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A residue formula for the index of a holomorphic flow

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Let $(\mathcal{V},P)$ be a normal, isolated complex analytic singularity of dimension $n \geqq 1$, and consider germs of holomorphic vector fields on $\mathcal{V}$ at P. One would like to be able to distinguish different germs.

If P is a regular point of $\mathcal{V}$, there are a number of invariants that sometimes allow us to distinguish germs of vector fields at P. However, for P singular, little is known.

If X is a holomorphic vector field on $\mathcal{V}$, singular only at P, then a natural way for studying the geometry, the topology and the dynamics of X near P, is to go up to a resolution of P and see what happens there, see for instance [HH]. More precisely, let

$$\pi : \tilde{\mathcal{V}} \to \mathcal{V},$$

be a resolution of P and let $E = \pi^{-1}(P)$ be the exceptional set of π. The solutions of X determine a non-singular holomorphic foliation $\mathcal{F}$ on $\mathcal{V} - \{P\}$ by immersed Riemann surfaces. The strict transform of $\mathcal{F}$ is a holomorphic foliation $\tilde{\mathcal{F}}$ of $\tilde{\mathcal{V}}$ by complex curves, with singular set S contained in E. Since π is a biholomorphism on $\tilde{\mathcal{V}} - E$, the vector field X lifts to a vector field $\tilde{X}$ on $\tilde{\mathcal{V}} - E$, tangent to $\tilde{\mathcal{F}}$. Thus one has several types of invariants to look at:

a) We may forget that $\tilde{X}$ is holomorphic and think of it as a C^∞-vector field. As such, we can approximate it by a sequence of vector fields $\{Y_t\}$, defined on all of $\tilde{\mathcal{V}}$ and having non-degenerate singularities. The total number of singular points of the Y_t's, counted with "signs", is independent of the way we approximate $\tilde{X}$: This is *the* (*total*) *index* $\mathrm{Ind}(\tilde{X},\tilde{\mathcal{V}})$ of $\tilde{X}$ in $\tilde{\mathcal{V}}$, introduced by Poincaré-Hopf (see [Mi]).

b) The bundle $T\tilde{\mathcal{F}}$, tangent to $\tilde{\mathcal{F}}$ on $\tilde{\mathcal{V}}-S$, can be extended to a holomorphic line bundle L over all of $\tilde{\mathcal{V}}$. Thus one has the Chern class $c_1(L) = c_1(T\tilde{\mathcal{F}}) \in H^2(\tilde{\mathcal{V}};\mathbb{Z})$. This is a topological invariant of $\tilde{\mathcal{F}}$, by [GSV]. One also

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has the *normal sheaf* Q of $\tilde{\mathcal{F}}$, defined by $Q = \frac{T\tilde{\mathcal{V}}}{L}$. The Chern classes of Q, $c_i(Q) \in H^{2i}(\tilde{\mathcal{V}}; \mathbb{Z})$, $i = 1, \ldots, n$, are linear combinations of products of the Chern classes of $T\tilde{\mathcal{V}}$ and the class $c_1(L)$, see [AH] or Sect. 2 below.

c) One may consider the top Chern class $c_n(Q)$ of the normal sheaf. We know, see Sect. 2 below, that the Chern class $c_n(Q)$ can be considered as a relative class in $H^{2n}(\tilde{\mathcal{V}}, \tilde{\mathcal{V}} - E; \mathbb{Z})$. Essentially by [BB], this class is localized at the singular set $S \subset E$:

$$c_n(Q) = \sum_{Z \subset S} \mu_* \mathrm{Res}_{c_n}(\tilde{\mathcal{F}}, Z),$$

where the sum runs over the connected components of S, $\mathrm{Res}_{c_n}(\tilde{\mathcal{F}}, Z)$ is a homology class in $H_0(Z; \mathbb{Z})$ that depends only on $\tilde{\mathcal{F}}$ near Z, and μ_* is given by the inclusion $H_0(Z; \mathbb{Z})$ into $H_0(\tilde{\mathcal{V}}; \mathbb{Z})$, followed by Poincaré-Lefschetz duality $H_0(\tilde{\mathcal{V}}; \mathbb{Z}) \cong H^{2n}(\tilde{\mathcal{V}}, \tilde{\mathcal{V}} - E; \mathbb{Z})$. If P_0 is an isolated singularity of $\tilde{\mathcal{F}}$, the term $\mathrm{Res}_{c_n}(\tilde{\mathcal{F}}, P_0)$ is the corresponding Grothendieck residue, by [BB]. Thus it equals the local Milnor number

$$\mu = \frac{\dim \mathcal{O}_{n, P_0}}{(F_1, \ldots, F_n)},$$

where the F_i's are holomorphic functions that define the germ of $\tilde{\mathcal{F}}$ at P_0 (cf. [CLS, CCS]).

Several questions arise:

Question 1. Is it possible to evaluate the index $\mathrm{Ind}(\tilde{X}, \tilde{\mathcal{V}})$ in terms of the above invariants of $\tilde{\mathcal{F}}$?.

Question 2. Can we use all these invariants to define something which depends only on the germ $(\mathcal{V}, P)$ and the vector field X, and not on the choice of resolution of P?.

Question 3. If so, what kind of invariant is this?.

In this work we study and partially answer these questions. Question 1 is a special case of a more general question: Let W be a connected, complex manifold, which is the interior of a compact, C^∞-manifold with boundary $M = \partial W$. Let X be a holomorphic vector field, defined and non-singular on a neighbourhood $\mathcal{U}$ of M in W. Assume further that X is tangent on $\mathcal{U}$ to a 1-dimensional holomorphic foliation $\mathcal{F}$ defined on W and with compact singular locus $S \subset W - \mathcal{U}$. Are the invariants, or "residues", of Baum-Bott [BB] related to the Poincaré-Hopf index of X?. Theorem I below gives an answer to this question, and also to question 1 above (see Sect. 2).

Theorem I. *Let X be a holomorphic vector field, defined and non-singular on a neighbourhood $\mathcal{U}$ of ∂W, the boundary of a complex manifold W which is the interior of a compact, smooth manifold with non-empty boundary. Suppose the holomorphic foliation defined by X on $\mathcal{U}$ can be extended to a holomorphic foliation $\mathcal{F}$ on W, which is full in the sense of* [BB]. *(So its singular set S has codimension at least* 2.*) Then the total (Poincaré-Hopf)*

index of X in W is:

$$\mathrm{Ind}(X,W) = \tilde{c}_1(\mathscr{F}) \cdot c_{n-1}(Q)[W] + \sum_{Z \subset S} \mathrm{Res}_{c_n}(\mathscr{F}, Z)$$

where $\tilde{c}_1(\mathscr{F})$ is the Chern class of $\mathscr{F}$ relative to X, $[W]$ is the fundamental cycle and the sum on the right runs over the connected components of the singular set S. In particular, if the singular set S consists of isolated points $P_1,\ldots,P_r$, then

$$\mathrm{Ind}(X,W) = \tilde{c}_1(\mathscr{F}) \cdot c_{n-1}(Q)[W] + \sum_{i=1}^{r} \mu_i ,$$

where μ_i is the local Milnor number of $\mathscr{F}$ at the singular point P_i, $i = 1,\ldots,r$.

Theorem II is of a different nature, closer to singularities theory. In fact Theorem II is a consequence of a deep theorem of Laufer, Steenbrink and Looijenga (see Sect. 1 below). This gives an answer to questions 2 and 3 above, for complete intersection germs of even dimension. The term Td^* is a rational linear combination of the Chern numbers of $\tilde{\mathscr{V}}$.

Theorem II. *Let $n = 2k$ be even and let $q_n = \dfrac{(-1)^{k-1}\mathscr{B}_k}{2k!}$, where $\mathscr{B}_k$ is the k^{th}-Bernoulli number, see [Hr]. Let $(\mathscr{V},P)$ be an isolated complete intersection singularity (ICIS) of dimension n, and let X be a continuous vector field on $\mathscr{V}$, non-singular away from P. If $\tilde{X}$ is the lifting of X to a resolution $\tilde{\mathscr{V}}$ of P, then the rational number*

$$\mathrm{Ind}(\tilde{X},\tilde{\mathscr{V}}) + \frac{1}{q_n}\,\mathrm{Td}^*[\tilde{\mathscr{V}}] ,$$

depends only on the germ of $\mathscr{V}$ at P and the homotopy class of the vector field X near P, independently of the choice of resolution of P; $\mathrm{Td}^*[\tilde{\mathscr{V}}]$ is (see Sect. 1 below) the n^{th} Todd polynomial Td_n in the relative Chern numbers of $\tilde{\mathscr{V}}$, but taking $c_n = 0$. Furthermore,

$$\mathrm{Ind}(X,\mathscr{V}) = \mathrm{Ind}(\tilde{X},\tilde{\mathscr{V}}) + \frac{1}{q_n}\,(\mathrm{Td}^*[\tilde{\mathscr{V}}] + p_g) ,$$

where $\mathrm{Ind}(X,\mathscr{V})$ is the index of X on $\mathscr{V}$ at P and p_g is the geometric genus of P.

If P is regular in $\mathscr{V}$, $\mathrm{Ind}(X,\mathscr{V})$ is the usual local index of Poincaré-Hopf; In general, if P is an ICIS, this is essentially the local index of a vector field on a singular variety, studied in [GSV, BG], see Sect. 1 below. The number q_n above is the coefficient of c_n in the n^{th}-Todd polynomial Td_n, by [BH]. For $n > 1$ odd, the coefficient of c_n in Td_n is 0 and this is why our method says nothing in these dimensions. For an ICIS of dimension 1 in $\mathbb{C}^k$, the statement corresponding to Theorem II is:

$$\mathrm{Ind}(X,\mathscr{V}) = \mathrm{Ind}((\tilde{X},\tilde{\mathscr{V}}) - 2\delta ,$$

where $\delta > 0$ "measures" the number of double points concentrated at P, see [Mi p. 85, or Se p. 68], and $\mathrm{Ind}(X, \mathcal{V})$ is the index of X on $\mathcal{V}$, so it is the element in $\pi_1(U(k)) \cong \mathbb{Z}$ determined by the map $(\mathcal{V} - \{P\}) \to U(k)$ given by X and the gradient vector fields of the functions that define the germ of $\mathcal{V}$ at P (cf. Sect. 1 below). (This follows from Milnor's formula,

$$\mu - 1 = 2\delta - r\,,$$

where μ is the Milnor number and r is the number of irreducible components of $\mathcal{V}$ at P, and the proof is the same as that of Theorem II, replacing the theorem of Laufer, Steenbrink and Looijenga by the above formula of Milnor.)

Theorem III is the sum of Theorems I and II:

Theorem III. *Let $n = 2k$ be even. Let q_n be as in theorem* II. *Let $(\mathcal{V}, P)$ be an ICIS of dimension n, and let X be a holomorphic vector field on $\mathcal{V}$, non-singular away from P. If $\tilde{\mathcal{F}}$ is the strict transform of the foliation on $\mathcal{V} - \{P\}$ determined by X, then the index of X on $\mathcal{V}$ at P is:*

$$\mathrm{Ind}(X, \mathcal{V}) = \sum_{Z \subset S} \mathrm{Res}_{c_n}(\tilde{\mathcal{F}}, Z) + \tilde{c}_1(\tilde{\mathcal{F}}) \cdot c_{n-1}(Q)[\tilde{\mathcal{V}}]$$

$$+ \frac{1}{q_n}(\mathrm{Td}^*[\tilde{\mathcal{V}}] + p_g)\,.$$

In particular, if the singularities of $\tilde{\mathcal{F}}$ are isolated, then the first term on the right hand side is the sum of the local Milnor numbers.

For instance, if $n = 2$, the singularities of $\tilde{\mathcal{F}}$ are necessarily isolated, the coefficient q_n is $\frac{1}{12}$ and $\mathrm{Td}^*[\tilde{\mathcal{V}}]$ is $\frac{K^2}{12}$, where K is the canonical class of the resolution. Thus the formula above becomes,

$$\mathrm{Ind}(X, \mathcal{V}) = \tilde{c}_1(\mathcal{F}) \cdot c_1(Q)[W] + \sum_{i=1}^{r} \mu_i + K^2 + 12 \cdot p_g\,.$$

In a subsequent work [SS], we study some natural generalizations of Sect. 2 above: What happens when the dimension of the leaves of $\mathcal{F}$ is larger than 1?. What if we have, on a neighbourhood of the boundary, several vector fields tangent to $\mathcal{F}$?, can we say anything for characteristic classes of $W, \mathcal{F}, Q$, other than c_n?. Lemma 2.1 below contains a result in this direction.

1 Vector fields and the Todd genus

We recall [Hr] that the Todd sequence $\{\mathrm{Td}_k\}$, $k \in \mathbb{N}$, is the (multiplicative) sequence of polynomials associated to the power series $\frac{x}{1 - e^{-x}}$. This is a sequence of polynomials in indeterminates $c_1, \ldots, c_k, \ldots$, such that Td_k is a rational linear combination of $c_1, \ldots, c_k$, and if we assign the weight i to c_i and weight $i_1 + \ldots + i_r$ to the cup product $c_{i_1} \cdot \ldots \cdot c_{i_r}$, then Td_k is homogeneous

of degree k. For example,

$$\mathrm{Td}_1 = \tfrac{1}{2}c_1\,,$$

$$\mathrm{Td}_2 = \tfrac{1}{12}(c_1^2 + c_2)\,,$$

$$\mathrm{Td}_3 = \tfrac{1}{24}c_1c_2\,,$$

$$\mathrm{Td}_4 = \tfrac{1}{720}(-c_4 + c_3c_1 + 3c_2^2 + 4c_2c_1^2 - c_1^4)\,,$$

and so on. Let us denote by Td_n^* the polynomial obtained from the n^{th}-Todd polynomial Td_n, replacing c_n by 0. In other words, we write $\mathrm{Td}_n = \mathrm{Td}_n^* + q_nc_n$, where q_n is the coefficient of c_n in Td_n. Thus, for instance, $\mathrm{Td}_2 = \frac{1}{12}\cdot c_1^2$. (We refer to [Hr or M3] for basic material about the Todd genus.)

If W is a closed, connected, C^∞-manifold of (real) even dimension $2n$, and a complex structure on its tangent bundle TW, its Todd genus is defined by,

$$\mathrm{Td}[W] = \mathrm{Td}_n(c_1,\ldots,c_n)[W]\,,$$

where $c_1,\ldots,c_n$ are the Chern classes of TW and $[W]$ is the fundamental cycle. If W has non-empty boundary M, then $H^{2n}(W) \cong 0$, so it is not interesting to speak of the Chern numbers of W. However, let us suppose we are given a C^∞-vector field X, defined and non-singular on a neighbourhood of the boundary M. We may look at the primary obstruction for extending this to a cross section of TW. This is a relative cohomology class $\tilde{c}_n(X) \in H^{2n}(W,M;\mathbb{Z})$ that maps to the usual Chern class of TW, and evaluated on the fundamental cycle $[W]$ equals the total index of X in W. There are as many liftings of c_n to a relative class as elements in $H^{2n-1}(M) \cong \mathbb{Z}$, parametrizing the possible indices that the vector fields as above can have. Moreover, if the bundle $TW|_M$ is trivial, then the Chern classes of TW vanish over the boundary M, hence each Chern class $c_i(W)$ can be lifted to a relative class $\tilde{c}_i(W)$. In general, there will be many different liftings of each $c_i(W)$; However, as noted in [LS] or [L1], if i,j are both $\geqq 1$ then the cup product $\tilde{c}_i(W)\cdot\tilde{c}_j(W)$ depends only on the absolute classes $c_i(W)$, $c_j(W)$. Therefore, the Chern numbers of W involving the lower dimensional classes $c_1,\ldots,c_{n-1}$, are well defined, depending only on W. In this situation one may define *the Todd genus of W relative to X* by

$$\mathrm{Td}[W;X] = \mathrm{Td}_n(\tilde{c}_1,\ldots,\tilde{c}_{n-1},\tilde{c}_n(X))[W]\,,$$

where Td_n is the n^{th}-Todd polynomial, $\tilde{c}_n(X)$ is the relative Chern class determined by X and the term on the right means that whenever we have a monomial of Td_n which is a product $c_{i_1}\cdot\ldots\cdot c_{i_r}$, we replace this by the lift $\tilde{c}_{i_1}\cdot\ldots\cdot\tilde{c}_{i_r}$.

Let us now denote by $(\mathcal{V},P)$ an isolated, complete intersection singularity $(\mathcal{V},P)$ of complex dimension $n \geqq 1$. For simplicity, we call such a germ an ICIS, following [L2]. Let $\tilde{\mathcal{V}}$ be a resolution of $(\mathcal{V},P)$; This will play the role of the above manifold W. By restricting the projection $\pi : \tilde{\mathcal{V}} \to \mathcal{V}$ to a compact neighbourhood of P, we may think of $\tilde{\mathcal{V}}$ as being a compact $2n$-manifold with boundary M, the link of P in $\mathcal{V}$, and with a complex structure on its interior.

Let $c_i \in H^{2i}(\tilde{\mathcal{V}};\mathbb{Z})$, $i = 1,\ldots,n$, be the Chern classes of $\tilde{\mathcal{V}}$. A vector field X on $(\mathcal{V},P)$ determines a vector field $\tilde{X}$ on a neighbourhood of the boundary of $\tilde{\mathcal{V}}$; The above discussion tells us that $\tilde{X}$ determines a lift $\tilde{c}_n(\tilde{X})$ of c_n to a relative class, with $\tilde{c}_n(\tilde{X})[\tilde{\mathcal{V}}] = \mathrm{Ind}(\tilde{X},\tilde{\mathcal{V}})$. The tangent bundle $T(\mathcal{V} - \{P\})$ is a (smoothly) trivial bundle, because P is an ICIS, hence $T\tilde{\mathcal{V}}|_M$ is smoothly trivial. Thus one has the *Todd genus of $\tilde{\mathcal{V}}$ relative to $\tilde{X}$*:

$$\mathrm{Td}[\tilde{\mathcal{V}};\tilde{X}] = \mathrm{Td}_n(\tilde{c}_1,\ldots,\tilde{c}_{n-1},\tilde{c}_n(\tilde{X}))[\tilde{\mathcal{V}}],$$

where Td_n is the n^{th}-Todd polynomial, $\tilde{c}_n(\tilde{X})$ is the relative Chern class determined by $\tilde{X}$ and the term on the right means that whenever we have a monomial of Td_n which is a decomposable product $c_{i_1}\cdot\ldots\cdot c_{i_r}$, we replace this by the lift $\tilde{c}_{i_1}\cdot\ldots\cdot\tilde{c}_{i_r}$.

We remark that $\mathrm{Td}[\tilde{\mathcal{V}},\tilde{X}]$ can be expressed as,

$$\mathrm{Td}[\tilde{\mathcal{V}},\tilde{X}] = q_n \cdot \mathrm{Ind}(\tilde{X},\tilde{\mathcal{V}}) + \mathrm{Td}^*[\tilde{\mathcal{V}}],$$

where $\mathrm{Td}^*[\tilde{\mathcal{V}}] = \mathrm{Td}_n^*(\tilde{c}_1,\ldots,\tilde{c}_{n-1},0)[\tilde{\mathcal{V}}]$ depends only on $\tilde{\mathcal{V}}$, not on X, and q_n is the coefficient of c_n in the Todd polynomial Td_n. If $n > 1$ is odd, then q_n is zero, by [Hr]. For $n = 1$, q_n is $\frac{1}{2}$ and for $n = 2k$, q_n is $\dfrac{(-1)^{k-1}\mathcal{B}_k}{2k!}$, where $\mathcal{B}_k$ is the k^{th}-Bernoulli number (see [Hr, BH]).

If $f_1,\ldots,f_r$ are holomorphic functions that define the germ $(\mathcal{V},P)$, then the gradient vector fields ∇f_i of the f_i's are linearly independent over $\mathbb{C}$ away from P, because P is an ICIS, and they are orthogonal to $\mathcal{V}$. Hence, up to homotopy these vector fields define a smooth map,

$$(X,\nabla f_1,\ldots,\nabla f_r)|_M : M \to W_{r+1,\,r+n},$$

where $W_{r+1,\,r+n}$ is the Stiefel manifold of complex orthonormal $(r+1)$-frames in $\mathbb{C}^{n+r}$; $W_{r+1,\,r+n}$ is $(2n-2)$-connected and its first non-zero homotopy group is $\pi_{2n-1}(W_{r+1,\,r+n}) \cong \mathbb{Z}$, by Bott's computations of the homotopy groups of the classical groups, see for instance [Hu]. Hence the maps from M into $W_{r+1,\,r+n}$ have a degree. Following [GSV], define the *local index of X on $\mathcal{V}$ at P* to be the degree of the above map $(X,\nabla f_1,\ldots,\nabla f_r)$. This definition of the local index on a singular variety was introduced in [GSV] for hypersurfaces, but the definition and its basic properties are essentially the same (cf. [BG]).

Theorem II. *Let $n = 2k$ be even and let q_n be the coefficient of c_n in Td_n. Let $(\mathcal{V},P)$ be an ICIS of dimension n, and let X be a vector field on $\mathcal{V}$, non-singular away from P. If $\tilde{X}$ is the lifting of X to a resolution $\tilde{\mathcal{V}}$ of P. Then the rational number*

$$\mathrm{Ind}(\tilde{X},\tilde{\mathcal{V}}) + \frac{1}{q_n}\mathrm{Td}^*[\tilde{\mathcal{V}}],$$

depends only on the germ of $\mathcal{V}$ at P and the homotopy class of the vector field X, independently of the choice of resolution of P. Furthermore,

$$\mathrm{Ind}(X,\mathcal{V}) = \mathrm{Ind}(\tilde{X},\tilde{\mathcal{V}}) + \frac{1}{q_n}(\mathrm{Td}^*[\tilde{\mathcal{V}}] + p_g),$$

where $\mathrm{Ind}(X,\mathcal{V})$ is the index of X at P and p_g is the geometric genus of P.

This theorem follows from the Laufer-Steenbrink-Looijenga theorem (see [L1]), that we re-formulate below without proof, together with Lemmas 1.2 and 1.3 below. The theorem in [L1] is in fact more general than we state below, but this is what we need.

Theorem [L1]. *Let $(\mathcal{V},P)$ be an ICIS of dimension n, let F be a Milnor fibre of P and let $\tilde{\mathcal{V}}$ be a resolution of P. Let $\chi(F)$ and $\chi(\tilde{\mathcal{V}})$ be the topological Euler-Poincaré characteristics of F and $\tilde{\mathcal{V}}$, respectively. Then,*

$$q_n \cdot \chi(F) = q_n \cdot \chi(\tilde{\mathcal{V}}) + \mathrm{Td}^*[\tilde{\mathcal{V}}] + (-1)^n p_g ,$$

where $\mathrm{Td}^*[\tilde{\mathcal{V}}]$ and p_g are as above.

1.2 Lemma. *Let X' be a vector field on $\mathcal{V}$, which is everywhere transversal to the link M of P in $\mathcal{V}$. Then,*

$$\mathrm{Ind}(X',\mathcal{V}) = \chi(F) .$$

Proof. The link M of P in $\mathcal{V}$ is, by definition, the intersection of $\mathcal{V}$ with a small sphere $\mathbb{S}_\varepsilon$ around P. From [M1], or using Ehresman fibration theorem, one knows that M is isotopic in $\mathbb{S}_\varepsilon$ to the intersection of $\mathbb{S}_\varepsilon$ with a non-singular fibre $F' = f_1^{-1}(t) \cap \ldots \cap f_r^{-1}(t)$, for suitable t. Thus, the vector field X' can be regarded as a continuous vector field on a neighbourhood of $F' \cap \mathbb{S}_\varepsilon$ in F'. Let F be $F' \cap \mathbb{D}_\varepsilon$, a Milnor fibre of P. Then the gradient vector fields $\nabla f_1, \ldots, \nabla f_r$ are linearly independent everywhere on F, because $(\mathcal{V},P)$ is an ICIS. Hence, the obstruction for extending the map,

$$(X', \nabla f_1, \ldots, \nabla f_r)|_M : M \to W_{r+1,\, r+n} ,$$

to all of F is the obstruction for extending X' to a vector field on F. Since the degree of this map is the index of X' on $\mathcal{V}$ at P, it follows that $\mathrm{Ind}(X,\mathcal{V})$ can be regarded as the obstruction for extending X' to a continuous vector field on F. The lemma now follows from the theorem of Poincaré-Hopf for manifolds with boundary, because X' is everywhere transversal to the boundary of F.

The following lemma is an immediate consequence of Lemma 1.2 and a well known result in differential topology, saying that the difference between the indices of two vector fields on a manifold with boundary is independent of the topology of the manifold away from a tubular neighbourhood of the boundary (see for instance [BG]).

1.3 Lemma. *Let X and X' be vector fields on $\mathcal{V}$. Then,*

$$\mathrm{Ind}(X,\mathcal{V}) - \mathrm{Ind}(X',\mathcal{V}) = \mathrm{Ind}(\tilde{X},\tilde{\mathcal{V}}) - \mathrm{Ind}(\tilde{X}',\tilde{\mathcal{V}})$$

where $\mathrm{Ind}(\tilde{X},\tilde{\mathcal{V}})$ and $\mathrm{Ind}(\tilde{X}',\tilde{\mathcal{V}})$ are, respectively, the indices of the liftings of X and X' to the resolution $\tilde{\mathcal{V}}$.

Theorem II now follows easily: By the theorem quoted above one has,

$$q_n \cdot \chi(F) = q_n \cdot \chi(\tilde{\mathscr{V}}) + \mathrm{Td}^*[\tilde{\mathscr{V}}] + (-1)^n p_g .$$

If X' is a vector field as in Lemma 1.2 and $\tilde{X}'$, is its lifting to $\tilde{\mathscr{V}}$, then the theorem of Poincaré-Hopf for manifolds with boundary [M2] implies $\mathrm{Ind}(\tilde{X}',\tilde{\mathscr{V}}) = \chi(\tilde{\mathscr{V}})$. Hence Lemma 1.2 yields,

$$q_n \cdot \mathrm{Ind}(X',\mathscr{V}) = q_n \cdot \mathrm{Ind}(\tilde{X}',\tilde{\mathscr{V}}) + \mathrm{Td}^*[\tilde{\mathscr{V}}] + (-1)^n p_g .$$

If n is even, then $q_n \neq 0$ and one has,

$$\mathrm{Ind}(X',\mathscr{V}) = \mathrm{Ind}(\tilde{X}',\tilde{\mathscr{V}}) + \frac{1}{q_n}(\mathrm{Td}^*[\tilde{\mathscr{V}}] + p_g) , \tag{1.4}$$

proving Theorem II for a vector field X' transversal to the link. If X is a vector field on $\mathscr{V}$ as in Theorem II, then

$$\mathrm{Ind}(X,\mathscr{V}) - \mathrm{Ind}(X',\mathscr{V}) = \mathrm{Ind}(\tilde{X},\tilde{\mathscr{V}}) - \mathrm{Ind}(\tilde{X}',\tilde{\mathscr{V}}) , \tag{1.5}$$

by Lemma 1.3. By 1.4 one has,

$$\mathrm{Ind}(\tilde{X}',\tilde{\mathscr{V}}) = \mathrm{Ind}(X',\mathscr{V}) - \frac{1}{q_n}(\mathrm{Td}^*[\tilde{\mathscr{V}}] + p_g) ,$$

which together with 1.5 implies,

$$\mathrm{Ind}(X,\mathscr{V}) = \mathrm{Ind}(\tilde{X},\tilde{\mathscr{V}}) + \frac{1}{q_n}(\mathrm{Td}^*[\tilde{\mathscr{V}}] + p_g) , \tag{1.6}$$

as claimed in Theorem II. Finally, that

$$\mathrm{Ind}(\tilde{X},\tilde{\mathscr{V}}) + \frac{1}{q_n}\mathrm{Td}^*[\tilde{\mathscr{V}}] ,$$

is independent of the choice of resolution follows from 1.6 and the fact that $\mathrm{Ind}(X,\mathscr{V})$ is defined intrinsically, together with the well known fact that the geometric genus p_g is an invariant of P.

2 Foliations on manifolds with boundary

Let $\overline{W}$ be a compact, connected manifold with boundary $\partial W \neq \emptyset$, and a complex structure on its interior W. So W is a complex manifold; We let n be the complex dimension of W and we assume $n > 1$. Let X be a nowhere-vanishing holomorphic vector field on a neighbourhood $\mathscr{U}$ of ∂W. So X defines a non-singular holomorphic foliation $\mathscr{F} = \mathscr{F}(X)$ on $\mathscr{U}$ by complex curves; We further assume that $\mathscr{F}$ is extended to a holomorphic foliation on W, with singular set S of codimension $\geqq 2$. In practice we will take W to be the resolution $\tilde{\mathscr{V}}$ of an ICIS $(\mathscr{V},P)$ and $\mathscr{F}$ the strict transform of the foliation on $\mathscr{V} - \{P\}$ determined by a holomorphic vector field (as in Sect. 1 above).

Let $T\mathcal{F}$ be the bundle tangent to $\mathcal{F}$ on $W - S$; so $T\mathcal{F}$ is a holomorphic line sub-bundle of $TW|_{W-S}$. This can always be extended to a holomorphic line bundle L over W, which is not a sub-bundle of TW on S: there is a bundle morphism $\pi : L \to TW$, which is not injective on S. The isomorphism class of L is determined by $\mathcal{F}$; In fact, as a C^∞-vector bundle, L is classified by its first Chern class, and this class depends only on the topological type of $\mathcal{F}$, by [GSV]. This is called *the Chern class of the foliation* and denoted $c_1(\mathcal{F})$. We note that the vector field X trivializes the bundle $L \cong T\mathcal{F}$ on a neighbourhood $\mathcal{U}$ of the boundary ∂W. Thus we have a connection D_1' for L on $\mathcal{U}$, flat relative to the frame X; D_1' can be extended to a connection D_1 for L on W, flat on $\mathcal{U}$. This determines a representative of the Chern class $c_1(\mathcal{F})$ that vanishes over ∂W, so it represents a class $\tilde{c}_1(\mathcal{F}) \in H^2(W, \partial W; \mathbb{Z})$. (The Chern classes of vector bundles defined via differential forms live in the cohomology with complex coefficients, but they are in fact integral classes.)

We now think of L and TW as being locally free sheaves on W of ranks 1 and n, respectively. The bundle map π induces an injection on the sheaf level and we have *the normal sheaf* Q, defined by the exact sequence of sheaves,

$$(*) \quad 0 \to L \to TW \to Q \to 0 .$$

We note that Q is not locally free at the singular set S, but the exact sequence above provides a resolution of Q by vector bundles. Thus [AH], the total Chern class of Q is:

$$c(Q) = \frac{c(TW)}{c(L)} ,$$

where $c(L) = 1 + c_1(\mathcal{F})$, because L has rank 1.

2.1 Lemma. *Let $\phi = \phi(c_1, \ldots, c_n)$ be a polynomial in indeterminates $c_1, \ldots, c_n$, with complex coefficients, homogeneous of degree n. Then, there exist relative cohomology classes $\tilde{\phi}(Q)$ and $\tilde{\phi}(TW)$ in $H^{2n}(W, \partial W; \mathbb{C})$, such that:*

a) *$\tilde{\phi}(Q)$ is localized at the singular set S of $\mathcal{F}$:*

$$\tilde{\phi}(Q) = \sum_{Z \subset S} \mu_*(\mathrm{Res}_\phi(\mathcal{F}, Z)) ,$$

where the sum runs over the connected components of the singular set, $\mathrm{Res}_\phi(\mathcal{F}, Z) \in H_0(Z; \mathbb{C})$ is the Baum-Bott residue corresponding to the polynomial ϕ; μ_* is the composition αi_*, where $i_* : H_0(Z; \mathbb{C}) \to H_0(W; \mathbb{C})$ is induced by the inclusion and $\alpha : H_0(W; \mathbb{C}) \to H^{2n}(W, \partial W; \mathbb{C})$ is Lefschetz duality.

b) *One has,*

$$\tilde{\phi}(TW) = \tilde{\phi}(Q) + \phi'(Q, \mathcal{F}) \cdot \tilde{c}_1(\mathcal{F}) ,$$

where $\phi'(Q, \mathcal{F})$ is a class in $H^{2n-2}(W; \mathbb{C})$ depending on $c_1(Q), \ldots, c_{n-1}(Q)$ and $\tilde{c}_1(\mathcal{F})$, and it is computable from ϕ.

c) *Let $\tilde{c}_n(TW; X) \in H^{2n}(W, M; \mathbb{Z})$ be the relative Chern class determined (in Sect. 1) by the vector field X. Then,*

$$\tilde{c}_n(TW; X) = \tilde{c}_n(Q) + c_{n-1}(Q) \cdot \tilde{c}_1(\mathcal{F}) .$$

One may also add that the image of $\tilde{\phi}(Q)$ and $\tilde{\phi}(TW)$ in the absolute cohomology are the corresponding classes $\phi(Q)$ and $\phi(TW)$, however $H^{2n}(W;\mathbb{C}) \cong 0$, so this observation is not interesting.

Proof. We first recall from [BB] the construction of a "basic connection". On $W - S$ the foliation $\mathcal{F}$ acts on the normal sheaf Q: If

$$\eta : TW \to Q ,$$

is the projection, defined on $W - S$,

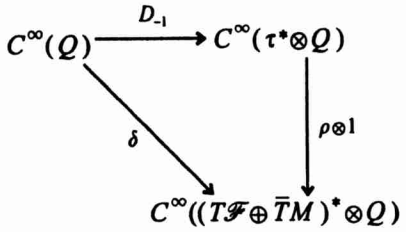
$$T\mathcal{F} \times Q \to Q$$

$$(u, \eta(v)) \to \eta([u,v]) = [u, \eta(v)] ,$$

which is well defined because $T\mathcal{F}$ is closed under Lie brackets. Thus one has a "partial connection" on Q,

$$\delta : C^\infty(Q) \to C^\infty((T\mathcal{F} \oplus \overline{T}W)^* \otimes Q) = C^\infty(\mathrm{Hom}(T\mathcal{F}, Q) \oplus (\overline{T}^* W \otimes Q)) ,$$

defined by $\delta(s) = (u \to [u,s], \bar{\partial}s)$, where $\overline{T}W$ is the conjugate of TW in its complexification $\tau = T_{\mathbb{R}}W \otimes \mathbb{C} \cong TW \oplus \overline{T}W$, and $\bar{\partial}$ is the $\bar{\partial}$-operator. It is easy to see [BB] that there exists a connection D_{-1} for Q on $W - S$ lifting δ:



where ρ is the canonical surjection. Any connection on Q lifting the partial connection δ is called [BB] a *basic connection.*

Now, on $W - S$ one has a splitting,

$$TW \cong T\mathcal{F} \oplus Q .$$

On a neighbourhood $\mathcal{U}$ of the boundary ∂W, $T\mathcal{F}$ has the flat connection D_1', determined by the vector field X. This can be extended to a connection D_1 on W, by [BB; 4.41], so one has a connection $D_0' = D_1 \oplus D_{-1}$ for TW on $W - S$.

Let $\{Z_\alpha\}$ be the connected components of the singular set S. For each component Z_α we choose a regular neighbourhood $\mathcal{U}_\alpha$ of Z_α and a compact set Σ_α, so that: i) $\mathcal{U}_\alpha \supset \Sigma_\alpha \supset \mathrm{Int}\,\Sigma_\alpha \supset Z_\alpha$, ii) $\mathcal{U}_\alpha \cap \mathcal{U} = \varnothing$ and iii) $\mathcal{U}_\alpha \cap \mathcal{U}_\beta = \varnothing$ if $\alpha \neq \beta$. Then by [BB; 4.41], there exists a connection D_0 on W for TW, which coincides with D_0' on $W - \Sigma$, where $\Sigma = \bigcup \Sigma_\alpha$.

Thus one has connections D_0, D_{-1} and D_1, such that:

i) D_{-1} is basic for Q on $W - S$.

ii) D_1 is a connection for L on all of W, and it is flat relative to the vector field X on a neighbourhood $\mathcal{U}$ of the boundary ∂W.

iii) D_0 is a connection for TW on W, and $D_0 = D_1 \oplus D_{-1}$ on $W - \Sigma$.

The connections D_j, $j = 0, 1, -1$, determine curvature matrices K_0, K_1 and K_{-1}, for TW, $T\mathcal{F}$ and Q, respectively. If $\sigma_i(K_j)$ is the i^{th}-symmetric function in the entries of K_j, and $[\sigma_i(K_j)]$ is its class in the de Rham cohomology, then the Chern classes of the corresponding bundles are given by,

$$c_i(TW) = \left(\frac{\sqrt{-1}}{2\pi}\right)^i \cdot [\sigma_i(K_0)]\,,$$

$$c_1(T\mathcal{F}) = \left(\frac{\sqrt{-1}}{2\pi}\right) \cdot [\sigma_1(K_1)]\,,$$

and on $W - \Sigma$,

$$c_i(Q) = \left(\frac{\sqrt{-1}}{2\pi}\right)^i \cdot [\sigma_i(K_{-1})]\,,$$

see [BB or Mi]. By iii) above, on $W - \Sigma$ one has,

$$\sigma_i(K_{-1}) = \sum_{j+k=1} (-\sigma_1(K_1))^j \cdot \sigma_k(K_0)\,.$$

We observe that the right hand side is actually defined on all of W, so we can think of it as extending $\sigma_i(K_{-1})$ over Σ. If we set

$$\omega_i = \sum_{j+k=1} (-\sigma_1(K_1))^j \cdot \sigma_k(K_0)$$

over W, then ω_i defines the Chern class $c_i(Q)$ of the sheaf Q on W, see [AH].

The class $\phi(Q)$ is represented by the form $\phi[\omega_1, \ldots, \omega_n]$, modulo the normalization constant $(\sqrt{-1}/2\pi)^n$, which coincides with $\phi(\sigma_1(K_{-1}), \ldots, \sigma_n(K_{-1}))$ on $W - \Sigma$. By Bott's vanishing theorem, the latter form vanishes identically on $W - \Sigma$. Thus $\phi(\omega_1, \ldots, \omega_n)$ defines a relative cohomology class $\tilde{\phi}(Q)$, which is the class in the statement of 2.1 above. Moreover, the restriction of $\phi(\omega_1, \ldots, \omega_n)$ to each $\mathcal{U}_\alpha$ defines a class in $H^{2n}(\mathcal{U}_\alpha, \partial\mathcal{U}_\alpha; \mathbb{C})$, and its image by the composition of Lefschetz duality $H^{2n}(\mathcal{U}_\alpha, \partial U_\alpha; \mathbb{C}) \to H_0(\mathcal{U}_\alpha; \mathbb{C})$ and the inverse of the natural isomorphism $H_0(Z_\alpha; \mathbb{C}) \to H_0(\mathcal{U}_\alpha; \mathbb{C})$, is the class $\mathrm{Res}_\phi(\mathcal{F}; Z_\alpha)$ by definition. Thus we have a). To prove b), note that

$$\sigma_i(K_0) = \omega_i + \omega_{i-1} \cdot \sigma_1(K_1)\,.$$

Since D_1 is flat on $\mathcal{U}$, $\sigma_1(K_1)$ vanishes on $\mathcal{U}$ and it represents the relative class $\tilde{c}_1(\mathcal{F})$. Hence we see that $\phi(\sigma_1(K_0), \ldots, \sigma_n(K_0))$ represents a relative cohomology class $\tilde{\phi}(TW)$ satisfying the identity of b). In particular, if $\phi = c_n$, we have

$$\tilde{c}_n(TW) = \tilde{c}_n(Q) + c_{n-1}(Q) \cdot \tilde{c}_1(\mathcal{F})\,.$$

Thus, to complete the proof of 2.1 we only need to show that the above class $\tilde{c}_n(TW)$ is the Chern class $\tilde{c}_n(TW, X)$ of Sect. 1. For this we recall, from

Sect. 1 above, that the difference $d(c_n, c_n')$ between two relative Chern classes c_n, c_n' is given by a contribution of the coboundary map,

$$\delta^* : H^{2n-1}(\partial W; \mathbb{Z}) \to H^{2n}(W, \partial W; \mathbb{Z}) .$$

On a neighbourhood $\mathcal{U}$ of ∂W the sheaf Q is actually a vector bundle, and one has a splitting $TW|_{\mathcal{U}} \cong Q|_{\mathcal{U}} \oplus T\mathcal{F}|_{\mathcal{U}}$. On $\mathcal{U}$, the class $\tilde{c}_n(Q)$ is identically 0, because Q has rank $n-1$. Hence on $\mathcal{U}$ one has,

$$\tilde{c}_n(TW) = \tilde{c}_1(\mathcal{F}) \cdot c_{n-1}(Q) .$$

The result follows because $\tilde{c}_1(\mathcal{F})c_{n-1}(Q) = \tilde{c}_n(TW, X)$ on $\mathcal{U}$, since Q is a vector bundle there, and in both cases the characteristic class in question is relative because it is the obstruction for extending the vector field X to the interior of W.

Note that since $\tilde{c}_n(TW)$, $\tilde{c}_1(\mathcal{F})$, and $c_{n-1}(Q)$ are integral classes, $\tilde{c}_n(Q)$ is also integral and the identity in c) holds in $H^{2n}(W, \partial W; \mathbb{Z})$.

We now have Theorem I, stated in the introduction:

Theorem I. *Let X be a holomorphic vector field, defined and non-singular on a neighbourhood of ∂W, the boundary of a complex manifold W which is the interior of a compact, smooth manifold with non-empty boundary. Then the total (Poincaré-Hopf) index of X in W is:*

$$\mathrm{Ind}(X, W) = \tilde{c}_1(\mathcal{F}) \cdot c_{n-1}(Q)[W] + \sum_{Z \subset S} \mathrm{Res}_{c_n}(\mathcal{F}, Z)$$

where $[W]$ is the fundamental cycle. Moreover, if the singular set S consists of isolated points $P_1, \ldots, P_r$, then

$$\mathrm{Ind}(X, W) = \tilde{c}_1(\mathcal{F}) \cdot c_{n-1}(Q)[W] + \sum_{i=1}^{r} \mu_i ,$$

where μ_i is the local Milnor number of $\mathcal{F}$ at the singular point P_i, $i = 1, \ldots, r$.

The first statement in this theorem follows from 2.1 together with the fact, noted in Sect. 1 above, that one has $\tilde{c}_n(TW)[W] = \mathrm{Ind}(X, W)$. The second claim follows from this and the fact [BB], that at each singular point P_i, the Baum-Bott residue agrees with the corresponding Grothendieck residue, which equals the intersection number μ_i. Alternatively, one can prove the second statement via differential topology: Away from the points $P_1, \ldots, P_r$, Q is a vector bundle; Therefore, away from these points, the class $c_{n-1}(Q)$ is the first obstruction for constructing a cross section σ of Q, ($C^\infty -$) transversal to the zero section of Q. Hence the zero-locus of σ is a smooth submanifold Σ_Q of $W - \{P_1, \ldots, P_r\}$ of real dimension 2. This submanifold may extend to the boundary of W, so it represents a homology class $[\Sigma_Q] \in H_2(W^*, \partial W; \mathbb{Z})$, where W^* is W minus small open balls $D_1, \ldots, D_r$ around the singular points. The natural map from $H_2(W^*, \partial W; \mathbb{Z})$ into $H_2(W, \partial W; \mathbb{Z})$ is an isomorphism,

because one has the exact sequence of the triple $(W, W^*, \partial W)$,

$$\cdots \to H_3(W, W^*) \to H_2(W^*, \partial W; \mathbb{Z}) \to H_2(W, \partial W; \mathbb{Z}) \to H_2(W, W^*, \mathbb{Z}) \to \cdots$$
$$\qquad\quad \| \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \|$$
$$\qquad\quad 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 0$$

Thus Σ_Q actually represents a homology class $[\Sigma_Q] \in H_2(W, \partial W; \mathbb{Z})$, which is isomorphic to $H^{2n-2}(W; \mathbb{Z})$, by Poincaré-Lefschetz duality. Its dual is the Chern class $c_{n-1}(Q)$. Similarly, the class $\tilde{c}_1(\mathcal{F})$ is dual to the $(2n-2)$-submanifold $\Sigma_{\mathcal{F}}$ of W, obtained by intersecting the zero-section of $T\mathcal{F}|_{W^*}$, with a non-zero section $\mathcal{S}$ of this bundle, which is transversal to the zero-section and restricts to X on ∂W. Since X is non-zero on ∂W, $\Sigma_{\mathcal{F}}$ actually represents a class in $H_{2n-2}(W^*; \mathbb{Z})$, which is isomorphic to $H_{2n-2}(W; \mathbb{Z}) \cong H^2(W, \partial W; \mathbb{Z})$. The above intersection number $\tilde{c}_1(\mathcal{F}) \cdot c_{n-1}(Q)[W]$ is the transverse intersection of Σ_Q with $\Sigma_{\mathcal{F}}$. We may extend $\mathcal{S}$ to the interior of the balls $D_1, \ldots, D_r$, being tangent to $\mathcal{F}$ and singular only at $P_1, \ldots, P_r$, so its local index at each P_i is the local Milnor number μ_i, by [CLS or GSV]. We may also multiply σ by an adequate bump function, to make it zero on ∂W and on the boundary spheres $\partial D_1, \ldots, \partial D_q$. Thus, adding up the vector fields σ and $\mathcal{S}$, we obtain a vector field $\mathcal{X}$ on W that agrees with X near ∂W, it is tangent to $\mathcal{F}$ near the singular points $P_1, \ldots, P_r$, and its remaining singularities are the intersection points $\Sigma_Q \cap \Sigma_{\mathcal{F}}$. Hence,

$$\mathrm{Ind}(X, W) = \mathrm{Ind}(\mathcal{X}, W) = \tilde{c}_1(\mathcal{F}) \cdot c_{n-1}(Q)[W] + \sum_{i=1}^{r} \mu_i ,$$

as stated.

We remark that this also provides a new proof of the well known fact, that the Grothendieck residue of $\mathcal{F}$ at an isolated singularity P_i, is the intersection number

$$\mu_i = \dim \frac{\mathcal{O}_{n,p}}{(F_1, \ldots, F_n)} ,$$

where $F = (F_1, \ldots, F_n)$ is a (local) holomorphic vector field on W at P, that spans $\mathcal{F}$ on a neighbourhood of P: In this case we take W to be a small neighbourhood D of P. Theorem II says,

$$\mathrm{Ind}(F; W) = (\mathrm{Res}_{c_n}(\mathcal{F}, P))$$

since $\tilde{c}_1(\mathcal{F})$ and $c_{n-1}(Q)$ vanish because D is contractible. The left hand side is the local index of F at P, and this equals the Milnor number μ. The right hand side equals the corresponding Grothendieck residue, by [BB].

Adding up theorems I and II we arrive to Theorem III, stated in the introduction:

Theorem III. *Let $n = 2k$ be even. Let q_n be the coefficient of c_n in Td_n. Let $(\mathcal{V}, P)$ be an ICIS of dimension n, and let X be a holomorphic vector field on $\mathcal{V}$, non-singular away from P. If $\tilde{\mathcal{F}}$ is the strict transform of the foliation*

on $\mathscr{V} - \{P\}$ determined by X, then the index of X on $\mathscr{V}$ at P is:

$$\mathrm{Ind}(X, \mathscr{V}) = \sum_{Z \subset S} \mathrm{Res}_{c_n}(\tilde{\mathscr{F}}, Z) + \tilde{c}_1(\mathscr{F}) \cdot c_{n-1}(Q)[\tilde{\mathscr{V}}]$$

$$+ \frac{1}{q_n}(\mathrm{Td}^*[\tilde{V}] + p_g) .$$

In particular, if the singularities of $\tilde{\mathscr{F}}$ are isolated, then the first term on the right hand side is the sum of the local Milnor numbers.

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References

[AH] M.F. Atiyah, F. Hirzebruch: Analytic cycles on complex manifolds. Topology **1**, 25–45 (1962)

[BB] P. Baum, R. Bott: Singularities of holomorphic foliations. J. Diff. Geom. **7**, 279–342 (1972)

[BG] Ch. Bonatti, X. Gomez-Mont: The index of holomorphic vector fields on singular varieties I. To be published in the proceedings of the International Symposium on Holomorphic Dynamical Systems, Rio de Janeiro, 1992

[BH] A. Borel, F. Hirzebruch: Characteristic classes and homogeneous spaces III. Am. J. Math. **82**, 491–504 (1969)

[CLS] C. Camacho, A. Lins, P. Sad: Topological invariants and equidesingularization for holomorphic vector fields. J. Diff. Geom. **20**, 143–174 (1984)

[CCS] C. Camacho, F. Cano, P. Sad: Absolutely isolated singularities of holomorphic vector fields. Inv. Math. **98**, 351–369 (1989)

[GSV] X. Gómez-Mont, J.A. Seade, A. Verjovsky: The index of a holomorphic flow with an isolated singularity. Math. Annalen **291**, 737–751 (1991)

[HH] G. Hector, U. Hirsch: Introduction to the geometry of foliations. Braunschweig Wiesbaden: Vieweg 1981

[Hr] F. Hirzebruch: Topological methods in algebraic geometry. Springer Verlag 1966

[Hu] D. Husemoller: Fibre bundles. Springer Verlag

[Ke] M. Kervaire: Relative characteristic classes. Am. J. Math. **79**, 517–558 (1957)

[La] H.B. Laufer: On μ for surface singularities. AMS Proc. Symp. Pure Math. **30**, part I, 45–49 (1977)

[LS] B. Li, J. Seade: Framings on algebraic knots. Quart. J. Math. Oxford (2), **38**, 297–306 (1987)

[L1] E. Looijenga: Isolated singular points on complete intersections. London Math. Soc., Lect. Notes **77** (1984)

[L2] E. Looijenga: Riemann Roch and smoothings of singularities. Topology **25**, 293–302 (1986)

[M1] J. Milnor: Topology from the differentiable viewpoint. Univ. Press of Virginia (1965)

[M2] J. Milnor: Singular points of complex hypersurfaces. Annals of Math. Studies 61. Princeton Univ. Press (1968)

[M3] J. Milnor: Characteristic classes. Annals of Math. Studies. Princeton Univ. Press

[SS] J. Seade, T. Suwa: Residues and singularities of holomorphic foliations on open manifolds. In preparation

[Se] J.P. Serre: Groupes algébriques et corpes de classes. Hermann, Paris, 1959

[St] J.H.M. Steenbrink: Mixed Hodge structures associated with isolated singularities. AMS Proc. Symp. Pure Maths. **40**, part 2, 513–536 (1983)