb{Z}} \sum_{k_j+n=l} g_{s+k_j}(w)b_{j,n}z^n$$

where I is finite, $k_j \in \mathbb{Z}$ and $g_{s+k_j}(w) = \sum_{(\alpha,\nu)=s+k_j} a_\alpha w^\alpha$. Since $v_l = \sum_{k_j+n=l} g_{s+k_j}(w)b_{j,n}z^n$ are eigenfunctions of ξ_0 it is sufficient to prove that v_l also vanish on X.

Let $(z,w) \in X$ be fixed. We have $f_s(\exp t\xi(z,w)) = 0$ for $t \in \mathbb{C}$. Using (29) and (30) this means that

$$\sum_{j\in I} e^{(s+k_j)t} Q_j(t)h_j(e^t z) = 0 \quad \text{for } t \in \mathbb{C},$$

where Q_j are polynomials, or equivalently

$$\sum_{j\in I} e^{k_j t} Q_j(t)h_j(e^t z) = 0 \quad \text{for } t \in \mathbb{C}.$$

From this identity it follows $\sum_{j\in I} e^{k_j t} Q_j(t + 2il\pi)h_j(e^t z) = 0$ for $t \in \mathbb{C}$ and $l \in \mathbb{Z}$. Therefore $\sum_{j\in I} e^{k_j t} Q_j(u)h_j(e^t z) = 0$ for any $u,t \in \mathbb{C}$ and this implies that

$$v_l = e^{s+l} \sum_{k_j+n=l} Q_j(t)b_{j,n}z^n \equiv 0$$

proving statement (ii).

Part (i) can be proved by an argument similar to the previous one which becomes much simpler because in the present situation one has to deal only with entire functions. □

Let M be a Hopf manifold described as the quotient of $\mathbb{C}^N - \{0\}$ by the group action generated by a holomorphic contracting automorphism φ of $\mathbb{C}^N$ fixing the origin and in normal form. It follows from Theorem 6 and Grauert-Remmert's extension theorem of analytic subsets (cf. [G-R], p. 181) that the analytic subsets of M, having any component of positive dimension, are the projection of algebraic φ-invariant subsets of $\mathbb{C}^N$. This was already proved by M. Kato in [Ka2]. In the case of a diagonal vector field $\xi = \xi_0$ any analytic subset of M of positive dimension is the intersection of hypersurfaces. In the general case Kato showed that given two subvarieties Y_1 and Y_2 of M of pure dimension and fulfilling $Y_1 \subset Y_2$ and $0 < \dim Y_1 < \dim Y_2$ then there is a chain of subvarieties $W_0, W_1, \ldots, W_p$ such that $Y_1 = W_0$, $Y_2 = W_p$, $W_i \subset W_{i+1}$ and $\dim W_{i+1} = 1 + \dim W_i$. All these properties can be extended to the subvarieties of $S_\xi^{m,n}$, where $m,n > 1$, whose dimension is big enough.

Proposition 15. *Let Y, Y_1, Y_2 denote analytic subsets of $S_\xi^{m,n}$ $(m,n > 1)$ whose irreducible components are all of dimension bigger or equal to* $\max(m,n)$. *Then*

(i) *Y is the projection of an algebraic subset $\tilde{Y}$ of $\mathbb{C}^{m+n}$ which is invariant by the flows associated to ξ and ξ_0. Moreover $\tilde{Y} = \bigcap_{i=1}^{k} P_i^{-1}(0)$ where P_i are polynomials such that $L_{\xi_0} P_i = b_i P_i$ with $b_i \in \mathbb{C}$. In the diagonal case $\xi = \xi_0$, Y is the intersection of hypersurfaces of $S_\xi^{m,n}$.*

(ii) *Suppose that Y_1 and Y_2 are subvarieties of pure dimension and fulfill $Y_1 \subset Y_2$ and $0 < \dim Y_1 < \dim Y_2$. Then there is a chain of subvarieties $W_0, W_1, \ldots, W_p$ of $S_\xi^{m,n}$ such that $Y_1 = W_0$, $Y_2 = W_p$, $W_i \subset W_{i+1}$ and $\dim W_{i+1} = 1 + \dim W_i$.*

Proof. The first statement is a corollary of Grauert-Remmert's theorem and Proposition 14(i) above. The second one is a consequence of the following lemma.

Lemma 5. *Let X and H be analytic subsets of $\mathbb{C}^N$ which are invariant by ξ and so by ξ_0. Assume that $X \subsetneq H$. Then there is an entire function f with $f_{|H} \not\equiv 0$ and $f_{|X} \equiv 0$ and fulfilling $\xi_0 f = sf$ and $\xi f = sf + u$ where $s \in \mathbb{C}$ and u is a function such that $u_{|H} \equiv 0$. In particular $X_1 = H \cap \{f = 0\}$ is invariant by ξ and by ξ_0.*

Proof. There is $h \in \mathcal{O}(\mathbb{C}^N)$ with $h_{|H} \not\equiv 0$ and $h_{|X} \equiv 0$. Using the decomposition of h with respect to the eigenspaces $\mathcal{P}_s = \{g \mid \xi_0 g = sg\}$ of ξ_0 we deduce from Proposition 14(i) that there is $g \in \mathcal{P}_s$ vanishing identically on X but not on H. Set $\mathcal{P}_s^X = \{g \in \mathcal{P}_s \mid g_{|X} \equiv 0\}$ and $\mathcal{I}_H = \{g \in \mathcal{P}_s^X \mid g_{|H} \equiv 0\}$. The vector space $E = \mathcal{P}_s^X / \mathcal{I}_H$ has finite dimension and does not reduce to zero. The vector field ξ acts on E as a linear operator whose semi-simple part is just ξ_0. Therefore ξ has an eigenvector in E. If $f \in \mathcal{P}_s^X$ is a representative of such an eigenvector then f fulfills the required conditions. □

Let $a \in \mathbb{C}^*$ be given. The diagonal vector field ξ_0^a defined by

$$\xi_0^a = \sum_{k=1}^{N} \mathrm{Re}(a\lambda_k)\zeta_k \frac{\partial}{\partial \zeta_k} \tag{31}$$

belongs to the center $Z(\mathfrak{g}_\lambda)$ of the Lie algebra $\mathfrak{g}_\lambda$. In fact, if $\lambda_s = (p, \lambda) = \sum p_k \lambda_k$ is a λ-resonance relation then $\mathrm{Re}(a\lambda_s) = \sum p_k \mathrm{Re}(a\lambda_k)$ and therefore given a resonant monomial $\zeta^p \partial/\partial\zeta_s$ we have

$$\left[\xi_0^a, \zeta^p \frac{\partial}{\partial \zeta_s}\right] = \left(\sum p_k \mathrm{Re}(a\lambda_k) - \mathrm{Re}(a\lambda_s)\right) \cdot \zeta^p \frac{\partial}{\partial \zeta_s} = 0 .$$

In particular ξ_0^a commutes with ξ and projects onto a vector field on $S_\xi^{m,n}$ that we denote by $\hat{\xi}_0^a$. Given any point $\zeta = (\zeta_1, \ldots, \zeta_N) \in V^{m,n}$ one can find $i \in \{1, \ldots, m\}$ and $j \in \{m+1, \ldots, N\}$ for which $\zeta_i \cdot \zeta_j \neq 0$ and such that the coefficients of $\partial/\partial\zeta_i$ and $\partial/\partial\zeta_j$ of the vector field ξ are $\lambda_i \zeta_i$ and $\lambda_j \zeta_j$ respectively. If there were $\alpha \in \mathbb{C}^*$ such that $\xi_0^a = \alpha \cdot \xi$ at the point ζ then we would have

$\mathrm{Re}(a\lambda_i) = \alpha \cdot \lambda_i$ and $\mathrm{Re}(a\lambda_j) = \alpha \cdot \lambda_j$ which implies $\lambda_j \in \mathbb{R} \cdot \lambda_i$ getting a contradiction. Therefore ξ_0^a and ξ are linearly independent at any point of $V^{m,n}$. It follows that $\hat{\xi}_0^a$ is a vector field without zeros.

Notice that if $\xi = \xi_0$ then all the vector fields $\hat{\xi}_0^a$ coincide up to multiplication by a constant. In the case of a Calabi-Eckmann manifold M this vector field defines the principal fibration of M by tori over $\mathbb{P}^m \times \mathbb{P}^m$ (cf. 1.4.3). Any Calabi-Eckmann manifold M has the property that all the analytic subsets of M are saturated by the fibers of the principal fibration. This property generalizes to any manifold $S_\xi^{m,n}$ for analytic subsets of dimension bigger or equal to $\max(m,n)$ when $m,n > 1$ and for any analytic subset of positive dimension when $m = 1$ or $n = 1$. This follows from Proposition 14 and the fact that if f is a function for which $\xi_0(f) = \tau \cdot f$ then $\xi_0^a(f) = \mathrm{Re}(a\tau) \cdot f$. We can collect the preceding results in the following statement.

Proposition 16. *The vector field ξ_0^a defined by* (31) *projects to a vector field $\hat{\xi}_0^a$ without zeros on $S_\xi^{m,n}$. Let Y be an analytic subset of $S_\xi^{m,n}$ of positive dimension. Assume that $m = 1$ or that $m,n > 1$ and Y has dimension bigger or equal to $\max(m,n)$. Then Y is saturated by the orbits of $\hat{\xi}_0^a$.*

Remark 9. In relation with the above proposition we should mention a result by M. Kato in [Ka2] stating that for a given Hopf manifold M there is a finite covering $\tilde{M}$ (which is itself a Hopf manifold) of M endowed with a flow which saturates all the analytic subsets of $\tilde{M}$ of positive dimension. To this end Kato considers M as a quotient of $\mathbb{C}^n - \{0\}$ by a contracting automorphism g of $\mathbb{C}^n$ fixing the origin and shows that there is a power g^k of g which can be embedded in a flow of $\mathbb{C}^n$. Then $\tilde{M}$ is the quotient of $\mathbb{C}^n - \{0\}$ by g^k. The vector field considered by Kato does not coincide with $\hat{\xi}_0^a$. We emphasize that in the construction of $\hat{\xi}_0^a$ on a Hopf manifold M, as well as in the proof of its saturating properties, we use in an essential way the description of M as a subset of the orbit space of the vector field ξ.

4 Classification

Let $X = \mathbb{C}^n - \{0\}/\langle h \rangle$ and $X' = \mathbb{C}^n - \{0\}/\langle h' \rangle$ be two Hopf manifolds where h and h' are contractions of $\mathbb{C}^n$, keeping fixed the origin and in normal form as in example 1.4.2. It is well known that the complex manifolds X and X' are isomorphic if and only if there is a holomorphic automorphism f of $\mathbb{C}^n$ such that $h' = h \circ f$. The aim of this section is to prove a statement analogous for the manifolds $S_\xi^{m,n}$ in the non Hopf case, i.e. when $m,n > 1$. More precisely we show in Theorem 7 that $S_\xi^{m,n}$ and $S_{\xi'}^{m,n}$ are isomorphic if and only if there is a holomorphic automorphism f of $\mathbb{C}^N$ such that $f_*\xi = \xi'$. Moreover we see that f belongs to the Lie group G_λ of automorphisms of $\mathbb{C}^N$ which keep fixed ξ_0. This is shown under an appropriate normalization condition on ξ and ξ' which is specified below. In the Hopf case the characterization of $S_\xi^{1,n}$ in terms of ξ is more complicated since, as it was noticed in 1.4.2, a λ-resonant

vector field ξ and a λ'-resonant vector field ξ' can give rise to two isomorphic Hopf manifolds although $\lambda \neq \lambda'$. Nevertheless under a suitable normalization condition on the vector fields the analogous of Theorem 7 for Hopf manifolds remains valid. The proof of this fact is sketched at the end of the section.

Proposition 17. *Let $g : S^{m,n}_{\xi} \to S^{m,n}_{\xi'}$ be a holomorphic isomorphism. There exist $a \in \mathbb{C}^*$ and a biholomorphic map $f : V^{m,n} \to V^{m,n}$ fulfilling $f_*\xi = a \cdot \xi'$ and such that the following diagram is commutative*

$$\begin{array}{ccc} V_{m,n} & \xrightarrow{f} & V_{m,n} \\ {\scriptstyle \pi_\xi}\downarrow & & \downarrow{\scriptstyle \pi_{\xi'}} \\ S^{m,n}_{\xi} & \xrightarrow{g} & S^{m,n}_{\xi'} \end{array}.$$

If $m,n > 1$ then f extends to a holomorphic automorphism of $\mathbb{C}^N$. If $1 = m < n$ then f can be extended to $\mathbb{C}^* \times \mathbb{C}^n$.

Proof. Recall that $V^{m,n}$ is the total space of the non trivial holomorphic $\mathbb{C}$-principal bundle $E'_1 \in H^1(S^{m,n}_{\xi'}, \mathcal{O})$ over $S^{m,n}_{\xi'}$ defined by ξ'. The pull-back $g^*E'_1$ is then a non trivial $\mathbb{C}$-principal bundle over $S^{m,n}_{\xi}$. By Proposition 12 there is an isomorphism of $\mathbb{C}$-principal bundles ψ from $g^*E'_1$ onto the bundle E_a associated to $\frac{1}{a}\xi$ for a suitable $a \in \mathbb{C}^*$. Then the composition f of ψ^{-1} with the natural isomorphism between $g^*E'_1$ and E'_1 fulfills the required conditions. The end of the proposition follows from Hartog's theorem. □

Until last paragraph in the section we will suppose that we are not dealing with Hopf manifolds, i.e. that $m,n > 1$.

The normalization condition (N). From now on we shall assume that the vector fields ξ, besides the normalization conditions (4) and (5), also fulfill (i) $\lambda_1 = 1$ and (ii) for a given eigenvalue λ_k the corresponding Jordan blocks of the linear part $\xi_l = \xi_0 + \xi_1$ of ξ are ordered by size. Under these assumptions two linear vector fields are linearly conjugated if and only if they are equal.

Let ξ and ξ' be two vector fields of types λ and λ' and normalized as before. Denote by $\Lambda(\xi, \xi')$ the set of holomorphic automorphisms f of $V^{m,n}$ such that $f_*\xi = \xi'$. Notice that any element $f \in \Lambda(\xi, \xi')$ extends to a holomorphic automorphisms of $\mathbb{C}^N$ still denoted by f and fulfilling $f_*\xi = \xi'$. Moreover f fixes the origin as it sends the singular point of ξ into the singular point of ξ'. Any element $f \in \Lambda(\xi, \xi')$ induces a holomorphic isomorphism from $S^{m,n}_{\xi}$ onto $S^{m,n}_{\xi'}$ that we denote by $\hat{f}$. We have

Proposition 18. *Let $g : S^{m,n}_{\xi} \to S^{m,n}_{\xi'}$ be a holomorphic isomorphism. Then*

(i) *There exist $f \in \Lambda(\xi, \xi')$ such that $g = \hat{f}$.*

(ii) *One has $\xi_l = \xi'_l$. In particular $\xi_0 = \xi'_0$ and therefore $\lambda = \lambda'$. The linear part f_l of f is in $\Lambda(\xi_l, \xi'_l)$ and also in $\Lambda(\xi_0, \xi'_0)$.*

(iii) *If $f_1, f_2 \in \Lambda(\xi, \xi')$ verify $\hat{f}_1 = \hat{f}_2$ then there exist $t_0 \in \mathbb{C}$ such that $f_1 = f_2 \circ \psi_{t_0}$ where ψ denotes the flow associated to ξ.*

Proof. Let f be the map given by the above proposition. Then $\hat{f} = g$. We write

$$\xi = B \cdot x + \text{ terms of degree } > 1 ,$$

$$\xi' = B' \cdot x + \text{ terms of degree } > 1 .$$

We also expand $f(x)$ as a sum $\sum_{k \geqq 1} f_k(x)$ of homogeneous polynomials of degree k and we set $f_1(x) = A \cdot x$. Then the relation $F_* \xi = a \cdot \xi'$ gives

$$A \cdot B \cdot x + \text{terms of degree } > 1 = a \cdot B' \cdot A \cdot x + \text{terms of degree } > 1 .$$

Hence the matrices B and $a \cdot B'$ are conjugated by A. By the normalization made this implies $a = 1$, $B = B'$, which means $\xi = \xi'$, and, by the unicity of the diagonal part in the Jordan decomposition, $\xi_0 = \xi_0'$. In particular $f \in \Lambda(\xi, \xi')$. So the linear map A belongs to $\Lambda(\xi_1, \xi_1')$ and by general linear algebra one also has $A \in \Lambda(\xi_0, \xi_0')$. This proves parts (i) and (ii).

Let us prove part (iii). Since the restriction to $V^{m,n}$ of the map $h = f_2^{-1} \circ f_1$ is an automorphism of the $\mathbb{C}$-principal bundle E_1 associated to ξ there is a holomorphic function r on $S_\xi^{m,n}$ such that $h(x) = \psi(r(\pi_\xi(x)); x)$ for any $x \in V^{m,n}$. But the compacity of $S_\xi^{m,n}$ implies that r must be a constant t_0. □

Remark 10. Let $1 < m, n$ and $m \neq n$. Assume that the vector fields ξ and ξ' fulfill the normalization conditions stated above. Suppose also that they are weakly hyperbolic of types (m, n) and (n, m) respectively. Then $S_\xi^{m,n}$ is not holomorphically isomorphic to $S_{\xi'}^{m,n}$. If there were such an isomorphism one could construct, reasoning as in Propositions 17 and 18, an holomorphic automorphism f of $\mathbb{C}^N$ such that f keeps fixed ξ_0 and, at the same time, sends the subspace $\mathbb{C}^m \times \{0\}$ into $\{0\} \times \mathbb{C}^m$ and the subspace $\mathbb{C}^n \times \{0\}$ into $\{0\} \times \mathbb{C}^n$ which is not possible.

Proposition 19. $\Lambda(\xi, \xi') \subset G_\lambda$.

Through the proof of the theorem we shall make use of the following two results. The first of them is contained in [Ha].

Lemma 6 [Ha]. *The Lie algebra C_ξ of holomorphic vector fields in $\mathbb{C}^N$ commuting with ξ is a Lie subalgebra of $\mathfrak{g}_\lambda$.*

Lemma 7. *The center $Z(\mathfrak{g}_\lambda)$ of the Lie algebra $\mathfrak{g}_\lambda$ is composed of diagonal vector fields, i.e. vector fields of the form $\sum a_k \zeta_k \dfrac{\partial}{\partial \zeta_k}$.*

Proof. Let $s \in \{1, \ldots, N\}$ be given. If $\lambda_s = \sum p_k \lambda_k$ is a non trivial resonance relation then $p_s = 0$. Thus for any λ-resonant monomial $\zeta^p \partial / \partial \zeta_s$ different from $\zeta_s \partial / \partial \zeta_s$ one has

$$\left[\zeta_s \frac{\partial}{\partial \zeta_s}, \zeta^p \frac{\partial}{\partial \zeta_s} \right] = -\zeta^p \frac{\partial}{\partial \zeta_s} .$$

One also has the relation

$$\left[\zeta_s \frac{\partial}{\partial \zeta_s}, \zeta^p \frac{\partial}{\partial \zeta_l}\right] = p_s \zeta^p \frac{\partial}{\partial \zeta_l}$$

for $l \neq s$. Since a given $\eta \in Z(\mathfrak{g}_\lambda)$ has to commute with any vector field $\zeta_s \partial/\partial \zeta_s$ it follows from the above relations that η must be diagonal. □

Proof of Proposition. Let $f \in A(\xi, \xi')$ be given. One has

$$f_* \xi_0 = \xi_0 + \text{ terms of degree } > 1 .$$

This comes from the fact that the linear part of $f_* \xi_0$ is conjugated to ξ_0 by the linear part f_l of f. But, by Proposition 18(ii), the linear map f_l keeps ξ_0 fixed. On the other hand, since ξ_0 commutes with ξ, one has $f_* \xi_0 \in C_{\xi'}$ and, by Lemma 6, $f_* \xi_0 \in \mathfrak{g}_\lambda$. Therefore $f_* \xi_0$ is in normal form with ξ_0 as linear part. Using Lemma 6 again one has $C_{f_* \xi_0} \subset \mathfrak{g}_\lambda$. This implies that f_* sends $\mathfrak{g}_\lambda$ into itself. As f is a biholomorphism the map f_* is in fact a Lie algebra automorphism of $\mathfrak{g}_\lambda$. Therefore $f_*(Z(\mathfrak{g}_\lambda)) = Z(\mathfrak{g}_\lambda)$. In particular $f_* \xi_0 \in Z(\mathfrak{g}_\lambda)$ and from Lemma 7 one deduces $f_* \xi_0 = \xi_0$. □

As a corollary we obtain

Theorem 7. *Let $m, n > 1$ and assume that ξ and ξ' are vector fields of types λ and λ' fulfilling the normalization condition (N). The complex manifolds $S_\xi^{m,n}$ and $S_{\xi'}^{m,n}$ are isomorphic if and only if there is an analytic automorphism f of $\mathbb{C}^N$ such that $f_* \xi = \xi'$. In this case $\lambda = \lambda'$ and $\xi_0 = \xi_0'$. Moreover the mapping f belongs to the Lie group G_λ and is uniquely determined up to composition with the flow of ξ. In the case ξ and ξ' are linear vector fields then the mapping f is also linear.*

Remark 11. Let $S_\xi^{m,n}$ and $S_{\xi'}^{m,n}$ be the Calabi-Eckmann manifolds (cf. 1.4.3) associated to the vector fields

$$\xi = \sum_{i=1}^{m} z_i \frac{\partial}{\partial z_i} + \nu \sum_{j=1}^{n} w_j \frac{\partial}{\partial w_j} \quad \text{and} \quad \xi' = \sum_{i=1}^{m} z_i \frac{\partial}{\partial z_i} + \nu' \sum_{j=1}^{n} w_j \frac{\partial}{\partial w_j}$$

where $\operatorname{Im} \nu$, $\operatorname{Im} \nu' > 0$. It follows from the above theorem that $S_\xi^{m,n} \cong S_{\xi'}^{m,n}$ if and only if $\nu = \nu'$.

We consider now the case of Hopf manifolds, that is $m = 1$. With this purpose we introduce the stronger normalization condition **(N′)**: a λ-resonant vector field ξ which is weakly hyperbolic of type $(1, n)$ satisfies (N′) if it fulfills (N) and the additional condition: (iii) $-1 < \operatorname{Re} \lambda_k \leqq 0$ for $2 \leqq k \leqq n+1$. It follows from Proposition 3 that condition (N′) can always be assumed just by substituting ξ by $\Phi_* \xi$ for an appropriate automorphism Φ of $\mathbb{C}^* \times \mathbb{C}^n$. With this assumption the Lie algebra $\mathfrak{g}_\lambda$ of vector fields on $\mathbb{C}^* \times \mathbb{C}^n$ commuting with ξ_0 is in fact a subalgebra of $H^0(\mathbb{C}^{n+1}, \Theta)$.

Theorem 7'. *Let $m = 1$, $n > 1$ and assume that ξ and ξ' are vector fields of types λ and λ' fulfilling the normalization condition (N'). The Hopf manifolds $S^{1,n}_{\xi}$ and $S^{1,n}_{\xi'}$ are isomorphic if and only if there is an analytic automorphism f of $\mathbb{C}^{n+1}$ such that $f_*\xi = \xi'$. In this case $\lambda = \lambda'$, $\xi_0 = \xi'_0$. Moreover the mapping f belongs to the Lie group G_λ and is uniquely determined up to composition with the flow of ξ. In the case ξ and ξ' are linear vector fields then the mapping f is also linear.*

Sketch of proof. We know from Proposition 17 that there is a holomorphic automorphism f of $\mathbb{C}^* \times \mathbb{C}^n$ and $a \in \mathbb{C}^*$ such that $f_*\xi = a \cdot \xi'$. One begins by showing that $f_*\xi_0$ is a diagonal vector field. In this setting the proof is more involved than in the non-Hopf case and uses the following fact: Let V be a finite dimensional vector subspace of $H^0(\mathbb{C}^* \times \mathbb{C}^n, \Theta)$ which is invariant by $ad\xi_0$ and $ad\eta$, where $\eta = \xi - \xi_0$. Then the restriction of these operators to V give the Jordan decomposition of $ad\xi$ in semi-simple and nilpotent parts. This is shown by proving first that $ad\xi_0$ is a semi-simple operator whose eigenvectors are functions on $\mathbb{C}^* \times \mathbb{C}^n$ which are polynomial in the variables w of $\mathbb{C}^n$. Then and by means of a suitable lexicographic order on the set of monomials in V one also proves that the operator $ad\eta$ is nilpotent.

The above statement is applied to $V = f_*^{-1}(\mathfrak{g}_{\lambda'})$ which is obviously invariant by $ad\xi$. The invariance by $ad\xi_0$, and so by $ad\eta$ comes from the fact that $f_*\xi_0$ is in the centralizer $C_{\xi'}$ of ξ' in $H^0(\mathbb{C}^* \times \mathbb{C}^n, \Theta)$. But $C_{\xi'}$ is contained in $\mathfrak{g}_{\lambda'}$ (this can be seen by the same argument used to prove Lemma 6). Therefore $ad\xi_0$ coincides with $ad(a \cdot \xi'_0)$ when acting on $\mathfrak{g}_{\lambda'}$ and thus $f_*\xi_0 = a \cdot \xi'_0$ is in the center of $\mathfrak{g}_{\lambda'}$. Then Lemma 7 implies that $f_*\xi_0$ is diagonal.

Set $\lambda = (1, \nu_1, \ldots, \nu_n)$. Using the last fact and the Laurent expansion of f one can directly check that

$$f_*\xi_0 = \kappa_0 z \frac{\partial}{\partial z} + \sum_{j=1}^{n} \kappa_j w_j \frac{\partial}{\partial w_j}$$

where $\kappa_0 = \pm 1$ and $\kappa_j = \nu_{\sigma(j)} + p_j$ where $p_j \in \mathbb{Z}$ and σ is a permutation of $\{1, \ldots, n\}$. So we can write $f = \varphi \circ g$ where

(1) g is a holomorphic automorphism of $\mathbb{C}^* \times \mathbb{C}^n$ with $g_*\xi_0 = \xi_0$. Since ξ fulfills the normalization condition (N') last relation implies that g is in fact the restriction of an automorphism of $\mathbb{C}^{n+1}$ belonging to G_λ. In particular g_* sends η into a sum of non diagonal λ-resonant vector fields.

(2) φ is a composition of maps of the following types: (i) $(z, w) \mapsto (1/z, w)$, (ii) $(z, w_1, \ldots, w_n) \mapsto (z, w_{\sigma(1)}, \ldots, w_{\sigma(n)})$ where σ is a permutation of $\{1, \ldots, n\}$ and (iii) maps $\Phi_{j,k}$ as defined in (10).

One sees directly that φ_* sends diagonal (resp. non diagonal) monomial vector fields into diagonal (resp. non diagonal) monomial vector fields. It follows that

$$a \cdot \xi_0 = f_*\xi_0 + f_*\eta = \varphi_*\xi_0 + \varphi_*(g_*\eta)$$

with $\varphi_*\xi_0$ diagonal and $\varphi_*(g_*\eta)$ a sum of non diagonal monomials. Since ξ' also fulfills condition (N') one has $\varphi = \mathrm{id}$ and the theorem follows. □

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