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Generalised Magnus modules over the braid group*

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1 Introduction

In this article it is shown that the class of Magnus representations of the braid group, cf. [Bir74], can naturally be generalised via group homology. The generalisation includes the topological representations of R.J. Lawrence [Law90]. The article is based on some of the ideas sketched in the author's thesis [Lüd92]. A more detailed version appeared in [Lüd95] in the form of lecture notes.

W. Magnus, in order to give a derivation for the well-known Burau representation of the braid group, described a class of representations of automorphism groups of free groups, see e.g. [Mag74]. J. Birman devoted a chapter of her book to this class of "Magnus representations", cf. [Bir74]. The Magnus representation modules (in the original sense of W. Magnus) can be understood as the first homology groups of the free group with particular coefficient modules. Using Artin's imbedding $B_n \hookrightarrow \mathrm{Aut}(F_n)$, a semidirect product $B_n \ltimes F_n$ of the braid group B_n with the free group F_n of rank n can be defined. By functoriality of H_* the quotient group $(B_n \ltimes F_n)/F_n \simeq B_n$ acts onto $H_1(F_n, M)$, if M is a module over $B_n \ltimes F_n$. In Sect. 2 we show that this construction generalises the approaches of Magnus and Birman which are the first nontrivial examples of a general procedure: any pair $K \lhd G$ of groups with quotient $G/K \simeq B_n$ together with a G module N leads to a B_n module $H_*(K, N)$.

A step in this direction has been undertaken by R.J. Lawrence, cf. [Law90], from a different and geometrical point of view. In an attempt to understand the Jones polynomial in geometrical terms, she constructs a vector bundle with a natural flat connection over a base space having the braid group B_n as its fundamental group. In this way she obtains braid representations from the holonomy of the connection, which under certain conditions factor through the Hecke

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algebra. The typical fiber of the bundle is the m-th homology $H_m(Y;\chi)$ of the configuration space Y of m distinct points in the n-fold punctured plane with a suitably chosen abelian local coefficient system $\chi \in \mathrm{Hom}(\pi_1(Y,y), \mathbf{C}\backslash\{0\})$. The class of representations obtained in this way e.g. can be used for the construction of the one-variable Jones polynomial [Law93]. For a short account on this approach, cf. [Ati90]. In Sect. 3 it is shown that Y is an Eilenberg-MacLane complex of type $(\pi_1(Y,y),1)$. So Lawrence's construction via Eilenberg's theorem on local coefficients is the topological approach to the construction of the homology of the group $\pi_1(Y,y)$ with coefficient module $(\chi,\mathbf{C})$. The fundamental group $\pi_1(Y,y)$ can be imbedded as a normal subgroup into a generalised braid group $B_{n,m}$ with quotient $B_{n,m}/\pi_1(Y,y) \simeq B_n$. Furthermore, $(\chi,\mathbf{C})$ has an extension to a $B_{n,m}$ module. Therefore Lawrence's approach fits into the algebraic setting sketched above. In particular, $\pi_1(Y,y) \simeq F_n$ for $m = 1$ and we recover (a subclass of) Magnus' representations.

In Sect. 4 we construct a free $\pi_1(Y,y)$ resolution $\partial \in \mathrm{Hom}_{\pi_1(Y,y)}(C)$ of the integers. This resolution is supposed by the recursive structure of $\pi_1(Y,y)$. It explicitly allows the computation of the B_n action onto chain complexes $N \otimes_{\pi_1(Y,y)} C$ by means of braid ring valued matrices.

These matrices are constructed in Sect. 5. A special version of them was found topologically in [Law90]. In their general form they are the "braid valued Burau matrices" derived independently and by different means in [CL92, Lüd92]. The name is due to the fact that the classical Burau matrices are images of the aforementioned ones under a ring homomorphism of the matrix elements. The braid valued matrices in a simple and unified form encode all the braid representations (among many others) that can be obtained by Magnus' and by Lawrence's approaches.

2 Magnus representations of submonoids of End(F)

This section presents a description of the "classical" situation in a more general formulation. The approaches of [Bir74] and [Mag74] will be linked together.

In the following, E is a general submonoid of the monoid End(F) of right endomorphisms of the free group F. A subscript indicates the finite rank n of F, if necessary. Since the endomorphisms are acting from the right, the product ab is the endomorphism $b \circ a$, b after a. The semidirect product $E \ltimes F$ is the monoid consisting of the set $E \times F$ equipped with the product $(a,f)(b,g) = (ab, b(f)g)$, $a,b \in E$, $f,g \in F$ and with unity $(1,1)$. We will identify E and F with their images in $E \ltimes F$ and therefore have $af = (a,1)(1,f) = (a,f)$, $E \ltimes F = EF$ and $afbg = abb(f)g$. F is normal in EF in the sense that for every $p \in EF$ and $f \in F$ there is a $g \in F$ uniquely determined by $fp = pg$. The linear extension of any group (or monoid) homomorphism $\phi \in \mathrm{Hom}(G,H)$ to the group ring $\mathbf{Z}G$ is the map ϕ_*. The relative augmentation ideal $I_{N,G} = \mathrm{Ker}(\phi_*) \lhd \mathbf{Z}G$ is generated by the subset $N-1 \subset \mathbf{Z}G$, where N is the normal subgroup $N = \mathrm{Ker}(\phi) \lhd G$. We set $I_{G,G} = I_G$ if we consider full augmentation ideals.

Theorem 1 (W. Magnus) *Let $U \lhd F$ be a normal subgroup. Let $E = St_{End(F)}(F/U) \leqq End(F)$ be the stabilizing monoid of F/U in $End(F)$. Then $U/[U,U]$ is an F/U-E bimodule: $[f] \in F/U$ maps $[u] \in U/[U,U]$ to $[fuf^{-1}], a \in E$ maps $[u]$ to $[a(u)]$, and the actions of E and F/U commute.*

Proof. This statement is taken from [Mag74], p. 471. The proof is a check of well-definedness of maps and of commutativity. □

Using the Fox derivation, Birman has given another construction related to the Magnus modules $U/[U,U]$. For any epimorphism $\psi \in \mathrm{Hom}(F,H)$ let M_ψ be the free left $\mathbf{Z}H$ module of rank n with basis $\{s_1,\ldots,s_n\}$. Let $\delta_\psi \in \mathrm{Der}(F,M_\psi)$ be the total Fox derivation, cf. [Bir74], pp. 104. It is a group derivation uniquely defined as a map by the equations

$$\delta_\psi(f_i) := s_i\,,$$
$$\delta_\psi(fg) := \delta_\psi(f) + \psi(f)\delta_\psi(g)\,,$$

for a free generating system $\{f_1,\ldots,f_n\}$ of F and $f,g \in F$. δ_ψ has an extension $\bar{\delta}_\psi \in \mathrm{Der}(\mathbf{Z}F, M_\psi)$ to a ring derivation, defined as the unique homomorphism of abelian groups with $\bar{\delta}_\psi(f) = \delta_\psi(f)$, $f \in F$. We drop the bar distinguishing both derivations. Let $\partial_{j,\psi} \in \mathrm{Der}(F,\mathbf{Z}H)$ be the partial Fox derivations with respect to the generators f_j, $\delta_\psi(f) = \sum_j \partial_{j,\psi}(f)s_j$, $f \in F$. When we consider the case $\psi = \mathrm{Id}$ we drop indices referring to ψ.

Theorem 2 (J. Birman) *Let $E \leqq End(F)$ be the maximal monoid satisfying $\psi \circ a = \psi$ for all $a \in E$. For every $a \in E$ let $\rho(a)$ be the endomorphism of M_ψ uniquely defined by*

$$s_i \mapsto \delta_\psi(a(f_i))\,.$$

Then $\rho \in Hom(E, End(M_\psi))$.

Proof. The proof makes use of a chain rule for the Fox derivation and of the E-invariance of ψ, cf. [Bir74], Theorem 3.9, p. 116. □

In order to understand the relationship between the Magnus modules $U/[U,U]$ and the Birman modules M_ψ, let $\mu_\psi \in \mathrm{Hom}(F, H \ltimes M_\psi)$ be the representation of F that is associated to the derivation $\delta_\psi \in \mathrm{Der}(F,M_\psi)$: the semidirect product of H and M_ψ is defined by $(g,p)(h,q)=(gh, p+gq)$ for $g,h \in H$ and $p,q \in M_\psi$ and μ_ψ maps $f \mapsto (\psi(f), \delta_\psi(f))$. Since δ_ψ is a group derivation, μ_ψ is a homomorphism (called the "Magnus ψ representation" of F in [Bir74]). It is characterised by

Theorem 3 (Blanchfield, Birman)

$$Ker(\mu_\psi) = [Ker(\psi), Ker(\psi)]\,,$$

$$Image(\mu_\psi) = \{(h, \textstyle\sum h_j s_j); h \in H, h_j \in \mathbf{Z}H, h-1 = \sum h_j(\psi(f_j)-1)\}\,.$$

Proof. By theorems of Blanchfield and Magnus the kernel is identified, cf. [Bir74], Theorem 3.5, pp. 107, Theorem 3.7, pp. 111. The image is determined by [Bir74], Theorem. 3.6, pp. 108. □

Both constructions are related by the following fact.

Theorem 4 *Given either a Magnus module $U/[U,U]$ or a Birman module M_ψ, there is a short exact sequence of $\mathbf{Z}H$-E bimodules,*

$$0 \to U/[U,U] \to M_\psi \to Q \to 0\,,$$

with $H \simeq F/U$, $\psi \in Hom(F,H)$ such that $Ker(\psi) = U$, and $E = St_{End(F)}(F/U)$.

Proof. Let $U \lhd F$ be given, $E = \mathrm{St}_{\mathrm{End}(F)}(F/U)$, giving rise to the Magnus module $U/[U,U]$ over F/U and E. Let $\psi \in \mathrm{Hom}(F, F/U)$ be the projection homomorphism. Then for any $a \in E$, $\psi(a(f)) = \psi(fp) = \psi(f)$, with some $p \in U$, since E acts as identity onto F/U. So the projection ψ is E-invariant and it defines a Birman module M_ψ. We must find a monomorphism from the F/U-E bimodule $U/[U,U]$ into M_ψ. The Magnus representation of F, μ_ψ, induces an isomorphism of the group $F/[U,U]$ onto its image. The restriction of this monomorphism to $U/[U,U]$ satisfies $\mu_\psi([u]) = (1, \delta_\psi(u))$, for $u \in U = \mathrm{Ker}(\psi)$. So $\delta_\psi : U/[U,U] \to M_\psi$ has the desired properties: it is an injective homomorphism of abelian groups due to Theorem (3). It intertwines the operation of $\mathbf{Z}(F/U)$ onto $U/[U,U]$ with the left multiplication in M_ψ,

$$\begin{aligned}\delta_\psi(fuf^{-1}) &= \delta_\psi(f) + \psi(f)\delta_\psi(u) - \psi(fuf^{-1})\delta_\psi(f)\\ &= \psi(f) \cdot \delta_\psi(u)\,.\end{aligned}$$

It also intertwines the right E actions on the modules, $\delta_\psi(a([u])) = \rho(a)(\delta_\psi([u]))$. Indeed, from $u = f_{i_1}^{\varepsilon_1} \dots f_{i_k}^{\varepsilon_k}$, $\varepsilon_j \in \{-1, 1\}$ follows

$$\delta_\psi(u) = \sum_{j=1}^{k} \psi(f_{i_1}^{\varepsilon_1} \dots f_{i_{j-1}}^{\varepsilon_{j-1}}) \varepsilon_j \psi(f_{i_j})^{(\varepsilon_j - 1)/2} s_{i_j}$$

and

$$\delta_\psi(a(u)) = \sum_{j=1}^{k} \psi(a(f_{i_1}^{\varepsilon_1} \dots f_{i_{j-1}}^{\varepsilon_{j-1}})) \varepsilon_j \psi(a(f_{i_j}^{(\varepsilon_j - 1)/2})) \delta_\psi(a(f_{i_j}))\,.$$

Now the E-invariance of ψ can be used and by comparison of the expressions the claim is proved.

Conversely, let ψ be given and be E-invariant. $U = \mathrm{Ker}(\psi)$ is an E-invariant (since $\psi(a(u)) = \psi(u) = 1$ for $u \in U$) normal subgroup of F. E acts as identity onto F/U, since $f^{-1}a(f) \in U$, so $U/[U,U]$ is a Magnus module with respect to E. Again the Fox derivation $\delta_\psi \in \mathrm{Der}(U, M_\psi)$ yields a monomorphism $U/[U,U] \to M_\psi$. □

We call this sequence the "Birman-Magnus sequence". The image $\delta_\psi(U)$ of the Magnus module in the Birman module "measures" how far the set $\{\psi(f_j) - 1\}$ is from being linearly independent over $\mathbf{Z}H$.

Theorem 5 *Image*$(\delta_\psi : Ker(\psi) \to M_\psi) = \{\sum_j h_j s_j ; h_j \in \mathbf{Z}H, \sum h_j(\psi(f_j) - 1) = 0\}$.

Proof. A consequence of Lyndon's theorem, [Bir74], Theorem 3.6, pp. 108. □

Now we will construct a short exact sequence of EF-E bimodules that generalises the Birman-Magnus sequence. Let $\eta \in \mathrm{Hom}(EF, E)$ be the projection, i.e. $\eta(af) = a$ for $a \in E$, $f \in F$. The extension $\eta_* \in \mathrm{Hom}(\mathbf{Z}(EF), \mathbf{Z}E)$ to the group ring defines the relative augmentation ideal $\mathrm{Ker}(\eta_*) = I_{F,EF}$ of F in $\mathbf{Z}(EF)$. We consider the ideal $I_{F,EF}$ as left EF right E bimodule. Since it is free as a left EF module, there is a representation of E in terms of $\mathbf{Z}(EF)$-valued matrices. In Sect. 5 we will find these matrices for the case of the braid group.

Theorem 6 *Let N be a right EF module. Then there is a short exact sequence of right E modules, $0 \to H_1(F, N) \to N \otimes_{EF} I_{F,EF} \to NI_{F,EF} \to 0$.*

Proof. We have a free resolution $0 \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon_*} \mathbf{Z} \to 0$ of the integers over the free group F, where $P_1 = I_F$ is the augmentation ideal, $P_0 = \mathbf{Z}F$ is the group ring and $\varepsilon_* : \mathbf{Z}F \to \mathbf{Z}$ is the augmentation homomorphism. By the exact sequence $0 \to H_1(F, N) \to N \otimes_F \partial_1(P_1) \to N \otimes_F P_0$, we have $H_1(F, N) \simeq \mathrm{Ker}(N \otimes_F I_F \to N \otimes_F \mathbf{Z}F)$. Finally we notice that $N \otimes_F I_F \simeq N \otimes_{EF} I_{F,EF}$ (on the right hand side, the structure as right E module is more obvious) and that the maps $N \otimes_F I_F \to N \otimes_F \mathbf{Z}F$ and $N \otimes_{EF} I_{F,EF} \to NI_{F,EF}$ have isomorphic kernels, the later map is onto and the kernel is preserved by the right E action. Thus, we obtain the short exact sequence of right E modules as claimed. □

We will show that this sequence specialises to the Birman-Magnus sequence. Let $\psi \in \mathrm{Hom}(F, H)$ be an epimorphism with $\psi \circ a = \psi$ for every $a \in E$. This condition allows us to construct an extension $\bar{\psi} \in \mathrm{Hom}(EF, H)$, by setting $\bar{\psi}(af) = \psi(f)$, $a \in E$, $f \in F$. We regard $\mathbf{Z}H$ as a right EF module using $\bar{\psi}$, i.e. $af : h \mapsto h\bar{\psi}(af) = h\psi(f)$.

Theorem 7 1. *The Birman module M_ψ as left $\mathbf{Z}H$ right E bimodule is isomorphic to $\mathbf{Z}H \otimes_{EF} I_{F,EF}$.*

2. *Let $U = \mathrm{Ker}(\psi)$. The corresponding Magnus module $U/[U, U]$ is isomorphic to $H_1(F, \mathbf{Z}H)$.*

Proof. 1) The restriction of δ_ψ to I_F is a homomorphism of left F modules, $\delta_\psi(f(f_i - 1)) = \delta_\psi(ff_i) - \delta_\psi(f) = \psi(f)\delta_\psi(f_i - 1)$. So there is a map

$$\mathbf{Z}H \otimes_{EF} I_{F,EF} \simeq \mathbf{Z}H \otimes_F I_F \xrightarrow{1 \otimes \delta_\psi} \mathbf{Z}H \otimes_F M_\psi \simeq M_\psi .$$

We claim it is an isomorphism of $\mathbf{Z}H$-E bimodules. Since $\mathbf{Z}H \otimes_F I_F$ and $\mathbf{Z}H \otimes_F M_\psi$ are free left $\mathbf{Z}H$ modules with bases $\{1 \otimes (f_j - 1)\}$ and $\{1 \otimes s_i\}$, respectively, and $(1 \otimes \delta_\psi)(1 \otimes (f_i - 1)) = 1 \otimes s_j$, the map is an isomorphism of left $\mathbf{Z}H$ modules. We now consider the right E action. In $\mathbf{Z}H \otimes_{EF} I_{F,EF}$ for $a \in E$ we have $1 \otimes (f_j - 1)a = 1 \otimes a(a(f_j) - 1)$ which is equal to $1 \otimes (a(f_j) - 1) \in \mathbf{Z}H \otimes_F I_F$ due to the condition $\bar{\psi}(a) = 1$. By $1 \otimes \delta_\psi$ the element $1 \otimes (a(f_j) - 1)$ is mapped to $1 \otimes \delta_\psi(a(f_j))$, such that indeed we have a homomorphism of right E modules.

2) The Birman-Magnus sequence together with Theorem (5) identifies the Magnus module $U/[U,U]$ with the submodule $\{\sum_j h_j s_j; h_j \in \mathbf{Z}H, \sum h_j \psi_*(f_j - 1) = 0\} \leqq M_\psi$ via δ_ψ. This submodule is isomorphic to $\mathrm{Ker}(\mathbf{Z}H \otimes_F I_F \to \mathbf{Z}H \otimes_F \mathbf{Z}F) \simeq H_1(F, \mathbf{Z}H)$ by the map $\sum h_j s_j \mapsto \sum h_j \otimes (f_j - 1)$. □

We thus have generalised the Birman-Magnus modules $\mathbf{Z}H \otimes_{EF} I_{F,EF}$ insofar, as we allow the condition $\psi \circ a = \psi$ to be dropped and any right EF module N be used instead of the ring $\mathbf{Z}H$, which is a particular right EF module.

There are no higher nontrivial homology groups than $H_1(F,N)$. But by choosing E as Artin's braid group B_n, one may use semidirect products of free groups in order to construct sequences of modules over B_n from higher homology groups. The homology module discussed so far is the first member of this sequence.

3 The braid group and Eilenberg-MacLane complexes

From now on we specialise our considerations from $E \leqq \mathrm{End}(F)$ to Artin's braid group. In this section we collect some well-known facts about it, taken from [Bir74], if not stated otherwise or proved explicitly. Eventually we show that the topological homology groups appearing as the fibers in the geometrical approach to braid representations in [Law90] can be understood as group homology.

The braid group B_n on n strings (over the Euclidean plane) is the group generated by the $n-1$-set $\{\tau_i; i \in \{1,\ldots,n-1\}\}$ according to the relations of Artin,

$$\tau_i \tau_j = \tau_j \tau_i, \quad \text{if } |i-j| \geqq 2,$$

$$\tau_i \tau_{1+i} \tau_i = \tau_{1+i} \tau_i \tau_{1+i}.$$

For $k \geqq 2$ let B_k be generated by the set $\{\tau_i^{(k)}; 1 \leqq i \leqq k-1\}$ according to Artin's relations. Then via the map $\tau_i^{(n)} \mapsto \tau_i^{(1+n)}$ we can imbed $B_n \hookrightarrow B_{1+n}$.

Let $\{t_i; 1 \leqq i \leqq n-1\}$ be the set of permutations mapping $t_i : i \mapsto i+1$, $t_i : i+1 \mapsto i$, $t_i : \{i, i+1\} \not\ni j \mapsto j$. These transpositions obey Artin's relations. So the map $\pi : \tau_i \mapsto t_i$ uniquely extends to an epimorphism $\pi \in \mathrm{Hom}(B_n, S_n)$ from the braid group onto the group S_n of permutations of the numbers $\{1,\ldots,n\}$ by mapping $\alpha\beta \mapsto \pi(\alpha)\pi(\beta)$. The normal subgroup $P_n := \mathrm{Ker}(\pi) \lhd B_n$ is called the pure or non-permuting braid group. The elements

$$\vartheta_{i,j} := \tau_i^{-1} \tau_{1+i}^{-1} \ldots \tau_{j-2}^{-1} \tau_{j-1}^2 \tau_{j-2} \ldots \tau_{1+i} \tau_i =: \vartheta_{j,i},$$

$i,j \in \{1,\ldots,n\}$, $i < j$, generate P_n with relations

$$\vartheta_{k,l}^{-\varepsilon} \vartheta_{i,j} \vartheta_{k,l}^{\varepsilon} = \begin{cases} \vartheta_{i,j}, & i < k \text{ or } l < i \\ \mathrm{Ad}((\vartheta_{k,j}\vartheta_{l,j})^{\varepsilon})(\vartheta_{i,j}), & i = k \text{ or } i = l, \\ \mathrm{Ad}([\vartheta_{l,j}^{\varepsilon}, \vartheta_{k,j}^{\varepsilon}]^{-\varepsilon})(\vartheta_{i,j}), & k < i < l \end{cases} \tag{1}$$

$i < j$, $k < l < j$, $\varepsilon \in \{-1, 1\}$, $\mathrm{Ad}(x)(y) := xyx^{-1}$, $[x, y] := xyx^{-1}y^{-1}$, for $x, y \in P_n$. Notice that in [Bir74], pp. 20, two pairs of parentheses are missing in lines 10 and 11.

A normal series for the pure braid group P_n solves the word problem for B_n.

Theorem 8 *Let $U_n \leqq P_n \leqq B_n$ be the subgroup of the non-permuting braid group generated by the set $\{\vartheta_{n,i}; i \in \{1, \ldots, n-1\}\}$. (This is the subgroup of the braid group B_n in which only the n-th string is not held fixed.) Then P_n is isomorphic to the semidirect product $P_n \simeq P_{n-1} \ltimes U_n$, where P_{n-1} is imbedded into P_n (by identifying the first $n-1$ strings): $\vartheta_{i,j}^{(n-1)} \mapsto \vartheta_{i,j}^{(n)}$. P_{n-1} acts onto U_n by conjugation. Furthermore, U_n is free of rank $n-1$ over the given generators.*

Proof. Cf. [Bir74]. □

Thus, we may identify P_n with the iterated semidirect product $F_2 \ltimes F_3 \ltimes \ldots F_{n-1}$ of free groups.

We want to formulate the geometrical approach to homological holonomy representations of [Law90] in algebraical terms. Let $X_n := \{(x_1, \ldots, x_n) \in \mathbf{C}^n;$ if $i \neq j$ then $x_i \neq x_j\}$ be the configuration space of n ordered distinct points in the plane. For $x \in X_n$ let $Y_{x,m} := \{(y_1, \ldots, y_m) \in X_m; y_i \neq x_j\}$ be the configuration space of m ordered distinct points in the plane with n points x_j removed.

Theorem 9 *$Y_{x,m}$ is an Eilenberg-MacLane space of type $(\pi_1(Y_{x,m}), 1)$.*

Proof. By a theorem of Faddell and Neuwirth, the projection $Y_{x,m+k} \to Y_{x,m}$ onto the first m components is a fiber bundle with the fiber over $y \in Y_{x,m}$ being $Y_{(x,y),k} = \{(z_1, \ldots, z_k) \in X_k; z_i \neq x_j, z_i \neq y_l\}$. For every k we choose $x^{(k)} \in X_k$ (e.g., $x^{(k)} := (1, 2, \ldots, k)$ such that the components are lying on the real line) and set $Y_{k,m} := Y_{x^{(k)},m}$. Then the fiber bundle $Y_{n+m,k} \to Y_{n,m+k} \to Y_{n,m}$ has the following exact homotopy sequence,

$$\ldots \xrightarrow{\partial} \pi_i(Y_{n+m,k}) \xrightarrow{i_*} \pi_i(Y_{n,m+k}) \xrightarrow{p_* \circ j} \pi_i(Y_{n,m}) \xrightarrow{\partial} \pi_{i-1}(Y_{n+m,k}) \ldots$$

By induction assumption, $\pi_i(Y_{n+m,k}) \simeq \{1\} \simeq \pi_i(Y_{n,m})$ for every $i > 1$, every $n \geqq 0$ and some m and k. This is true for $k = m = 1$ at least, since in this case $Y_{n+m,k}$ and $Y_{n,m}$ are Euclidean planes with $n+m$ and n punctures, respectively. We conclude that also the homotopy groups $\pi_i(Y_{n,m+k})$ vanish for $i > 1$. □

We have to determine the structure of the fundamental group $\pi_1(Y_{x,m})$. By using appropriate homotopy sequences, it can be proved that the non-permuting braid group P_n is isomorphic to the fundamental group $\pi_1(X_n)$. Moreover, this isomorphism extends to an isomorphism $B_n \simeq \pi_1(X_n/S_n)$.

Let $P_n^n := \{\alpha \in B_n; \pi(\alpha)(n) = n\} \leqq B_n$ be the subgroup of the braid group that does not permute the n-th string (the n-pure braid group). Then P_n^n is isomorphic to the semidirect product $P_n^n \simeq B_{n-1} \ltimes U_n$, where B_{n-1} is imbedded into B_n by identifying the first $n-1$ strings and it acts onto U_n by conjugation.

This is an extension of the isomorphism $P_n \simeq P_{n-1} \ltimes U_n$ that we introduced before. In the following, F_n will be the free group generated by $\{f_1, \ldots, f_n\}$. Subscripts may be dropped, if the rank of the group is clear or immaterial. The braid action onto the free group can be made explicit and shown to be faithful.

Theorem 10 (E. Artin) *Let $\iota \in Hom(U_{1+n}, F_n)$, $\vartheta_{1+n,i} \mapsto f_i$ be an isomorphism of free groups. Let $\psi \in Hom(B_n, Aut(F_n))$ be such that $\psi(\alpha)(f) = \iota(\alpha^{-1}\iota^{-1}(f)\alpha)$, $\alpha \in B_n$, $f \in F_n$. Then ψ is a monomorphism and*

$$\psi(\tau_i) : f_j \mapsto \begin{cases} f_i f_{i+1} f_i^{-1}, & j = i\,, \\ f_i, & j = i+1\,, \\ f_j, & j \notin \{i, i+1\}\,. \end{cases}$$

Proof. Cf. [Bir74]. □

Theorem 11 *The action of the elements $\vartheta_{n,i} \in B_n$ generating the free group U_n of rank $n-1$ in B_n onto the generators of the free group F_n is given by ($\varepsilon \in \{-1, 1\}$)*

$$\vartheta_{n,i}^{\varepsilon}(f_j) = \begin{cases} f_j, & j < i \\ Ad((f_i f_n)^{\varepsilon})(f_j), & j \in \{i, n\} \\ Ad([f_n^{\varepsilon}, f_i^{\varepsilon}]^{-\varepsilon})(f_j), & i < j < n \end{cases},$$

where $Ad(x)(y) := xyx^{-1}$, $[x, y] := xyx^{-1}y^{-1}$ for $x, y \in F_n$.

Proof. This can be inferred from equation (1) by imbedding $U_n \leqq P_n \hookrightarrow P_{1+n}$ and $F_n \simeq U_{1+n} \leqq P_{1+n}$. Equivalently, Theorem (10) can be used. □

The procedure of taking a semidirect product $B_n \ltimes F_n$ and identifying it with a subgroup of the braid group B_{1+n}, $B_n \ltimes F_n \simeq P_{1+n}^{1+n} \leqq B_{1+n}$, can be iterated. We set

$$B_{n,m} := B_n \ltimes F_n \ltimes \ldots \ltimes F_{n+m-1}\,,$$

$$F_{n,m} := F_n \ltimes \ldots \ltimes F_{n+m-1}\,,$$

such that $B_{n,m} = B_n \ltimes F_{n,m} \leqq B_{n+m}$. In this iterated product, F_k acts onto F_{1+k} according to Theorem (11). $F_{n,m}$ may be identified with the subgroup of the pure braid group P_{n+m}, in which only the last m strings are not held fixed.

Theorem 12 *The fundamental group of the space $Y_{x,m} = \{(y_1, \ldots, y_m) \in X_m;\ y_i \neq x_j\}$ for $x \in X_n$ is isomorphic to the group $F_{n,m}$.*

Proof. The fiber bundle $Y_{n,m} \to X_{n+m} \to X_n$ due to vanishing of the higher homotopy groups yields the exact sequence $1 \to \pi_1(Y_{n,m}) \to \pi_1(X_{n+m}) \to \pi_1(X_n) \to 1$. The exact sequence $1 \to F_{n,m} \to P_{n+m} \to P_n \to 1$ can be derived from the structure theorem (8) of P_{n+m}. We know, $\pi_1(X_n) \simeq P_n$ and $\pi_1(X_{n+m}) \simeq P_{n+m}$. Due to topological reasons, the last isomorphism restricts to a homomorphism $\pi_1(Y_{n,m}) \to F_{n,m}$. By the five lemma, $F_{n,m} \simeq \pi_1(Y_{n,m})$. □

Theorem 13 *The fiber* $Y_{x,m} = \{(y_1,\ldots,y_m) \in X_m; y_i \neq x_j\}$ *over* $x \in X_n$ *has homology* $H_*(Y_{x,m},M) \simeq H_*(F_{n,m},M)$ *for any* $F_{n,m}$ *module* M.

Proof. We have shown, $Y_{x,m}$ is an Eilenberg-MacLane space of type $(F_{n,m},1)$. As a smooth manifold, it is homotopic to a cell complex, cf. [Mil73], Theorem 3.5, pp. 20 and p. 36. So the claim follows e.g. from [Bro82], par. III.1, pp. 56, using Eilenberg's theorem in [Whi78]. □

Now we are in the position to show, in which way the holonomy representations of B_n on $H_*(Y_{x,m},\chi)$ derived in [Law90] can be constructed in purely algebraic terms. The imbedding $B_n \hookrightarrow \mathrm{Aut}(F_n)$ allows the application of Sect. 2 to the braid group. The structure theorem (8) leads us to replace the free group F_n of Magnus' approach by the sequence of groups $F_{n,m}$.

4 Resolution of the integers over $F_{n,m}$

The homology $H_*(Y_{x,m};M)$ is isomorphic to group homology $H_*(F_{n,m};M)$. So we construct a free $F_{n,m} \simeq \pi_1(Y_{x,m})$ resolution C of the integers. It is chosen in such a way that there is a particular braid action onto complexes $N \otimes_{F_{n,m}} C$ and onto their homology (in which case the braid action is the natural parallel transport on the homology bundle) for any right $B_{n,m}$ module N. This action is described by the braid valued Burau matrices that will be derived in the next section.

For fixed integers n,m and for $k \in \{1,\ldots,m\}$ let $F^k := F_{n+k-1}$ be the free group freely generated by the subset $\{f_i^{(k)}; i \in \{1,\ldots,n+k-1\}\} \subset F^k$. Let I^k be the relative augmentation ideal of F^k in the group ring of $F_{n,k} = F^1 \ltimes \ldots \ltimes F^k$. I^k is freely generated as a left module by $\{(f_i^{(k)} - 1); i \in \{1,\ldots,n+k-1\}\}$. We distinguish I^k from its image in $\mathbf{Z}F_{n,k}$. Therefore we write s_i^k when we consider the generators in the ideal I^k rather than in $\mathbf{Z}F_{n,k}$. The natural imbedding $\imath : I^k \hookrightarrow \mathbf{Z}F_{n,k}$ thus sends $s_i^k \mapsto (f_i^{(k)} - 1)$. We consider the ideal I^k as an $F_{n,k}$ bimodule (and therefore as a bimodule over $F_{n,j} \leqq F_{n,k}$, $j \leqq k$). We sometimes write s^k for $(f^{(k)} - 1) \in I^k$, with $f^{(k)} \in F^k$.

Theorem 14 *Let*

$$C_k^m := \oplus_i \mathbf{Z}F_{n,m} \otimes_{F_{n,i_1}} I^{i_1} \otimes_{F_{n,i_2}} \ldots \otimes_{F_{n,i_k}} I^{i_k}$$

for $k \in \{1,\ldots,m\}$, $1 \leqq i_k < i_{k-1} < \ldots < i_1 \leqq m$ *and let* $C_0^m := \mathbf{Z}F_{n,m}$ *with the natural structures as left* $F_{n,m}$ *modules. Let* $\partial_k^m : C_k^m \to C_{k-1}^m$ *be given by* $\partial_k^m := \sum_{l=1}^k (-)^{1+l} \partial_{k,l}^m$, *with*

$$\partial_{k,l}^m : \mathbf{Z}F_{n,m} \otimes_{F_{n,i_1}} I^{i_1} \ldots \otimes_{F_{n,i_{l-1}}} I^{i_{l-1}} \otimes_{F_{n,i_l}} I^{i_l} \otimes_{F_{n,i_{l+1}}} \ldots I^{i_k}$$

$$\to \mathbf{Z}F_{n,m} \otimes_{F_{n,i_1}} I^{i_1} \ldots \otimes_{F_{n,i_{l-1}}} I^{i_{l-1}} \imath(I^{i_l}) \otimes_{F_{n,i_{l+1}}} \ldots I^{i_k},$$

where $\partial^m_{k,l} := 1 \otimes \ldots \otimes \imath \otimes \ldots \otimes 1$ *and the imbedding* $\imath : I^{i_l} \hookrightarrow \mathbf{Z}F_{n,i_l}$ *occurs at the* $(l+1)$*-st position in the* $(k+1)$*-fold tensor product. Then*

$$0 \to C^m_m \xrightarrow{\partial^m_m} C^m_{m-1} \xrightarrow{\partial^m_{m-1}} \ldots \xrightarrow{\partial^m_1} C^m_0 \xrightarrow{\varepsilon} \mathbf{Z} \to 0$$

with the augmentation $\varepsilon : \mathbf{Z}F_{n,m} \to \mathbf{Z}$ *sending* $F_{n,m}$ *to* 1 *is a complex of free left* $F_{n,m}$ *modules, augmented over the trivial left* $F_{n,m}$ *module* $\mathbf{Z}$.

Proof. We will fix m and drop corresponding superscripts. Each C_k is a free left $F_{n,m}$ module, since all the factors in the defining tensor products are free over the respective rings. ∂_k is a homomorphism of left $F_{n,m}$ modules and $\mathrm{Image}(\partial_1)$ is contained in the augmentation ideal of $F_{n,m}$ in $\mathbf{Z}F_{n,m}$, s.t. $\mathrm{Image}(\partial_1) \leqq \mathrm{Ker}(\varepsilon)$. Furthermore for $k \in \{2,\ldots,m\}$, $\mathrm{Image}(\partial_k) \leqq \mathrm{Ker}(\partial_{k-1})$, i.e. $\partial_{k-1} \circ \partial_k = 0$, since in the sum expressing $\partial_{k-1} \circ \partial_k$, each tensor product with $\imath$ at the i-th and at the j-th position, $i \neq j$, occurs twice, once with sign $(-)^i(-)^{j-1}$ and once with sign $(-)^j(-)^i$. □

In explicit terms, the components of the boundary operator act as

$$\partial^m_{k,l}(s^{i_1} \otimes \ldots s^{i_{l-1}} \otimes s^{i_l} \otimes \ldots s^{i_k}) = s^{i_1} \otimes \ldots s^{i_{l-1}}(f^{(i_l)} - 1) \otimes \ldots s^{i_k} ,$$

where the product $s^{i_{l-1}}(f^{(i_l)} - 1)$ has to be understood as the right action of $I^{i_l} \leqq \mathbf{Z}F_{n,i_l}$ onto the bimodule $I^{i_{l-1}}$. This boundary is equivalent to the topological boundary matrix in [Law90], Sect. 3.2, p. 152.

The complexes (C^m, ∂^m) can be described in recursive terms: for $k > 0$, $m > 0$, the modules are given by $C^m_0 = \mathbf{Z}F_{n,m}$, $C^1_1 \simeq I^1$, $C^1_{1+k} = \{0\}$,

$$C^{1+m}_k \simeq I^{1+m} \otimes_{F_{n,m}} C^m_{k-1} \oplus \mathbf{Z}F_{n,1+m} \otimes_{F_{n,m}} C^m_k .$$

The homomorphisms are determined by the augmentation $\varepsilon^m : C^m_0 \to \mathbf{Z}$, by $\partial^m_1 : C^m_1 \to C^m_0$, which imbeds the direct summands $I^k \hookrightarrow \mathbf{Z}F_{n,k}$ for all $k \in \{1,\ldots,m\}$ and by the recursive step

$$\partial^{1+m}_k(j \otimes c + f \otimes d) = (\partial^{1+m}_1(j) \otimes c - j \otimes \partial^m_{k-1}(c)) + f \otimes \partial^m_k(d) ,$$

for $0 < k \leqq m$, $j \in I^{1+m}$, $c \in C^m_{k-1}$, $f \in F_{n,1+m}$, $d \in C^m_k$.

Theorem 15 *The complex* $0 \to C^m_m \xrightarrow{\partial^m_m} C^m_{m-1} \xrightarrow{\partial^m_{m-1}} \ldots \xrightarrow{\partial^m_1} C^m_0 \xrightarrow{\varepsilon} \mathbf{Z} \to 0$ *of left* $F_{n,m}$ *modules is exact.*

Proof. The proof given in [Lüd95] emphasises the connection between the complex and the structure of P_n. Here we give a shorter proof. We have shown already, ∂ is a complex, which means, $\mathrm{Image}(\partial^m_{1+k}) \leqq K^m_k := \mathrm{Ker}(\partial^m_k)$ and $\mathrm{Image}(\partial^m_1) \leqq K^m_0 := \mathrm{Ker}(\varepsilon^m)$. By induction on m we will prove the converse inclusions. For $m = 1$ we find the resolution $0 \to I_{F_{n,1}} \to \mathbf{Z}F_{n,1} \to \mathbf{Z} \to 0$. Thus, $K^1_0 = I_{F_{n,1}} = \mathrm{Image}(I_{F_{n,1}} \to \mathbf{Z}F_{n,1})$ and $K^1_1 = \{0\} = \mathrm{Image}(\{0\} \to I_{F_{n,1}})$. Now assume, our claims are valid for some $m \geqq 1$. The image of ∂^{1+m}_1 is $I_{F_{n,1+m}}$ since the augmentation ideal as a left module is generated by $(f^{(I)}_i - 1)$. So is

the kernel of ε^{1+m}. Let $x \in K_k^{1+m} \leqq C_k^{1+m} \simeq I^{1+m} \otimes_{F_{n,m}} C_{k-1}^m \oplus \mathbf{Z}F_{n,1+m} \otimes_{F_{n,m}} C_k^m$, $k \geqq 1$. Thus, $x = \sum_{i,j} c_{i,j} s_i^{1+m} \otimes_{F_{n,m}} m_j + d$ for some $c_{i,j} \in \mathbf{Z}F^{1+m}$ (we have absorbed coefficients from $\mathbf{Z}F_{n,m}$ into the m_j), $m_j \in C_{k-1}^m$, $d \in \mathbf{Z}F_{n,1+m} \otimes_{F_{n,m}} C_k^m$. From

$$\partial_k^{1+m}(x) = \sum_{i,j} c_{i,j}((f_i^{(1+m)} - 1) \otimes m_j - s_i^{1+m} \otimes \partial_{k-1}^m(m_j)) + \partial_k^m(d) = 0$$

by linear independence we find, $\partial_{k-1}^m(m_j) = 0$ and, using the induction assumption, $m_j = \partial_k^m(x_j)$ for some $x_j \in C_k^m$. Thus, $\sum_{i,j} c_{i,j}(f_i^{(1+m)} - 1) \otimes m_j + \partial_k^m(d) = 0$. We can therefore express x as $x = \sum_{i,j} c_{i,j}(s_i^{1+m} \otimes \partial_k^m(x_j) - (f_i^{(1+m)} - 1) \otimes x_j) + y$ for $y = \sum_{i,j} c_{i,j}(f_i^{(1+m)} - 1) \otimes x_j + d \in \mathbf{Z}F_{n,1+m} \otimes_{F_{n,m}} K_k^m$. In other words, $x = \partial_{1+k}^{1+m}(-(\sum_{i,j} c_{i,j} s_i^{1+m} \otimes x_j) + z)$ for some $z \in C_{1+k}^{1+m}$, where again we have used the assumption. □

For every right $B_{n,m}$ module N, the tensor complex $N \otimes_{F_{n,m}} C$ and its homology $H_*(F_{n,m}, N)$ have a structure as a right B_n module. The right action onto the complex is given by $\alpha : n \otimes_{F_{n,m}} c \mapsto n\alpha \otimes_{F_{n,m}} \alpha(c)$, where $\alpha \in B_n$ acts via Artin's automorphisms Theorem (10) onto $c \in C^k$. We prefer a different point of view. Instead of the augmentation ideals $I^l = I_{F_{n+l-1},F_{n,l}}$ used in the construction of the resolution C, we may consider the ideals $\bar{I}^l := I_{F_{n+l-1},B_{n,l}}$, with $B_{n,l} = B_n \ltimes F_{n,l}$. $\bar{I}^l$ as a left $B_{n,l}$ module is isomorphic to $\mathbf{Z}B_{n,l} \otimes_{F_{n,l}} I^l$ and as an ideal in the ring $\mathbf{Z}B_{n,l}$, it carries a structure as right $B_{n,l}$ module, too. Let the complex $\bar{C}$ be constructed similar to C in Theorem (14) by setting

$$\bar{C}_k^m := \oplus_t \mathbf{Z}B_{n,m} \otimes_{B_{n,i_1}} \bar{I}^{i_1} \otimes_{B_{n,i_2}} \ldots \otimes_{B_{n,i_k}} \bar{I}^{i_k} .$$

This complex is not a free resolution of $\mathbf{Z}$ anymore but it is still exact and for any right $B_{n,m}$ module N we may identify $N \otimes_{F_{n,m}} C \simeq N \otimes_{B_{n,m}} \bar{C}$ as right B_n modules, where the B_n action on $N \otimes_{B_{n,m}} \bar{C}$ is more obvious. Indeed, it is induced by the multiplication from the right in $\bar{I}^l = I_{F_{n+l-1},B_{n,l}}$. This action is known as soon as the representations on the ideals I_{F_k,B_kF_k} for all k are known, because $I_{F_{n+l-1},B_{n,l}}$ is imbedded into I_{F_k,B_kF_k} with $k = n + l - 1$. This action will be found in the next section.

5 The braid valued Burau matrices

The relative augmentation ideal I_{F_n,B_nF_n} is a right B_n module and is free as a left B_nF_n module. The right B_n action therefore is determined by matrices which also determine the action onto the complex $N \otimes_{F_{n,m}} C$ and onto its homology for any right $B_{n,m}$ module N. On homology the braid action is the natural parallel transport in the homology bundle of [Law90]. These matrices generalise the Burau matrices.

The Burau representation of the braid group is a "classical" Birman-Magnus representation in the sense of Sect. 2, cf. [Bir74, Mag74]. Let $\psi \in \mathrm{Hom}(F_n, F_1)$,

with $F_1 := \langle t \rangle$. This map has the kernel $U = \{f_{i_1}^{\varepsilon_1} \cdots f_{i_k}^{\varepsilon_k}; \sum \varepsilon_l = 0\}$. According to Theorem (10), for every braid $\alpha \in B_n$ and every $f \in F_n$, $\psi(\alpha(f)) = \psi(f)$. So by Theorems (2) and (7) there is a Birman module $M_\psi \simeq \mathbf{Z}[t] \otimes_{B_n F_n} I_{F_n, B_n F_n}$ carrying a right action of B_n,

$$s_j \tau_i = \delta_\psi(\tau_i(f_j)) = \begin{cases} \delta_\psi(f_i f_{i+1} f_i^{-1}) = (1-t)s_i + t s_{i+1}, & j = i \\ \delta_\psi(f_i) = s_i, & j = i+1 \\ \delta_\psi(f_j) = s_j, & j \notin \{i, i+1\} \end{cases}.$$

This means the right action is determined by the well-known Burau matrices

$$\tau_i \mapsto \begin{pmatrix} \mathbf{1}_{i-1} & 0 & 0 & 0 \\ 0 & (1-t) & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_{n-i-1} \end{pmatrix}.$$

The Magnus submodule $\delta_\psi(U) \simeq U/[U,U] \simeq H_1(F_n, \mathbf{Z}[t])$ according to Theorem (5) is

$$\{\sum g_j s_j; g_j \in \mathbf{Z}[t], \sum g_j(\psi(f_j) - 1) = 0\} = \{\sum g_j s_j; g_j \in \mathbf{Z}[t], \sum g_j = 0\},$$

since $\psi(f_j) = t$ and the polynomial ring $\mathbf{Z}[t]$ is free of zero divisors. A complementary geometrical point of view can be found in [Ati90], whereas the algebraical approach is stressed in [Jon91, Mag74].

The modules constructed in Sect. 2 according to Theorems (6) and (7) are induced by the representation on the relative augmentation ideal $I_{F,EF}$. In Sect. 4 we have seen that the right B_n action onto $N \otimes_{F_{n,m}} C$ for any right $B_{n,m}$ module N also is determined by the ideals $I_{F,BF}$. So we obtain the braid valued Burau representation, [CL92, Lüd92].

Theorem 16 *Let M be a free left BF module of rank n with basis $\{s_1, \ldots, s_n\}$. Then the map $\rho : \{\tau_1, \ldots, \tau_{n-1}\} \to End_{BF}(M)$ defined by*

$$\rho(\tau_i)(s_j) = \tau_i \begin{cases} (1 - f_i f_{i+1} f_i^{-1}) s_i + f_i s_{i+1}, & j = i \\ s_i, & j = i+1 \\ s_j, & j \notin \{i, i+1\} \end{cases},$$

uniquely extends to a monomorphism $\rho \in Hom(B, End_{BF}(M))$.

Proof. $I_{F,BF}$ is a free left BF module over the set $\{(f_i - 1); i \in \{1, \ldots, n\}\}$, so we identify M with $I_{F,BF}$ via $s_i \mapsto (f_i - 1)$. By multiplication from the right $I_{F,BF}$ is a module over B, $(f_j - 1) \mapsto (f_j - 1)\tau_i = \tau_i(\tau_i(f_j) - 1)$. The element $\tau_i(f_j)$ is determined by Artin's Theorem (10) so we obtain the equation as claimed. □

In terms of matrices over the ring $\mathbf{Z}(BF)$ the representation of τ_i is given by

$$\tau_i \mapsto \tau_i \begin{pmatrix} \mathbf{1}_{i-1} & 0 & 0 & 0 \\ 0 & (1 - f_i f_{i+1} f_i^{-1}) & f_i & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_{n-i-1} \end{pmatrix}. \tag{2}$$

With the help of the imbedding $B_n F_n \hookrightarrow B_{\mathbf{1}+n}$, $\tau_i f_j \mapsto \tau_i \vartheta_{\mathbf{1}+n,j}$, the semidirect product $B_n F_n$ and therefore the matrix elements of eqn. (2) can again be represented by the braid valued Burau matrices. By m-fold iteration we obtain representations of B_n (or of subgroups $S \leqq B_n$) in terms of matrices with values in the ring $\mathbf{Z}(B_n F_n F_{1+n} \dots F_{n+m-1})$ ($\mathbf{Z}(SF_n F_{1+n} \dots F_{n+m-1})$, respectively).

Finally we show that the local coefficient system chosen in [Law90] enables us to identify the recursion matrices given there with images of the braid valued Burau matrices. We imbed $F_{n,m} = F_n \ltimes \dots F_{n+m-1}$ into B_{n+m}. The generators $\{f_i^{(k)}; i \in \{1, \dots, n+k-1\}\}$ of F_{n+k-1} are mapped as $f_i^{(k)} \mapsto \vartheta_{k+n,i}$.

Theorem 17 *Let a homomorphism $\chi \in Hom(F_{n,m}, \mathbf{C}\backslash\{0\})$ be given by the map $\vartheta_{l,i} \mapsto q_{l,i}$, for $l \in \{1+n, \dots, m+n-1\}$, $i < l$. If $q_{l,i} = q_{l,j}$ for $i,j \in \{1, \dots, n\}$, then χ extends to a map $\bar{\chi} \in Hom(B_n \ltimes F_{n,m}, \mathbf{C}\backslash\{0\})$ by setting $\bar{\chi}(\alpha\vartheta) = \chi(\vartheta)$, $\alpha \in B_n$, $\vartheta \in F_{n,m}$.*

Proof. Eqn. (1) shows that any map $\vartheta_{l,i} \mapsto q_{l,i}$ of the generators into an abelian group extends to a homomorphism of $F_{n,m}$. This is also the content of Lemma 2.1, p. 145, [Law90]. The relations defining the semidirect product $B_{n,m} = B_n \ltimes F_{n,m}$ are those in B_n, in $F_{n,m}$ and in addition those between B_n and $F_{n,m}$:

$$\tau_i^{-1} \vartheta_{l,j} \tau_i = \begin{cases} \vartheta_{l,i} \vartheta_{l,i+1} \vartheta_{l,i}^{-1}, & j = i \\ \vartheta_{l,i}, & j = i+1 \\ \vartheta_{l,j}, & j \notin \{i, i+1\} \end{cases}.$$

So the conditions $q_{l,i} = q_{l,j}$ allow the extension of χ to $B_{n,m}$ by mapping B_n to 1. □

From the following matrices, holonomy representations of B_n on the homology $H_m(Y_{x,m}, \chi)$ are derived in [Law90].

Theorem 18 (R.J. Lawrence) *For $r \in \{0, 1, \dots, m\}$, $l \in \{1+r, \dots, m+n-1\}$, let the matrices $A_l^{(r)}$ be recursively defined by the equations*

$$A_i^{(0)} = 1\,,$$

$$A_i^{(k)} = \begin{pmatrix} \mathbf{1}_{m+n-k-i} & 0 & 0 & 0 \\ 0 & 0 & b_{k,1+i}^{(k-1)} & 0 \\ 0 & 1 & (1 - (b_{k,1+i}^{(k-1)})^{-1} b_{k,i}^{(k-1)} b_{k,1+i}^{(k-1)}) & 0 \\ 0 & 0 & 0 & \mathbf{1}_{i-2} \end{pmatrix} A_i^{(k-1)},$$

where for $q_{1+r,p} \in \mathbf{C}\backslash\{0\}$, $p > 1+r$ *we set*

$$b^{(r)}_{1+r,\,p} = A^{(r)2}_{1+r,}/q_{1+r,\,p}\,,$$
$$A^{(r)}_{1+r,p} = A^{(r)}_{1+r} A^{(r)}_{2+r} \ldots A^{(r)}_{p-1} (A^{(r)}_{p-2})^{-1} \ldots (A^{(r)}_{1+r})^{-1}\,.$$

If $q_{j,s} = q_{j,t}$ *for* $j \in \{1,\ldots,m\}$, $s,t \in \{1+m,\ldots,n+m\}$, *then the matrices* $A^{(m)}_l$, $l \in \{1+m,\ldots,m+n-1\}$ *generate a homomorphic image of the braid group* B_n.

Proof. This is Theorem 3.4, pp. 156, [Law90], where it is proved by computing the braid action on a cell complex. We have only given the recursion matrices $A^{(r)}_l = A^{(r)}_{l,1+l}$ instead of the more complicated ones $A^{(r)}_{l,p}$, because the matrices $A^{(r)}_l$ generate the others. We have furthermore used Lemma 3.3, p. 153, parts (i) and (ii) to slightly change the form of $A^{(r)}_l$ compared to the expression in loc. cit. □

The theorem can be understood in our algebraic approach. We compare $A^{(r)}_l$ to the braid valued Burau matrix, eqn. (2). Lawrence's matrices act by left multiplication with column vectors, the generalised Burau matrices act by right multiplication with row vectors. Furthermore, the conventions in [Law90] on the imbedding $B_{n,m} \hookrightarrow B_{n+m}$ are different from ours. This accounts for the reversal of products of matrices, for rearranging of rows and columns and for reindexing of the generators. Apart from this both matrices have the same appearance. The lowest order matrices due to lemma (17) yield a representation $\bar{\chi} \in \mathrm{Hom}(B_{n,m}, \mathbf{C}\backslash\{0\})$, $\tau_{m+i} \to A^0_{m+i} = 1$, $\vartheta_{l,p} \mapsto q^-_{l,\,p}$, $i \in \{1,\ldots,n-1\}$, $l \in \{1,\ldots,m\}$, $l < p$ (where now we use the imbedding of [Law90]). By the braid valued Burau matrices and the similarity of the recursion matrix with these, the matrices $A^{(m)}_l$ for $l \in \{1+n,\ldots,n+m-1\}$ represent B_n, if the matrices $A^{(m-1)}_l$, $l \in \{1+m,\ldots,n+m-1\}$ and $b^{(m-1)}_{m,\,p}$, $p \in \{1+m,\ldots,n+m-1\}$ represent B_nF_n. So finally we are led to a representation of $B_nF_n \ldots F_{n+m-1}$ which is given by $\bar{\chi}$.

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