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Autor: Li, Jian-Shu; Zhu, Chen-Bo

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Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

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On the decay of matrix coefficients for exceptional groups

Jian-Shu Li[1,*], **Chen-Bo Zhu**[2,**]

[1] Department of Mathematics, University of Maryland, College Park, MD 20742, USA
[2] Department of Mathematics, National University of Singapore, Kent Ridge, Singapore 0511

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0 Introduction and main result

For a locally compact group G, there is the so-called Fell topology ([Fe]) defined on $\hat{G}$, the set of equivalence classes of irreducible unitary representations of G. In [Ka], Kazhdan proved that, if G is a simple group defined over a local field F and has F-split rank at least 2, then the trivial representation of G is isolated in $\hat{G}$. In other words, G has property (T).

Let G be a locally compact group with compact center. A unitary representation ρ of G is said to be strongly L^p if, for a dense set of vectors v in the space of ρ, the matrix coefficients $x \mapsto (\rho(x)v, v)$ all lie in $L^p(G)$. Let $p(G)$ be the smallest real number such that any irreducible infinite dimensional unitary representation of G is strongly L^p for any $p > p(G)$. Any good classification of the unitary dual of G should of course give $p(G)$ as a corollary. In [Ho] Howe computed the number $p(G)$ for symplectic groups over local fields, for which the unitary dual is not yet known (except in the complex case [Ba]). Using more or less similar methods, the first author recently obtained $p(G)$ for almost all the classical groups [Li]. In view that the Fell topology is defined in terms of matrix coefficients, such a result provides a quantitative version of Kazhdan's result mentioned above.

The purpose of the present article is to establish upper bounds of $p(G)$ for exceptional groups G. More precisely, we have

Theorem 0.1. *Let F be a local field, not of characteristic 2. Let G be one of the five simple groups of exceptional type split over F. Let ρ be a uni-*

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tary representation of G not containing the trivial representation. Then ρ is strongly $L^{p+\varepsilon}$ for any $\varepsilon > 0$, with

$$p = \begin{cases} 6, & G = G_2 , \\ \frac{44}{5}, & G = F_4 , \\ 22, & G = E_6 , \\ 14, & G = E_7 , \\ \frac{810}{77}, & G = E_8 . \end{cases}$$

The proof of the above result is based on a strategy of Howe (Sect. 3). We shall first develop some initial results concerning the decay of matrix coefficients of the restriction to certain reductive subgroups of G (Sect. 2). In turn, this relies on the fact that G contains a "canonical" maximal parabolic subgroup with unipotent radical a Heisenberg group (Sect. 1); note that the same group played an important role in the construction of the "smallest" representation of G by Kazhdan and Savin [KS]. Section 4 gives detailed proofs for each of the exceptional groups.

1 A Heisenberg algebra

In this section, G is one of the simple, simply connected groups split over F, not of type "A". Fix a minimal parabolic subgroup B of G. Let $A \subset B$ be a maximal split torus, and $U \subset B$ be the unipotent radical of B. The action of A on U by conjugation gives rise to a collection Φ^+ of positive roots of A. The set of simple roots of Φ^+ is denoted by Δ. Thus each root is a rational character of A, a homomorphism from A to $F^\times$. We shall denote such a root by χ_α, so that the set of α's forms an additive root system.

Let $\mathfrak{g}$ (resp. $\mathfrak{a}$) be the Lie algebra of G (resp. A). Thus we have a root space decomposition

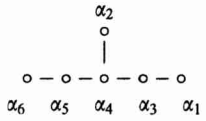
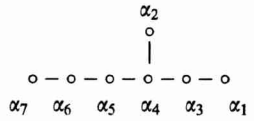
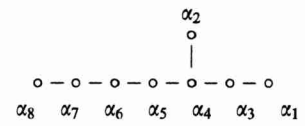
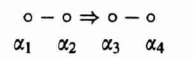
$$\mathfrak{g} = \mathfrak{a} \oplus \sum_{\alpha \in \Phi} F g_\alpha .$$

The following lemma can be easily verified.

Lemma 1.1. *Let Φ be an irreducible root system, not of type "A". Let Δ be the set of simple roots. Let γ be the highest root. Then there exists a unique $\tilde{\alpha} \in \Delta$ such that $(\tilde{\alpha}, \gamma) \neq 0$.*

Remark. If Φ has rank ≥ 2, then $\tilde{\alpha} \neq \gamma$.

We have

$\mathfrak{g}$	Dynkin diagram	$\tilde{\alpha}$
e_6		α_2
e_7		α_1
e_8		α_8
f_4		α_1
g_2		α_2

By the usual procedure, we construct a maximal parabolic subgroup of G corresponding to the set of simple roots $\Delta\backslash\{\tilde{\alpha}\}$, denoted by $P = P_{\Delta\backslash\{\tilde{\alpha}\}}$. Thus we have the Levi decomposition $P = MN$. Let A_s be the split part of M, let $\mathfrak{m}$, $\mathfrak{a}_s$, $\mathfrak{n}$ be the Lie algebras of M, A_s, N, respectively. Then $\mathfrak{a}_s$ is the subalgebra of $\mathfrak{a}$ annihilated by $\Delta\backslash\{\tilde{\alpha}\}$, $\mathfrak{m}$ is the centralizer of $\mathfrak{a}_s$ in $\mathfrak{g}$. Let Φ_M be the set of roots α such that $\alpha|_{\mathfrak{a}_s} = 0$, Q be the set of positive roots α such that $\alpha|_{\mathfrak{a}_s} \neq 0$. We have

$$Fg_\alpha \subset \mathfrak{m}, \quad \text{if and only if } \alpha \in \Phi_M ,$$

and

$$\mathfrak{n} = \sum_{\alpha \in Q} Fg_\alpha .$$

Also $\Delta\backslash\{\tilde{\alpha}\}$ is a set of simple roots for Φ_M. Notice that $\gamma \in Q$, i.e., $g_\gamma \in \mathfrak{n}$.

Lemma 1.2. $\mathfrak{n}$ *is a Heisenberg algebra with center* $Z = Fg_\gamma$.

Proof. Take any $\alpha \in Q$ or equivalently $g_\alpha \in \mathfrak{n}$. Clearly, $(\alpha, \gamma) > 0$.

By the Schwarz inequality, we have

$$\langle \alpha, \gamma \rangle = \frac{2(\alpha, \gamma)}{(\gamma, \gamma)} \leqq \frac{2|\alpha||\gamma|}{|\gamma|^2} = \frac{2|\alpha|}{|\gamma|} .$$

Since γ is always a long root, we have $\langle \alpha, \gamma \rangle \leqq 2$, and the equality holds if and only if α is a positive multiple of γ, which is equivalent to $\alpha = \gamma$.

Since $\langle \alpha, \gamma \rangle$ is a positive integer, either $\langle \alpha, \gamma \rangle = 2$ and so $\alpha = \gamma$ or $\langle \alpha, \gamma \rangle = 1$.

Let

$$\Sigma = Q \backslash \{\gamma\} .$$

Thus Σ is precisely the set of positive roots β such that $\langle \beta, \gamma \rangle = 1$. We have

$$\mathfrak{n} = Fg_\gamma \oplus \sum_{\beta \in \Sigma} Fg_\beta .$$

For any $\beta_1, \beta_2 \in \Sigma$, either $\beta_1 + \beta_2$ is not a root, in which case $[g_{\beta_1}, g_{\beta_2}] = 0$, or $\beta_1 + \beta_2$ is a root, in which case $\beta_1 + \beta_2 = \gamma$, since $\langle \beta_1 + \beta_2, \gamma \rangle = \langle \beta_1, \gamma \rangle + \langle \beta_2, \gamma \rangle = 2$. Thus $[g_{\beta_1}, g_{\beta_2}] = \lambda(\beta_1, \beta_2) g_\gamma$, for some scalar $\lambda(\beta_1, \beta_2)$.

Also it is clear that g_γ is in the center of $\mathfrak{n}$, for γ is the highest root.

It follows that $\mathfrak{n}$ is a Heisenberg algebra with center $Z = Fg_\gamma$. □

Let

$$W = \mathfrak{n}/Z = \sum_{\beta \in \Sigma} Fg_\beta ,$$

the bilinear form $B(\cdot, \cdot)$ defined by

$$[g_{\beta_1}, g_{\beta_2}] = B(g_{\beta_1}, g_{\beta_2}) g_{\beta_1 + \beta_2}$$

is a nondegenerate symplectic form on W.

Since M acts on $\mathfrak{n}$ via automorphisms, it preserves the center Z.

Let M^0 be the subgroup of M which acts trivially on Z. We have

$$M^0 \cong \begin{cases} SL_2, & G = G_2 , \\ Sp_6, & G = F_4 , \\ SL_6, & G = E_6 , \\ SO(6,6), & G = E_7 , \\ E_7, & G = E_8 . \end{cases}$$

M^0 acts on the symplectic vector space $W = \mathfrak{n}/Z$, via symplectic automorphisms.

Notice that if $\beta \in \Sigma$, then $\gamma - \beta \in \Sigma$.

Remark. Since $\tilde{\alpha} \in \Sigma$, we have $\gamma - \tilde{\alpha} \in \Sigma$, i.e., $g_{\gamma - \tilde{\alpha}} \in W$. In fact, W is an irreducible representation of M^0 with the highest weight $\gamma - \tilde{\alpha}$.

Let $*$ be the natural involution of the set Σ noted above:

$$\beta^* = \gamma - \beta .$$

Fix a subset Π of Σ such that Σ is a disjoint union of Π and Π^*.

Let

$$V = \sum_{\beta \in \Pi} Fg_\beta , \qquad V^* = \sum_{\beta \in \Pi} Fg_{\beta^*} .$$

Then $W = V \oplus V^*$ gives a complete polarization, namely, both V and V^* are maximal isotropic subspaces of W with respect to the symplectic form $B(\cdot, \cdot)$.

2 An estimate for the matrix coefficients

Let (W,B) be a symplectic vector space over F, $Sp(W)$ be the isometry group of the symplectic form B. Let $\tilde{S}p(W)$ be the unique two-fold cover of $Sp(W)$ (let $\tilde{S}p(W) = Sp(W)$ for $F = \mathbb{C}$). For a subgroup E of $Sp(W)$, let $\tilde{E}$ be the preimage of E under the projection map $\tilde{S}p(W) \mapsto Sp(W)$.

Let $H(W)$ be the associated Heisenberg group, namely $H(W) = W \oplus F$, which has the group law

$$(w,s)(w',s') = \left(w + w',\, s + s' + \frac{1}{2}B(w,w')\right)$$

with $w,w' \in W$, $s,s' \in F$. Let $Z(H)$ be the center of $H(W)$, which can be identified with F. An arbitrary unitary character of F, and thus of $Z(H)$, has the following form

$$\chi_t(s) = \chi_1(ts), \qquad t,s \in F\,,$$

where χ_1 is a non-trivial unitary character of F, which we choose once and for all. Let π_t be the irreducible unitary representation of $H(W)$ with central character χ_t, and let ω_t be the corresponding oscillator representation of $\tilde{S}p(W)$ ([Sh], [We]).

In the following, let G be one of the simple, simply connected groups split over F, not of type "A". Recall in Sect. 1, we constructed a maximal parabolic subgroup $P = MN$ of G such that N is a Heisenberg group. The group $M^0 \subseteq M$ acts on the associated symplectic vector space $W = \mathfrak{n}/Z$, and $M^0 \hookrightarrow Sp(W)$. We also fixed a complete polarization

$$W = V \oplus V^*\,,$$

where

$$V = \sum_{\beta \in \Pi} Fg_\beta, \qquad V^* = \sum_{\beta \in \Pi} Fg_{\beta^*}$$

by fixing a subset Π of Σ such that $\Pi \cup \Pi^* = \Sigma$.

If ρ is a unitary representation of G not containing the trivial representation, then $\rho|_N$ does not contain the trivial representation of N by a well known result of Howe and Moore ([HM]). Thus the only representations of $N \cong H(W)$ occurring in $\rho|_N$ are the representations π_t given by the Stone-Von Neumann Theorem. Since $\tilde{M}^0$ acts on N by conjugation, it acts on representations of N. The isotropic group of π_t under $\tilde{M}^0$ is trivial and we can extend π_t to a representation of $\tilde{M}^0N$ by way of the oscillator representation $\omega_t|_{\tilde{M}^0}$, still denoted by ω_t. By Mackey Theory, the irreducible unitary representations of $\tilde{M}^0N$ which are not trivial on N are of the form $\omega_t \otimes \sigma_t$, where σ_t is an irreducible unitary representation of $\tilde{M}^0$. Since $\tilde{M}^0N$ is of type one, any unitary representation can be written as a direct integral over irreducible ones. Now we can view $\rho|_{M^0N}$ as a unitary representation of $\tilde{M}^0N$ by composing with $\tilde{M}^0 \mapsto M^0$, and so we can write it as a direct integral over $\omega_t \otimes \sigma_t$, for $t \in F^\times$ and certain σ_t. In view of the fact that the isomorphism class of $\omega_t|_{\tilde{M}^0}$ depends only on $t \in F^\times/(F^\times)^2$, we conclude

Proposition 2.1. *Let ρ be a unitary representation of G on a Hilbert space $\mathscr{H}$. Assume that ρ does not contain the trivial representation of G. Then there exists a unitary representation μ_t of $\tilde{M}^0$, for $t \in F^\times/(F^\times)^2$, such that*

$$\rho|_{M^0} \cong \sum_{t \in F^\times/(F^\times)^2} \omega_t|_{\tilde{M}^0} \otimes \mu_t \,.$$

Here the representation $\omega_t|_{\tilde{M}^0} \otimes \mu_t$ of $\tilde{M}^0$ factors through $\tilde{M}^0 \mapsto M^0$.

Let $A(M^0) = A \cap M^0$. It is a maximal split torus of M^0, which consists of $d \in A$ such that $\chi_\gamma(d) = 1$. We have the following embedding $A(M^0) \hookrightarrow GL(V) \hookrightarrow Sp(W)$. Here the first embedding is given by

$$d \mapsto D(d), \quad \text{where } D(d) = diag(\chi_\beta(d)),\ \beta \in \Pi \,,$$

and the second embedding is given by

$$D \mapsto \begin{pmatrix} D & 0 \\ 0 & D^{t-1} \end{pmatrix}, \quad \text{where } D \in GL(V) \,.$$

Remark. $\omega_t|_{\tilde{M}^0}$ factors through $\tilde{M}^0 \mapsto M^0$ if and only if

$$\prod_{\beta \in \Pi} \chi_\beta(d) \in (F^\times)^2, \quad \forall d \in A(M^0) \,.$$

Using the Schrödinger realization of ω_t on $L^2(V)$ and the unitarity of μ_t, we obtain the following estimate:

Proposition 2.2. *In the hypothesis of Proposition* 2.1, *there exists a positive function ϕ on M^0 such that*

$$\phi|_{A(M^0)}(d) = \prod_{\beta \in \Pi} max(|\chi_\beta|(d), |\chi_\beta|(d)^{-1})^{-\frac{1}{2}}, \quad d \in A(M^0) \,,$$

and a dense subspace of $\mathscr{H}$ such that for x, y in this subspace, the matrix coefficient $\phi_{x,y}$ satisfies

$$\|\phi_{x,y}\|_{M^0} \leqq C \cdot \phi \,,$$

for some positive constant C depending on x, y.

Remark. ϕ is essentially the absolute value of a matrix coefficient of $\omega_t|_{\tilde{M}^0}$.

The proof of the following proposition is by tedious calculations. The details will be given in Sect. 4.

Proposition 2.3. *Assume G is of exceptional type. Then there is a real number $q > 0$ such that $\phi \in L^{q+\varepsilon}(M^0)$, but $\phi \notin L^q(M^0)$.*

$$q = \begin{cases} 1, & G = G_2 \,, \\ \frac{10}{3}, & G = F_4 \,, \\ 4, & G = E_6 \,, \\ 5, & G = E_7 \,, \\ \frac{17}{3}, & G = E_8 \,. \end{cases}$$

This implies

Proposition 2.4. *Let ρ be a unitary representation of G not containing the trivial representation. Then $\rho|_{M^0}$ is strongly $L^{q+\varepsilon}$. Here q is as in Proposition* 2.3.

Let $L \cong SL_2$ be the subgroup of G generated by the roots $g_{\pm\gamma}$. M^0 commutes with L, since $\beta \pm \gamma$ is never a root, for any simple root $\beta \neq \tilde{\alpha}$. Thus we can form the subgroup $M^0 \times L$ of G.

Proposition 2.5. *Let ρ be a unitary representation of G not containing the trivial representation. Then $\rho|_L$ is strongly $L^{1+\varepsilon}$.*

Proof. We first observe that $L \cong SL_2$ is contained in a subgroup $E \cong SL_3$. This is obvious except when G is of type "G_2". For G of type "G_2", one can take E to be the subgroup generated by the six long roots, which is isomorphic to SL_3.

We know $\rho|_E$ does not contain the trivial representation of E. We then apply the following result which follows easily from [Sc], Theorem 2.2.

Let SL_2 be the subgroup of SL_3 consisting of the matrices $\begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}$. Let ρ be a unitary representation of SL_3 not containing the trivial representation, then $\rho|_{SL_2}$ is strongly $L^{1+\varepsilon}$.

Since the subgroups of SL_3 which are isomorphic to the above SL_2 are all conjugate, the proposition follows. □

3 A strategy of Howe

Let G be a reductive group with compact center. Let B be a minimal parabolic subgroup of G, A a maximal split torus contained in B, and $A^+ \subset A$ the positive Weyl chamber determined by B. Let H be a reductive subgroup of G with compact center, such that $B(H) = B \cap H$ is a minimal parabolic subgroup of H, and $A(H) = A \cap H$ is a maximal split torus of H. Then the corresponding Weyl chamber $A^+(H)$ in $A(H)$ contains A^+. Let δ and $\delta(H)$ be the modulus functions for B and $B(H)$, respectively.

Suppose that $H = H_1 \times \cdots \times H_m$, a product of reductive subgroups. Then in obvious notations one has $A(H) = A(H_1) \times \cdots \times A(H_m)$ and $\delta(H) = \prod_{j=1}^m \delta(H_j)$.

Theorem 3.1. *Let ρ be a unitary representation of G. Suppose that*

(a) *For $1 \leqq j \leqq m$, $\rho|_H$ is strongly $L^{p_j+\varepsilon}$. Let k_j be a positive integer such that $p_j \leqq 2k_j$.*

(b) *For $a \in A^+$, one has*

$$\prod_{j=1}^{m} \delta(H_j)(a)^{\frac{1}{2k_j}} \geqq \delta(a)^{\frac{1}{p}}$$

for a positive number p.

Then ρ is strongly $L^{p+\varepsilon}$.

For the proof of this result we refer the reader to [Ho] and [Li].

In the cases we are considering, we shall pick an appropriate Weyl group element w and take H_1 (resp. H_2) to be the w conjugate of M^0 (resp. L). Then set $H = H_1 \times H_2$. Set

$$k_1 = \begin{cases} 1, & G = G_2 , \\ 2, & G = F_4 , \\ 2, & G = E_6 , \\ 3, & G = E_7 , \\ 3, & G = E_8 , \end{cases}$$

so that $q \leqq 2k_1$, and $k_2 = 1$ (see Propositions 2.4 and 2.5). We then show by elementary comparisons that

$$\delta(H_1)^{\frac{1}{2k_1}}(a)\delta(H_2)^{\frac{1}{2}}(a) \geqq \delta(G)^{\frac{1}{p}}(a), \quad a \in A^+ .$$

Here p is as in Theorem 0.1. Theorem 3.1 then implies our main result. We shall leave details of these comparisons to the next section.

4 Details of proof

In this section, we give the L^p estimate of the function ϕ (Proposition 2.3), and finish the proof of Theorem 0.1 for each of the five exceptional groups.

4.1. $G = G_2$

Let G_2 be the split form of the exceptional Lie group of type G_2, $\mathfrak{g}_2$ its Lie algebra. It is known that $\mathfrak{g}_2$ has the following realization [HS]:

$$\mathfrak{g}_2 = \mathfrak{sl}_3(F) \oplus F^3 \oplus (F^3)^* . \tag{4.1.1}$$

Let $\mathfrak{a}$ be the Cartan subalgebra of $\mathfrak{sl}_3$ consisting of diagonal matrices of trace zero, A be the corresponding maximal torus. For $d = diag(a_1, a_2, a_3) \in \mathfrak{a}$, let $\varepsilon_i \in \mathfrak{a}^*$ be $\varepsilon_i(d) = a_i$. Thus we always have $\sum_{i=1}^3 \varepsilon_i = 0$. With respect to $\mathfrak{a}$, the roots of the decomposition (4.1.1) are:

$$\pm(\varepsilon_i - \varepsilon_j), \quad 1 \leqq i < j \leqq 3 ,$$

$$\pm \varepsilon_i, \quad 1 \leqq i \leqq 3 .$$

A set Δ of simple roots can be taken as

$$\alpha_1 = \varepsilon_2, \qquad \alpha_2 = \varepsilon_1 - \varepsilon_2 .$$

Remark. The inner product on $\mathfrak{a}^*$ induced by the Killing form is a constant multiple of the following

$$(\varepsilon_1, \varepsilon_1) = (\varepsilon_2, \varepsilon_2) = \frac{2}{3}, \qquad (\varepsilon_1, \varepsilon_2) = -\frac{1}{3} .$$

For $\varepsilon_3 = -(\varepsilon_1 + \varepsilon_2)$, we also have $(\varepsilon_3, \varepsilon_3) = \frac{2}{3}$, $(\varepsilon_1, \varepsilon_3) = (\varepsilon_2, \varepsilon_3) = -\frac{1}{3}$.

The highest root $\gamma = \varepsilon_1 - \varepsilon_3 = 2\varepsilon_1 + \varepsilon_2$. Notice that $\gamma - \alpha_2$ is still a root, i.e., $\tilde{\alpha} = \alpha_2$ (c.f. Lemma 1.1).

Consider the parabolic subgroup $P = P_{\Delta\setminus\{\alpha_2\}}$ of G_2 corresponding to the set of simple roots $\Delta\setminus\{\alpha_2\} = \{\alpha_1\}$, $P = MN$. We have $g_\alpha \in \mathfrak{m}$ if and only if $\alpha = \pm\varepsilon_2$ and $M^0 \cong SL_2$.

$\mathfrak{n}$ is spanned by all positive root vectors g_α, where α belongs to the following set Q of positive roots:

$$A_1 = \varepsilon_1 - \varepsilon_2, \qquad B_1 = \varepsilon_2 - \varepsilon_3 ,$$
$$A_2 = \varepsilon_1, \qquad B_2 = -\varepsilon_3 ,$$
$$\gamma = \varepsilon_1 - \varepsilon_3 .$$

It is a Heisenberg algebra with the center $Z = Fg_\gamma$.

M^0 acts on the symplectic vector space $W = \mathfrak{n}/Z$, which is of dimension 4. In fact, W is an irreducible representation of $M^0 \cong SL_2$ with the highest weight $\gamma - \tilde{\alpha} = \varepsilon_2 - \varepsilon_3$. It is the so-called symmetric cube representation of SL_2.

The set Π specifying the complete polarization $W = V \oplus V^*$ can be taken as $A_1 = \varepsilon_1 - \varepsilon_2$ and $A_2 = \varepsilon_1$.

$A(M^0)$, the maximal torus of M^0, consists of $d = diag(a, b, a)$ such that $a^2 b = 1$. The positive Weyl chamber $A^+(M^0)$ is determined by the condition $|b| \geqq 1$. Thus on $A^+(M^0)$, we have $|\chi_{A_1}| \leqq 1$, $|\chi_{A_2}| \leqq 1$.

Thus given any unitary representation ρ of G_2 not containing the trivial representation, its matrix coefficients are dominated by

$$\phi(d) = |\chi_{A_1}|^{\frac{1}{2}} |\chi_{A_2}|^{\frac{1}{2}} = \left|\frac{a}{b}\right|^{\frac{1}{2}} |a|^{\frac{1}{2}} = \frac{1}{|b|}, \quad \text{for } d = diag(a, b, a) \in A^+(M^0) .$$

Since $\delta(M^0) = |b|$, and $\int_1^\infty \phi^p \delta(M^0) d|b|^\times = \int_1^\infty \frac{1}{|b|^{p-1}} d|b|^\times$ is convergent for $p > 1$, we have

Proposition 2.4 [G_2]. *Let ρ be a unitary representation of G_2 without trivial constituents, then $\rho|_{M^0}$ is strongly $L^{1+\varepsilon}$, for any $\varepsilon > 0$.*

Theorem 0.1 [G_2]. *Let ρ be a unitary representation of G_2 without trivial constituents, then ρ is strongly $L^{6+\varepsilon}$, for any $\varepsilon > 0$.*

Proof. We apply Theorem 3.1 for $H = H_1 \times H_2 = M^0 \times L$, and $k_1 = 1$, $k_2 = 1$.

Recall

$$A^+(G_2) = \{d : |\chi_{\varepsilon_2}|(d) \geqq 1,\ |\chi_{\varepsilon_1 - \varepsilon_2}|(d) \geqq 1\}$$
$$= \{d = diag(a_1, a_2, a_3) : |a_1| \geqq |a_2| \geqq 1,\ a_1 a_2 a_3 = 1\} .$$

On $A^+(G_2)$, we compute to get

$$\tilde{\delta} = \delta(M^0)^{\frac{1}{2}} \delta(L)^{\frac{1}{2}} = |\chi_{\varepsilon_2}|^{\frac{1}{2}} |\chi_\gamma|^{\frac{1}{2}} = \left(|a_2| \left|\frac{a_1}{a_3}\right|\right)^{\frac{1}{2}} = |a_1||a_2|$$

and

$$\delta(G_2) = |a_2| \left|\frac{a_1}{a_3}\right|^3 = |a_1|^6 |a_2|^4 .$$

Therefore,

$$\tilde{\delta}(d) \geqq \delta(G_2)^{\frac{1}{6}}(d), \quad \text{for } d \in A^+(G_2)\,.$$

Theorem 3.1 implies the desired result. □

4.2. $G = F_4$

Let F_4 be the split form of the exceptional Lie group of type F_4, $\mathfrak{f}_4$ be its Lie algebra. It is known that $Spin(4,4) \subseteq F_4$. Let $A \cong (F^\times)^4$ be a maximal torus of $Spin(4,4)$, $\mathfrak{a}$ be the corresponding Cartan subalgebra.

We have the covering map

$$\pi : Spin(4,4) \to SO(4,4)\,,$$

whose restriction to A is given by:

$$(a_1, a_2, a_3, a_4) \to (a_1^2 a_2 a_3 a_4, a_2, a_3, a_4), \qquad a_i \in F^\times\,.$$

$\bar{A} = \pi(A)$ is a maximal torus of $SO(4,4)$. Let $\chi_{\bar{\varepsilon}_i}$ be the following character of $\bar{A}$:

$$\chi_{\bar{\varepsilon}_i}(b_1, b_2, b_3, b_4) = b_i, \qquad 1 \leqq i \leqq 4\,.$$

For a character $\chi_{\bar{\alpha}}$ of $\bar{A}$, let χ_α be the character of A induced by π, namely, $\chi_\alpha = \chi_{\bar{\alpha}} \circ \pi|_A$.

Then with respect to $\mathfrak{a}$, $\mathfrak{f}_4$ has the root space decomposition:

$$\mathfrak{f}_4 = \mathfrak{a} \oplus \sum_{a \in \Phi} F g_\alpha\,, \tag{4.2.1}$$

where Φ is the following set of roots:

$$\begin{aligned} &\pm \varepsilon_i, \quad 1 \leqq i \leqq 4\,, \\ &\pm \varepsilon_i \pm \varepsilon_j, \quad 1 \leqq i < j \leqq 4\,, \\ &\frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)\,. \end{aligned} \tag{4.2.2}$$

A set Δ of simple roots can be taken as

$$\alpha_1 = \varepsilon_2 - \varepsilon_3, \qquad \alpha_2 = \varepsilon_3 - \varepsilon_4, \qquad \alpha_3 = \varepsilon_4, \qquad \alpha_4 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)\,.$$

The characters corresponding to the simple roots are

$$\chi_{\alpha_1}(d) = \frac{a_2}{a_3}, \qquad \chi_{\alpha_2}(d) = \frac{a_3}{a_4}, \qquad \chi_{\alpha_3}(d) = a_4, \qquad \chi_{\alpha_4}(d) = a_1\,,$$

for $d = (a_1, a_2, a_3, a_4) \in A$.

Remark. The inner product on $\mathfrak{a}^*$ induced by the Killing form is a constant multiple of the following

$$(\varepsilon_i, \varepsilon_j) = \delta_{ij}, \quad 1 \leqq i, j \leqq 4\,.$$

The highest root $\gamma = \varepsilon_1 + \varepsilon_2$. Notice that $\gamma - \alpha_1$ is still a root, i.e., $\tilde{\alpha} = \alpha_1$ (c.f. Lemma 1.1).

Consider the parabolic subgroup $P = P_{\Delta\setminus\{\alpha_1\}}$ of F_4 corresponding to the set of simple roots $\Delta\setminus\{\alpha_1\} = \{\alpha_2, \alpha_3, \alpha_4\}$, $P = MN$. $g_\alpha \in \mathfrak{m}$ if and only if α belongs to the following set of roots:

$$(4.2.3) \qquad \pm\varepsilon_3,\ \pm\varepsilon_4,\ \pm(\varepsilon_1 - \varepsilon_2),\ \pm(\varepsilon_3 \pm \varepsilon_4),\ \pm\frac{1}{2}(\varepsilon_1 - \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)\,.$$

Recall the maximal torus $A(M^0)$ of M^0, which consists of $d = (a_1, a_2, a_3, a_4)$ such that $a_1^2 a_2^2 a_3 a_4 = 1$. There is an isomorphism

$$A(M^0) \cong (F^\times)^3 : d \mapsto b = (b_1, b_2, b_3)$$

given by

$$\begin{cases} \dfrac{b_1}{b_2} = a_1 \\ \dfrac{b_2}{b_3} = a_4 \\ b_3 = a_1 a_2 a_3\,. \end{cases}$$

Let $\chi_{\beta_i}(b_1, b_2, b_3) = b_i$, $1 \leqq i \leqq 3$. Then the $A(M^0)$ roots of M^0 are

$$\frac{\chi_{\beta_i}}{\chi_{\beta_j}}, \quad 1 \leqq i \neq j \leqq 3\,,$$

$$\chi_{\beta_i}\chi_{\beta_j},\ \chi_{\beta_i}^{-1}\chi_{\beta_j}^{-1}, \quad 1 \leqq i \leqq j \leqq 3\,.$$

Thus we have $M^0 \cong Sp_6$.

$\mathfrak{n}$ is spanned by all positive root vectors g_α, where α belongs to the following set Q of positive roots:

$$A_1 = \varepsilon_2 - \varepsilon_3, \qquad B_1 = \varepsilon_1 + \varepsilon_3\,,$$

$$A_2 = \varepsilon_1 - \varepsilon_3, \qquad B_2 = \varepsilon_2 + \varepsilon_3\,,$$

$$A_3 = \varepsilon_2, \qquad B_3 = \varepsilon_1\,,$$

$$A_4 = \varepsilon_2 + \varepsilon_4, \qquad B_4 = \varepsilon_1 - \varepsilon_4\,,$$

$$A_5 = \varepsilon_2 - \varepsilon_4, \qquad B_5 = \varepsilon_1 + \varepsilon_4\,,$$

$$A_6 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4), \qquad B_6 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)\,,$$

$$A_7 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \varepsilon_4), \qquad B_7 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4)\,,$$

$$\gamma = \varepsilon_1 + \varepsilon_2\,.$$

It is a Heisenberg algebra with the center $Z = Fg_\gamma$.

M^0 acts on the symplectic vector space $W = \mathfrak{n}/Z$, which is of dimension 14. In fact, W is an irreducible representation of $M^0 \cong Sp_6$ with the highest weight

$\gamma - \tilde{\alpha} = \varepsilon_1 + \varepsilon_3 = \beta_1 + \beta_2 + \beta_3$. W is isomorphic to $H(\Lambda^3(F^6)) \subset \Lambda^3(F^6)$, the space of "$Sp_6$-harmonics" in $\Lambda^3(F^6)$. But we shall not use these facts.

The set Π specifying the complete polarization $W = V \oplus V^*$ can be taken as A_i, $1 \leqq i \leqq 7$.

Recall the positive Weyl chamber $A^+(M^0)$, which consists of $d = (a_1, a_2, a_3, a_4)$, with $a_1^2 a_2^2 a_3 a_4 = 1$, and $|a_3| \geqq |a_4| \geqq 1$, $|a_1| \geqq 1$.

It is straightforward to verify that except $|\chi_{A_2}| = |a_1^2 a_2 a_4| = |a_1| |\frac{a_4}{a_3}|^{\frac{1}{2}}$, all the other $|\chi_{A_i}|$ is less than or equal to one on $A^+(M^0)$.

Thus given any unitary representation ρ of F_4 not containing the trivial representation, its matrix coefficients are dominated by

$$\begin{aligned}
\phi(d) &= \left(\left|\frac{a_2}{a_3}\right| |a_2||a_2 a_4| \left|\frac{a_2}{a_4}\right| |a_1 a_2||a_1 a_2 a_4| \right)^{\frac{1}{2}} \\
&\quad \cdot max\left(|a_1| \left|\frac{a_4}{a_3}\right|^{\frac{1}{2}}, \frac{1}{|a_1| \left|\frac{a_4}{a_3}\right|^{\frac{1}{2}}} \right)^{-\frac{1}{2}} \\
&= |a_1||a_2|^3 \left|\frac{a_4}{a_3}\right|^{\frac{1}{2}} \cdot max\left(|a_1| \left|\frac{a_4}{a_3}\right|^{\frac{1}{2}}, \frac{1}{|a_1| \left|\frac{a_4}{a_3}\right|^{\frac{1}{2}}} \right)^{-\frac{1}{2}} \\
&= \frac{1}{|a_1|^2 |a_3|^2 |a_4|} \cdot max\left(|a_1| \left|\frac{a_4}{a_3}\right|^{\frac{1}{2}}, \frac{1}{|a_1| \left|\frac{a_4}{a_3}\right|^{\frac{1}{2}}} \right)^{-\frac{1}{2}},
\end{aligned}$$

for $d = diag(a_1, a_2, a_3, a_4) \in A^+(M^0)$. We have

$$\phi(d) = \begin{cases} \frac{1}{|a_1|^{\frac{3}{2}} |a_3|^{\frac{9}{4}} |a_4|^{\frac{3}{4}}}, & |a_1| \leqq \left|\frac{a_3}{a_4}\right|^{\frac{1}{2}}, \\ \frac{1}{|a_1|^{\frac{5}{2}} |a_3|^{\frac{7}{4}} |a_4|^{\frac{5}{4}}}, & |a_1| \geqq \left|\frac{a_3}{a_4}\right|^{\frac{1}{2}}. \end{cases}$$

The modular function of $B(M^0)$ is computed to be

$$\delta(M^0) = |b_1|^6 |b_2|^4 |b_3|^2 = |a_1|^6 |a_3|^6 |a_4|^4 .$$

Let $D = \{(a_1, a_3, a_4) : |a_1| \geqq 1, |a_3| \geqq |a_4| \geqq 1\}$. We have $D = D_1 \cup D_2$, where

$$\begin{aligned}
D_1 &= D \cap \left\{ (a_1, a_3, a_4) : |a_1| \leqq \left|\frac{a_3}{a_4}\right|^{\frac{1}{2}} \right\} \\
&= \{(a_1, a_3, a_4) : |a_3| \geqq |a_1|^2 |a_4|, |a_1| \geqq 1, |a_4| \geqq 1\}
\end{aligned}$$

and

$$D_2 = D \cap \left\{ (a_1, a_3, a_4) : |a_1| \geqq \left| \frac{a_3}{a_4} \right|^{\frac{1}{2}} \right\}$$

$$= \{(a_1, a_3, a_4) : |a_1| \geqq |a_3|^{\frac{1}{2}} |a_4|^{-\frac{1}{2}}, |a_3| \geqq |a_4| \geqq 1\} .$$

Let

$$I_1 = \int\int\int_{D_1} \phi^p \delta_{B(M^0)} d|a_1|^\times d|a_3|^\times d|a_4|^\times$$

and

$$I_2 = \int\int\int_{D_2} \phi^p \delta_{B(M^0)} d|a_1|^\times d|a_3|^\times d|a_4|^\times .$$

Using the explicit formulas of ϕ in D_1 and D_2, we see that I_1 and I_2 are both convergent integrals if and only if $p > \frac{10}{3}$. Therefore

Proposition 2.4 [F_4]. *Let ρ be a unitary representation of F_4 without trivial constituents, then $\rho|_{M^0}$ is strongly $L^{\frac{10}{3}+\varepsilon}$, for any $\varepsilon > 0$.*

Remark. One can improve the above result as follows: We have

$$\rho|_{M^0} \cong \sum_{t \in F^\times/(F^\times)^2} \omega_t|_{\tilde{M}^0} \otimes \mu_t .$$

Since

$$\prod_{\beta \in \Pi} \chi_\beta(d) = \frac{a_1^4 a_2^7 a_4^2}{a_3} \quad (a_1^2 a_2^2 a_3 a_4 = 1)$$

can take values which are not in $(F^\times)^2$, we see ω_t is a genuine representation of $\tilde{M}^0$, and thus μ_t is also a genuine representation of $\tilde{M}^0 \cong \tilde{S}p_6$. As such, μ_t is strongly $L^{12+\varepsilon}$ ([Ho]). In view of the Hölder's inequality, we conclude $\rho|_{M^0}$ is strongly $L^{r+\varepsilon}$, where $\frac{1}{r} = \frac{1}{\frac{10}{3}} + \frac{1}{12}$, i.e., $r = \frac{60}{23}$.

Theorem 0.1 [F_4]. *Let ρ be a unitary representation of F_4 without trivial constituents, then ρ is strongly $L^{\frac{44}{5}+\varepsilon}$, for any $\varepsilon > 0$.*

Proof. Let w be the following Weyl group element:

$$w : \varepsilon_1 \mapsto \varepsilon_1, \ \varepsilon_2 \mapsto -\varepsilon_4, \ \varepsilon_3 \mapsto \varepsilon_3, \ \varepsilon_4 \mapsto -\varepsilon_2 .$$

Let H_1 (resp. H_2) be the w conjugate of M^0 (resp. L).

We apply Theorem 3.1 to the subgroup $H = H_1 \times H_2$, and $k_1 = 2$, $k_2 = 1$.

The positive roots of H_1 are

$$\varepsilon_3, \varepsilon_2, \varepsilon_1 + \varepsilon_4, \varepsilon_2 \pm \varepsilon_3, \frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 + \varepsilon_4) .$$

Thus, $\delta(H_1) = |\chi_{3\varepsilon_1+3\varepsilon_2+\varepsilon_3+3\varepsilon_4}|$.

We also have $\delta(H_2) = |\chi_{\varepsilon_1-\varepsilon_4}|$.

Let $\tilde{\delta} = \delta(H_1)^{\frac{1}{4}}\delta(H_2)^{\frac{1}{2}} = \left|\chi_{\frac{5}{4}\varepsilon_1+\frac{3}{4}\varepsilon_2+\frac{1}{4}\varepsilon_3+\frac{1}{4}\varepsilon_4}\right| = |a_1|^{\frac{5}{2}}|a_2|^{\frac{4}{2}}|a_3|^{\frac{3}{2}}|a_4|^{\frac{3}{2}}$.
We compute to get

$$\delta(F_4) = \left|\chi_{11\varepsilon_1+5\varepsilon_2+3\varepsilon_3+\varepsilon_4}\right| = |a_1|^{22}|a_2|^{16}|a_3|^{14}|a_4|^{12} .$$

Recall the positive Weyl Chamber of F_4:

$$A^+(F_4) = \{d = (a_1, a_2, a_3, a_4) : |a_1| \geqq 1,\ |a_2| \geqq |a_3| \geqq |a_4| \geqq 1\} .$$

We have

$$\tilde{\delta}(d) \geqq \delta(F_4)^{\frac{5}{44}}(d), \qquad \text{for } d \in A^+(F_4) .$$

Theorem 3.1 implies the desired result. □

Because of the fact that $E_6 \hookrightarrow E_7 \hookrightarrow E_8$, we shall arrange the order of remaining discussions as follows.

4.3. $G = E_8$

Let E_8 be the split form of the exceptional Lie group of type E_8, $\mathfrak{e}_8$ be its Lie algebra. It is known that $\mathfrak{e}_8$ has the following realization [HS]:

$$\mathfrak{e}_8 = \mathfrak{sl}_9(F) \oplus \Lambda^3 F^9 \oplus (\Lambda^3 F^9)^* . \tag{4.3.1}$$

Let $\mathfrak{a}$ be the Cartan subalgebra of $\mathfrak{sl}_9$ consisting of diagonal matrices of trace zero, A be the corresponding maximal torus. For $d = diag(a_1, a_2, \ldots, a_8, a_9) \in \mathfrak{a}$, let $\varepsilon_i \in \mathfrak{a}^*$ be $\varepsilon_i(d) = a_i$. Thus we always have $\sum_{i=1}^{9} \varepsilon_i = 0$. With respect to $\mathfrak{a}$, the roots of the decomposition (4.3.1) are:

$$\begin{aligned} &\pm(\varepsilon_i - \varepsilon_j), \quad 1 \leqq i < j \leqq 9 , \\ &\pm(\varepsilon_i + \varepsilon_j + \varepsilon_k), \quad 1 \leqq i < j < k \leqq 9 . \end{aligned} \tag{4.3.2}$$

A set Δ of simple roots can be taken as

$$\alpha_1 = \varepsilon_7 - \varepsilon_8, \quad \alpha_2 = \varepsilon_6 + \varepsilon_7 + \varepsilon_8, \quad \alpha_3 = \varepsilon_6 - \varepsilon_7, \quad \alpha_4 = \varepsilon_5 - \varepsilon_6 ,$$

$$\alpha_5 = \varepsilon_4 - \varepsilon_5, \quad \alpha_6 = \varepsilon_3 - \varepsilon_4, \quad \alpha_7 = \varepsilon_2 - \varepsilon_3, \quad \alpha_8 = \varepsilon_1 - \varepsilon_2 .$$

Here we number the simple roots according to the usual notation of [He].

Remark. The inner product on $\mathfrak{a}^*$ induced by the Killing form is a constant multiple of the following

$$(\varepsilon_i, \bar{\varepsilon}_i) = \frac{8}{9}, \qquad (\varepsilon_i, \varepsilon_j) = -\frac{1}{9}, \quad 1 \leqq i < j \leqq 8 .$$

For $\varepsilon_9 = -\sum_{i=1}^{8} \varepsilon_i$, we also have $(\varepsilon_9, \varepsilon_9) = \frac{8}{9}$, $(\varepsilon_i, \varepsilon_9) = -\frac{1}{9}$, for $1 \leqq i \leqq 8$.

The highest root $\gamma = \varepsilon_1 - \varepsilon_9 = \varepsilon_1 + (\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_8)$, and out of all simple roots α_i, only $\gamma - \alpha_8$ is still a root, i.e., $\tilde{\alpha} = \alpha_8$ (c.f. Lemma 1.1).

Consider the parabolic subgroup $P = P_{\Delta\setminus\{\alpha_8\}}$ of E_8 corresponding to the set of simple roots $\Delta\setminus\{\alpha_8\}$, $P = MN$. A_s, the split part of M, is the subgroup of A annihilated by $\Delta\setminus\{\alpha_8\} = \{\alpha_1, \alpha_2, \ldots, \alpha_7\}$. Thus, it consists of

$d = diag(a_1, 1, \ldots, 1, a_9)$ such that $a_1 a_9 = 1$. $g_\alpha \in \mathfrak{m}$ if and only if $\alpha|_{\mathfrak{a}_s} = 0$, thus it is one of the following roots:

$$\begin{aligned} &\pm(\varepsilon_i - \varepsilon_j), \quad 2 \leqq i < j \leqq 8, \\ &\pm(\varepsilon_i + \varepsilon_j + \varepsilon_k), \quad 2 \leqq i < j < k \leqq 8, \\ &\mp(\varepsilon_1 + \varepsilon_i + \varepsilon_9), \quad 2 \leqq i \leqq 8. \end{aligned} \tag{4.3.3}$$

From [He], we have $M^0 \cong E_7$, and the simple roots of M^0 are α_i, $1 \leqq i \leqq 7$.

$\mathfrak{n}$ is spanned by all positive root vectors g_α such $\alpha|_{\mathfrak{a}_s} \neq 0$. Thus $\mathfrak{n} = \sum_{\alpha \in Q} F g_\alpha$, where Q is the following set of positive roots:

$$\begin{aligned} &A_i = \varepsilon_1 - \varepsilon_i, \qquad B_i = \varepsilon_i - \varepsilon_9, \quad 2 \leqq i \leqq 8, \\ &C_{ij} = \varepsilon_1 + \varepsilon_i + \varepsilon_j, \qquad D_{ij} = -(\varepsilon_i + \varepsilon_j + \varepsilon_9), \quad 2 \leqq i < j \leqq 8, \\ &\gamma = \varepsilon_1 - \varepsilon_9 . \end{aligned}$$

It is a Heisenberg algebra with the center $Z = Fg_\gamma$.

M^0 acts on the symplectic vector space $W = \mathfrak{n}/Z$, which is of dimension $2[\binom{7}{1} + \binom{7}{2}] = 56$. In fact, W is an irreducible representation of M^0 with the highest weight $\gamma - \tilde{\alpha} = \varepsilon_2 - \varepsilon_9$.

Recall the positive Weyl chamber $A^+(M^0)$:

$$\begin{aligned} &A^+(M^0) \\ &= \{d \in A(M^0) : |\chi_{\alpha_i}|(d) \geqq 1, \ 1 \leqq i \leqq 7\} \\ &= \{d = (t, a_2, \ldots, a_8, t) : t^2 a_2 \ldots a_8 = 1, \ |a_2| \geqq \cdots \geqq |a_8|, \ |a_6 a_7 a_8| \geqq 1\}. \end{aligned}$$

Given any unitary representation ρ of E_8 not containing the trivial representation, its matrix coefficients are dominated by

$$\phi(d) = \prod_{2 \leqq i \leqq 8} max\left(\left|\frac{t}{a_i}\right|, \left|\frac{a_i}{t}\right|\right)^{-\frac{1}{2}} \prod_{2 \leqq i < j \leqq 8} max\left(|ta_i a_j|, \left|\frac{1}{ta_i a_j}\right|\right)^{-\frac{1}{2}},$$

for $d = diag(t, a_2, a_3, \ldots, a_8, t) \in A^+(M^0)$.

A simple computation gives

$$\begin{aligned} \phi(d) = \Bigg\{ &\left(\frac{|a_2|}{|t|^{21}|a_3|^3|a_4|^3|a_5|^4|a_6|^4|a_7|^4|a_8|^4}\right) \\ &\times max\left(|ta_3 a_4|, \frac{1}{|ta_3 a_4|}\right) \prod_{3 \leqq j \leqq 8} max\left(|ta_2 a_j|, \frac{1}{|ta_2 a_j|}\right) \Bigg\}^{-\frac{1}{2}}. \end{aligned}$$

Let $D = \{(a_2, a_3, \ldots, a_8) : |a_2| \geqq |a_3| \geqq \cdots \geqq |a_8|, |a_6 a_7 a_8| \geqq 1\}$.

We have $D = \bigcup_{i=2}^{8} D_i$, where

$$\begin{aligned} &D_2 = D \cap \{|ta_3 a_4| \geqq 1\}, \\ &D_i = D \cap \{|ta_3 a_4| < 1, \ |ta_2 a_i| \geqq 1, \ |ta_2 a_{i+1}| < 1\}, \quad 3 \leqq i \leqq 7, \\ &D_8 = D \cap \{|ta_3 a_4| < 1, \ |ta_2 a_8| \geqq 1\}, \end{aligned}$$

and

$$\phi|_{D_2} = \frac{1}{|a_2a_3a_4|^5|a_5a_6a_7a_8|^3},$$

$$\phi|_{D_i} = \frac{|a_{i+1}|\dots|a_8|}{|a_2|^{i-3}|a_2\dots a_8|^{\frac{13-i}{2}}}, \quad 3 \leqq i \leqq 7,$$

$$\phi|_{D_8} = \frac{1}{|a_2|^{\frac{15}{2}}|a_3a_4a_5a_6a_7a_8|^{\frac{5}{2}}}.$$

We compute the modular function $\delta(M^0)$ of $B(M^0)$:

$$\begin{aligned}\delta(M^0) &= \prod_{2\leqq i<j\leqq 8}|\chi_{\varepsilon_i-\varepsilon_j}| \prod_{2\leqq i<j<k\leqq 8}|\chi_{\varepsilon_i+\varepsilon_j+\varepsilon_k}| \prod_{2\leqq i\leqq 8}|\chi_{-(\varepsilon_1+\varepsilon_i+\varepsilon_9)}| \\ &= \prod_{2\leqq i<j\leqq 8}\left|\frac{a_i}{a_j}\right| \prod_{2\leqq i<j<k\leqq 8}|a_ia_ja_k| \prod_{2\leqq i\leqq 8}\left|\frac{1}{a_1a_ia_9}\right| \\ &= \frac{\prod_{i=2}^8|a_i|^{24-2i}}{|a_1a_9|^7}.\end{aligned}$$

c.f. (4.3.3). Thus, $\delta(M^0)|_{A^+(M^0)} = \dfrac{\prod_{i=2}^8|a_i|^{24-2i}}{|t^2|^7} = \prod_{i=2}^8|a_i|^{31-2i}$.

Let

$$I = \int_D \phi^p\delta(M^0)d|a_2|^\times d|a_3|^\times\dots d|a_8|^\times.$$

Under the following change of variables,

$$b_8 = a_6a_7a_8, \qquad b_i = \frac{a_i}{a_{i+1}}, \quad 2 \leqq i \leqq 7,$$

the region D goes into $E = \{(b_2,\dots,b_8) : |b_i| \geqq 1\}$. Using the explicit formulas of ϕ in D_i, and

$$a_8 = \left(\frac{b_8}{b_6b_7^2}\right)^{\frac{1}{3}}, \qquad a_i = b_i\dots b_7a_8, \quad 2 \leqq i \leqq 7,$$

we see that I is convergent if and only if $p > \dfrac{17}{3}$. Therefore we have

Proposition 2.4 [E_8]. *Let ρ be a unitary representation of E_8 without trivial constituents, then $\rho|_{M^0}$ is strongly $L^{\frac{17}{3}+\varepsilon}$, for any $\varepsilon > 0$.*

Theorem 0.1 [E_8]. *Let ρ be a unitary representation of E_8 without trivial constituents, then ρ is strongly $L^{\frac{810}{77}+\varepsilon}$, for any $\varepsilon > 0$.*

Proof. Let w be the following Weyl group element:

$$\begin{aligned}w :&\varepsilon_1 \mapsto \varepsilon_4,\ \varepsilon_2 \mapsto \varepsilon_1,\ \varepsilon_3 \mapsto \varepsilon_2,\ \varepsilon_4 \mapsto \varepsilon_3,\ \varepsilon_5 \mapsto \varepsilon_5,\ \varepsilon_6 \mapsto \varepsilon_6,\\ &\varepsilon_7 \mapsto \varepsilon_8,\ \varepsilon_8 \mapsto \varepsilon_9,\ \varepsilon_9 \mapsto \varepsilon_7.\end{aligned}$$

Let H_1 (resp. H_2) be the w conjugate of M^0 (resp. L).

We apply Theorem 3.1 for $H = H_1 \times H_2$, and $k_1 = 3$, $k_2 = 1$.
The roots belonging to H_1 are

$$\pm(\varepsilon_i - \varepsilon_j), \quad i < j,$$
$$\pm(\varepsilon_i + \varepsilon_j + \varepsilon_k), \quad i < j < k,$$
$$\pm(\varepsilon_4 + \varepsilon_7 + \varepsilon_i),$$

where $i,j,k = 1,2,3,5,6,8,9$. Thus by a straightforward computation,

$$\delta(H_1) = |a_1|^{34}|a_2|^{32}|a_3|^{30}|a_4|^{27}|a_5|^{28}|a_6|^{26}|a_7|^{27}|a_8|^{24}.$$

We also have $\delta(H_2) = |\chi_{\varepsilon_4-\varepsilon_7}| = |\frac{a_4}{a_7}|$.
Let $\tilde{\delta} = \delta(H_1)^{\frac{1}{6}}\delta(H_2)^{\frac{1}{2}} = |a_1|^{\frac{17}{3}}|a_2|^{\frac{16}{3}}|a_3|^{\frac{15}{3}}|a_4|^{\frac{15}{3}}|a_5|^{\frac{14}{3}}|a_6|^{\frac{13}{3}}|a_7|^{\frac{12}{3}}|a_8|^{\frac{12}{3}}$.
Using (4.3.2), we compute to get

$$\delta(E_8) = |a_1|^{58}|a_2|^{56}|a_3|^{54}|a_4|^{52}|a_5|^{50}|a_6|^{48}|a_7|^{46}|a_8|^{44}.$$

Recall the positive Weyl Chamber of E_8:

$$A^+(E_8) = \{d \in A : |\chi_{\alpha_i}|(d) \geqq 1,\ 1 \leqq i \leqq 8\}$$
$$= \{d = (a_1,\ldots,a_9) : a_1 \ldots a_9 = 1,\ |a_1| \geqq \cdots \geqq |a_8|,\ |a_6a_7a_8| \geqq 1\}.$$

We have (via an elementary comparison)

$$\tilde{\delta}(d) \geqq (E_8)^{\frac{77}{810}}(d), \quad \text{for } d \in A^+(E_8).$$

Note that $\frac{77}{810} = \frac{\frac{17}{3}+\frac{16}{3}+\frac{15}{3}+\frac{15}{3}+\frac{14}{3}}{58+56+54+52+50}$.
Theorem 3.1 implies the desired result. □

4.4. $G = E_7$

Let E_7 be the split form of the exceptional Lie group of type E_7, $\mathfrak{e}_7$ be its Lie algebra. We know that $E_7 \hookrightarrow E_8$, and the roots of $\mathfrak{e}_7$ consist of the following:

$$\pm(\varepsilon_i - \varepsilon_j), \quad 2 \leqq i < j \leqq 8,$$
$$\pm(\varepsilon_i + \varepsilon_j + \varepsilon_k), \quad 2 \leqq i < j < k \leqq 8, \tag{4.4.1}$$
$$\mp(\varepsilon_1 + \varepsilon_i + \varepsilon_9), \quad 2 \leqq i \leqq 8.$$

A set Δ of simple roots can be taken as

$$\alpha_1 = \varepsilon_7 - \varepsilon_8, \quad \alpha_2 = \varepsilon_6 + \varepsilon_7 + \varepsilon_8, \quad \alpha_3 = \varepsilon_6 - \varepsilon_7, \quad \alpha_4 = \varepsilon_5 - \varepsilon_6,$$
$$\alpha_5 = \varepsilon_4 - \varepsilon_5, \quad \alpha_6 = \varepsilon_3 - \varepsilon_4, \quad \alpha_7 = \varepsilon_2 - \varepsilon_3.$$

The highest root $\gamma = -(\varepsilon_1 + \varepsilon_8 + \varepsilon_9) = \varepsilon_2 + \cdots + \varepsilon_7$, and out of all simple roots α_i, $1 \leqq i \leqq 7$, only $\gamma - \alpha_1$ is still a root, i.e., $\tilde{\alpha} = \alpha_1$ (c.f. Lemma 1.1).

Consider the parabolic subgroup $P = P_{\Delta\setminus\{\alpha_1\}}$ of E_7 corresponding to the set of simple roots $\Delta\setminus\{\alpha_1\}$, $P = MN$. Then $g_\alpha \in \mathfrak{m}$ if and only if α is one of

the following roots:

$$(4.4.2) \qquad \pm(\varepsilon_i - \varepsilon_j),\ \pm(\varepsilon_i + \varepsilon_j + \varepsilon_8),\quad 2 \leqq i < j \leqq 7\,.$$

A set of simple roots for M^0 can be taken as $\alpha_i, 2 \leqq i \leqq 7$.

$\mathfrak{n}$ is spanned by all positive root vectors g_α, where α belongs to the following set Q of positive roots:

$$\begin{aligned} A_i &= \varepsilon_i - \varepsilon_8, \qquad B_i = -(\varepsilon_1 + \varepsilon_i + \varepsilon_9), \quad 2 \leqq i \leqq 7, \\ C_{ijk} &= \varepsilon_i + \varepsilon_j + \varepsilon_k, \quad 2 \leqq i < j < k \leqq 7, \\ \gamma &= -(\varepsilon_1 + \varepsilon_8 + \varepsilon_9)\,. \end{aligned}$$

It is a Heisenberg algebra with the center $Z = Fg_\gamma$.

M^0 acts on the symplectic vector space $W = \mathfrak{n}/Z$, which is of dimension $2\binom{6}{1} + \binom{6}{3} = 32$. In fact, W is an irreducible representation of M^0 with the highest weight $\gamma - \tilde{\alpha} = -(\varepsilon_1 + \varepsilon_7 + \varepsilon_9)$.

The positive Weyl chamber $A^+(M^0)$ consists of $d = (t, a_2, a_3, \ldots, a_7, a_8, t)$ with $t^2 a_8 = 1, a_2 a_3 \ldots a_7 = 1$, and $|a_2| \geqq |a_3| \geqq \cdots \geqq |a_7|, |a_6 a_7 a_8| \geqq 1$.

Given any unitary representation ρ of E_7 not containing the trivial representation, its matrix coefficients are dominated by

$$\phi(d) = \prod_{2 \leqq i \leqq 7} max\left(\left|\frac{a_i}{a_8}\right|, \left|\frac{a_8}{a_i}\right|\right)^{-\frac{1}{2}} \prod_{2 \leqq i<j<k \leqq 7}^{(i,j,k) \in \Pi} max\left(|a_i a_j a_k|, \left|\frac{1}{a_i a_j a_k}\right|\right)^{-\frac{1}{2}},$$

for $d = diag(t, a_2, a_3, \ldots, a_7, a_8, t) \in A^+(M^0)$. Here Π is a set of (i,j,k) such that $\Pi \cup \Pi^* = \{2,3,4,5,6,7\}$.

We have

$$\begin{aligned} \phi(d) \leqq \phi_1(d) &= \left\{ \left(\prod_{2 \leqq i \leqq 7} \left|\frac{a_8}{a_i}\right| \right) |a_2 a_3 a_4||a_2 a_3 a_5||a_2 a_3 a_6||a_2 a_3 a_7| \right. \\ &\qquad \left. \times |a_2 a_4 a_5||a_2 a_4 a_6||a_2 a_4 a_7||a_3 a_4 a_5||a_3 a_4 a_6||a_3 a_4 a_7| \right\}^{-\frac{1}{2}} \\ &= \frac{1}{|a_8|^3 |a_2 a_3 a_4|^2}\,. \end{aligned}$$

We compute the modular function of $B(M^0)$:

$$\begin{aligned} \delta(M^0) &= \prod_{2 \leqq i<j \leqq 7} |\chi_{\varepsilon_i - \varepsilon_j}| \prod_{2 \leqq i<j \leqq 7} |\chi_{\varepsilon_i + \varepsilon_j + \varepsilon_8}| \\ &= \prod_{2 \leqq i<j \leqq 7} \left|\frac{a_i}{a_j}\right| \prod_{2 \leqq i<j \leqq 7} |a_i a_j a_8| \\ &= |a_8|^{15} \prod_{i=2}^{7} |a_i|^{14-2i}\,. \end{aligned}$$

Let $D = \{(a_2, a_3, \ldots, a_6, a_8) : |a_2| \geqq |a_3| \geqq \cdots \geqq |a_7|, a_2 a_3 \ldots a_7 = 1, |a_6 a_7 a_8| \geqq 1\}$.

Let

$$I = \int_D \phi^p \delta(M^0) d|a_2|^\times d|a_3|^\times \dots d|a_6|^\times d|a_8|^\times$$

and

$$I_1 = \int_D \phi_1^p \delta(M^0) d|a_2|^\times d|a_3|^\times \dots d|a_6|^\times d|a_8|^\times .$$

Using the following change of variables, $b_i = \dfrac{a_i}{a_{i+1}}$, $2 \leqq i \leqq 6$, it is not difficult to see that I_1 is convergent if and only if $p > 5$.

On the other hand, let $D_1 = D \cap \{|a_8| \geqq |a_2|\}$. We have

$$\phi|_{D_1} = \frac{1}{|a_8|^3} \psi(a_2, \dots, a_7)$$

for some function ψ. Thus $\int_{D_1} \phi^p \delta_{B(M^0)} d|a_2|^\times d|a_3|^\times \dots d|a_6|^\times d|a_8|^\times$ is not convergent unless $p > 5$.

Thus, I is a convergent integral if and only if $p > 5$. This implies

Proposition 2.4 [E_7]. *Let ρ be a unitary representation of E_7 without trivial constituents, then $\rho|_{M^0}$ is strongly $L^{5+\varepsilon}$, for any $\varepsilon > 0$.*

Theorem 0.1 [E_7]. *Let ρ be a unitary representation of E_7 without trivial constituents, then ρ is strongly $L^{14+\varepsilon}$, for any $\varepsilon > 0$.*

Proof. Let w be the following Weyl group element:

$$w : \varepsilon_1 \mapsto \varepsilon_1, \ \varepsilon_2 \mapsto \varepsilon_3, \dots, \varepsilon_7 \mapsto \varepsilon_8, \ \varepsilon_8 \mapsto \varepsilon_2, \ \varepsilon_9 \mapsto \varepsilon_9 .$$

Let H_1 (resp. H_2) be the w conjugate of M^0 (resp. L).

We apply Theorem 3.1 to the subgroup $H = H_1 \times H_2$, and $k_1 = 3$, $k_2 = 1$.

The roots of H_1 are

$$\pm(\varepsilon_i - \varepsilon_j), \quad 3 \leqq i < j \leqq 8 ,$$
$$\pm(\varepsilon_i + \varepsilon_j + \varepsilon_2), \quad 3 \leqq i < j \leqq 8 .$$

Thus by a straightforward computation,

$$\delta(H_1) = |a_2|^{15}|a_3|^{10}|a_4|^8|a_5|^6|a_6|^4|a_7|^2 .$$

We also have $\delta(H_2) = |\chi_{-(\varepsilon_1+\varepsilon_2+\varepsilon_9)}| = |a_3\, a_4\, a_5\, a_6\, a_7\, a_8|$.

Let $\tilde{\delta} = \delta(H_1)^{\frac{1}{6}} \delta(H_2)^{\frac{1}{2}} = |a_2|^{\frac{15}{6}} |a_3|^{\frac{13}{6}} |a_4|^{\frac{11}{6}} |a_5|^{\frac{9}{6}} |a_6|^{\frac{7}{6}} |a_7|^{\frac{5}{6}} |a_8|^{\frac{3}{6}}$.

Recall the positive Weyl Chamber of E_7:

$$A^+(E_7) = \{d = (t, a_2, \dots, a_8, t) : t^2 a_2 \dots a_8 = 1,$$
$$|a_2| \geqq \dots \geqq |a_8|, |a_6 a_7 a_8| \geqq 1\} .$$

Since

$$\delta(E_7) = |a_2|^{27}|a_3|^{25}|a_4|^{23}|a_5|^{21}|a_6|^{19}|a_7|^{17}|a_8|^{15} ,$$

we have (via an elementary comparison)

$$\tilde{\delta}(d) \geqq \delta(E_7)^{\frac{1}{14}}(d) \quad \text{for } d \in A^+(E_7)\,.$$

Theorem 3.1 implies the desired result. □

4.5. $G = E_6$

Let E_6 be the split form of the exceptional Lie group of type E_6, $\mathfrak{e}_6$ be its Lie algebra. It is known that $E_6 \hookrightarrow E_7 \hookrightarrow E_8$, and the roots of $\mathfrak{e}_6$ consist of the following:

$$\begin{aligned} &\pm(\varepsilon_i - \varepsilon_j), \quad 3 \leqq i < j \leqq 8\,, \\ &\pm(\varepsilon_i + \varepsilon_j + \varepsilon_k), \quad 3 \leqq i < j < k \leqq 8\,, \\ &\mp(\varepsilon_1 + \varepsilon_2 + \varepsilon_9)\,. \end{aligned} \tag{4.5.1}$$

A set Δ of simple roots can be taken as

$$\alpha_1 = \varepsilon_7 - \varepsilon_8, \quad \alpha_2 = \varepsilon_6 + \varepsilon_7 + \varepsilon_8, \quad \alpha_3 = \varepsilon_6 - \varepsilon_7, \quad \alpha_4 = \varepsilon_5 - \varepsilon_6\,,$$
$$\alpha_5 = \varepsilon_4 - \varepsilon_5, \quad \alpha_6 = \varepsilon_3 - \varepsilon_4\,.$$

The highest root $\gamma = -(\varepsilon_1 + \varepsilon_2 + \varepsilon_9) = \varepsilon_3 + \cdots + \varepsilon_8$, and out of all simple roots $\alpha_i, 1 \leqq i \leqq 6$, only $\gamma - \alpha_2$ is still a root, i.e., $\tilde{\alpha} = \alpha_2$ (cf. Lemma 1.1).

Consider the parabolic subgroup $P = P_{\Delta\setminus\{\alpha_2\}}$ of E_6 corresponding to the set of simple roots $\Delta\setminus\{\alpha_2\}, P = MN$. Then $g_\alpha \in \mathfrak{m}$ if and only if α is one of the following roots:

$$\pm(\varepsilon_i - \varepsilon_j), \quad 3 \leqq i < j \leqq 8\,. \tag{4.5.2}$$

A set of simple roots for M^0 can be taken as α_i, $3 \leqq i \leqq 7$. We have $M^0 \cong SL_6$.

$\mathfrak{n}$ is spanned by all positive root vectors g_α, where α belongs to the following set Q of positive roots:

$$\begin{aligned} &C_{ijk} = \varepsilon_i + \varepsilon_j + \varepsilon_k, \quad 3 \leqq i < j < k \leqq 8\,, \\ &\gamma = -(\varepsilon_1 + \varepsilon_8 + \varepsilon_9)\,. \end{aligned}$$

It is a Heisenberg algebra with the center $Z = Fg_\gamma$.

M^0 acts on the symplectic vector space $W = \mathfrak{n}/Z$, which is of dimension $\binom{6}{3} = 20$. In fact, W is an irreducible representation of M^0 with the highest weight $\gamma - \tilde{\alpha} = \varepsilon_3 + \varepsilon_4 + \varepsilon_5$.

The positive Weyl chamber $A^+(M^0)$ consists of $d = (1, 1, a_3, \ldots, a_7, a_8, 1)$, with $a_3 a_4 \ldots a_8 = 1$, and $|a_3| \geqq |a_4| \geqq \cdots \geqq |a_8|, |a_6 a_7 a_8| \geqq 1$.

Given any unitary representation ρ of E_6 not containing the trivial representation, its matrix coefficients are dominated by

$$\phi(d) = \prod_{3 \leqq i<j<k \leqq 8}^{(i,j,k) \in \Pi} max\left(|a_i a_j a_k|, \left|\frac{1}{a_i a_j a_k}\right|\right)^{-\frac{1}{2}}\,,$$

for $d = diag(1, 1, a_3, \ldots, a_7, a_8, 1) \in A^+(M^0)$. Here Π is a set of (i, j, k) such that $\Pi \cup \Pi^* = \{3, 4, 5, 6, 7, 8\}$.

Let $D = \{(a_3, \ldots, a_6, a_7) : |a_3| \geqq \cdots \geqq |a_7| \geqq |a_8|, a_3 \ldots a_7 a_8 = 1\}$.

We have $D = \bigcup_{i=1}^{6} D_i$, where

$$D_1 = D \cap \{|a_4 a_5 a_8| \geqq 1\}, \qquad D_2 = D \cap \{|a_4 a_5 a_6| \leqq 1\},$$

$$D_3 = D \cap \{|a_4 a_5 a_6| > 1, |a_4 a_5 a_7| \leqq 1\},$$

$$D_4 = D \cap \{|a_4 a_5 a_7| > 1, |a_4 a_5 a_8| \leqq 1, |a_3 a_5 a_8| > 1\},$$

$$D_5 = D \cap \{|a_3 a_5 a_8| \leqq 1, |a_3 a_4 a_8| > 1\}, \qquad D_6 = D \cap \{|a_3 a_4 a_8| \leqq 1\},$$

and

$$\phi|_{D_1} = \frac{1}{|a_3|^2 |a_4|^2 |a_5|^2}, \qquad \phi|_{D_2} = \frac{1}{|a_3|^3},$$

$$\phi|_{D_3} = \frac{1}{|a_3|^3 |a_4| |a_5| |a_6|}, \qquad \phi|_{D_4} = \frac{1}{|a_3|^3 |a_4|^2 |a_5|^2 |a_6| |a_7|},$$

$$\phi|_{D_5} = \frac{1}{|a_3|^3 |a_4|^3 |a_5|^2 |a_6|^2 |a_7|^2}, \qquad \phi|_{D_6} = \frac{1}{|a_3|^3 |a_4|^3 |a_5|^3 |a_6|^3 |a_7|^3}.$$

Recall the modular function of $B(M^0)$:

$$\delta(M^0) = \prod_{3 \leqq i < j \leqq 8} |\chi_{\varepsilon_i - \varepsilon_j}| = \prod_{3 \leqq i < j \leqq 8} \left| \frac{a_i}{a_j} \right| = \prod_{i=3}^{8} |a_i|^{11-2i}.$$

Thus, $\delta(M^0)|_{A^+(M^0)} = |a_3|^{10} |a_4|^8 |a_5|^6 |a_6|^4 |a_7|^2$.

Let

$$I = \int_D \phi^p \delta(M^0) d|a_3|^\times d|a_4|^\times \ldots d|a_7|^\times .$$

Using the explicit formulas of ϕ in D_i, we see that I is convergent if and only if $p > 4$. Thus we have

Proposition 2.4 [E_6]. *Let ρ be a unitary representation of E_6 without trivial constituents, then $\rho|_{M^0}$ is strongly $L^{4+\varepsilon}$, for any $\varepsilon > 0$.*

Theorem 0.1 [E_6]. *Let ρ be a unitary representation of E_6 without trivial constituents, then ρ is strongly $L^{22+\varepsilon}$, for any $\varepsilon > 0$.*

Proof. We apply Theorem 3.1 to the subgroup $H = H_1 \times H_2 = M^0 \times L$, and $k_1 = 2$, $k_2 = 1$.

We have

$$\delta(M^0) = |a_3|^5 |a_4|^3 |a_5| |a_6|^{-1} |a_7|^{-3} |a_8|^{-5}$$

and $\delta(L) = |\chi_{-(\varepsilon_1 + \varepsilon_2 + \varepsilon_9)}| = |a_3 a_4 a_5 a_6 a_7 a_8|$.

Let

$$\tilde{\delta} = \delta(M^0)^{\frac{1}{4}} \delta(L)^{\frac{1}{2}} = |a_3|^{\frac{7}{4}} |a_4|^{\frac{5}{4}} |a_5|^{\frac{3}{4}} |a_6|^{\frac{1}{4}} |a_7|^{-\frac{1}{4}} |a_8|^{-\frac{3}{4}}.$$

We compute to get

$$\delta(E_6) = |a_3|^{16}|a_4|^{14}|a_5|^{12}|a_6|^{10}|a_7|^8|a_8|^6 .$$

Recall the positive Weyl Chamber of E_6:

$$A^+(E_6) = \{d = (t, t, a_3, \ldots, a_8, t) : t^3 a_3 \ldots a_8 = 1,$$
$$|a_3| \geqq \cdots \geqq |a_8|,\ |a_6 a_7 a_8| \geqq 1\} .$$

We have (via an elementary comparison)

$$\tilde{\delta}(d) \geqq \delta(E_6)^{\frac{1}{22}}(E_6)(d), \quad \text{for } d \in A^+(E_6) .$$

Theorem 3.1 implies the desired result. □

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