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Coupling of universal monodromy representations of Galois-Teichmüller modular groups

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1 Introduction

Let (g,n) be a pair of nonnegative integers satisfying the condition $2-2g-n<0$, and let $M_{g,n}$ be the moduli stack over $\mathbb{Q}$ of the smooth projective curves of genus g with n marked points. Fix a rational prime l. Then there arises, in a canonical way, a tower of profinite etale covers $\{M_{g,n}(m)\}_{m=1}^{\infty}$ over $M_{g,n}$ depending on l. Here $M_{g,n}(m)$ is, roughly speaking, the moduli of the punctured curves equipped with certain 'weight $-m$ structures' on their pro-l fundamental groups (see Sect. 2 (2.3) for the precise definition.) The field of definition $\mathbb{Q}_{g,n}(m)$ of $M_{g,n}(m)$ is known to be a pro-l Galois extension over the l-cyclotomic field $\mathbb{Q}(\mu_{l^\infty})$ unramified outside l. The field tower $\{\mathbb{Q}_{g,n}(m)\}_{m=1}^{\infty}$ generalizes Ihara's field tower $\{\mathbb{Q}(m)\}_{m=1}^{\infty}$ (e.g. [Ih1-2]) together with Oda's field tower $\{K(M_g,\text{class } m)\}_{m=1}^{\infty}$ ([O3]) in the following ways:

$$\mathbb{Q}_{0,3}(2m)=\mathbb{Q}(m),\quad \mathbb{Q}_{g,0}(m)=K(M_g,\text{class } m)\quad (m\geqq 1,\ g\geqq 2)\,.$$

In [NTU], we studied some properties of these field towers $\{\mathbb{Q}_{g,n}(m)\}_{m=1}^{\infty}$ and especially showed that they are independent of n at least when $n\geqq 1$. The purpose of this paper is then to relate these field towers of different genera. We prove

Theorem A. *Assume $n\geqq 1$. Then we have $\mathbb{Q}_{0,3}(m)\subset\mathbb{Q}_{g,n}(m)\subset\mathbb{Q}_{1,1}(m)$ for $m\geqq 1$. More precisely, it holds that*

$$\mathbb{Q}_{gr,n}(m)\subset\mathbb{Q}_{g,n}(m)$$

for $2-2g-n<0$, $2-2gr-n<0$, $n\geqq 1$ and $r\geqq 0$.

Our method for proving Theorem A is inspired from A. Grothendieck's note [G1], and depends on his fundamental work with J.P. Murre [GM] on

the tame fundamental groups of formal neighborhoods of normal crossing divisors. We shall give a device for coupling universal monodromy representations of possibly different genera (Theorem 3.10) which shows, for example, $\mathfrak{Q}_{g_1,n_1}(m)\mathfrak{Q}_{g_2,n_2}(m) \supset \mathfrak{Q}_{g_1+g_2,n_1+n_2-2}(m)$. The above theorem is obtained by arguments involving this coupling device together with the result of [NTU].

We have an application of Theorem A in the theory of exterior Galois representations of hyperbolic affine curves. Let X be a smooth projective curve of genus g over a number field k, S a subset of closed points of X with $S(\bar{k}) = n$, and let $C = X \setminus S$. Then we have a natural Galois representation

$$\varphi_C : G_k \to \mathrm{Out}\,\pi_1(\bar{C})(l)$$

of the absolute Galois group G_k into the outer automorphism group of the maximal pro-l quotient of the fundamental group of $\bar{C} = C \otimes \bar{k}$. We denote by $\mathrm{Out}_{G_k}\pi_1(\bar{C})(l)$ the centralizer of the Galois image $\varphi_C(G_k)$ in the target group, and call it the (pro-l) Galois centralizer of C. Assume that C is of hyperbolic type, i.e., $2-2g-n<0$. There exists then a natural injective homomorphism

$$\Phi_C : \mathrm{Aut}_k C \to \mathrm{Out}_{G_k}\pi_1(\bar{C})(l)$$

which can be expected to give a bijection (the pro-l anabelian Tate conjecture).

Theorem B. *If $n \geqq 1$, then the Galois centralizer of C can be embedded in $Sp(2g,\mathbf{Z}_l) \times S_n$. Moreover, if the endomorphism ring of the Jacobian of X is isomorphic to $\mathbf{Z}$, then the Galois centralizer of C is a finite group isomorphic to a subgroup of $\{\pm 1\} \times S_n$.*

In fact, φ_C has its image in the pro-l mapping class group $\Gamma_{g,[n]} \subset \mathrm{Out}\,\pi_1(\bar{C})(l)$ equipped with a natural weight filtration $\Gamma_{g,[n]} \supset \Gamma_{g,n}(1) \supset \Gamma_{g,n}(2) \supset \ldots$ (cf. [K], [NT1]). In Sect. 4 (4.6), (4.7), we shall see that φ_C has non-trivial Galois images in certain portion of $\mathrm{gr}^m\Gamma_{g,n} = \Gamma_{g,n}(m)/\Gamma_{g,n}(m+1)$ for every even m except 2, 4, 8, 12 (which are, in some sense, generated by Soule's cyclotomic elements [S] in degrees $m = 4k+2$ $(k = 1,2,\ldots)$). Theorem B is a consequence of the application of these Galois images to a standard argument of the graded coordinate system of $\Gamma_{g,[n]}$ (cf. [NT1] Sect. 3). We also should remark that Theorem B generalizes Corollary (5.2) of our previous work [N2], although the proof given there depends on essentially different kinds of Galois images via the elliptic analogue of Ihara's power series (See [N2]).

The Galois images obtained in the present paper, in a rough sense, appear in the complements to the images of the Johnson homomorphisms

$$\iota_m \otimes \mathbf{Q}_l : \mathrm{gr}^m\Gamma_{g,n}^{top} \otimes_{\mathbf{Z}} \mathbf{Q}_l \to \mathrm{gr}^m\Gamma_{g,n} \otimes_{\mathbf{Z}_l} \mathbf{Q}_l, \quad (m \geqq 1)$$

where $\mathrm{gr}^m\Gamma_{g,n}^{top}$ are graded quotient modules of the topological mapping class group of type (g,n) (cf. [AN]). In [AN], we generalized definitions of Johnson homomorphisms ι_m (over $\mathbf{Z}$) to any pair (g,n) with $2-2g-n<0$, and discussed which types of $\mathrm{Sp}(2g,\mathbf{Q}) \times S_n$-irreducible components can be contained in the image of the Johnson homomorphism $\iota_m \otimes \mathbf{Q}$. The problem of determining the images of ι_m has fairly topological features, and can be formulated

in a purely group theoretical way. So it would be surprising that the above existence of Galois images forces us to obtain nontrivial cokernels of $\iota_m \otimes \mathbb{Q}$ by the logic of arithmetic:

Theorem C. *If $n \geqq 1$, then the Johnson homomorphism $\iota_m \otimes \mathbb{Q}$ has nontrivial obstruction* $\mathrm{Coker}(\iota_m \otimes \mathbb{Q})$ *to its isomorphy for even m other than* 2, 4, 8, 12.

See also Theorem (4.8) for more precise estimates of the 'Galois obstructions' obtained by our method. We should note that S. Morita [Mo] gave different kinds of obstructions for odd $m \geqq 3$ by using his theory of trace mappings. It is also remarkable that T. Oda gave an effective lower bound for the rank of $\mathrm{gr}^m \Gamma_{g,n}^{top}$ ([O2]).

The organization of this paper is as follows. In Sect. 2, we formulate basic objects for introducing our field tower $\{\mathbb{Q}_{g,n}(m)\}_{m=1}^{\infty}$. We also compare two natural filtrations of $\pi_1(M_{g,n})$, and show their coincidence when $g \neq 1$. In Sect. 3, our basic theorem for coupling two universal monodromy representations of possibly different genera is proved (Theorem 3.10). As a series of its corollaries, we conclude the proof of Theorem A. In Sect. 4, we apply weight arguments to relate effectively graded Lie algebras associated to our field towers. The proofs of Theorems B, C together with a precise description of our Galois obstructions to the isomorphy of the Johnson homomorphism will be given there.

2 Basic setup and preliminaries

In the following, we fix a rational prime number l, and an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$.

(2.1) We first recall basic properties concerning universal pro-l monodromy representations. It is known that the natural morphism $f_M : M_{g,n+1} \to M_{g,n}$ obtained by forgetting the $(n+1)$-th marked point gives the universal family of n-point punctured smooth curves over $M_{g,n}$. Let ξ be a geometric point of $M_{g,n+1}$. The kernel of the induced homomorphism $p_M(\xi) : \pi_1(M_{g,n+1}, \xi) \to \pi_1(M_{g,n}, f_M(\xi))$ is isomorphic to $\hat{\Pi}_{g,n}$, the profinite fundamental group of the n-point punctured smooth curve of genus g corresponding to $f_M(\xi)$ (cf. [O1]). Suppose that ξ' is another geometric point of $M_{g,n+1}$. Then we can choose a chemin between ξ and ξ' which induces compatible isomorphisms $\pi_1(M_{g,n+1}, \xi) \cong \pi_1(M_{g,n+1}, \xi')$, $\pi_1(M_{g,n}, f_M(\xi)) \cong \pi_1(M_{g,n}, f_M(\xi'))$ and $\ker(p_M(\xi)) \cong \ker(p_M(\xi'))$. So, in the following, for avoiding nonessential complexities, we shall frequently omit references to base points and chemins for π_1 as far as arguments do not suffer from the omittion. In principle, in commutative diagrams of fundamental groups, the reader should agree that necessary base points and chemins are firstly taken in the origin sides of arrows, and in the target sides of arrows they are taken as natural projection images.

(2.2) Let $\Pi_{g,n}$ denote the maximal pro-l quotient of $\hat{\Pi}_{g,n}$. Then the map p_M of (2.1) induces an exterior monodromy representation into the pro-l mapping

class group $\Gamma_{g,n}$ (cf. [O3], [NTU] see [K], [NT1] for $\Gamma_{g,n}$):

$$\phi_{g,n} : \pi_1(M_{g,n}) \to \Gamma_{g,n} \subset \mathrm{Out}\,\Pi_{g,n} .$$

Recall that $\Gamma_{g,n}$ is equipped with a natural weight filtration $\Gamma_{g,n} \supset \Gamma_{g,n}(1) \supset \Gamma_{g,n}(2) \supset \ldots$ such that $\Gamma_{g,n}/\Gamma_{g,n}(1) = \mathrm{GSp}(2g, \mathbb{Z}_l)$ with $\mathrm{gr}^m \Gamma_{g,n}$ $(m \geqq 1)$ being a free $\mathbb{Z}_l$-module of finite rank.

(2.3) **Definition.** *Let $\pi_1(M_{g,n})(m)$ denote the inverse image of $\Gamma_{g,n}(m)$ by $\phi_{g,n}$ and let $M_{g,n}(m)$ denote the profinite etale cover of $M_{g,n}$ corresponding to it. We define a number field $\mathbb{Q}_{g,n}(m)$ to be the constant field of the function field of $M_{g,n}(m)$. It is easy to see that if $p_{g,n} : \pi_1(M_{g,n}) \to G_{\mathbb{Q}}$ is the natural augmentation homomorphism, then $G_{\mathbb{Q}_{g,n}(m)} = p_{g,n}(\pi_1(M_{g,n})(m))$.*

(2.4) We next introduce another filtration $\pi_1(M_{g,n+1})[m]$ $(m \geqq 1)$ as follows. By conjugation, we have a natural action of $\pi_1(M_{g,n+1})$ on the maximal pro-l quotient of $\ker(p_M) \cong \hat{\Pi}_{g,n}$ which has its image in the pro-l braid-like automorphism group $\tilde{\Gamma}_{g,n}$:

$$\psi_{g,n} : \pi_1(M_{g,n+1}) \to \tilde{\Gamma}_{g,n} \subset \mathrm{Aut}\,\Pi_{g,n} .$$

Recall that there is a natural weight filtration $\{\tilde{\Gamma}_{g,n}(m)\}_{m=1}^{\infty}$. Then, we shall define

$$\pi_1(M_{g,n+1})[m] := \psi_{g,n}^{-1}(\tilde{\Gamma}_{g,n}(m)) .$$

We also introduce the pro-l weight filtration $\{\hat{\Pi}_{g,n}(m)\}_m$ to be the pull-back of $\{\Pi_{g,n}(m)\}_m$ via the natural map $\hat{\Pi}_{g,n} \to \Pi_{g,n}$.

(2.5) **Proposition.** *The map $p_M : \pi_1(M_{g,n+1}) \to \pi_1(M_{g,n})$ induces the following exact sequence*

$$1 \to \hat{\Pi}_{g,n}(m) \to \pi_1(M_{g,n+1})[m] \to \pi_1(M_{g,n})(m) \to 1 .$$

Proof. The left injectivity and the right surjectivity are obvious from definitions. The middle exactness follows from e.g. [NT1] Sect. 2 Theorem (1.14). □

(2.6) In order to compare $\pi_1(M_{g,n})(m)$ and $\pi_1(M_{g,n})[m]$, we need a more precise argument involving base points. Choose a geometric base point ξ of $M_{g,n+2}$ valued in $\mathbb{C}$, and let $f_M^{(i)} : M_{g,n+2} \to M_{g,n+1}$ $(i = 1, 2)$ be the morphism obtained by forgetting the $(n+i)$-th marked point, and by substituting the $(n+2)$-th marked point for the $(n+1)$-th one when $i = 1$. Put $\xi_i = f_M^{(i)}(\xi)$, $\xi_0 = f_M(\xi_i)$ and $p_M^{(i)} = \pi_1(f_M^{(i)})$ $(i = 1, 2)$. Then we have a commutative diagram

$$\begin{array}{ccc} \pi_1(M_{g,n+1}, \xi_1) & \xleftarrow{p_M^{(1)}} & \pi_1(M_{g,n+2}, \xi) \\ {\scriptstyle p_M(\xi_1)}\downarrow & & \downarrow{\scriptstyle p_M^{(2)}} \\ \pi_1(M_{g,n}, \xi_0) & \xleftarrow{p_M(\xi_2)} & \pi_1(M_{g,n+1}, \xi_2) . \end{array} \tag{2.7}$$

Let $(R; x_1, \ldots, x_{n+2})$ be the $(n+2)$-pointed Riemann surface corresponding to ξ, and identify $\pi_1(\overline{M}_{g,n+2}, \xi)$ with the profinite completion of the mapping class

group of $(R; x_1, \ldots, x_{n+2})$ (cf. [O1]). If we denote by τ the image of the Dehn twist along a simple closed curve on R bounding a disk Δ containing only x_{n+1} and x_{n+2} from the x_i $(1 \leqq i \leqq n+2)$, then we see that $\tau \in \ker(p_M^{(i)})$ $(i = 1, 2)$. Moreover it follows that the normalizer N_τ of $\langle \tau \rangle$ in $\pi_1(M_{g,n+2})$ is surjectively mapped by $p_M^{(i)}$ while $\langle \tau \rangle$ is self-normalizing in each of $\ker(p_M^{(i)})$ $(i = 1, 2)$.

(2.8) We define a map $\iota : \pi_1(M_{g,n+1}, \xi_1) \to \pi_1(M_{g,n+1}, \xi_2)$ by

$$\iota(\sigma) = p_M^{(2)}(p_M^{(1)\,-1}(\sigma) \cap N_\tau)$$

for $\sigma \in \pi_1(M_{g,n+1}, \xi_1)$. It is well-defined by the last remark in (2.6). By construction and Proposition (2.5), we have

$$\iota(\pi_1(M_{g,n+1}, \xi_1)(m)) \subset \pi_1(M_{g,n+1}, \xi_2)[m] \quad (m \geqq 1)\,. \tag{2.8.1}$$

On the other hand,

$$\iota(\pi_1(M_{g,n+1}, \xi_1)(m)) = \pi_1(M_{g,n+1}, \xi_2)(m)\,. \tag{2.8.2}$$

In fact, we first show the identity in the geometric level, i.e., for $\overline{M}_{g,n+1} = M_{g,n+1} \otimes \overline{\mathbf{Q}}$. Then, ι has the following topological interpretation. Namely, let σ be an isotopy class of orientation preserving diffeomorphisms of $(R; x_1, \ldots, \check{x}_{n+1}, x_{n+2})$. It is possible to choose a diffeomorphism from σ which is identity on Δ. Then it gives a class of diffeomorphism of $(R; x_1, \ldots, x_{n+1})$ which is nothing but $\iota(\sigma)$. So, in this sense, ι is just a tautological identification of the mapping class groups of $(R; x_1, \ldots, \check{x}_{n+1}, x_{n+2})$ and of $(R; x_1, \ldots, x_{n+1})$. Thus ι can be regarded as an isomorphism via a chemin constructed topologically as above from ξ_1 to ξ_2 at least in the geometric level. But since ι is $\pi_1(M_{g,n}, \xi_0)$-compatible, and since $\ker(p_M(\xi_1) \circ p_M^{(1)})$ has trivial center, we see that ι coincides with the isomorphism of the whole $\pi_1(M_{g,n+1}, \xi_1)$ onto $\pi_1(M_{g,n+1}, \xi_2)$ induced from that chemin (cf. e.g. [N] 1.5). Therefore (2.8.2) follows.

From (2.8.1) and (2.8.2), we get

$$\pi_1(M_{g,n+1})(m) \subset \pi_1(M_{g,n+1})[m]\,. \tag{2.8.3}$$

(2.9) **Lemma.** *If $g \neq 1$, then we have $\pi_1(M_{g,n+1})(m) = \pi_1(M_{g,n+1})[m]$.*

Proof. Before going to the proof of (2.9), we shall show that

$$\hat{\Pi}_{g,n}(m) \subset \pi_1(M_{g,n+1})(m)\,. \tag{2.10}$$

Here $\hat{\Pi}_{g,n}$ denotes the kernel of a forgetful homomorphism with respect to a marked point. Consider the diagram (2.7) and identify $\hat{\Pi}_{g,n} \cong \ker(p_M(\xi_1))$. Recall that the maximal pro-l quotient of $p_M(\xi_i) \circ p_M^{(i)}$ is isomorphic to $\Pi_{g,n}^{(2)}$ studied in [NTU] Sect. 2 which has a good weight filtration inducing an exact sequence

$$0 \to \mathrm{gr}^m \Pi_{g,n+1} \to \mathrm{gr}^m \Pi_{g,n}^{(2)} \to \mathrm{gr}^m \Pi_{g,n} \to 0\,.$$

If $\sigma \in \hat{\Pi}_{g,n}(m)$, then the exterior action of σ on the maximal pro-l quotient $\ker(p_M^{(1)})(l) \cong \Pi_{g,n+1}$ comes from the conjugate action of an element of $\Pi_{g,n}^{(2)}(m)$. Hence $\sigma \in \pi_1(M_{g,n+1})(m)$ and (2.10) follows.

We next show the following

(2.11) **Claim.** *If $g \neq 1$, $n \geqq 1$, then p_M maps $\pi_1(M_{g,n+1})(m)$ surjectively onto $\pi_1(M_{g,n})(m)$.*

Proof. By [NTU], we know that $\mathbb{Q}_{g,n}(m) = \mathbb{Q}_{g,n+1}(m)$. Hence it suffices to see the surjectivity of the geometric part. Consider the commutative diagram

$$\begin{array}{ccc} \pi_1(\overline{M}_{g,n+1})(m) & \xrightarrow{\phi_{g,n+1}} & \Gamma_{g,n+1}(m) \\ \downarrow & & \downarrow \\ \pi_1(\overline{M}_{g,n})(m) & \xrightarrow{\phi_{g,n}} & \Gamma_{g,n}(m)\,. \end{array}$$

where $\overline{M}$ means $M \otimes \overline{\mathbb{Q}}$. In [A], Asada showed that if $g \neq 1$, then the image of $\pi_1(\overline{M}_{g,n})(m)$ in $\mathrm{gr}^m \Gamma_{g,n}$ is the l-adic closure of the image of the discrete mapping class group $\Gamma_{g,n}^{top}(m)$. We also know the surjectivity of $\Gamma_{g,n+1}^{top}(m) \to \Gamma_{g,n}^{top}(m)$ at least for $n \geqq 1$ ([A]). Applying these facts and (2.10) to an easy snake-lemma argument, we conclude the claim. □

Proof of Lemma (2.9). Let $\sigma \in \pi_1(M_{g,n+1})[m]$. By Proposition (2.5) and Claim (2.11) there exists $\sigma' \in \pi_1(M_{g,n+1})(m)$ such that $p_M(\sigma) = p_M(\sigma')$. Then by Proposition (2.5) again, we have $\sigma'\sigma^{-1} \in \hat{\Pi}_{g,n}(m)$. Thus from (2.10) follows that $\sigma \in \pi_1(M_{g,n+1})(m)$. □

(2.12) We next introduce a filtration in $\pi_1(M_{g_1,n_1} \times M_{g_2,n_2})$. The forgetful morphisms $f_{M,i}^* : M_{g_i,n_i+1} \to M_{g_i,n_i}$ obtained by forgetting the n_i-th point and substituting the (n_i+1)-th point for the n_i-th one $(i = 1,2)$ yield two families of punctured curves

$$f_{M,1} = f_{M,1}^* \otimes id : M_{g_1,n_1+1} \times M_{g_2,n_2} \to M_{g_1,n_1} \times M_{g_2,n_2}\,,$$

$$f_{M,2} = id \otimes f_{M,2}^* : M_{g_1,n_1} \times M_{g_2,n_2+1} \to M_{g_1,n_1} \times M_{g_2,n_2}\,.$$

Combining pro-l monodromy representations induced from the above two families, we obtain a new representation

$$\phi_{\mathcal{D}} : \pi_1(M_{g_1,n_1} \times M_{g_2,n_2}) \to \Gamma_{g_1,n_1} \times \Gamma_{g_2,n_2}\,.$$

For a pair of positive integers (m_1, m_2), let $\pi_1(M_{g_1,n_1} \times M_{g_2,n_2})(m_1,m_2)$ be the inverse image of $\Gamma_{g_1,n_1}(m_1) \times \Gamma_{g_2,n_2}(m_2)$.

The following proposition is easy, but will appear a crucial observation in our later investigation.

(2.13) **Proposition.** *Let $p_{\mathcal{D}} : \pi_1(M_{g_1,n_1} \times M_{g_2,n_2}) \to G_{\mathbb{Q}}$ be the natural augmentaion map. Then*

$$p_{\mathcal{D}}(\pi_1(M_{g_1,n_1} \times M_{g_2,n_2})(m_1,m_2)) = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}_{g_1,n_1}(m_1)\mathbb{Q}_{g_2,n_2}(m_2))\,.$$

Proof. Let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}_{g_1,n_1}(m_1)\mathbb{Q}_{g_2,n_2}(m_2))$. Then there exists $\sigma_i \in \pi_1(M_{g_i,n_i})(m_i)$ with $p_{g_i,n_i}(\sigma_i) = \sigma$ $(i = 1,2)$. Consider the commutative diagram

$$\begin{array}{ccc} \pi_1(M_{g_1,n_1} \times M_{g_2,n_2}) & \xrightarrow{p_1} & \pi_1(M_{g_1,n_1}) \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_{g_1,n_1}} \\ \pi_1(M_{g_2,n_2}) & \xrightarrow{p_{g_2,n_2}} & G_{\mathbb{Q}}\,. \end{array}$$

Since the above p_1 maps $p_2^{-1}(\sigma_2)$ bijectively onto $p_{g_1,n_1}^{-1}(\sigma)$, there exists $\tilde{\sigma} \in \pi_1(M_{g_1,n_1} \times M_{g_2,n_2})$ with $p_i(\tilde{\sigma}) = \sigma_i$ $(i = 1,2)$. Hence the proposition follows (the converse direction is obvious). □

3 Coupling of monodromy representations

We fix two pairs (g_1,n_1), (g_2,n_2) of nonnegative integers with $2 - 2g_i - n_i < 0$, $n_i \geqq 1$ $(i = 1,2)$. Put $g = g_1 + g_2$, $n = n_1 + n_2 - 2$ and assume that $g \geqq 2$, $n \geqq 1$.

(3.1) Let us begin with a preliminary argument on pro-l mapping class groups. We fix standard presentations of Π_{g_i,n_i} $(i = 1,2)$ and $\Pi_{g,n}$ as follows:

$$\Pi_{g_1,n_1} = \left\langle \begin{matrix} x_1,\ldots,x_{2g_1} \\ z_1,\ldots,z_{n_1} \end{matrix} \middle| [x_1,x_{g_1+1}]\cdots[x_{g_1},x_{2g_1}]z_1\cdots z_{n_1} = 1 \right\rangle ,$$

$$\Pi_{g_2,n_2} = \left\langle \begin{matrix} y_1,\ldots,y_{2g_2} \\ w_1,\ldots,w_{n_2} \end{matrix} \middle| [y_1,y_{g_2+1}]\cdots[y_{g_2},y_{2g_2}]w_1\cdots w_{n_2} = 1 \right\rangle ,$$

$$\Pi_{g,n} = \left\langle \begin{matrix} x_1,\ldots,x_{2g_1},y_1,\ldots,y_{2g_2} \\ z_1,\ldots,z_{n_1-1},w_1,\ldots,w_{n_2-1},\delta \end{matrix} \middle| \begin{matrix} \delta = [x_1,x_{g_1+1}]\cdots[x_{g_1},x_{2g_1}]z_1\cdots z_{n_1-1} \\ = [y_1,y_{g_2+1}]\cdots[y_{g_2},y_{2g_2}]w_1\cdots w_{n_2-1} \end{matrix} \right\rangle .$$

Here 'standard' means that z_j, w_j give generators of some inertia subgroups over punctures of curves. Noticing that these three groups are free pro-l groups, we may consider Π_{g_i,n_i} $(i = 1,2)$ are subgroups of $\Pi_{g,n}$ with amalgamation $\delta = z_{n_1}^{-1} = w_{n_2}^{-1}$.

(3.2) Let $\Gamma^*_{g_i,n_i}$ be the subgroup of $\tilde{\Gamma}_{g_i,n_i}$ whose elements stabilize the cyclic group $\langle\delta\rangle$ (cf. [NT1] Sect. 2). Then for each pair $(\sigma_1',\sigma_2') \in \Gamma^*_{g_1,n_1} \times \Gamma^*_{g_2,n_2}$, we can define $\partial'(\sigma_1',\sigma_2') \in \tilde{\Gamma}_{g,n}$ by $\partial'(\sigma_1',\sigma_2')(\gamma) = \sigma_i'(\gamma)$ for $\gamma \in \Pi_{g_i,n_i}$ $(i = 1,2)$. On the other hand, since $\mathrm{Int}(\delta)$ gives a nontrivial element of $\mathrm{gr}^2\Gamma^*_{g_i,n_i}$ and $\langle\delta\rangle$ is self-normalizing in Π_{g_i,n_i} for each $i = 1,2$, it follows that $\Gamma^*_{g_i,n_i}(3) \cong \Gamma_{g_i,n_i}(3)$. Thus we get a natural mapping

$$\partial : \Gamma_{g_1,n_1}(3) \times \Gamma_{g_2,n_2}(3) \to \Gamma_{g,n}(3)\,.$$

(3.3) **Lemma.** *The above map ∂ preserves strictly weight filtration. Namely,*

$$\partial^{-1}(\Gamma_{g,n}(m)) = \Gamma_{g_1,n_1}(m) \times \Gamma_{g_2,n_2}(m) \quad (m \geqq 3)\,.$$

Proof. Let $\sigma_i \in \Gamma_{g_i,n_i}(m_i)$ with $m_i \geqq 3$ and suppose that $\partial(\sigma_1,\sigma_2) \in \Gamma_{g,n}(m)$ $(m \geqq 3)$. There exists a unique lift $\sigma_i' \in \Gamma^*_{g_i,n_i}(m_i)$ of σ_i. Without loss of generality, we may assume $m_1 \leqq m_2$ and $\sigma_1 \notin \Gamma_{g_1,n_1}(m_1+1)$. We will deduce a contradiction from an additional hypothesis that $m_1 < m$. Consider the m_1-th stage coordinates $c_{m_1}(\sigma_i') \in C^{m_1}(2g_i, n_i)$ for $i = 1,2$. Then their last coordinates 'at the δ-positions' are by definition 0 and the m_1-th stage coordinates of $\partial'(\sigma_1',\sigma_2')$ is just a composite arrangement of $c_{m_1}(\sigma_1')$ and $c_{m_1}(\sigma_2')$ after omitting those 0-coordinates at the δ-positions. One can observe then that the image of the 'diagonal' embedding of $\mathrm{gr}^{m_1}\Pi_{g,n}$ into $C^{m_1}(2g,n)$ cannot cancel out the nontrivial coordinates of $c_{m_1}(\sigma_1')$. Thus the reduced coordinates of $\partial(\sigma_1,\sigma_2)$ remains nontrivial. This contradicts our assumption $m_1 < m$. □

(3.4) Let $\mathscr{H}_g$ $(g \geqq 2)$ be the moduli scheme over $\mathbb{Q}$ of all the tri-canonically embedded stable curves introduced by Deligne-Mumford [DM] which is known to be a smooth irreducible scheme of finite type over $\mathbb{Q}$. Let $\mathscr{M}_{g,n}$ be the moduli stack of n-pointed stable curves of genus g studied by Deligne-Mumford [DM] and Knudsen [K] (for simplicity we write $\mathscr{M}_{g,0} = \mathscr{M}_g$.) The forgetful morphism $h_g : \mathscr{H}_g \to \mathscr{M}_g$ is representable, smooth and surjective with fibres $PGL(5g-6)$-torsors. Put $\mathscr{H}_{g,n} = \mathscr{H}_g \times_{\mathscr{M}_g} \mathscr{M}_{g,n}$ and denote by $h_{g,n} : \mathscr{H}_{g,n} \to \mathscr{M}_{g,n}$ the canonical projection. We write $H_{g,n} \subset \mathscr{H}_{g,n}$ for the pullback of $M_{g,n} \subset \mathscr{M}_{g,n}$ via $h_{g,n}$. It is an open subscheme of $\mathscr{H}_{g,n}$ with its complement being normal crossing divisors. It is known that the fundamental group of $PGL(5g-6,\mathbb{C})$ is cyclic of order $5g-6$, hence there is an exact sequence of profinite groups

$$\mathbb{Z}/(5g-6)\mathbb{Z} \longrightarrow \pi_1(H_{g,n}) \xrightarrow{\mu_{g,n}} \pi_1(M_{g,n}) \longrightarrow 1\,.$$

The natural forgetful map $f_H : H_{g,n+1} \to H_{g,n}$ induced from $f_M : M_{g,n+1} \to M_{g,n}$ gives a family of n-point punctured smooth curves of genus g over $H_{g,n}$. From the surjectivity of $\mu_{g,n}$, we see that the kernels of $\pi_1(f_H)$ and $\pi_1(f_M)$ are isomorphic, and so we get a monodromy representation

$$\varphi_{g,n} : \pi_1(H_{g,n}) \to \Gamma_{g,n}$$

which obviously factors through $\phi_{g,n} : \pi_1(M_{g,n}) \to \Gamma_{g,n}$. Define $\pi_1(H_{g,n})(m) := \varphi_{g,n}^{-1}(\Gamma_{g,n}(m))$. Then the profinite etale cover $H_{g,n}(m)$ over $H_{g,n}$ pulled back from $M_{g,n}(m)$ has fundamental group $\pi_1(H_{g,n})(m)$ and is defined over $\mathbb{Q}_{g,n}(m)$ as $\mu_{g,n}$ is surjective.

(3.5) Next we consider certain smooth divisors $\mathscr{E} \subset \mathscr{M}_{g,n}$, $\mathscr{E}_i \subset \mathscr{M}_{g,n+1}$ $(i = 1,2)$. These divisors are the images of Knudsen's clutching morphisms

$$\beta_{g_1,g_2,H,K} : \mathscr{M}_{g_1,n_1} \times \mathscr{M}_{g_2,n_2} \to \mathscr{M}_{g,n}\,,$$

$$\beta_{g_1,g_2,H_1,K} : \mathscr{M}_{g_1,n_1+1} \times \mathscr{M}_{g_2,n_2} \to \mathscr{M}_{g,n+1}\,,$$

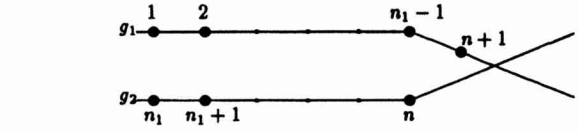
$$\beta_{g_1,g_2,H,K_2} : \mathscr{M}_{g_1,n_1} \times \mathscr{M}_{g_2,n_2+1} \to \mathscr{M}_{g,n+1}$$

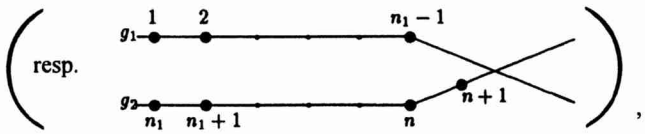
with $H = \{1,\dots,n_1-1\}, K = \{n_1,\dots,n\}, H_1 = \{1,\dots,n_1-1,n+1\}$ and $K_2 = \{n_1,\dots,n+1\}$ respectively. Notice that under our assumptions, these morphisms are closed immersions ([Kn] Corollary 3.9). Recall also that the substack $\mathscr{S}_{g,n}$ of $\mathscr{M}_{g,n}$ consisting of singular curves is a divisor with normal crossing. We define $\Sigma_{g,n}$ (resp. $\Sigma_{g,n+1}$) to be the union of those divisors in $\mathscr{S}_{g,n}$ (resp. $\mathscr{S}_{g,n+1}$) other than $\mathscr{E}$ (resp. $\mathscr{E}_1, \mathscr{E}_2$) and set

$$\mathscr{M}^{\sim}_{g,n} = \mathscr{M}_{g,n}\backslash\Sigma_{g,n}, \quad \mathscr{M}^{\sim}_{g,n+1} = \mathscr{M}_{g,n+1}\backslash\Sigma_{g,n+1},$$

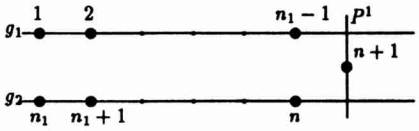
$$\mathscr{D} = \mathscr{E}\backslash\Sigma_{g,n}, \quad \mathscr{D}_i = \mathscr{E}_i\backslash\Sigma_{g,n+1} \ (i=1,2), \quad \mathscr{D}_0 = \mathscr{D}_1 \cap \mathscr{D}_2 .$$

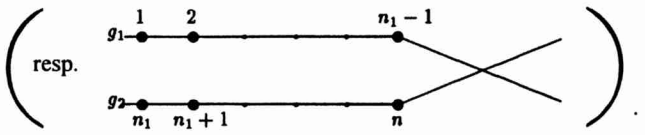
Then the points of $\mathscr{D}_1\backslash\mathscr{D}_0$ (resp. $\mathscr{D}_2\backslash\mathscr{D}_0$) parametrize the stable curves of type





and the points of $\mathscr{D}_0$ (resp. $\mathscr{D}$) parametrize the following type stable curves





Moreover we have the following commutative diagrams

(3.5.1)

$$\begin{array}{ccccccc}
M_{g_1,n_1+1}\times M_{g_2,n_2} & \xrightarrow{\sim} & \mathscr{D}_1\backslash\mathscr{D}_0 & \hookrightarrow & \mathscr{M}^{\sim}_{g,n+1} & \hookleftarrow & \mathscr{D}_2\backslash\mathscr{D}_0 \xleftarrow{\sim} M_{g_1,n_1}\times M_{g_2,n_2+1} \\
\downarrow{\scriptstyle f_{M,1}} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f_{M,2}} \\
M_{g_1,n_1}\times M_{g_2,n_2} & \xrightarrow{\sim} & \mathscr{D} & \hookrightarrow & \mathscr{M}^{\sim}_{g,n} & \hookleftarrow & \mathscr{D} \xleftarrow{\sim} M_{g_1,n_1}\times M_{g_2,n_2}
\end{array}$$

and

(3.5.2)

$$\begin{array}{ccccc}
M_{g_1,n_1}\times M_{g_2,n_2} & \xrightarrow{\sim} & \mathscr{D}_0 & \xrightarrow{\text{immer.}} & \mathscr{M}^{\sim}_{g,n+1} \\
\| & & \downarrow & & \downarrow \\
M_{g_1,n_1}\times M_{g_2,n_2} & \xrightarrow{\sim} & \mathscr{D} & \xrightarrow{\text{immer.}} & \mathscr{M}^{\sim}_{g,n},
\end{array}$$

where vertical arrows other than $f_{M,i}$ $(i = 1,2)$ (see 2.12) are forgetful (and contraction) morphisms with respect to the $(n+1)$-th marked point. See also Fig. A, B for moduli theoretic understanding of the commutativities of (3.5.1) and (3.5.2).

(3.6) We then consider schemes

$$H^{\sim}_{g,n} = h^{-1}_{g,n}(\mathscr{M}^{\sim}_{g,n})$$
$$H^{\sim}_{g,n+1} = h^{-1}_{g,n+1}(\mathscr{M}^{\sim}_{g,n+1})$$

together with the divisors $D = h^{-1}_{g,n}(\mathscr{D})$, $D_i = h^{-1}_{g,n+1}(\mathscr{D}_i)$ $(i = 1,2)$ on them. Obviously we have $H^{\sim}_{g,n} = H_{g,n} \cup D$ and $H^{\sim}_{g,n+1} = H_{g,n+1} \cup D_1 \cup D_2$. Moreover if we put $D_0 = D_1 \cap D_2$, then we see that

$$D_0 \cong D\,,$$
$$D_1 \backslash D_0 = h^{-1}_{g,n+1}(\beta_{g_1,g_2,H_1,K}(M_{g_1,n_1+1} \times M_{g_2,n_2}))\,,$$
$$D_2 \backslash D_0 = h^{-1}_{g,n+1}(\beta_{g_1,g_2,H,K_2}(M_{g_1,n_1} \times M_{g_2,n_2+1}))$$

where β_* are clutching morphisms in (3.5). Put $D_3 = D_1 \cup D_2$ and let $f_{D,i}$: $D_i \to D$ denote the forgetful (and contraction) morphism with respect to the $(n+1)$-th marked point. Then the restrictions of $f_{D,i}$ to $D_i \backslash D_0$ give homotopy

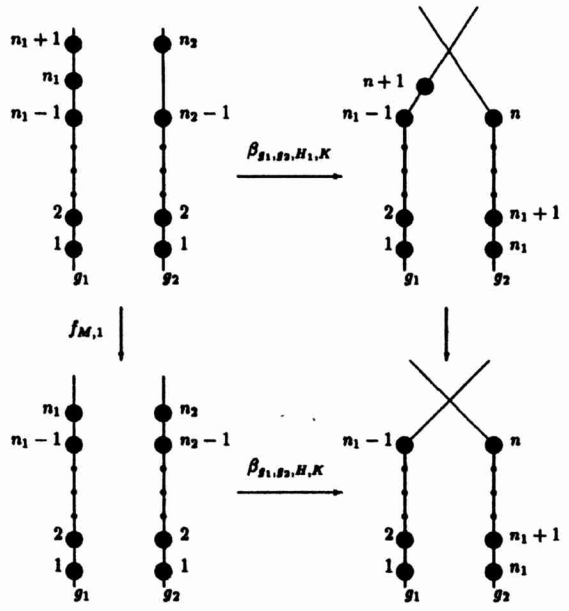


Fig. A. (Left) The commutativity of (3.5.1)

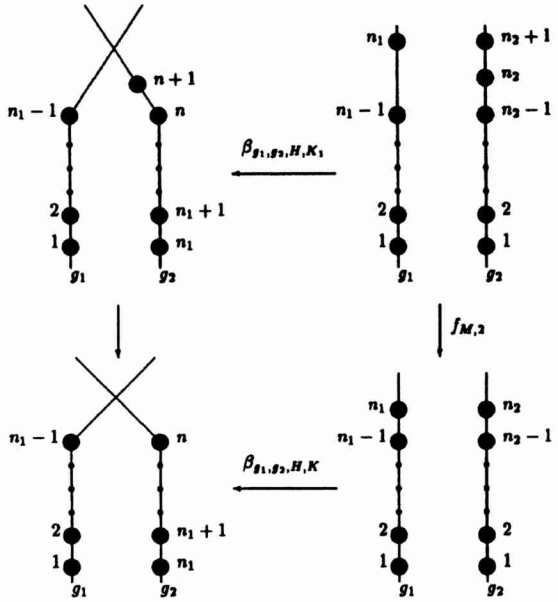


Fig. A. (Right) The commutativity of (3.5.1)

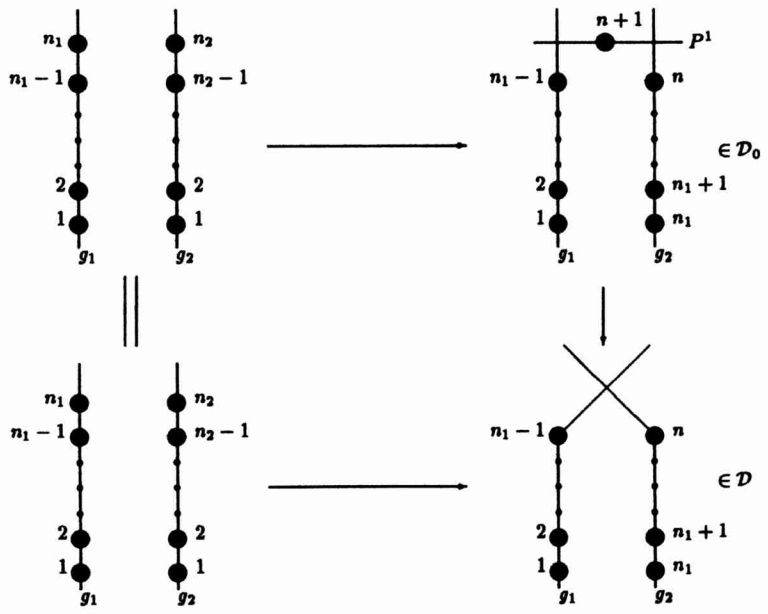


Fig. B. The commutativity of (3.5.2)

exact sequences

$$1 \to \hat{\Pi}_{g_i, n_i} \to \pi_1(D_i - D_0) \xrightarrow{p_{D,i}} \pi_1(D) \to 1 \quad (i = 1,2)$$

together with a pro-l monodromy representation

$$\varphi_D : \pi_1(D) \to \Gamma_{g_1, n_1} \times \Gamma_{g_2, n_2}$$

in a similar way to (2.12). Define, for a pair of positive integers (m_1, m_2), $\pi_1(D)(m_1, m_2)$ to be the inverse image of $\Gamma_{g_1, n_1}(m_1) \times \Gamma_{g_2, n_2}(m_2)$ via φ_D. Obviously φ_D is the composite of the natural map $\mu_D : \pi_1(D) \to \pi_1(M_{g_1, n_1} \times M_{g_2, n_2})$ with $\phi_{\mathscr{D}}$ of (2.12). Therefore we see from (2.13) that

$$p_D(\pi_1(D)(m_1, m_2)) = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}_{g_1, n_1}(m_1)\mathbb{Q}_{g_2, n_2}(m_2)),$$

where $p_D : \pi_1(D) \to G_{\mathbb{Q}}$ denotes the canonical augmentation map ($p_D = p_{\mathscr{D}} \circ \mu_D$).

(3.7) Let $H^{\wedge}_{/D_i}$ (resp. $H^{\wedge}_{/D}$) denote the formal completion of $H^{\sim}_{g, n+1}$ (resp. $H^{\sim}_{g, n}$) along the divisor D_i ($i = 0, 1, 2, 3$) (resp. D). Then by the theory of Grothendieck-Murre [GM], we have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \langle \tau_i \rangle & \longrightarrow & \pi_1^{D_3}(H^{\wedge}_{/D_i}) & \xrightarrow{\varepsilon_{D,i}} & \pi_1(D_i - D_0) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow \hat{p}_{D,i} & & \downarrow p_{D,i} & & \\ 1 & \longrightarrow & \langle \tau \rangle & \longrightarrow & \pi_1^{D}(H^{\wedge}_{/D}) & \xrightarrow{\varepsilon_D} & \pi_1(D) & \longrightarrow & 1 \end{array} \tag{3.7.1}$$

for $i = 1, 2$. Here π_1^* denotes the fundamental group tamely ramified along (the image of) $*$, and $\langle \tau \rangle$ and $\langle \tau_i \rangle$ are isomorphic to $\hat{\mathbb{Z}}(1)$. We can take generators τ, τ_i of the latter groups so that they correspond to (profinite) conjugates of Dehn twists along certain separating simple closed curves on marked Riemann surfaces. (The left injectivities of the above sequences can follow from this topological observation, because they have full procyclic images through monodromy representations.)

(3.8) There are natural homomorphisms

$$\partial_{D_i} : \pi_1^{D_3}(H^{\wedge}_{/D_i}) \to \pi_1(H_{g, n+1}) \quad (i = 1, 2, 3, 4),$$
$$\partial_D : \pi_1^{D}(H^{\wedge}_{/D}) \to \pi_1(H_{g, n})$$

making the following diagram commutes (these maps are expected to be injective, but the author does not know whether or not):

$$\begin{array}{ccccccc} \pi_1^{D_3}(H^{\wedge}_{/D_0}) & \xrightarrow{q_{0,i}} & \pi_1^{D_3}(H^{\wedge}_{/D_i}) & \xrightarrow{q_{i,3}} & \pi_1^{D_3}(H^{\wedge}_{/D_3}) & \xrightarrow{\partial_{D_3}} & \pi_1(H_{g,n+1}) \\ \hat{p}_{D,0} \downarrow & & \hat{p}_{D,i} \downarrow & & \hat{p}_{D,3} \downarrow & & p_H \downarrow \\ \pi_1^{D}(H^{\wedge}_{/D}) & = & \pi_1^{D}(H^{\wedge}_{/D}) & = & \pi_1^{D}(H^{\wedge}_{/D}) & \xrightarrow{\partial_D} & \pi_1(H_{g,n}). \end{array} \tag{3.8.1}$$

Since the natural map $\pi_1^{D_3}(H^{\wedge}_{/D_0}) \to \pi_1(D_0)$ is surjective (cf. [GM] p. 51), and since τ_i can be taken already from $\pi_1^{D_3}(H^{\wedge}_{/D_0})$, the map $\hat{p}_{D,0}$ is surjective. Let $\hat{\delta} = q_{1,3}(\tau_1)q_{2,3}(\tau_2)^{-1}$. Then, observing that $\ker(p_H) \cong \hat{\Pi}_{g,n}$ is an amalgamated product of two free profinite groups isomorphic respectively to $\ker(\hat{p}_{D,i}) \cong \hat{\Pi}_{g_i,n_i}$ (3.7.1) $(i = 1,2)$ along $\langle \partial_{D_3}(\hat{\delta}) \rangle$, we see that $q_{i,3}$ $(i = 1,2)$ are both injective. Thus, we shall often consider $\pi_1^{D_3}(H^{\wedge}_{/D_i})$ $(i = 1,2)$ as a subgroup of $\pi_1^{D_3}(H^{\wedge}_{/D_3})$. We also henceforth identify $\ker(\hat{p}_{D,3})$ with $\ker(p_H)$ and introduce standard generator systems of $\hat{\Pi}_{g,n}$ and $\hat{\Pi}_{g_i,n_i}$ $(i = 1,2)$ so that they fit into the presentations (3.1) in their maximal pro-l quotients with $\hat{\delta} = \tau_1\tau_2^{-1}$ mapped to $\delta = z_{n_1}^{-1} = w_{n_2}^{-1}$ in the presentations. Moreover, it follows easily that $\mathrm{Im}(q_{0,1}) = \mathrm{Im}(q_{0,2})$ is the normalizer of $\langle \hat{\delta} \rangle$ in $\pi_1^{D_3}(H^{\wedge}_{/D_3})$.

(3.9) Applying [GM] to $\pi_1^{D_0}(D^{\wedge}_{i/D_0})$, we have the following commutative diagram of exact sequences $(i = 1,2)$:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \langle \hat{\delta} \rangle & \longrightarrow & \pi_1^{D_0}(D^{\wedge}_{i/D_0}) & \xrightarrow{\varepsilon_{i,0}} & \pi_1(D_0) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle \lambda_{D,i}} & & \downarrow{\scriptstyle p_{D,0}} \cong & & \\
1 & \longrightarrow & \hat{\Pi}_{g_i,n_i} & \longrightarrow & \pi_1(D_i - D_0) & \xrightarrow{p_{D,i}} & \pi_1(D) & \longrightarrow & 1\,.
\end{array}
\tag{3.9.1}
$$

We shall then relate $\pi_1^{D_3}(H^{\wedge}_{/D_0})$ and $\pi_1^{D_0}(D^{\wedge}_{i/D_0})$ $(i = 1,2)$ as follows. Observe first that both $\mathrm{Im}(\varepsilon_{D,i} \circ q_{0,i})$ and $\mathrm{Im}(\lambda_{D,i})$ coincide with the normalizer of $\langle \hat{\delta} \rangle$ in $\pi_1(D_i - D_0)$. Hence by the injectivity of $\lambda_{D,i}$ (3.9.1), we obtain a natural homomorphism $\varepsilon_{H,i} : \pi_1^{D_3}(H^{\wedge}_{/D_0}) \to \pi_1^{D_0}(D^{\wedge}_{i/D_0})$ making the following diagram commutes:

(3.9.2)

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \langle \tau_i \rangle & \longrightarrow & \pi_1^{D_3}(H^{\wedge}_{/D_0}) & \xrightarrow{\varepsilon_{H,i}} & \pi_1^{D_0}(D^{\wedge}_{i/D_0}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle q_{0,i}} & & \downarrow{\scriptstyle \lambda_{D,i}} & & \\
1 & \longrightarrow & \langle \tau_i \rangle & \longrightarrow & \pi_1^{D_3}(H^{\wedge}_{/D_i}) & \xrightarrow{\varepsilon_{D,i}} & \pi_1(D_i - D_0) & \longrightarrow & 1 \text{ (exact)}\,.
\end{array}
$$

Here the left injectivity and the right surjectivity of the former sequence are easy to see, while its middle exactness is not pursued here.

In the following theorem, recall that $\varepsilon_D : \pi_1^D(H^{\wedge}_{/D}) \to \pi_1(D)$ is the canonical homomorphism introduced in (3.7.1) and that $\mu_D : \pi_1(D) \to \pi_1(\mathscr{D})$ and $\mu_{g,n} : \pi_1(H_{g,n}) \to \pi_1(M_{g,n})$ are natural ones induced from the forgetful morphism $h_{g,n} : \mathscr{H}_{g,n} \to \mathscr{M}_{g,n}$.

(3.10) **Theorem** (Coupling of universal monodromies). *There exists a closed subgroup N of $\pi_1^D(H^{\wedge}_{/D})$ which is surjectively mapped onto $\pi_1(D)(3,3)$ by ε_D making the following diagram commutes:*

$$
\begin{array}{ccccc}
N & \xrightarrow{\mu_D \circ \varepsilon_D | N} & \pi_1(M_{g_1,n_1} \times M_{g_2,n_2})(3,3) & \xrightarrow{\phi_{\mathscr{D}}} & \Gamma_{g_1,n_1}(3) \times \Gamma_{g_2,n_2}(3) \\
{\scriptstyle \partial_{D|N}}\downarrow & & & & \downarrow{\scriptstyle \partial} \\
\pi_1(H_{g,n}) & \xrightarrow{\mu_{g,n}} & \pi_1(M_{g,n}) & \xrightarrow{\phi_{g,n}} & \Gamma_{g,n}\,.
\end{array}
$$

Proof. In this proof, we use the following convention. Given a homomorphism $p : G \to H$ of profinite groups, we can construct its 'pro-l truncation' $p' : G' \to H$ as follows. Namely, let K be the kernel of the natural homomorphism of $\ker(p)$ onto its maximal pro-l quotient $\ker(p)(l)$. Then we set $G' = G/K$ and define $p' : G' \to H$ to be the induced homomorphism. Applying the pro-l truncation functor to the vertical arrows of (3.8.1), (3.7.1) and to the right horizontal arrows of (3.9.1), we obtain for $i = 1, 2$ the following commutative diagram (3.10.i):

$$\begin{array}{ccccccccc}
\pi_1^{D_0}(D^{\wedge}_{i/D_0})' & \xrightarrow{\lambda'_{D,i}} & \pi_1(D_i - D_0)' & \xleftarrow{\varepsilon'_{D,i}} & \pi_1^{D_3}(H^{\wedge}_{/D_i})' & \xrightarrow{q'_{i,3}} & \pi_1^{D_3}(H^{\wedge}_{/D_3})' & \xrightarrow{\partial_{D_3}} & \pi_1(H_{g,n+1})' \\
\downarrow{\scriptstyle \varepsilon'_{i,0}} & & \downarrow{\scriptstyle p'_{D,i}} & & \downarrow{\scriptstyle \hat{p}'_{D,i}} & & \downarrow{\scriptstyle \hat{p}'_{D,3}} & & \downarrow{\scriptstyle p'_H} \\
\pi_1(D_0) & \xrightarrow[p_{D,0}]{\sim} & \pi_1(D) & \xleftarrow{\varepsilon_D} & \pi_1^D(H^{\wedge}_{/D}) & = & \pi_1^D(H^{\wedge}_{/D}) & \xrightarrow{\partial_D} & \pi_1(H_{g,n}) .
\end{array}$$

Recall that we have now identifications

$$\begin{aligned}
\Pi_{g,n} &= \ker(p'_H) = \ker(\hat{p}'_{D,3}) , \\
\Pi_{g_i,n_i} &= \ker(p'_{D,i}) = \ker(\hat{p}'_{D,i}), \quad (i = 1,2) , \\
\langle\delta\rangle &= \ker(\varepsilon'_{i,0}) .
\end{aligned}$$

For each $i = 1, 2$, there exists a unique subgroup N_i of $\mathrm{Im}(\lambda'_{D,i}) \subset \pi_1(D_i - D_0)'$ such that $p'_{D,i}(N_i) = \pi_1(D)(3,3)$ and the conjugate action of N_i on Π_{g_i,n_i} is contained in $\Gamma^*_{g_i,n_i}(3)$. Put

$$N_3 := q'_{1,3}(\varepsilon'^{-1}_{D,1}(N_1)) \cap q'_{2,3}(\varepsilon'^{-1}_{D,2}(N_2)), \quad \text{and} \quad N := \hat{p}'_{D,3}(N_3) .$$

Then by construction, $N \cong N_3$. Since the monodromy representation $\varphi_{g,n} = \phi_{g,n} \circ \mu_{g,n}$ and $\varphi_D = \phi_{\mathscr{D}} \circ \mu_D$ come from p'_H, $p'_{D,i}$ $(i = 1,2)$ respectively, the commutativities of (3.10.1-2) assure the commutativity of the diagram in Theorem (3.10). Thus we have only to show that $\varepsilon_D(N) = \pi_1(D)(3,3)$. Let $\sigma \in \pi_1(D)(3,3)$ and take $\sigma_i \in N_i$ such that $p'_{D,i}(\sigma_i) = \sigma$. By (3.7.1), we see that

$$\varepsilon'^{-1}_{D,i}(\sigma_i) = \{\tilde{\sigma}_i \bar{\tau}_i^{\alpha} \mid \alpha \in \hat{\mathbf{Z}}\} ,$$

where $\tilde{\sigma}_i$ is an element of it, and $\bar{\tau}_i$ is the image of the Dehn twist τ_i of (3.7.1) in the pro-l truncated π'_1. Then there exists $a \in \hat{\mathbf{Z}}$ such that

$$\hat{p}'_{D,1}(\tilde{\sigma}_1) = \hat{p}'_{D,2}(\tilde{\sigma}_2)\tau^a .$$

We next consider the commutative diagram induced from (3.9.2):

$$\begin{array}{ccc}
\pi_1^{D_3}(H^{\wedge}_{/D_0})' & \xrightarrow{q'_{0,i}} & \pi_1^{D_3}(H^{\wedge}_{/D_i})' \\
\downarrow{\scriptstyle \varepsilon'_{H,i}} & & \downarrow{\scriptstyle \varepsilon'_{D,i}} \\
\pi_1^{D_0}(D^{\wedge}_{i/D_0})' & \xrightarrow{\lambda'_{D,i}} & \pi_1(D_i - D_0)' ,
\end{array}$$

where the vertical arrows are surjective and the pro-l truncations are taken with respect to arrows to $\pi_1^D(H_{/D}^{\wedge})$ or $\pi_1(D)$. From this we see that $\varepsilon_{D,i}'^{-1}(\sigma_i)$ is contained in the image of $q_{0,i}'$ which is the normalizer of δ in $\pi_1^{D_3}(H_{/D_i}^{\wedge})'$ $(i = 1, 2)$ (hence that in $\pi_1^{D_3}(H_{/D_3}^{\wedge})$). In particular, $\varepsilon_{D,1}'^{-1}(\sigma_1) \subset \pi_1^{D_3}(H_{/D_2}^{\wedge})'$. It is possible then to find $b \in \hat{\mathbb{Z}}$ (determined up to $\prod_{p \neq l} \mathbb{Z}_p$) such that

$$\varepsilon_{D,2}'(\tilde{\sigma}_1)\delta^b = \sigma_2 .$$

At this stage, we see that $(\alpha, \beta) = (b, a + b)$ gives a solution of the equation

$$\tilde{\sigma}_1 \bar{\tau}_1^{\alpha} = \tilde{\sigma}_2 \bar{\tau}_2^{\beta} \left(\overset{def}{=} : \tilde{\sigma}\right)$$

in $\pi_1^{D_3}(H_{/D_3}^{\wedge})$, because we can check

$$\hat{p}_{D,1}'(\tilde{\sigma}_1 \bar{\tau}_1^{\alpha}) = \hat{p}_{D,2}'(\tilde{\sigma}_2 \bar{\tau}_2^{\beta}),$$
$$\varepsilon_{D,2}'(\tilde{\sigma}_1 \bar{\tau}_1^{\alpha}) = \varepsilon_{D,2}'(\tilde{\sigma}_2 \bar{\tau}_2^{\beta})$$

with noticing that $\delta = \bar{\tau}_1 \bar{\tau}_2^{-1}$ for the latter. Since the image of the above element $\tilde{\sigma}$ in $\pi_1^D(H_{/D}^{\wedge})$ by $\hat{p}_{D,3}'$ gives an element of $\varepsilon_D^{-1}(\sigma) \cap N$, the proof of Theorem (3.10) is completed. □

(3.11) **Corollary.** *Assumptions being as in the beginning of this section, we have* $\mathbb{Q}_{g_1,n_1}(m)\mathbb{Q}_{g_2,n_2}(m) \supset \mathbb{Q}_{g,n}(m)$ $(m \geqq 3)$.

Proof. Let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}_{g_1,n_1}(m)\mathbb{Q}_{g_2,n_2}(m))$. Then by Proposition (2.13), there exists $\tilde{\sigma} \in p_{\mathscr{G}}^{-1}(\sigma)$ with $\tilde{\sigma} \in \pi_1(M_{g_1,n_1} \times M_{g_2,n_2})(m, m)$. Since $\mu_D \circ \varepsilon_D|_N$ is surjective, we can take its inverse image $\sigma' \in N$ and obtain $\sigma'' = \mu_{g,n} \circ \partial_D(\sigma')$. Since ∂ preserves weight filtration, we conclude $\sigma'' \in \pi_1(M_{g,n})(m)$. As the homomorphisms in the diagram of (3.10) are all Galois compatible, we conclude $\sigma \in G_{\mathbb{Q}_{g,n}(m)}$. □

(3.12) **Corollary.** $\mathbb{Q}_{gr,n}(m) \subset \mathbb{Q}_{g,n}(m)$ *for* $m \geqq 1$, $g \geqq 1$, $r \geqq 1$ *and* $n \geqq 1$.

Proof. Since we know that $\mathbb{Q}_{g,n}(1) = \mathbb{Q}_{g,n}(2) = \mathbb{Q}(\mu_{l^\infty})$ (cf. [NTU] Sect. 6), we may suppose $m \geqq 3$. Note also that we need not take into consideration the value $n \geqq 1$ as our fields $\mathbb{Q}_{g,n}(m)$ do not depend on it by [NTU]. First if we put $g_1 = g_2$, then we see from (3.11) that $\mathbb{Q}_{2g,n}(m) \subset \mathbb{Q}_{g,n}(m)$. Then next putting $g_1 = 2g$, $g_2 = g$, we get

$$\mathbb{Q}_{3g,n}(m) \subset \mathbb{Q}_{2g,n}(m)\mathbb{Q}_{g,n}(m) \subset \mathbb{Q}_{g,n}(m) .$$

By repeating this argument, we obtain the conclusion. □

(3.13) **Corollary.** $\mathbb{Q}_{0,3}(m) \subset \mathbb{Q}_{g,n}(m)$ *for* $m \geqq 1$, $n \geqq 1$, $2 - 2g - n < 0$.

Proof. As in the proof of (3.12), we may assume $m \geqq 3$. By (3.12) and [NTU] Theorem A(2), we may assume $g \geqq 2$ without loss of generality. Furthermore by [NTU] Theorem A(1), we may assume n is sufficiently

large. Let $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}_{g,n}(m))$ for $m \geqq 3$. Then automatically, we have $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}_{0,3}(3))$, as it is known that $\mathbf{Q}_{0,3}(1) = \mathbf{Q}_{0,3}(6) = \mathbf{Q}(\mu_{l^\infty})$ ([Ih2]). So there exists $\tilde{\sigma} \in \pi_1(M_{g,n-1} \times M_{0,3})(m,3)$ with $p_{\mathscr{D}}(\tilde{\sigma}) = \sigma$. Let $\phi_{\mathscr{D}}(\tilde{\sigma}) = (\sigma_1, \sigma_2)$. It suffices to show $\sigma_2 \in \Gamma_{0,3}(m)$. Since $M_{0,3} = \mathrm{Spec}\,\mathbf{Q}$, we see $\tilde{\sigma} \in \pi_1(M_{g,n-1})(m)$ and $p_{g,n-1}(\tilde{\sigma}) = \sigma$. We shall apply Theorem (3.10) with $(g_1, n_1) = (g, n-1)$, $(g_2, n_2) = (0,3)$. Then there exists an inverse image $\sigma' \in N$ of $\tilde{\sigma}$ such that $\sigma'' = \partial_D(\sigma')$ is contained in $\pi_1(H_{g,n})(3)$ while $\varepsilon_D(\sigma') \in \pi_1(H_{g,n-1})(m)$. Let p denote the forgetful homomorphism $\pi_1(H_{g,n}) \to \pi_1(H_{g,n-1})$ obtained by forgetting the $(n-1)$-th marked point and substituting the n-th marked point for the $(n-1)$-th one. Then, in the present case, we can add the extra arrow p to the bottom sequence of (3.7.1) to get the commutative diagram

$$\begin{array}{ccccccccc} & & & & \pi_1(H_{g,n}) & \xrightarrow{p} & \pi_1(H_{g,n-1}) & & \\ & & & & {\scriptstyle\partial_D}\uparrow & & \| & & \\ 1 & \longrightarrow & \langle\tau\rangle & \longrightarrow & \pi_1^D(H_{/D}^{\wedge}) & \xrightarrow{\varepsilon_D} & \pi_1(D) & \longrightarrow & 1\,. \end{array}$$

From this follows that $p(\sigma'') = \varepsilon_D(\sigma') \in \pi_1(H_{g,n-1})(m)$. Moreover we see that the 'Dehn twist' τ is contained in the kernel of p and is centralized by σ''. This and the fact that $\mu_{g,n}(\sigma'') \in \pi_1(M_{g,n})[3]$ (2.8.3) assure $\mu_{g,n}(\sigma'') \in \pi_1(M_{g,n})[m]$. By Lemma (2.9) then follows that $\mu_{g,n}(\sigma'') \in \pi_1(M_{g,n})(m)$, hence $\sigma'' \in \pi_1(H_{g,n})(m)$. Since ∂ is strictly compatible with the weight filtration (3.3), we conclude from $\partial(\sigma_1, \sigma_2) = \varphi_{g,n}(\sigma'')$ that $\sigma_2 \in \Gamma_{0,3}(m)$. This completes the proof of the corollary. □

The proof of Theorem A is thus completed by these corollaries.

4 Graded Lie algebras

In this section, we shall study connections among the graded Lie algebras

$$\mathrm{Gr}_l\mathscr{G}_{g,n} = \bigoplus_{m=1}^{\infty} \mathrm{gr}_l^m \mathscr{G}_{g,n} \quad (2-2g-n<0)$$

over $\mathbf{Q}_l$ where $\mathrm{gr}_l^m\mathscr{G}_{g,n} = \mathrm{Gal}(\mathbf{Q}_{g,n}(m+1)/\mathbf{Q}_{g,n}(m)) \otimes \mathbf{Q}_l$. By [NTU], it is independent of n provided that $n \geqq 1$. Our Theorem A implies that there exists a natural homomorphism of graded Lie algebras

$$\Phi_{g,n}^{(r)} : \mathrm{Gr}_l\mathscr{G}_{g,n} \to \mathrm{Gr}_l\mathscr{G}_{gr,n}$$

for $r \geqq 0$, $n \geqq 1$, $2-2gr-n<0$.

(4.1) **Lemma.** *The above map $\Phi_{g,n}^{(r)}$ gives a surjection.*

Proof. Let $G_0 = \mathrm{Gal}(\mathbf{Q}(\mu_{l^\infty})/\mathbf{Q})$ and $\chi : G_0 \to \mathbf{Z}_l^\times$ denote the cyclotomic character. We fix $m \geqq 1$ and put $G = \mathrm{Gal}(\mathbf{Q}_{gr,n}(m+1)/\mathbf{Q})$,

$$G(i) = \mathrm{Gal}(\mathbf{Q}_{gr,n}(m+1)/\mathbf{Q}_{gr,n}(i))\,,$$
$$G[i] = \mathrm{Gal}(\mathbf{Q}_{gr,n}(m+1)/\mathbf{Q}_{gr,n}(m+1) \cap \mathbf{Q}_{g,n}(i)) \quad (1 \leqq i \leqq m)\,.$$

Then $G[i] \subset G(i)$ and it suffices to show that the index of $G[m]$ in $G(m)$ is finite. Notice that the filtrations $\{G(i)\}_i$, $\{G[i]\}_i$ of $G(1) = G[1]$ are both central, and that G_0 acts on their graded quotients $G[2i]/G[2i+1]$, $G(2i)/G(2i+1)$ by conjugation via multiplication by χ^i. (Recall that $G(2i-1)/G(2i)$ is known to be finite. cf. [O3], [NTU].) If we assume $G(m)/G[m]$ is not finite, then m is even, and we have $\sigma \in G(m)$ with $\langle\sigma\rangle \cong \mathbb{Z}_l$, $\langle\sigma\rangle \cap G[m] = \{1\}$. Then there exists $2i \leqq m-1$ with $\langle\sigma\rangle \cap G[2i+1] = \{1\}$, $\langle\sigma\rangle \cap G[2i] \neq \{1\}$. After replacing σ by its finite l-power (if necessary), we may furthermore assume $\sigma \in G[2i]$. Then by conjugation, G_0 acts on $\sigma(\mathrm{mod}\ G[2i+1])$ via χ^i, while it acts on $\sigma \in G(m)$ via $\chi^{m/2}$. This is a contradiction, as $\sigma(\mathrm{mod}\ G[2i+1])$ is still not torsion in $G[2i]/G[2i+1]$. □

(4.2) **Corollary.** *The free $\mathbb{Z}_l$-ranks $r_{g,n}(m)$ of $\mathrm{Gal}(\mathbb{Q}_{g,n}(m+1)/\mathbb{Q}_{g,n}(m))$ $(m \geqq 1)$ satisfy the following inequalities*:

$$r_{0,3}(m) \leqq r_{g,n}(m) \leqq r_{1,1}(m)\,.$$

Proof. This follows from (4.1) directly, as $r_{g,n}(m)$ is nothing but the dimension of $\mathrm{gr}_l^m \mathscr{G}_{g,n}$. □

(4.3) The ranks $r_{0,3}(m)$, $r_{1,1}(m)$ have been estimated in detail by Ihara [Ih1], Drinfeld [Dr], Matsumoto [M], Tsunogai [NT2]. For small m, we have the following table

m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$r_{0,3}(m)$	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	1	0	1	0	1
$r_{1,1}(m)$	0	0	0	0	0	1	0	0	0	$\leqq 3$	0	0	0	$\leqq 11$	0	$\leqq 10$	0	$\leqq 57$	0	$\leqq 108$

More generally, Y. Ihara showed that $\mathrm{Gr}_l^{4k+2}\mathscr{G}_{0,3}$ $(k = 1, 2, \ldots)$ are nontrivial by combining his theory of the universal power series for Jacobi sums arising from $\mathbf{P}^1 - \{0, 1, \infty\}$ with Soule's nonvanishing result on their coefficient characters (cf. [Ih2], [De], [S]). The nontriviality of other degrees are deduced by calculating Lie brackets of the Galois elements of degrees $4k+2$ $(k = 1, 2, \ldots)$ ([Ih1], [M]). In particular, we can learn that $r_{0,3}(m) > 0$ for even m other than 2, 4, 8, 12 from [Ih1] and that $r_{0,3}(2m) \to \infty$ $(m \to \infty)$ from [M]. The upper bounds for $r_{1,1}(m)$ above come from computer calculations of the ranks of SL_2 invariants of $\mathrm{gr}^m \Gamma_{1,1}$ ([NT2]). Since $-1 \in SL_2$ acts on $\mathrm{gr}^m \Gamma_{1,1}$ via multiplication by $(-1)^m$, it is easy to see that $r_{1,1}(m) = 0$ for odd m.

(4.4) We next consider the exterior Galois representations arising from pro-l fundamental groups of curves over a number field k. Let X be a smooth projective curve over k, S a finite subset of closed points of X with $|S(\bar{k})| = n$. Assume $2 - 2g - n < 0$, $n \geqq 1$ and put $C = X - S$. We shall say that C is a (smooth) curve of type (g, n). Identifying the pro-l fundamental group π_1 of $C \otimes \bar{k}$ with $\Pi_{g,n}$ in a standard way, we get an exterior Galois representation

$$\varphi_C : G_k \to \Gamma_{g,[n]} \subset \mathrm{Out}\,\pi_1\,.$$

We are interested in the field tower $\{k_C(m)\}_m$ defined by

$$\varphi_C^{-1}(\Gamma_{g,n}(m)) = G_{k_C(m)}$$

together with the graded Lie algebra

$$\mathrm{Gr}_l\mathscr{G}_C = \bigoplus_{m=1}^{\infty} \mathrm{gr}_l^m\mathscr{G}_C := \bigoplus_{m=1}^{\infty} \mathrm{Gal}(k_C(m+1)/k_C(m)) \otimes \mathbb{Q}_l\,.$$

equipped with a natural $G_C := \mathrm{Gal}(k_C(1)/k)$-action. It is easy to see that $\mathbb{Q}_{g,n}(m) \subset k_C(m)$ and hence that there is a natural homomorphism

$$\Phi_C : \mathrm{Gr}_l\mathscr{G}_C \to \mathrm{Gr}_l\mathscr{G}_{g,n}$$

of graded Lie algebras over $\mathbb{Q}_l$. Remark also that $\mathrm{Gr}_l\mathscr{G}_C$ is invariant under the finite base change of k.

(4.5) **Lemma.** *The map $\Phi_C : \mathrm{Gr}_l\mathscr{G}_C \to \mathrm{Gr}_l\mathscr{G}_{g,n}$ is surjective.*

Proof. By the above remark, we may assume $S(\bar{k}) = S(k)$. The proof can be done in a similar way to that of (4.1) except for one-point remark. Let $m \geqq 1$ and put $G = \mathrm{Gal}(\mathbb{Q}_{g,n}(m+1)/\mathbb{Q})$,

$$G(i) = \mathrm{Gal}(\mathbb{Q}_{g,n}(m+1)/\mathbb{Q}_{g,n}(i))\,,$$
$$G[i] = \mathrm{Gal}(\mathbb{Q}_{g,n}(m+1)/\mathbb{Q}_{g,n}(m+1) \cap k_C(i))$$

for $1 \leqq i \leqq m$. Assume $G(m)/G[m]$ is infinite, and take $\sigma \in G(m)$ with $\langle\sigma\rangle \cong \mathbb{Z}_l$, $\langle\sigma\rangle \cap G[m] = \{1\}$. In this case, $G[1]$ is not necessarily equal to $G(1)$. If $\sigma \in G[1]$, then a similar argument as in the proof of (4.1) can be applied, as the action of G_C on $(G[i]/G[i+1]) \otimes \mathbb{Q}_l$ through $G/G[1]$ is pure of weight $(-i)$ for $i \geqq 1$, and deduces a contradiction. In case $\sigma \notin G[1]$, it is sufficient to find a suitably small $\tau \in G\backslash G(1)$ with $\tau\sigma\tau^{-1} \equiv \sigma \bmod G[1]$. But if T_lJ denotes the l-adic Tate module of the Jacobian of X, then G_C is the Galois image in $\mathrm{GSp}(T_lJ)$ which is known to have infinitely many scalar matrices by Bogomolov [B]. Thus the above τ is easy to find. □

(4.6) **Corollary.** *Let C be a noncomplete hyperbolic curve of type (g, n) over a number field k. Then the exterior Galois representation $\varphi_C : G_k \to \Gamma_{g,[n]}$ has Galois images in $\mathrm{gr}^m\Gamma_{g,n}$ whose free $\mathbb{Z}_l$-rank is at least $r_{0,3}(m)$ $(m \geqq 1)$.*

Proof. By [NTU], we know that $\mathrm{Gr}_l\mathscr{G}_{g,n}$ is independent of n. The above corollary is then a consequence of the surjectivity of

$$\mathrm{Gr}_l\mathscr{G}_C \xrightarrow{\Phi_C} \mathrm{Gr}_l\mathscr{G}_{g,n} = \mathrm{Gr}_l\mathscr{G}_{g,3} \xrightarrow{\Phi_{g,3}^{(0)}} \mathrm{Gr}_l\mathscr{G}_{0,3}$$

assured by Lemmas (4.1), (4.5). □

(4.7) Now we are ready to enter the proofs of Theorems B, C. After replacing k by its finite extension if necessary, we may assume that S is pointwise k-rational. Let us fix a numbering of the elements of S, and $\xi_C : \mathrm{Spec}\,k \to M_{g,n}$

be the morphism associated with C and the numbering of S. It induces an injection of G_k into $\pi_1(M_{g,n})$ and the Galois representation φ_C factors through $\phi_{g,n}$. The map $\phi_{g,n}$ gives embeddings of graded Lie algebras

$$\bigoplus_{m=1}^{\infty} \pi_1(M_{g,n})(m)/\pi_1(M_{g,n})(m+1) \hookrightarrow \mathrm{Gr}\Gamma_{g,n} ,$$

$$\bigoplus_{m=1}^{\infty} \pi_1(\overline{M}_{g,n})(m)/\pi_1(\overline{M}_{g,n})(m+1) \hookrightarrow \mathrm{Gr}\Gamma_{g,n} ,$$

whose images $\otimes \mathbb{Q}_l$ in $\mathrm{Gr}_l\Gamma_{g,n} := \mathrm{Gr}\Gamma_{g,n} \otimes \mathbb{Q}_l$ are denoted by $\mathrm{Gr}_l\Gamma_{g,n}^{univ}$, $\mathrm{Gr}_l\Gamma_{g,n}^{geom}$ respectively. These two Lie algebras are stable under the action of $\mathrm{GSp}_{2g,n} = \mathrm{GSp}(2g, \mathbb{Q}_l) \times S_n$, and give the following $\mathrm{GSp}_{2g,n}$-equivariant exact sequences

$$0 \longrightarrow \mathrm{Gr}_l\Gamma_{g,n}^{geom} \longrightarrow \mathrm{Gr}_l\Gamma_{g,n}^{univ} \xrightarrow{\mathrm{Gr}_l(p_{g,n})} \mathrm{Gr}_l\mathscr{G}_{g,n} \longrightarrow 0 . \tag{4.7.1}$$

Here the action of $\mathrm{GSp}_{2g,n}$ on $\mathrm{Gr}_l\mathscr{G}_{g,n}$ factors through the similitude character $\chi : \mathrm{GSp}(2g, \mathbb{Q}_l) \to \mathbb{Q}_l^{\times}$ as pointed out by T. Oda in a slightly different formulation (cf. [O2], see also [NTU] 6.1). On the other hand, the representation φ_C gives an embedding

$$\mathrm{Gr}_l\mathscr{G}_C \hookrightarrow \mathrm{Gr}_l\Gamma_{g,n}^{univ} , \tag{4.7.2}$$

and Lemma (4.5) assures that the projection of its image into $\mathrm{Gr}_l\mathscr{G}_{g,n}$ via $\mathrm{Gr}_l(p_{g,n})$ is surjective.

Proof of Theorem B. Let λ be an element of the Galois centralizer $\mathrm{Out}_{G_k}\pi_1$ regarded as an element of $\mathrm{GSp}(2g, \hat{\mathbb{Z}}_l) \times S_n \subset \mathrm{GSp}_{2g,n}$ (by weight argument, cf. [NT1] Sect. 3). Then it must fix the images of (4.7.2). Now we know from (4.2), (4.3) that $\mathrm{gr}_l^6\mathscr{G}_{g,n}$ and $\mathrm{gr}_l^{10}\mathscr{G}_{g,n}$ are nontrivial. So by Lemma (4.5), we choose an element $y_m \in \mathrm{gr}_l^m\mathscr{G}_C$ such that $z_m = \mathrm{Gr}_l(p_{g,n})(y_m) \neq 0$ in $\mathrm{gr}_l^m\mathscr{G}_{g,n}$ $(m = 6, 10)$. Since $\lambda(y_m) = y_m$ implies $\chi(\lambda)^{m/2} z_m = z_m$, we have $\chi(\lambda)^3 = \chi(\lambda)^5 = 1$. Hence $\chi(\lambda) = 1$, i.e., $\lambda \in \mathrm{Sp}(2g, \hat{\mathbb{Z}}_l) \times S_n$.

The statement for the case $\mathrm{End}_k(J) \cong \mathbb{Z}$ is now easy to deduce from [NT1] Sect. 3 Lemma (3.1). □

(4.8) **Theorem.** *The cokernel of the Johnson homomorphism $\iota_m \otimes \mathbb{Q}$ has dimension greater than or equal to $r_{0,3}(m)$ $(m \geqq 1)$.*

Proof. The image of $\mathrm{gr}^m\Gamma_{g,n}^{top}$ via the Johnson homomorphism is contained in $\mathrm{gr}_l^m\Gamma_{g,n}^{geom}$. By applying Corollary (4.2) to the sequence (4.7.1), we see that

$$\dim \mathrm{Coker}(\iota_m \otimes \mathbb{Q}) \geqq \dim \mathrm{gr}_l^m\mathscr{G}_{g,n} = r_{g,n}(m) \geqq r_{0,3}(m) .$$

Thus the theorem is proved. □

Finally, Theorem C of Introduction follows from the above Theorem (4.8) together with Ihara's estimates of $r_{0,3}(m)$ described in (4.3).

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