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Diffusion processes and second order elliptic operators with singular coefficients for lower order terms

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1 Introduction

In this paper, we study the connection between diffusion processes and second order elliptic differential operators of the form

$$\begin{aligned} L &= \frac{1}{2}\nabla\cdot(a\nabla) + b\cdot\nabla + q \\ &= \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{d} b_i\frac{\partial}{\partial x_i} + q \end{aligned} \tag{1.1}$$

in a d-dimensional Euclidean domain D, where $a : D \to \mathbb{R}^d \otimes \mathbb{R}^d$ is a measurable, symmetric matrix-valued function which satisfies the uniform elliptic condition

$$\lambda^{-1} I \leqq a(\cdot) \leqq \lambda I \tag{1.2}$$

for some $\lambda \geqq 1, b : D \to \mathbb{R}^d$ and $q : D \to (-\infty, 0]$ are measurable functions which could be singular such that

$$1_D|b|^2 \in K_d \quad \text{and} \quad 1_D q \in K_d\,. \tag{1.3}$$

Here a real-valued measurable function f on $\mathbb{R}^d$ is said to be in class K_d if and only if

$$\lim_{\alpha\downarrow 0}\left[\sup_x \int_{|y-x|\leqq\alpha} |x-y|^{-(d-2)}|f(y)|\,dy\right] = 0, \quad \text{if } d \geqq 3\,, \tag{1.4}$$

$$\lim_{\alpha\downarrow 0}\left[\sup_x \int_{|y-x|\leqq\alpha} (-\ln|x-y|)|f(y)|\,dy\right] = 0, \quad \text{if } d = 2\,, \tag{1.5}$$

and

$$\sup_x \int_{|y-x|\leqq 1} |f(y)|\, dy < \infty, \quad \text{if } d = 1 . \tag{1.6}$$

One of the important properties of a function f in class K_d is that for any $\varepsilon > 0$, there exists a constant $A(\varepsilon) > 0$ such that (cf. [20])

$$\int_{\mathbb{R}^d} |f| u^2\, dx \leqq \varepsilon \int_{\mathbb{R}^d} |\nabla u|^2\, dx + A(\varepsilon) \int_{\mathbb{R}^d} |u|^2\, dx \tag{1.7}$$

for all square integrable function u on $\mathbb{R}^d$ whose distributional derivatives are also square integrable.

Our primary interest is in the probabilistic representation of solutions for the Dirichlet boundary value problem

$$\begin{cases} Lu = 0 & \text{in } D \\ u = \phi & \text{on } \partial D . \end{cases} \tag{1.8}$$

Let m denote the Lebesgue measure in D and

$$W^{1,2}(D) = \{f \in L^2(D,m) : \nabla f \in L^2(D,m)\}$$

where the gradient ∇f are understood in the distributional sense. A function f is in $W^{1,2}_{\text{loc}}(D)$ if $f \in W^{1,2}(D_1)$ for every domain D_1 with compact closure in D.

Definition 1.1 *A function u in $W^{1,2}_{\text{loc}}(D)$ is called a weak (or local) solution of $Lu = 0$ in D if*

$$\frac{1}{2}\sum_{i,j=1}^{d} \int a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial \psi}{\partial x_j}\, dm - \sum_{i=1}^{d} \int b_i \frac{\partial u}{\partial x_i} \psi\, dm - \int q u \psi\, dm = 0 \tag{1.9}$$

for all $\psi \in C^1_c(D)$, where $C^1_c(D)$ is the space of continuously differentiable functions with compact support in D.

Trudinger [30] proved that for $\phi \in W^{1,2}(D)$, there exists a unique function u in $W^{1,2}(D)$ satisfying

$$Lu = 0 \quad \text{in } D$$

and

$$u - \phi \in W^{1,2}_0(D), \tag{1.10}$$

where $W^{1,2}_0(D)$ is the completion of $C^1_c(D)$ with respect to the norm

$$\|f\|_{1,2} = \left(\int |\nabla f|^2\, dx + \int |f|^2\, dx\right)^{\frac{1}{2}} . \tag{1.11}$$

He also proved a weak maximum principle for such u.

By setting $a = I, b = 0$ and $q = 0$ off D, we may assume that the operator L is defined on $\mathbb{R}^d$. A continuous strong Markov process $(\Omega, X, \zeta, \{P^x, x \in \mathbb{R}^d\})$

on $\mathbb{R}^d$ with lifetime ζ can be constructed so that its infinitesimal generator is L with domain

$$\text{Domain}(L) = \{f \in W^{1,2}(\mathbb{R}^d) : Lf \in L^2(\mathbb{R}^d, dx)\} . \tag{1.12}$$

Let

$$\tau(D) = \inf\{t > 0 : X_t \notin D\} \tag{1.13}$$

be the first exit time from D. One of our main results in this paper is:

Theorem 1.1 *Suppose D is a bounded domain in $\mathbb{R}^d$ and $\phi \in C(\partial D)$. Then*

$$u(x) = E^x[\phi(X_{\tau(D)})] \tag{1.14}$$

is the (unique) weak solution of $Lu = 0$ which is continuous in D and

$$\lim_{\substack{x \to y \\ x \in D}} u(x) = \phi(y) \tag{1.15}$$

for $y \in \partial D$ which is regular for the Laplace operator $\left(\frac{1}{2}\Delta, D\right)$.

It can be shown by using Theorem 1.1 that every locally bounded weak solution of $Lu = 0$ has a continuous version in D.

Theorem 1.1 can be regarded as the most general form of the probabilistic representation for the Dirichlet boundary value problem of second order linear elliptic equations. Since the probabilistic representation for the classical Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u = f & \text{on } \partial D \end{cases}$$

was established by Kakutani [19] in 1944, it has been developed along two directions. One is to extend the Laplacian Δ to the general elliptic operators: a) in the non-divergence form $(a\nabla)\cdot\nabla$, it was done by using Ito's stochastic calculus; b) in the divergence form $\nabla(a\nabla)$, the connection was made by the theory of Dirichlet spaces (see [14]). The other direction of the development is to add lower order terms $b\cdot\nabla$ and q to the second order term, which was established by adapting famous Cameron-Martin-Girsanov formula and Feynman-Kac formula, respectively, into their "stopping time" versions (see [11, 13]). For Schrödinger operator $\frac{\Delta}{2} + q$, the Dirichlet boundary problem was investigated by Chung and Rao [6] for bounded q, which is allowed to change signs. Aizenman and Simon [3] established several properties for the Schrödinger operator with potential q in class K_d. The recent development of the probabilistic methods for the Schrödinger operators can be found in a forthcoming book [7] of Chung and Zhao.

For operator $\frac{1}{2}\Delta + b\cdot\nabla$ with singular drift coefficient b, the probabilistic connection was first investigated by Cranston and Zhao [5], in which the powerful John-Nirenberg inequality was used instead of Khasminskii's lemma in the Schrödinger operator case. John-Nirenberg inequality (its probability version) also plays an important role in this paper. There has been considerable work on symmetric Dirichlet spaces applied to the gradient case, see [2] and references in there.

Unlike that for operators of the form $\frac{1}{2}(a\nabla)\cdot\nabla + b\cdot\nabla + q$, the process corresponding to L of (1.1) is in general not a semimartingale, since $a(\cdot)$ is only assumed to be measurable. This makes the use of Ito calculus more delicate. On the other hand, one can not use non-symmetric Dirichlet space theory, especially its additive functional theory, to the operator L directly, since the semigroup of the adjoint operator $\hat{L}$ of L may not be sub-Markovian. The idea of this paper is first to use symmetric Dirichlet space theory for divergent operator

$$L^0 = \frac{1}{2}\nabla\cdot(a\nabla) = \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right). \tag{1.16}$$

with

$$\mathrm{Domain}(L^0) = \{u \in W_0^{1,2}(D) : L^0 u \in L^2(D,m)\}\,. \tag{1.17}$$

It follows from Fukushima's additive functional theory in the context of Dirichlet spaces that the continuous Hunt process $(\Omega, X, \zeta, \{P_0^x, x \in D\})$ associated with L^0 has the decomposition

$$X_t = X_0 + \int_0^t \sigma(X_s)\cdot dW_s + N_t, \quad 0 \leqq t < \zeta\,,$$

P_0^x-a.e. for $x \in D$, where σ is the positive, symmetric square root of a, W is a d-dimensional Brownian motion up to time ζ, N is a d-dimensional continuous additive functional of X locally having zero energy (that is, there exists a sequence of stopping time T_n with $T_n \uparrow \infty$ such that for each $n, \lim_{t\downarrow 0}\frac{1}{t}\int_D E_0^x[N_{t\wedge T_n}^2]m(dx) = 0$). We then show by using John-Nirenberg inequality that

$$\left.\frac{dQ^x}{dP_0^x}\right|_{\mathscr{F}_t} = \exp\left(\int_0^t \sigma^{-1}b(X_s)\,dW_s - \frac{1}{2}\int_0^t |\sigma^{-1}b|^2(X_s)\,ds\right), \quad t > 0 \tag{1.18}$$

defines a family of probability measures $\{Q^x, x \in D\}$ on Ω, where $\{\mathscr{F}_t\}_{t>0}$ is the minimal complete admissible σ-fields generated by $(\Omega, X, P_0^x, x \in D)$. Let $\{\Omega, P^x, x \in D\}$ be the family of probability measures determined by

$$E^x[f(X_t)] = E_Q^x\left[e^{\int_0^t q(X_s)\,ds} f(X_t)\right], \quad t > 0 \tag{1.19}$$

for every bounded Borel function f on D. For $\alpha > 0$, let

$$G_\alpha^0 f(x) = E_0^x\left[\int_0^\infty e^{-\alpha t} f(X_t)\,dt\right], \tag{1.20}$$

$$G_\alpha^b f(x) = E_Q^x\left[\int_0^\infty e^{-\alpha t} f(X_t)\,dt\right], \tag{1.21}$$

and
$$G_\alpha f(x) = E^x\left[\int_0^\infty e^{-\alpha t} f(X_t)\,dt\right]. \tag{1.22}$$

It is well known (cf. [14]) that $G_\alpha^0 f$ is in $W_0^{1,2}(D)$ for $f \in L^2(D,m)$. We prove

Theorem 1.2 *There exists $\alpha_0 > 0$ such that for $\alpha > \alpha_0$ and $f \in L^2(D,m)$, $G_\alpha^b f, G_\alpha f \in W_0^{1,2}(D)$ and*

$$G_\alpha^b f = \sum_{n=0}^\infty G_\alpha^0 (b\cdot\nabla G_\alpha^0)^n f\,, \tag{1.23}$$

$$G_\alpha f = \sum_{n=0}^\infty G_\alpha^b (q G_\alpha^b)^n f\,. \tag{1.24}$$

The series are convergent in $(W_0^{1,2}(D), \|\cdot\|_{1,2})$. And

$$G_\alpha^b f = G_\alpha^0 f + G_\alpha^b(b\cdot\nabla G_\alpha^0 f) = G_\alpha^0 f + G_\alpha^0(b\cdot\nabla G_\alpha^b f)\,, \tag{1.25}$$

$$G_\alpha f = G_a^b f + G_\alpha(q G_\alpha^b f) = G_\alpha^b f + G_\alpha^b(q G_\alpha f)\,. \tag{1.26}$$

It follows from Theorem 1.2 that $(\Omega, X, \zeta, \{Q^x, x \in D\})$ is the continuous Hunt process on D whose infinitesimal generator is

$$L^b = L^0 + b\cdot\nabla = \frac{1}{2}\nabla\cdot(a\nabla) + b\cdot\nabla \tag{1.27}$$

with
$$\text{Domain}(L^b) = \{u \in W_0^{1,2}(D) : L^b u \in L^2(D,m)\} \tag{1.28}$$

and that $(\Omega, X, \zeta, \{P^x, x \in D\})$ is the continuous Hunt process on D whose infinitesimal generator is

$$L = \tfrac{1}{2}\nabla\cdot(a\nabla) + b\cdot\nabla + q$$

with
$$\text{Domain}(L) = \{u \in W_0^{1,2}(D) : Lu \in L^2(D,m)\}\,. \tag{1.29}$$

By setting $a = I, b = 0$ and $q = 0$ off D, we may assume that X is a continuous process on $\mathbb{R}^d$. For $\phi \in C(\partial D)$, let

$$u_0(x) = E_0^x[\phi(X_{\tau(D)})], \quad x \in D\,,$$
$$u_1(x) = E_Q^x[\phi(X_{\tau(D)})], \quad x \in D\,,$$

and
$$u(x) = E^x[\phi(X_{\tau(D)})], \quad x \in D\,.$$

We finally prove that u_0, u_1 and u are weak solutions for $L^0 u_0 = 0, L^b u_1 = 0$ and $Lu = 0$ in D respectively. Furthermore,

$$u_1(x) = u_0(x) + \lim_{t\uparrow\infty} E_0^x\left[\int_0^{t\wedge\tau(D)} b(X_s)\cdot\nabla u_0(X_s)\,ds\right], \tag{1.30}$$

$$u(x) = u_1(x) + E_Q^x\left[\int_0^{\tau(D)} q(X_s)u_1(X_s)\,ds\right]$$
$$= u_1(x) + G^{b,D}(qu_1)(x)\,, \tag{1.31}$$

where $G^{b,D}$ is the Green operator of L^b on D. Then Theorem 1.1 will follow.

2 Elliptic operators in divergent form $\frac{1}{2}\nabla\cdot(a\nabla)$

Let D be a d-dimensional domain and m the Lebesgue measure on D. In this section, we study operator

$$L^0 = \frac{1}{2}\nabla\cdot(a\nabla) = \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right) \tag{2.1}$$

on D and its associated diffusion process, where $a : D \to \mathbb{R}^d \otimes \mathbb{R}^d$ is a measurable, symmetric matrix-valued function which satisfies the uniform ellipticity condition (1.2).

Define a symmetric bilinear form $\mathscr{E}^0$ on $W_0^{1,2}(D)$ by

$$\mathscr{E}^0(f,g) = \frac{1}{2}\int_D \nabla f\cdot a\nabla g\, dm, \quad f,g \in W_0^{1,2}(D) \tag{2.2}$$

and let $\mathscr{E}_1^0 = \mathscr{E}^0 + (\ ,\ )$, where $(\ ,\ )$ is the standard inner product in $L^2(D,m)$. $W_0^{1,2}(D)$ is a Hilbert space with the inner product $\mathscr{E}_1^0$. If $f \in W_0^{1,2}(D)$, $g = 0 \vee f \wedge 1$ is also in $W_0^{1,2}(D)$ and $\mathscr{E}^0(g,g) \leqq \mathscr{E}^0(f,f)$. Therefore $(W_0^{1,2}(D), \mathscr{E}^0)$ is a symmetric regular Dirichlet space on $L^2(D,m)$ (cf. [14]). By the general theory of symmetric Dirichlet spaces, there is a m-symmetric continuous Hunt process $(\Omega, X, \zeta, \{P_0^x, x \in D\})$ on D such that

$$\left\{f \in L^2(D,m) : \lim_{t\downarrow 0}\frac{1}{t}(f - P_t^0 f, f) < \infty\right\} = W_0^{1,2}(D) \tag{2.3}$$

and

$$\lim_{t\downarrow 0}\frac{1}{t}(f - P_t^0 f, g) = \mathscr{E}^0(f,g), \quad f,g \in W_0^{1,2}(D)\,. \tag{2.4}$$

Here $\{P_t^0\}_{t>0}$ is the transition semigroup of $(X, \{P_0^x, x \in D\})$ and Ω is chosen to be the canonical sample space. X is called the Hunt process associated with $(W_0^{1,2}(D), \mathscr{E}^0)$. The process X can be modified so that it starts at each point x in D, because the transition density function $p_0(t,x,y)$ of X (or equivalently, the fundamental solution of L^0 with Dirichlet boundary condition) has the upbound estimate [1]

$$p_0(t,x,y) \leqq \frac{M}{t^{d/2}}\exp\left(-\frac{|y-x|^2}{Mt}\right), \quad (t,x,y) \in (0,\infty)\times D\times D \tag{2.5}$$

for some constant $M = M(\lambda, d) \geqq 1$. The infinitesimal generator of $(\Omega, X, \zeta, \{P_0^x, x \in D\})$ is L^0 with domain (1.17). It is known [14] that every function in $W_0^{1,2}(D)$ has a quasi-continuous version and in the sequel we always assume that any function f in $W_0^{1,2}(D)$ is represented by its quasi-continuous version with $f(\hat{\Delta}) = 0$, where $\hat{\Delta}$ is the point added to D as a one-point compactification.

For $f \in W_0^{1,2}(D)$, $f(X)$ has the following decomposition under P_0^x for $x \in D$

$$f(X_t) = f(X_0) + M_t^{[f]} + N_t^{[f]}, \quad t \geqq 0 \tag{2.6}$$

where $M^{[f]}$ is a continuous martingale additive functional of X whose quadratic variation

$$\langle M^{[f]}, M^{[f]} \rangle (t) = \int_0^t \nabla f \cdot (a \nabla f)(X_s)\, ds, \quad t \geqq 0 \tag{2.7}$$

and $N^{[f]}$ is a continuous additive functional of zero energy. $M^{[f]}$ and $N^{[f]}$ have local property in the sense that if f, $g \in W_0^{1,2}(D)$ and $f = g$ m-a.e. on a open set $D_1 \subset D$, then

$$M_t^{[f]} = M_t^{[g]} \text{ and } N_t^{[f]} = N_t^{[g]}, \quad \text{for } t \leqq \tau(D_1)\,, \tag{2.8}$$

where $\tau(D_1) = \inf\{t > 0 : X_t \notin D_1\}$. Thus the decomposition (2.6) together with (2.7) can be extended to functions in $W_{\text{loc}}^{1,2}(D)$, except that $M^{[f]}$ would only be local martingale and $N^{[f]}$ be continuous additive functional locally of zero energy. In particular since coordinate functions $\{x_i, 1 \leqq i \leqq d\}$ are in $W_{\text{loc}}^{1,2}(D)$,

$$X_t^i = X_0^i + M_t^{[x_i]} + N_t^{[x_i]}, \quad 0 \leqq t < \zeta\,. \tag{2.9}$$

Let σ be the positive symmetric square root matrix of a and $M = (M^{[x_1]}, \ldots, M^{[x_d]})^{tr}$, $N = (N^{[x_1]}, \ldots, N^{[x_d]})^{tr}$, where the superscript tr denotes vector transpose. Then

$$W = \begin{pmatrix} W^1 \\ \vdots \\ W^d \end{pmatrix} = \int_0^{\bullet} \sigma^{-1}(X_s)\, dM_s \tag{2.10}$$

is a d-dimensional local martingale additive functional of X with

$$\langle W^i, W^j \rangle (t) = \delta_{ij} t \wedge \zeta, \quad t \geqq 0\,, \tag{2.11}$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Therefore W is a d-dimensional Brownian motion stopped at time ζ and

$$X_t = X_0 + \int_0^t \sigma(X_s)\, dW_s + N_t, \quad 0 \leqq t < \zeta\,. \tag{2.12}$$

It follows from [14, Theorem 5.4.4] that

$$M_t^{[u]} = \int_0^t (\nabla u)(X_s)\, dM_s = \int_0^t (\sigma \nabla u)(X_s)\, dW_s, \quad t \geqq 0 \tag{2.13}$$

for $u \in W_0^{1,2}(D)$ and therefore for $u \in W_{\text{loc}}^{1,2}(D)$ by (2.8). It follows from [15, Theorem A2] that if u is a weak solution of $L^0 u = 0$ in D (see Definition 1.1), then $N^{[u]} = 0$ and therefore

$$u(X_t) = u(X_0) + \int_0^t (\sigma \nabla u)(X_s)\, dW_s, \quad t \geqq 0 \tag{2.14}$$

is a P_0^x-local martingale for each $x \in D$.

Suppose that D_1 is a relatively compact open subset of D and $\phi \in W_0^{1,2}(D)$. Then by the classical Hilbert space approach there exists a unique weak solution of $L^0 u = 0$ in D_1 such that $u - \phi|_{D_1} \in W_0^{1,2}(D_1)$. By setting $u = \phi$ off D_1, $u \in W_0^{1,2}(D)$ and

$$u(X_t) - u(X_0) = \int_0^t (\sigma \nabla u)(X_s)\, dW_s + N_t^{[u]}, \quad t \geqq 0 . \tag{2.15}$$

It follows from [15, Theorem A2] that $N_t^{[u]} = 0$ for $t \leqq \tau(D_1)$ and therefore

$$u(X_{t\wedge\tau(D_1)}) - u(X_0) = \int_0^{t\wedge\tau(D_1)} (\sigma \nabla u)(X_s)\, dW_s, \quad t \geqq 0 \tag{2.16}$$

is a P_0^x-uniformly integrable martingale for $x \in D_1$ in view of (2.5) and Lemma II.1.2 in [28]. Thus

$$u(x) = E_0^x[u(X_{\tau(D_1)})] = E_0^x[\phi(X_{\tau(D_1)})] . \tag{2.17}$$

Since the restrictions to ∂D_1 of functions in $C_c^1(D)$ are uniformly dense in $C(\partial D_1)$, by Theorem 3.1 and Theorem 9.1 of [22] we get the following

Theorem 2.1 *Let D_1 be a bounded subdomain of D. Then for $\phi \in C(\partial D_1)$,*

$$u(x) = E_0^x[\phi(X_{\tau(D_1)})] \tag{2.18}$$

is the unique weak solution of $L^0 u = 0$ in D_1 such that

$$\lim_{\substack{x \to y \\ x \in D_1}} u(x) = \phi(y) \tag{2.19}$$

for every boundary point $y \in \partial D_1$ which is regular for the Dirichlet boundary value problem of $(\frac{1}{2}\Delta, D_1)$.

Remark 1. As we noted in Sect. 1, the weak solution u of (L^0, ϕ, D_1) does not depend on the value of a outside D_1. If $L^0 = \frac{1}{2}\nabla\cdot(a\nabla)$ is originally defined only on D_1, then we extend it to $D \supset \bar{D}_1$ by letting $a = I$ off D_1.

Remark 2. $y \in \partial D_1$ is said to be regular for $(\frac{1}{2}\Delta, D_1)$ if and only if

$$\lim_{\substack{x \to y \\ x \in D_1}} E^x[\phi(B_{\tau(D_1)})] = \phi(y), \quad \forall \phi \in C(\partial D_1) , \tag{2.20}$$

where $(B, P^x, x \in \mathbb{R}^d)$ is standard d-dimensional Brownian motion, which is also equivalent to

$$P^y[\inf\{t > 0 : B_t \notin D_1\} = 0] = 1 .$$

A boundary point $y \in \partial D_1$ is said to be regular for (L^0, D_1) if and only if (2.19) holds for each $\phi \in C(\partial D_1)$. A famous theorem of Littman, Stampacchia and Weinberger [22] states that a boundary point $y \in \partial D_1$ is regular for (L^0, D_1) if and only if it is regular for $(\frac{1}{2}\Delta, D_1)$. It can be shown in a similar way as that of Proposition 3.6 in [25] that $y \in \partial D_1$ is regular for (L^0, D_1) if and only if

$$P_0^y[\tau(D_1) = 0] = 1 . \tag{2.21}$$

We now turn our attention to the continuity of transition density function $p_0(t,x,y)$ of X in D.

Lemma 2.2 *Let D_1 be a relatively compact open subset of D and $T_2 > T_1 > 0$ be two fixed constants. There exists $\alpha > 0$ and $A > 0$ which only depend on the dimension d, ellipticity constant λ, T_1, T_2 and D_1, D such that for every $f \in L^2(D,m)$*

$$|u(t,x) - u(s,y)| \leqq A\|u\|_{\infty,D_1\times[T_1,T_2]}(|x-y| \vee |t-s|^{\frac{1}{2}})^{\alpha} \tag{2.22}$$

for $x, y \in D$ and $t,s \in [T_1,T_2]$, where $u(t,x) = \int_D p_0(t,x,y)f(y)dy$ and

$$\|u\|_{\infty,D_1\times[T_1,T_2]} = \operatorname*{ess\,sup}_{(t,x)\in D_1\times[T_1,T_2]} |u(t,x)| . \tag{2.23}$$

Proof. For $f \in L^2(D,m)$, $u(t,x) = \int_D p_0(t,x,y)f(y)dy$ is a weak solution of $\frac{\partial u}{\partial t} = L^0 u$. The Lemma follows directly from a corollary of Moser's Harnack Inequality for parabolic equations [23, p. 109]. □

Let $\phi \in C_c^{\infty}(\mathbb{R}^d)$ such that $\phi \geqq 0$, $\phi \equiv 1$ on $B(0,\frac{1}{2}) = \{x \in \mathbb{R}^d : |x| < \frac{1}{2}\}$, $\phi \equiv 0$ off $B(0,1)$ and $\int \phi dx = 1$. For $z \in D_1$, put

$$f_n(y) = n^d\phi(n(y-z)) . \tag{2.24}$$

For sufficiently large n, supp $[f_n] \subset D$ and by (2.5)

$$u_n(t,x) = \int_D p_0(t,x,y)f_n(y)dy, \quad n \geqq 1$$

are uniformly bounded on $D_1 \times [T_1,T_2]$. Applying (2.23) to u_n and then letting $n \to \infty$, we get

Corollary 2.3 *There exist positive constants α and A which only depend on d, λ, T_1, T_2 and D_1, D such that*

$$|p_0(t,x,z) - p_0(s,y,z)| \leqq A(|x-y| \vee |t-s|^{\frac{1}{2}})^{\alpha} \tag{2.25}$$

for $x, y \in D_1$ and $t,s \in [T_1,T_2]$.

Theorem 2.4 *If $x_0 \in \partial D$ is a regular point for $(\frac{1}{2}\Delta, D)$, then*

$$\lim_{\substack{x\to x_0\\ x\in D}} \sup_{z\in D} p_0(t,x,z) = 0 . \tag{2.26}$$

Proof. We extend operator L^0 to $\mathbb{R}^d$ by putting $a = I$ off D. Let $(Y, P_0^x, x \in \mathbb{R}^d)$ be the continuous Hunt process on $\mathbb{R}^d$ associated with L^0. Let

$$Z_t = \begin{cases} Y_t, & t < \tau(D) \\ \hat{\Delta}, & t \geqq \tau(D) . \end{cases}$$

Then $(Z, P_0^x, x \in D)$ is the continuous Hunt process associated with L^0 on D.

We prove this theorem by using a technique from Port and Stone [25]. It is known from Corollary 2.3 that $(Y, P_0^x, x \in \mathbb{R}^d)$ has jointly continuous transition density kernel $p_0^{\mathbb{R}^d}(t,x,y)$. Since

$$\begin{aligned} &P_0^x[Y_r \in \mathbb{R}^d \backslash D \text{ for some } r \in (s,t]] \\ &= \int_{\mathbb{R}^d} p_0^{\mathbb{R}^d}(s,x,y) P_0^y[\tau(D) \leqq t-s]\, dy \end{aligned} \tag{2.27}$$

which is continuous in x and increases to $P_0^x[\tau(D) \leqq t]$ as $s \downarrow 0$, $P_0^x[\tau(D) \leqq t]$ is lower semicontinuous in x. Thus by Remark 2 of Theorem 2.1, if $x_0 \in \partial D$ is a regular point for $(\frac{1}{2}\Delta, D)$, then

$$\lim_{\substack{x \to x_0 \\ x \in D}} P_0^x[\tau(D) \leqq t] = 1 . \tag{2.28}$$

For bounded Borel function f on D,

$$\begin{aligned} &\int_D p_0(t,x,y) f(y)\, dy \\ &= E^x[f(Z_t)] \\ &= E^x[P_{\frac{t}{2}}^0 f(Z_{\frac{t}{2}})] \quad \text{where } P_{\frac{t}{2}}^0 f(x) = E^x[f(Z_{\frac{t}{2}})] \\ &= E^x\left[P_{\frac{t}{2}}^0 f(Y_{\frac{t}{2}}); \frac{t}{2} < \tau(D)\right] \\ &\leqq \|P_{\frac{t}{2}}^0 f\|_\infty P^x\left[\frac{t}{2} < \tau(D)\right] . \end{aligned}$$

For $z \in D$, applying the above inequality to f_n of (2.24) and then letting $n \to \infty$, by (2.5) there is a constant $C > 0$ such that

$$p_0(t,x,z) \leqq \frac{C}{t^{\frac{d}{2}}} P^x\left[\frac{t}{2} < \tau(D)\right] . \tag{2.29}$$

Hence

$$\lim_{\substack{x \to x_0 \\ x \in D}} \sup_{z \in D} p_0(t,x,z) \leqq \frac{C}{t^{\frac{d}{2}}} \lim_{\substack{x \to x_0 \\ x \in D}} P^x\left[\tau(D) > \frac{t}{2}\right] = 0 . \quad \square$$

3 Change of measures and diffusions with generator $\frac{1}{2}\nabla\cdot(a\nabla) + b\cdot\nabla$

In this section, we will use a change of probability measure to construct diffusion process on D with infinitesimal generator

$$L^b = L^0 + b\cdot\nabla = \frac{1}{2}\nabla\cdot(a\nabla) + b\cdot\nabla \tag{3.1}$$

where $b : D \to \mathbb{R}^d$ is a measurable function such that $1_D|b|^2 \in K_d$.

Let X, W be the processes in (2.12) and let

$$\alpha_0(t) = \sup_{x \in D} E_0^x\left[\int_0^t |\sigma^{-1} b|^2(X_s)\, ds\right] . \tag{3.2}$$

Since $1_D|\sigma^{-1}b|^2 \in K_d$, by (2.5) and [3, Theorem 4.5],

$$\lim_{t \downarrow 0} \alpha_0(t) = 0 . \tag{3.3}$$

Let

$$M_t = \exp\left(\int_0^t \sigma^{-1}b(X_s)\,dW_s - \frac{1}{2}\int_0^t |\sigma^{-1}b|^2(X_s)\,ds\right), \quad t \geqq 0 . \tag{3.4}$$

$(M_t, \mathscr{F}_t, P_0^x)$ is the exponential continuous local martingale such that

$$M_t = 1 + \int_0^t M_s \sigma^{-1} b(X_s)\,dW_s, \quad t \geqq 0 . \tag{3.5}$$

Theorem 3.1 *$(M_t, \mathscr{F}_t, P_0^x)$ is a positive martingale with $E_0^x[M_t^2] < \infty$ for $t > 0$. Furthermore for $k > 0$, let η_k be a positive number such that $4k(\sqrt{\alpha_0(\eta_k)} + \frac{1}{2}\alpha_0(\eta_k)) < 1$, then*

$$E_0^x\left[\sup_{0 \leqq t \leqq \eta_k} (M_t)^k\right] \leqq \frac{1}{1 - 4k(\sqrt{\alpha_0(\eta_k)} + \frac{1}{2}\alpha_0(\eta_k))} . \tag{3.6}$$

Proof. Let

$$\xi_t = \int_0^t \sigma^{-1}b(X_s)\,dW_s - \frac{1}{2}\int_0^t |\sigma^{-1}b|^2(X_s)\,ds, \quad t \geqq 0 . \tag{3.7}$$

For an $\{\mathscr{F}_t\}_{t \geqq 0}$-stopping time S,

$$\left\{\left|\int_0^{t+r} \sigma^{-1}b(X_s)\,dW_s - \int_0^{S\wedge t} \sigma^{-1}b(X_s)\,dW_s\right|\right\}_{r \geqq 0}$$

is a $\{\mathscr{F}_{t+r}\}_{r>0}$-submartingale. Therefore

$$\begin{aligned}
&E_0^x\left[|\xi_t - \xi_{t\wedge S}|\,\middle|\,\mathscr{F}_{t\wedge S}\right] \\
&\leqq E_0^x\left[\left|\int_{t\wedge S}^{t+t\wedge S} \sigma^{-1}b(X_s)\,dW_s\right| + \frac{1}{2}\int_{t\wedge S}^{t+t\wedge S} |\sigma^{-1}b|^2(X_s)\,ds\,\middle|\,\mathscr{F}_{t\wedge S}\right] \\
&= E_0^{X_{t\wedge S}}\left[\left|\int_0^t \sigma^{-1}b(X_s)\,dW_s\right| + \frac{1}{2}\int_0^t |\sigma^{-1}b|^2(X_s)\,ds\right] \\
&\leqq \sqrt{E_0^{X_{t\wedge S}}\left[\int_0^t |\sigma^{-1}b|^2(X_s)\,ds\right]} + \frac{1}{2}E_0^{X_{t\wedge S}}\left[\int_0^t |\sigma^{-1}b|^2(X_s)\,ds\right] \\
&\leqq \sqrt{\alpha_0(t)} + \frac{1}{2}\alpha_0(t) .
\end{aligned}$$

Thus by John-Nirenberg Inequality (see [8]),

$$E_0^x\left[\sup_{0 \leqq t \leqq \eta_k} (M_t)^k\right] \leqq \frac{1}{1 - 4k(\sqrt{\alpha_0(\eta_k)} + \frac{1}{2}\alpha_0(\eta_k))}$$

whenever $4k(\sqrt{\alpha_0(\eta_k)} + \frac{1}{2}\alpha_0(\eta_k)) < 1$. In particular, there exists $\eta_2 > 0$ such that

$$E_0^x\left[\sup_{0\leqq t\leqq\eta_2} M_t^2\right] < \infty ,$$

which by the Markov property of X implies that $E_0^x[M_t^2] < \infty$ for each $t > 0$. Since by (3.5)

$$E_0^x\left[\int_0^t M_s^2|\sigma^{-1}b|^2(X_s)\,ds\right] = E_0^x[M_t^2] < \infty ,$$

$(M_t, \mathscr{F}_t, P_0^x)$ is a positive martingale for each $x \in D$. □

By Kolmogorov Theorem, the formula

$$\left.\frac{dQ^x}{dP_0^x}\right|_{\mathscr{F}_t} = M_t, \quad t \geqq 0 \tag{3.8}$$

uniquely defines a family of probability measures $\{Q^x, x \in D\}$ on Ω. In particular, $\{M_t, \mathscr{F}_t\}_{t\geqq 0}$ is a P_0^x-uniformly integrable martingale for each $x \in D$. We use E_Q^x to denote integration with respect to the probability measure Q^x.

The main result of this section is the following.

Theorem 3.2 *$(\Omega, X, \zeta, Q^x, x \in D)$ is a continuous Hunt process whose infinitesimal generator is L^b with*

$$Domain(L^b) = \{f \in W_0^{1,2}(D) : L^b f \in L^2(D,m)\} , \tag{3.9}$$

where $L^b f$ is understood in the sense of distribution.

Before proving this theorem, we need to identify resolvents of $(\Omega, X, \zeta, Q^x, z \in D)$. For $\alpha > 0$, let

$$G_\alpha^b f(x) = E_Q^x\left[\int_0^\infty e^{-\alpha t} f(X_t)\,dt\right] \tag{3.10}$$

and

$$G_\alpha^0 f(x) = E_0^x\left[\int_0^\infty e^{-\alpha t} f(X_t)\,dt\right] \tag{3.11}$$

whenever they are defined. It follows from Lemma 3.1 and the Markov property of X that there exists a constant $\alpha_0 > 1$ such that

$$\sup_{x\in D} E_0^x[M_t^2] \leqq e^{\frac{1}{2}\alpha_0 t}, \quad t \geqq 0 . \tag{3.12}$$

Let P_t^b be the semigroup of $(\Omega, X, \zeta, Q^x, x \in D)$. Then for $f \in L^2(D,m)$

$$|P_t^b f(x)| = |E_0^x[M_t f(X_t)]| \leqq (E_0^x[M_t^2])^{\frac{1}{2}}(E_0^x[f^2(X_t)])^{\frac{1}{2}} .$$

Thus

$$\|P_t^b f\|_2^2 \leqq e^{\frac{1}{2}\alpha_0 t}\|P_t^b f^2\|_1 \leqq e^{\frac{1}{2}\alpha_0 t}\|f\|_2^2\,, \tag{3.13}$$

where $\|\cdot\|_1$ denotes the L^1-norm in $L^1(D,m)$. Thus $\|P_t^b\|_2 \leqq e^{\frac{1}{4}\alpha_0 t} \leqq e^{\alpha_0 t}$ for $t > 0$. G_α^b is then a bounded operator in $L^2(D,m)$ for $\alpha > \alpha_0$ with

$$\|G_\alpha^b\|_{2,2} \leqq \frac{1}{\alpha - \alpha_0}\,. \tag{3.14}$$

Recall that $\mathscr{E}^0$ is the symmetric bilinear form defined on $W_0^{1,2}(D)$ by (2.2).

Lemma 3.3 *There exists a constant* $\alpha_1 > \alpha_0$ *such that for* $\alpha > \alpha_1$ *and* $f \in L^2(D,m)$,

$$\phi_n = G_\alpha^0(b\cdot\nabla G_\alpha^0)^n f, \quad n = 0,1,2,\dots\,, \tag{3.15}$$

is in $W_0^{1,2}(D)$. $\sum_{n=0}^\infty \phi_n$ *converges in* $W_0^{1,2}(D)$ *with respect to the norm* $\|\cdot\|_{1,2}$ *and for each* $n \geqq 1$,

$$\mathscr{E}_\alpha^0(\phi_n, g) = (b\cdot\nabla\phi_{n-1},\, g), \quad \forall g \in W_0^{1,2}(D)\,. \tag{3.16}$$

Proof. Since $1_D|b|^2 \in K_d$, by (2.5) and [27, Proposition A.2.3], we may assume that

$$\sup_{x\in D} E_0^x\left[\int_0^\infty e^{-\alpha_0 t}|b|^2(X_t)\,dt\right] \leqq C < \infty\,. \tag{3.17}$$

For $f \in L^2(D,m), G_\alpha^0 f$ is in $W_0^{1,2}(D)$ and $G_\alpha^0(b\cdot\nabla G_\alpha^0 f)$ is well defined for $\alpha > \alpha_0$ since

$$\begin{aligned}
&E_0^x\left[\int_0^\infty e^{-\alpha s}|b(X_s)\cdot\nabla G_\alpha^0 f(X_s)|\,ds\right]\\
&\leqq \left(E_0^x\left[\int_0^\infty e^{-\alpha s}|b|^2(X_s)\,ds\right]\right)^{\frac{1}{2}}\left(E_0^x\left[\int_0^\infty e^{-\alpha s}|\nabla G_\alpha^0 f|^2(X_s)\,ds\right]\right)^{\frac{1}{2}}\\
&\leqq \sqrt{C}\left(E_0^x\left[\int_0^\infty e^{-\alpha s}|\nabla G_\alpha^0 f|^2(X_s)\,ds\right]\right)^{\frac{1}{2}}\\
&< \infty\,.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\|\phi_1\|_2^2 &= \|G_\alpha^0(b\cdot\nabla G_\alpha^0 f)\|_2^2\\
&\leqq C\int_0^\infty e^{-\alpha s}\|P_s^0(|\nabla G_\alpha^0 f|^2)\|_1\,ds\\
&\leqq \frac{C}{\alpha}\|\nabla G_\alpha^0 f\|_2^2\,.
\end{aligned} \tag{3.18}$$

For $\beta > 0$, define

$$\mathscr{E}^{(\beta)}(f,g) = \beta(f - \beta G_\beta^0 f, g), \quad \text{for } f,g \in L^2(D,m)\,. \tag{3.19}$$

It is known [21] that

$$\mathscr{E}^{(\beta)}(f,f) \geqq \mathscr{E}^0(\beta G^0_\beta f, \beta G^0_\beta f), \quad f \in L^2(D,m) \tag{3.20}$$

and $f \in W^{1,2}_0(D)$ if and only if

$$\sup_{\beta>0} \mathscr{E}^{(\beta)}(f,f) < \infty \tag{3.21}$$

and in this case

$$\mathscr{E}^0(f,g) = \lim_{\beta\to\infty} \mathscr{E}^{(\beta)}(f,g), \quad \text{for } g \in W^{1,2}_0(D)\,. \tag{3.22}$$

By the resolvent identity, $G^0_\alpha - \beta G^0_\beta G^0_\alpha = \frac{\beta}{\beta-\alpha} G^0_\beta - \frac{\alpha}{\beta-\alpha} G^0_\alpha$. Therefore

$$\mathscr{E}^{(\beta)}(\phi_1,\phi_1) = \frac{\beta}{\beta-\alpha}(b\cdot\nabla G^0_\alpha f,\ \beta G^0_\beta \phi_1) - \frac{\alpha\beta}{\beta-\alpha}\|\phi_1\|_2^2\,. \tag{3.23}$$

Since $1_D|b|^2 \in K_d$, by (1.7) for $\varepsilon > 0$

$$\begin{aligned}
&|(b\cdot\nabla G^0_\alpha f,\ \beta G^0_\beta \phi_1)| \\
&\leqq \frac{1}{2}\|\nabla G^0_\alpha f\|_2^2 + \frac{1}{2}\int |b|^2 (\beta G^0_\beta \phi_1)^2\, m(dx) \\
&\leqq \frac{1}{2}\|\nabla G^0_\alpha f\|_2^2 + \frac{\varepsilon}{2}\|\nabla(\beta G^0_\beta \phi_1)\|_2^2 + \frac{A(\varepsilon)}{2}\|\beta G^0_\beta \phi_1\|_2^2 \\
&\leqq \frac{1}{2}\|\nabla G^0_\alpha f\|_2^2 + \frac{\lambda\varepsilon}{2}\mathscr{E}^0(\beta G^0_\beta\phi_1, \beta G^0_\beta \phi_1) + \frac{A(\varepsilon)}{2}\|\phi_1\|_2^2\,.
\end{aligned}$$

Hence by (3.23)

$$\begin{aligned}
&\left(1 - \frac{\lambda\varepsilon}{2}\frac{\beta}{\beta-\alpha}\right)\mathscr{E}^{(\beta)}(\phi_1,\ \phi_1) \\
&\leqq \frac{\beta}{\beta-\alpha}\left(\frac{1}{2}\|\nabla G^0_\alpha f\|_2^2 + \frac{A(\varepsilon)}{2}\|\phi_1\|_2^2\right) - \frac{\alpha\beta}{\beta-\alpha}\|\phi_1\|_2^2\,.
\end{aligned}$$

By letting $\varepsilon = \lambda^{-1}$,

$$\begin{aligned}
\sup_{\beta>0}\mathscr{E}^{(\beta)}(\phi_1,\phi_1) &= \lim_{\beta\to\infty}\mathscr{E}^{(\beta)}(\phi_1,\phi_1) \\
&= \|\nabla G^0_\alpha f\|_2^2 + (A(\lambda^{-1}) - \alpha)\|\phi_1\|_2^2 \\
&< \infty\,.
\end{aligned}$$

Thus $\phi_1 = G^0_\alpha(b\cdot\nabla G^0_\alpha f) \in W^{1,2}_0(D)$. Since $\beta G^0_\beta \phi_1$ converges to ϕ in $\mathscr{E}^0_1$-norm as $\beta \to \infty$ and $(f,g) \mapsto \int (b\cdot\nabla f) g\, dm$ is a continuous bilinear form in $W^{1,2}_0(D)$ with respect to $\mathscr{E}^0_1$-norm,

$$\mathscr{E}^0(\phi_1,\phi_1) = \lim_{\beta\to\infty}\mathscr{E}^{(\beta)}(\phi_1,\phi_1) = (b\cdot\nabla G^0_\alpha f, \phi_1) - \alpha\|\phi_1\|_2^2\,.$$

Thus

$$\mathscr{E}^0_\alpha(\phi_1,\phi_1) = (b\cdot\nabla G^0_\alpha f, \phi_1)\,. \tag{3.24}$$

A similar argument shows that for $\alpha > \alpha_0$,

$$\mathscr{E}_\alpha^0(\phi_1, g) = (b\cdot\nabla G_\alpha^0 f, g), \quad \forall g \in W_0^{1,2}(D). \tag{3.25}$$

Since $1_D|b|^2 \in K_d$, by (1.7) there exists $\alpha_1 > \alpha_0 > 0$ such that

$$\int |b|^2(x) u^2(x) m(dx) \leqq \frac{1}{2\lambda}\mathscr{E}_{\alpha_1}^0(u,u), \quad u \in W_0^{1,2}(D). \tag{3.26}$$

For $\alpha > \alpha_1$, by (3.24)

$$\mathscr{E}_\alpha^0(\phi_1,\phi_1) \leqq \|\nabla G_\alpha^0 f\|_2 \left(\int_D |b|^2 \phi_1^2 m(dx)\right)^{\frac{1}{2}}$$

$$\leqq (\lambda \mathscr{E}_\alpha^0(G_\alpha^0 f, G_\alpha^0 f))^{\frac{1}{2}} \left(\frac{1}{2\lambda}\mathscr{E}_\alpha^0(\phi_1,\phi_1)\right)^{\frac{1}{2}}.$$

Hence

$$\mathscr{E}_\alpha^0(\phi_1,\phi_1) \leqq \frac{1}{2}\mathscr{E}_\alpha^0(G_\alpha^0 f, G_\alpha^0 f) = \frac{1}{2}\mathscr{E}_\alpha^0(\phi_0,\phi_0). \tag{3.27}$$

By induction, we have $\phi_n \in W_0^{1,2}(D)$ for $n \geqq 2$,

$$\mathscr{E}_\alpha^0(\phi_n,\phi_n) \leqq \frac{1}{2}\mathscr{E}_\alpha^0(\phi_{n-1},\phi_{n-1}), \tag{3.28}$$

and

$$\mathscr{E}_\alpha^0(\phi_n, g) = (b\cdot\nabla\phi_{n-1}, g), \quad \text{for } g \in W_0^{1,2}(D). \tag{3.29}$$

It follows immediately from (3.28) that $\sum_{n=0}^\infty \phi_n$ converges in $(W_0^{1,2}(D), \|\cdot\|_{1,2})$. □

Theorem 3.4 *Let α_1 be the positive constant in Lemma 3.3. For $\alpha > \alpha_1$ and $f \in L^2(D,m)$, we have $G_\alpha^b f \in W_0^{1,2}(D)$ and*

$$G_\alpha^b f = G_\alpha^0 f + G_\alpha^b(b\cdot\nabla G_\alpha^0 f), \tag{3.30}$$

$$G_\alpha^b f = \sum_{n=0}^\infty G_\alpha^0 (b\cdot\nabla G_\alpha^0)^n f, \tag{3.31}$$

$$G_\alpha^b f = G_\alpha^0 f + G_\alpha^0(b\cdot\nabla G_\alpha^b f), \tag{3.32}$$

and

$$\mathscr{E}_\alpha^0(G_\alpha^0(b\cdot\nabla G_\alpha^b f), g) = (b\cdot\nabla G_\alpha^b f, g), \quad \textit{for } g \in W_0^{1,2}(D). \tag{3.33}$$

Proof. Since $G_\alpha^0 f \in W_0^{1,2}(D)$, by (2.6) and (2.13)

$$G_\alpha^0 f(X_t) = G_\alpha^0 f(X_0) + \int_0^t (\sigma\cdot\nabla G_\alpha^0 f)(X_s)\,dW_s + \int_0^t (\alpha G_\alpha^0 f(X_s) - f(X_s))\,ds.$$

Using integration by parts,

$$e^{-\alpha t} G_\alpha^0 f(X_t) = G_\alpha^0 f(X_0) + \int_0^t e^{-\alpha s}(\sigma\cdot\nabla G_\alpha^0 f)(X_s)\,dW_s - \int_0^t e^{-\alpha s} f(X_s)\,ds.$$

Since $M_t = 1 + \int_0^t M_s(\sigma^{-1}b)(X_s)\,dW_s$,

$$E_0^x\left[M_t\left(e^{-\alpha t}G_\alpha^0 f(X_t) - G_\alpha^0 f(X_0) + \int_0^t e^{-\alpha s} f(X_s)\,ds\right)\right]$$
$$= E_0^x\left[\int_0^t e^{-\alpha s} M_s(b\cdot\nabla G_\alpha^0 f)(X_s)\,ds\right].$$

Hence

$$E_0^x[M_t e^{-\alpha t} G_\alpha^0 f(X_t)] + E_0^x\left[\int_0^t M_s e^{-\alpha s} f(X_s)\,ds\right]$$
$$= G_\alpha^0 f(x) + E_0^x\left[\int_0^t e^{-\alpha s} M_s(b\cdot\nabla G_\alpha^0 f)(X_s)\,ds\right]. \tag{3.34}$$

By integration by parts,

$$e^{-\frac{1}{2}\alpha t}M_t = 1 + \int_0^t e^{-\frac{1}{2}\alpha s} M_s(\sigma^{-1}b)(X_s)\,dW_s - \frac{\alpha}{2}\int_0^t e^{-\frac{1}{2}\alpha s} M_s\,ds.$$

Thus for $\alpha > \alpha_0$, by (3.12)

$$E_0^x\left[\int_0^t e^{-\alpha s} M_s^2 |\sigma^{-1}b|^2(X_s)\,ds\right]$$
$$= E_0^x\left[\left(e^{-\frac{1}{2}\alpha t}M_t - 1 + \frac{\alpha}{2}\int_0^t e^{-\frac{1}{2}\alpha s} M_s\,ds\right)^2\right]$$
$$\leqq 3E_0^x\left[e^{-\alpha t}M_t^2 + 1 + \frac{\alpha}{2}\int_0^\infty e^{-\frac{1}{2}\alpha s} M_s^2\,ds\right]$$
$$\leqq 3e^{-(\alpha - \frac{1}{2}\alpha_0)t} + 3 + \frac{3\alpha}{\alpha - \alpha_0}.$$

Therefore

$$E_0^x\left[\int_0^\infty e^{-\alpha s} M_s^2 |\sigma^{-1}b|^2(X_s)\,ds\right] \leqq 6 + \frac{3\alpha}{\alpha - \alpha_0} < \infty. \tag{3.35}$$

Since

$$E_0^x\left[\int_0^\infty e^{-\alpha s} M_s |b\cdot\nabla G_\alpha^0 f|(X_s)\,ds\right]$$
$$\leqq \left(E_0^x\left[\int_0^\infty e^{-\alpha s} M_s^2 |\sigma^{-1}b|^2(X_s)\,ds\right]\right)^{\frac{1}{2}} \left(E_0^x\left[\int_0^\infty e^{-\alpha s} |\sigma\cdot\nabla G_\alpha^0 f|^2(X_s)\,ds\right]\right)^{\frac{1}{2}}$$
$$\leqq \left(6 + \frac{3\alpha}{\alpha - \alpha_0}\right)^{\frac{1}{2}} \left(E_0^x\left[\int_0^\infty e^{-\alpha s} |\sigma\cdot\nabla G_\alpha^0 f|^2(X_s)\,ds\right]\right)^{\frac{1}{2}}$$
$$< \infty, \tag{3.36}$$

$G_\alpha^b(b\cdot\nabla G_\alpha^0 f)$ is well defined and

$$\begin{aligned} &\|G_\alpha^b(b\cdot\nabla G_\alpha^0 f)\|_2^2 \\ &\leqq \left(6+\frac{3\alpha}{\alpha-\alpha_0}\right)\int_0^\infty e^{-\alpha s}\|\sigma\cdot\nabla G_\alpha^0 f\|_2^2\,ds \\ &\leqq \frac{\lambda}{\alpha}\left(6+\frac{3\alpha}{\alpha-\alpha_0}\right)\|\nabla G_\alpha^0 f\|_2^2\,. \end{aligned} \tag{3.37}$$

By (3.12) and Cauchy-Schwartz inequality, $E_0^x\left[\int_0^\infty e^{-\alpha s}M_s|f(X_s)|\,ds\right] < \infty$ and $\lim_{t\to\infty} E_0^x[M_t e^{-\alpha t}G_\alpha^0 f(X_t)] = 0$. Let $t \to \infty$ in (3.34), we have

$$E_0^x\left[\int_0^\infty M_s e^{-\alpha s} f(X_s)\,ds\right] = G_\alpha^0 f(x) + E_0^x\left[\int_0^\infty e^{-\alpha s}M_s(b\cdot\nabla G_\alpha^0 f)(X_s)\,ds\right].$$

That is,

$$G_\alpha^b f(x) = G_\alpha^0 f + G_\alpha^b(b\cdot\nabla G_\alpha^0 f)(x)\,.$$

By the same argument, we have

$$\begin{aligned} &G_\alpha^b(b\cdot\nabla G_\alpha^0 f)(x) \\ &= G_\alpha^0(b\cdot\nabla G_\alpha^0 f)(x) + G_\alpha^b(b\cdot\nabla G_\alpha^0(b\cdot\nabla G_\alpha^0 f)) \\ &= G_\alpha^0(b\cdot\nabla G_\alpha^0 f)(x) + G_\alpha^b(b\cdot\nabla G_\alpha^0)^2 f(x) \\ &= \cdots \\ &= \sum_{k=1}^n G_\alpha^0(b\cdot\nabla G_\alpha^0)^k f(x) + G_\alpha^b(b\cdot\nabla G_\alpha^0)^{n+1} f(x)\,. \end{aligned}$$

If we set $\phi_k = G_\alpha^0(b\cdot\nabla G_\alpha^0)^k f$, which is known to be in $W_0^{1,2}(D)$ by Lemma 3.3 for $\alpha > \alpha_1$, then

$$G_\alpha^b(b\cdot\nabla G_\alpha^0 f)(x) = \sum_{k=1}^n \phi_k(x) + G_\alpha^b(b\cdot\nabla\phi_n)(x)\,. \tag{3.38}$$

By the same reasoning as that for (3.37), we have

$$\|G_\alpha^b(b\cdot\nabla\phi_n)\|_2^2 \leqq \frac{\lambda}{\alpha}\left(6+\frac{3\alpha}{\alpha-\alpha_0}\right)\|\nabla\phi_n\|_2^2\,,$$

which converges to zero as $n \to \infty$ by Lemma 3.3. Hence

$$G_\alpha^b(b\cdot\nabla G_\alpha^0 f)(x) = \sum_{k=1}^\infty \phi_k(x)$$

for $\alpha > \alpha_1$, which proves (3.31). (3.32) follows immediately from (3.31). (3.33) follows from (3.31) and (3.16). □

Definition 3.1 *A bilinear form* $(\mathscr{D}(\mathscr{E}),\mathscr{E})$ *(not necessarily symmetric) on* $L^2(D,m)$ *is called a closed form if*

(a) $\mathscr{E}$ *is bounded from below, i.e., there exists a constant* $\beta_0 \geqq 0$ *such that*

$$\mathscr{E}_{\beta_0}(u,u) \geqq 0 \quad \text{for } u \in \mathscr{D}(\mathscr{E}),$$

where $\mathscr{E}_{\beta_0} = \mathscr{E} + \beta_0(\ ,\ )$.

(b) $(\mathscr{D}(\mathscr{E}),\mathscr{E})$ *satisfies the sector condition: there exists a constant* $K > 0$ *such that*

$$|\mathscr{E}(u,v)| \leqq K\sqrt{\mathscr{E}_{\beta_0}(u,u)}\sqrt{\mathscr{E}_{\beta_0}(v,v)}, \quad u,v \in \mathscr{D}(\mathscr{E}), \tag{3.39}$$

(c) $\mathscr{D}(\mathscr{E})$ *is a dense linear subspace of* $L^2(D,m)$ *and is complete with respect to the norm* $|||\cdot|||_\alpha$ *for some (and hence for all)* $\alpha > \beta_0$, *where*

$$|||u|||_\alpha = \sqrt{\mathscr{E}_\alpha(u,u)}, \quad u \in \mathscr{D}(\mathscr{E}).$$

Definition 3.2 *A closed form* $(\mathscr{D}(\mathscr{E}),\mathscr{E})$ *on* $L^2(D,m)$ *is called a Dirichlet space if for each* $u \in \mathscr{D}(\mathscr{E})$, $u^+ \wedge 1 \in \mathscr{D}(\mathscr{E})$ *and*

$$\mathscr{E}(u^+ \wedge 1, u - u^+ \wedge 1) \geqq 0 \tag{3.40}$$

where $u^+ = u \vee 0$. *A Dirichlet space* $(\mathscr{D}(\mathscr{E}),\mathscr{E})$ *on* $L^2(D,m)$ *is called regular if* $\mathscr{D}(\mathscr{E}) \cap C_c(D)$ *is* $|||\cdot|||_\alpha$*-dense in* $\mathscr{D}(\mathscr{E})$ *for some* $\alpha > \beta_0$ *and uniformly dense in* $C_c(D)$.

It is well known now [4] that for a regular Dirichlet space $(\mathscr{D}(\mathscr{E}),\mathscr{E})$ on $L^2(D,m)$, there exists a unique Hunt process $(Y, P^x, x \in D)$ associated with $(\mathscr{D}(\mathscr{E}),\mathscr{E})$ such that for $\beta > \beta_0$ and $f \in L^2(D,m)$,

$$G_\beta f(x) = E^x\left[\int_0^\infty e^{-\beta s} f(X_s)\,ds\right]$$

is in $\mathscr{D}(\mathscr{E})$ and satisfies

$$\mathscr{E}_\beta(G_\beta f, g) = (f,g), \quad \text{for } g \in \mathscr{D}(\mathscr{E}).$$

For $f,g \in W_0^{1,2}(D)$, let

$$\begin{aligned}\mathscr{E}^b(f,g) &= \frac{1}{2}\int_D (a\nabla f)\cdot\nabla g\, m(dx) - \int_D (b\cdot\nabla f) g m(dx) \\ &= \frac{1}{2}\sum_{i,j=1}^d \int_D a_{ij}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}\, m(dx) - \sum_{i=1}^d \int_D b_i \frac{\partial f}{\partial x_i} g m(dx),\end{aligned} \tag{3.41}$$

where a and b are as in (3.1). By (1.7), it is easy to check that $(W_0^{1,2}(D),\mathscr{E}^b)$ is a regular non-symmetric Dirichlet space on $L^2(D,m)$.

Theorem 3.5 $(\Omega, X, \zeta, Q^x, x \in D)$ *is the continuous Hunt process in* D *associated with the regular Dirichlet space* $(W_0^{1,2}(D),\mathscr{E}^b)$.

Proof. This follows from Theorem 3.4 since for $\alpha > \alpha_1$ and $f \in L^2(D,m), G_\alpha^b f \in W_0^{1,2}(D)$ and for $g \in W_0^{1,2}(D)$

$$\begin{aligned}
&\mathscr{E}_\alpha^b(G_\alpha^b f, g) \\
&= \mathscr{E}_\alpha^0(G_\alpha^0 f + G_\alpha^0(b\cdot\nabla G_\alpha^b f), g) - (b\cdot\nabla G_\alpha^b f, g) \\
&= (f, g) + (b\cdot\nabla G_\alpha^b f, g) - (b\cdot\nabla G_\alpha^b f, g) \\
&= (f, g) . \quad \square
\end{aligned}$$

Theorem 3.2 now is a direct consequence of Theorem 3.5 (c.f. [21]). By [21, Lemma 3.2], we have

Corollary 3.6

$$W_0^{1,2}(D) = \left\{ f \in L^2(D,m) : \sup_{\alpha>\alpha_1} \alpha(f - \alpha G_\alpha^b f, f) < \infty \right\} \tag{3.42}$$

$$\mathscr{E}(f, g) = \lim_{\alpha\to\infty} \alpha(f - \alpha G_\alpha^b f, g), \quad f, g \in W_0^{1,2}(D) . \tag{3.43}$$

Proposition 3.7 *For a bounded Borel function ϕ on $D, x \to E_Q^x[\phi(X_t)]$ is a continuous function in D. If $y \in \partial D$ is a regular point for $(\frac{1}{2}\Delta, D)$, then*

$$\lim_{\substack{x\to y\\ x\in D}} E_Q^x[\phi(X_t)] = 0, \quad \textit{for } t > 0 . \tag{3.44}$$

Proof. Denote $E_Q^x[\phi(X_t)]$ by $P_t^Q\phi(x)$. Then for $s < t$

$$\begin{aligned}
&E_Q^x[\phi(X_t)] \\
&= E_Q^x[E^{X_s}[\phi(X_{t-s})]] \\
&= E_Q^x[P_{t-s}^Q\phi(X_s)] \\
&= E_0^x[P_{t-s}^Q\phi(X_s)] + E_0^x[(M_s - 1)P_{t-s}^Q\phi(X_s)] .
\end{aligned}$$

Since by (3.6) and (3.3), $\lim_{s\downarrow 0} \sup_{x\in D} E_0^x[M_t^2] = 1$, thus

$$\begin{aligned}
&\limsup_{s\downarrow 0}\sup_{x\in D} E_0^x[(M_s - 1)P_{t-s}^Q\phi(X_s)] \\
&\leqq \limsup_{s\downarrow 0}\sup_{x\in D} \|\phi\|_\infty E_0^x[(M_s - 1)^2] \\
&= \limsup_{s\downarrow 0}\sup_{x\in D} \|\phi\|_\infty (E_0^x[M_s^2] - 1) \\
&= 0 .
\end{aligned}$$

Therefore $x \to E_Q^x[\phi(X_t)]$ is a continuous function on D satisfying (3.44) since $x \to E_0^x[P_{t-s}^Q\phi(X_s)]$ is so for each $s \in (0, t)$. $\square$

4 Existence and representation of solutions for Dirichlet boundary value problems of $\frac{1}{2}\nabla\cdot(a\nabla) + b\cdot\nabla$

In this section, we will give an existence and representation theorem for solutions of Dirichlet boundary value problem

$$\begin{cases} L^b u = 0 & \text{in } D_1 \\ u = \phi & \text{on } \partial D_1 \end{cases} \tag{4.1}$$

where D_1 is a relatively compact open subset of D.

For a bounded function ϕ in $W_0^{1,2}(D)$, let

$$u(x) = E_Q^x[\phi(X_{\tau(D_1)})], \quad x \in D_1 \tag{4.2}$$

and

$$u_0(x) = E_0^x[\phi(X_{\tau(D_1)})], \quad x \in D_1 \ . \tag{4.3}$$

By (2.17), u_0 is the unique weak solution of $L^0 u_0 = 0$ in D_1 such that $u_0 - \phi|_{D_1} \in W_0^{1,2}(D_1)$. By setting $u_0 = \phi$ off D_1, $u_0 \in W_0^{1,2}(D)$.

Lemma 4.1

$$u(x) = u_0(x) + \lim_{t\to\infty} E_Q^x\left[\int_0^{t\wedge\tau(D_1)} b\cdot\nabla u_0(X_s)\,ds\right], \quad x \in D_1 \ . \tag{4.4}$$

Proof. By (2.16),

$$u_0(X_{t\wedge\tau(D_1)}) = u_0(X_0) + \int_0^{t\wedge\tau(D_1)} (\sigma\nabla u_0)(X_s)\,dW_s, \quad t \geqq 0$$

is a bounded P_0^x-martingale for $x \in D_1$. Since $M_{t\wedge\tau(D_1)} = 1 + \int_0^{t\wedge\tau(D_1)} M_s\sigma^{-1}b(X_s) \cdot dW_s$,

$$E_0^x[(u_0(X_{t\wedge\tau(D_1)}) - u_0(X_0))M_{t\wedge\tau(D_1)}] = E_0^x\left[\int_0^{t\wedge\tau(D_1)} M_s(b\cdot\nabla u_0)(X_s)\,ds\right] .$$

Hence

$$E_Q^x[u_0(X_{t\wedge\tau(D_1)})] = u_0(x) + E_Q^x\left[\int_0^{t\wedge\tau(D_1)} (b\cdot\nabla u_0)(X_s)\,ds\right], \quad t > 0 \ . \tag{4.5}$$

(4.4) follows from the bounded convergence theorem and [14, Theorem 4.3.2]. □

Theorem 4.2 *Suppose ϕ is a bounded function in $W_0^{1,2}(D)$. Then $u \in W^{1,2}(D_1) \cap C(D_1)$ and u is the unique weak solution of $L^b u = 0$ such that $u - \phi|_{D_1} \in W_0^{1,2}(D_1)$.*

Proof. Let

$$f(x) = \lim_{t\to\infty} E_Q^x\left[\int_0^{t\wedge\tau(D_1)} (b\cdot\nabla u_0)(X_s)\,ds\right], \tag{4.6}$$

which is a bounded function by (4.4). Denote by Y the killed process of X upon leaving D_1, that is,

$$Y_t = \begin{cases} X_t, & t < \tau(D_1) \\ \hat{\Delta}, & t \geqq \tau(D_1)\,. \end{cases}$$

$(\Omega, Y, Q^x, x \in D_1)$ is the Hunt process associated with the regular Dirichlet space $(W_0^{1,2}(D_1), \mathscr{E}^b)$ (cf. [4]). Let $G_\alpha^{D_1}$ be the α-resolvent of Y and let

$$\mathscr{E}^{(\beta)}(h,g) = \beta(h - \beta G_\beta^{D_1} h, g)_{L^2(D_1)}, \quad h, g \in L^2(D_1)\,. \tag{4.7}$$

Note that

$$\begin{aligned} & f - \beta G_\beta^{D_1} f \\ &= E_Q^x\left[\beta\int_0^\infty e^{-\beta t}(f(Y_0) - f(Y_t))\,dt\right] \\ &= E_Q^x\left[\beta\int_0^\infty e^{-\beta t}\left(\int_0^t (b\cdot\nabla u_0)(Y_s)\,ds\right) dt\right] \\ &= E_Q^x\left[\int_0^\infty e^{-\beta s}(b\cdot\nabla u_0)(Y_s)\,ds\right]. \end{aligned}$$

Thus for sufficiently large $\beta > 0$

$$\mathscr{E}^{(\beta)}(f,f) = (b\cdot\nabla u_0, \beta\hat{G}_\beta^{D_1} f)_{L^2(D_1)}, \tag{4.8}$$

where $\hat{G}_\beta^{D_1}$ is the adjoint of $G_\beta^{D_1}$ in $L^2(D_1, dx)$. Since $|b|^2 \in K_d$ and $\hat{G}_\beta^{D_1} f \in W_0^{1,2}(D_1)$, for $\varepsilon > 0$

$$\begin{aligned} & \mathscr{E}^{(\beta)}(f,f) \\ &\leqq \tfrac{1}{2}\int_{D_1} |\nabla u_0|^2\,dx + \tfrac{1}{2}\int_{D_1} |b|^2(\beta\hat{G}_\beta^{D_1} f)^2\,dx \\ &\leqq \tfrac{1}{2}\int_{D_1} |\nabla u_0|^2\,dx + \tfrac{1}{2}\left(\varepsilon\int_{D_1} |\nabla\beta\hat{G}_\beta^{D_1} f|^2\,dx + A(\varepsilon)\int_{D_1} |\beta\hat{G}_\beta^{D_1} f|^2\,dx\right). \end{aligned} \tag{4.9}$$

Note that there exists a constant $\alpha > 1$ such that

$$\int_{D_1} |\nabla g|^2\,dx \leqq \alpha\mathscr{E}^b(g,g) + \alpha^2\int_{D_1} |g|^2\,dx, \quad \text{for } g \in W_0^{1,2}(D)\,. \tag{4.10}$$

Thus by selecting $\varepsilon = \frac{1}{\alpha}$ in (4.9) we have by [21, Lemma 3.1(i)] and (3.14) that

$$\mathscr{E}^{(\beta)}(f,f) \leqq \frac{1}{2}\int_{D_1} |\nabla u_0|^2\,dx + \frac{1}{2}\mathscr{E}^b(\beta\hat{G}_\beta^{D_1}f, \beta\hat{G}_\beta^{D_1}f)$$
$$+ \frac{\beta}{\beta-\alpha_0}\left(A\left(\frac{1}{\alpha}\right)+\frac{\alpha}{2}\right)\int_{D_1}|f|^2\,dx$$
$$\leqq \frac{1}{2}\int_{D_1}|\nabla u_0|^2\,dx + \frac{1}{2}\mathscr{E}^{(\beta)}(f,f) + \frac{\beta}{\beta-\alpha_0}\left(A\left(\frac{1}{\alpha}\right)+\frac{\alpha}{2}\right)\int_{D_1}|f|^2\,dx$$

for $\beta > \alpha_0 > 0$. Therefore $\sup_{\beta>2\alpha_0}\mathscr{E}^{(\beta)}(f,f) < \infty$ and by (3.42), $f \in W_0^{1,2}(D_1)$. So $u = u_0 + f$ is in $W^{1,2}(D_1)$ and $u - \phi|_{D_1} = (u_0 - \phi|_{D_1}) + f \in W_0^{1,2}(D_1)$. Since $\beta\hat{G}_\beta^{D_1}f$ converges to f in $(W_0^{1,2}(D_1), \|\cdot\|_{1,2})$ as $\beta \to \infty$, after letting $\beta \to \infty$ in (4.8) we get

$$\mathscr{E}^b(f,f) = (b\cdot\nabla u_0, f)_{L^2(D_1)}\,. \tag{4.11}$$

A similar argument yields

$$\mathscr{E}^b(f,g) = (b\cdot\nabla u_0, g)_{L^2(D_1)}\,. \tag{4.12}$$

For any test function $\psi \in C_c^\infty(D_1)$,

$$\begin{aligned}&\mathscr{E}^b(u,\psi)\\ &= \mathscr{E}^b(u_0,\psi) + \mathscr{E}^b(f,\psi)\\ &= \mathscr{E}^0(u_0,\psi) - \int_{D_1}(b\cdot\nabla u_0)\psi\,dx + \int_{D_1}(b\cdot\nabla u_0)\psi\,dx\\ &= 0\,.\end{aligned}$$

Hence u is a weak solution for $L^b u = 0$. The uniqueness comes from [30]. We now show that f is continuous in D_1. For $s > 0$, by the Markov property of X

$$f(x) = E_Q^x\left[\int_0^s (b\cdot\nabla u_0)(Y_r)\,dr\right] + E_Q^x[f(X_s)]$$

which by (4.5)

$$\begin{aligned}&= E_Q^x[u_0(X_{s\wedge\tau(D_1)})] - u_0(x) + E_Q^x[f(X_s)]\\ &= E_0^x[(M_s-1)u_0(X_{s\wedge\tau(D_1)})] + E_Q^x[f(X_s)]\,.\end{aligned} \tag{4.13}$$

Hence by (3.6) and (3.3)

$$\begin{aligned}&\lim_{s\downarrow 0}\sup_{x\in D_1}|f(x) - E_Q^x[f(X_s)]|\\ &\leqq \lim_{s\downarrow 0}\left(\sup_{x\in\partial D_1}|\phi(x)|\right)\sup_{x\in D_1}(E_0^x M_s^2 - 1)^{\frac{1}{2}}\\ &= 0\,.\end{aligned} \tag{4.14}$$

Thus by Proposition 3.7, $f \in C(D_1)$ and therefore $u \in C(D_1)$ since u_0 is known to be Hölder continuous in D_1 (cf. [22]). □

Corollary 4.3 *Any locally bounded weak solution of $L^b u = 0$ in D has a continuous version.*

Proof. Suppose that u is a weak solution of $L^b u = 0$ in D which is locally bounded. For an arbitrary point y in D, let $r > 0$ such that $\overline{B(y,r)} \subset D$ where $B(y,r) = \{x \in \mathbb{R}^d : |x-y| < r\}$. Applying Theorem 4.2 to domain $B(y,r)$, $x \to E^x_Q[u(X_{\tau(B(y,r))})]$ is a continuous version of u in $B(y,r)$. Therefore u has a continuous version by using partition of unity. □

Lemma 4.4 *For any weak solution u of $L^b u = 0$ in D, the following inequality holds*

$$\int_{B(y,r)} |\nabla u|^2\,dx \leqq C \int_{B(y,R)} u^2\,dx \tag{4.15}$$

with $0 < r < R$ such that $\overline{B(y,R)} \subset D$, where $C > 0$ is a constant which only depends on λ in (1.2), the constants in (1.7) for $|b|^2$ and $R - r$.

Proof. Suppose that u is a weak solution of $L^b u = 0$ in D. Then $\mathscr{E}^b(u,\phi) = 0$ for $\phi \in C_c^\infty(D)$ and therefore $\mathscr{E}^b(u,\phi) = 0$ for any $\phi \in W_0^{1,2}(D)$ with compact support in D. Let $\gamma : [0,\infty) \to [0,\infty)$ be defined as

$$\gamma(s) = \begin{cases} 1, & 0 \leqq s \leqq r \\ \dfrac{R-s}{R-r}, & r < s \leqq R \\ 0, & s > R \end{cases} \tag{4.16}$$

and let $\psi(x) = \gamma(|x-y|)$. Then $\phi = \psi^2 u$ is in $W_0^{1,2}(D)$ with compact support in D.

It follows from $\mathscr{E}^b(u,\phi) = 0$ that for $\varepsilon > 0$

$$\begin{aligned}
&\frac{\lambda^{-1}}{2}\int_D \psi^2 |\nabla u|^2\,dm \\
&\leqq \frac{1}{2}\sum_{i,j=1}^d \int_D a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\psi^2\,dm \\
&= -\sum_{i,j=1}^d \int_D a_{ij}\psi u\frac{\partial u}{\partial x_i}\frac{\partial \psi}{\partial x_j}\,dm + \sum_{i=1}^d \int_D b_i\frac{\partial u}{\partial x_i}u\psi^2\,dm \\
&\leqq \frac{\varepsilon}{2}\sum_{i,j=1}^d \int_D \psi^2 a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\,dm + \frac{1}{2\varepsilon}\sum_{i,j=1}^d \int_{B(y,R)} a_{ij}u^2\frac{\partial \psi}{\partial x_i}\frac{\partial \psi}{\partial x_j}\,dm \\
&\quad + \frac{\varepsilon}{2}\int_D \psi^2|\nabla u|^2\,dm + \frac{1}{2\varepsilon}\int_D |b|^2 u^2\psi^2\,dm \\
&\leqq \frac{\varepsilon\lambda}{2}\int_D \psi^2|\nabla u|^2\,dm + \frac{\lambda}{2\varepsilon}\frac{1}{(R-r)^2}\int_{B(y,R)} u^2\,dm \\
&\quad + \frac{\varepsilon}{2}\int_D \psi^2|\nabla u|^2\,dm + \frac{1}{2\varepsilon}\left(\varepsilon^2\int_D |\nabla(u\psi)|^2\,dm + A(\varepsilon^2)\int_{B(y,R)} u^2\psi^2\,dm\right)
\end{aligned}$$

$$\leqq \left(\frac{\lambda}{2}+\frac{3}{2}\right)\varepsilon\int_D \psi^2|\nabla u|^2\,dm+\left(\frac{\lambda}{2\varepsilon}\frac{1}{(R-r)^2}+\frac{A(\varepsilon^2)}{2\varepsilon}\right)\int_{B(y,R)} u^2\,dm$$
$$+\varepsilon\int_{B(y,R)}|u|^2|\nabla\psi|^2 dm\,.$$

Hence

$$\left(\frac{1}{2\lambda}-\frac{\lambda+3}{2}\varepsilon\right)\int_D \psi^2|\nabla u|^2\,dm \leqq \left(\left(\frac{\lambda}{2\varepsilon}+\varepsilon\right)\frac{1}{(R-r)^2}+\frac{A(\varepsilon^2)}{2\varepsilon}\right)\int_{B(y,R)} u^2\,dm\,.$$

Let $\varepsilon=\frac{1}{2\lambda(\lambda+3)}$ in above inequality and set

$$C=4\lambda\left(\left(\frac{\lambda}{2\varepsilon}+\varepsilon\right)\frac{1}{(R-r)^2}+\frac{A(\varepsilon^2)}{2\varepsilon}\right),$$

(4.15) is thus proved. □

Theorem 4.5 *Let D_1 be a relatively compact open subset of D and $\phi\in C(\partial D_1)$. Then*

$$u(x)=E^x_Q[\phi(X_{\tau(D_1)})]$$

is the unique weak solution of $L^b u=0$ in D_1 such that

$$\lim_{\substack{x\to y\\ x\in D_1}} u(x)=\phi(y) \tag{4.17}$$

for every boundary point $y\in\partial D_1$ which is regular for $(\frac{1}{2}\Delta, D_1)$. u is continuous in D_1.

Proof. Let $\phi_n\in C^2(\partial D_1)$ such that ϕ_n converges uniformly to ϕ on ∂D_1, and let $u_n(x)=E^x_Q[\phi_n(X_{\tau(D_1)})]$. By Theorem 4.2 together with Lemma 4.1, Theorem 2.1, Proposition 3.7 and (4.14), $u_n\in W^{1,2}(D_1)\cap C(D_1)$ is the unique weak solution of $L^b u=0$ such that (4.17) holds for (u_n,ϕ_n). Clearly, u_n uniformly converges to u on D_1 and by Lemma 4.4, u_n converges to u in $(W^{1,2}(D_2),\|\cdot\|_{1,2})$ for any open set D_2 with $\bar{D}_2\subset D_1$. Therefore u is the weak solution of $L^b u=0$ such that (4.17) holds. □

Remark. By a minor modification of the arguments given in [25], it can be shown that $y\in\partial D_1$ is regular for (L^b, D_1) if and only if it is regular for $(\frac{1}{2}\Delta, D_1)$.

5 Operators of the form $\frac{1}{2}\nabla\cdot(a\nabla) + b\cdot\nabla + q$

Let

$$L = L^b + q = \frac{1}{2}\nabla\cdot(a\nabla) + b\cdot\nabla + q \tag{5.1}$$

be defined on $D \subseteq \mathbb{R}^d$ with $q \leqq 0$, $1_D q \in K_d$ and $1_D|b|^2 \in K_d$. For $f, g \in W_0^{1,2}(D)$, let

$$\mathscr{E}(f,g) = \frac{1}{2}\sum_{i,j=1}^{d} \int_D a_{ij}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}\,dm - \sum_{i=1}^{d}\int_D b_i \frac{\partial f}{\partial x_i} g\,dm - \int_D qfg\,dm\,. \tag{5.2}$$

Then $(W_0^{1,2}(D), \mathscr{E})$ is a regular (non-symmetric) Dirichlet space on $L^2(D,m)$ and there exist $\alpha_0 > 0$ and $M > 1$ such that

$$\frac{1}{M}\|f\|_{1,2}^2 \leqq \mathscr{E}_{\alpha_0}(f,f) \leqq M\|f\|_{1,2}^2, \quad f \in W_0^{1,2}(D)\,, \tag{5.3}$$

where $\mathscr{E}_{\alpha_0} = \mathscr{E} + \alpha_0(\,,\,)$ and $\|\cdot\|_{1,2}$ is defined in (1.11). As a direct consequence of (5.3) and (1.7), we have

Lemma 5.1 *Suppose that $1_D g \in K_d$. Then for any $\delta \in (0,1)$ there exist constants $\beta_0 = \beta_0(\delta, g) > 0$, $C_1 = C_1(\delta, g) > 0$ and $C_2 = C_2(\delta, g) > 0$ such that for $u, v \in W_0^{1,2}(D)$*

$$\int_D |g(x)u(x)v(x)|\,m(dx) \leqq \delta\sqrt{\mathscr{E}_{\beta_0}(u,u)}\sqrt{\mathscr{E}_{\beta_0}(v,v)}\,, \tag{5.4}$$

$$\int_D |g(x)u(x)v(x)|\,m(dx) \leqq \delta\sqrt{\mathscr{E}^b_{\beta_0}(u,u)}\sqrt{\mathscr{E}^b_{\beta_0}(v,v)}\,, \tag{5.5}$$

$$\int_D |g(x)u(x)v(x)|\,m(dx) \leqq C_1\|u\|_{1,2} + C_2\|v\|_2 + \delta\mathscr{E}(u,v)\,, \tag{5.6}$$

$$\int_D |g(x)u(x)v(x)|\,m(dx) \leqq C_1\|u\|_{1,2} + C_2\|v\|_2 + \delta\mathscr{E}^b(u,v)\,. \tag{5.7}$$

For each $x \in D$, let P^x be the unique probability measure on Ω such that

$$E^x[f(X_t)] = E^x_Q\left[e^{\int_0^t q(X_s)\,ds} f(X_t)\right], \quad t > 0\,, \tag{5.8}$$

for each bounded Borel function f on D. Clearly, $(\Omega, X, \zeta, \{P^x, x \in D\})$ is a continuous Hunt process on D. Let G_α be its α-resolvent and G^b_α be the α-resolvent for $(\Omega, X, \zeta, Q^x, x \in D)$. Since $q \leqq 0$

$$\|G_\alpha\|_{2,2} \leqq \|G^b_\alpha\|_{2,2} < \frac{1}{\alpha - \alpha_0} \tag{5.9}$$

for $\alpha > \alpha_0$ where α_0 is the constant in (3.12). By Theorem 3.4, there exists a constant $\alpha_1 > \alpha_0$ such that $G^b_\alpha f \in W_0^{1,2}(D)$ for $\alpha > \alpha_1$ and $f \in L^2(D,m)$.

Theorem 5.2 *For $\alpha > \alpha_1$ and $f \in L^2(D,m)$, $G_\alpha f \in W_0^{1,2}(D)$.*

$$G_\alpha f = G_\alpha^b f + G_\alpha(qG_\alpha^b f), \tag{5.10}$$
$$G_\alpha f = G_\alpha^b f + G_\alpha^b(qG_\alpha f), \tag{5.11}$$

and

$$\mathscr{E}_\alpha^b(G_\alpha f, g) = (f,g) + (qG_\alpha f, g), \quad g \in W_0^{1,2}(D). \tag{5.12}$$

Proof. It suffices to prove results for nonnegative f in $L^2(X,m)$. By the Markov property of X and Fubini theorem,

$$\begin{aligned}
&G_\alpha f(x)\\
&= E^x\left[\int_0^\infty e^{-\alpha t} f(X_t)\,dt\right]\\
&= E_Q^x\left[\int_0^\infty e^{-\alpha t} e^{\int_0^t q(X_s)\,ds} f(X_t)\,dt\right]\\
&= E_Q^x\left[\int_0^\infty e^{-\alpha t}\left(1+\int_0^t q(X_s)e^{\int_0^s q(X_u)\,du}\,ds\right) f(X_t)\,dt\right]\\
&= G_\alpha^b f(x) + E_Q^x\left[\int_0^\infty e^{-\alpha t}\left(\int_0^t q(X_s)e^{\int_0^s q(X_u)\,du}\,ds\right) f(X_t)\,dt\right]\\
&= G_\alpha^b f(x) + E_Q^x\left[\int_0^\infty q(X_s)e^{\int_0^s q(X_u)\,du}\left(\int_s^\infty e^{-\alpha t} f(X_t)\,dt\right) ds\right]\\
&= G_\alpha^b f(x) + E_Q^x\left[\int_0^\infty q(X_s)e^{\int_0^s q(X_u)\,du} e^{-\alpha s} G_\alpha^b f(X_s)\,ds\right]\\
&= G_\alpha^b f(x) + G_\alpha(qG_\alpha^b f)(x).
\end{aligned}$$

This proves (5.10). On the other hand,

$$\begin{aligned}
&G_\alpha^b(qG_\alpha f)(x)\\
&= E_Q^x\left[\int_0^\infty e^{-\alpha t} q(X_t) G_\alpha f(X_t)\,dt\right]\\
&= E_Q^x\left[\int_0^\infty e^{-\alpha t} q(X_t) E_Q^{X_t}\left[\int_0^\infty e^{-\alpha s} e^{\int_0^s q(X_s)\,du} f(X_s)\,ds\right]\right]\\
&= E_Q^x\left[\int_0^\infty e^{-\alpha t} q(X_t) e^{\alpha t} e^{-\int_0^t q(X_u)\,du}\left(\int_t^\infty e^{-\alpha s} e^{\int_0^s q(X_u)\,du} f(X_s)\,ds\right) dt\right]\\
&= E_Q^x\left[\int_0^\infty e^{-\alpha s} e^{\int_0^s q(X_u)\,du} f(X_s)\left(\int_0^s q(X_t) e^{-\int_0^t q(X_u)\,du}\,dt\right) ds\right]\\
&= E_Q^x\left[\int_0^\infty e^{-\alpha s} e^{\int_0^s q(X_u)\,du} f(X_s)(1-e^{-\int_0^s q(X_u)\,du})\,ds\right]\\
&= G_\alpha f(x) - G_\alpha^b f(x),
\end{aligned}$$

which proves (5.11). Note that $f \geqq 0$, $q \leqq 0$, $(qG_\alpha f) \leqq 0$. Hence

$$\begin{aligned}&\beta(G_\alpha f - \beta G^b_\beta G_\alpha f,\ G_\alpha f)\\&= \beta(G^b_\alpha f - \beta G^b_\beta G^b_\alpha f,\ G_\alpha f) + \beta(G^b_\alpha(qG_\alpha f) - \beta G^b_\beta G^b_\alpha(qG_\alpha f),\ G_\alpha f)\\&\leqq \beta(G^b_\alpha f - \beta G^b_\beta G^b_\alpha f,\ G^b_\alpha f) + \frac{\alpha\beta}{\beta-\alpha}[(G^b_\alpha f, G^b_\alpha f) - (G_\alpha f, G_\alpha f)]\,.\end{aligned}$$

Therefore

$$\begin{aligned}&\sup_{\beta>\alpha_1}\beta(G_\alpha f - \beta G^b_\beta G_\alpha f,\ G_\alpha f)\\&\leqq \sup_{\beta>\alpha_1}\beta(G^b_\alpha f - \beta G^b_\beta G^b_\alpha f,\ G^b_\alpha f) + \frac{\alpha\alpha_1}{\alpha_1-\alpha}[(G^b_\alpha f, G^b_\alpha f) - (G_\alpha f,\ G_\alpha f)]\\&< \infty\,.\end{aligned}$$

By Corollary 3.6, $G_\alpha f \in W^{1,2}_0(D)$ and

$$\mathscr{E}^b_\alpha(G_\alpha f,\ G_\alpha f) \leqq \mathscr{E}^b_\alpha(G^b_\alpha f,\ G^b_\alpha f)\,. \tag{5.13}$$

Thus $G^b_\alpha(qG_\alpha f) = G_\alpha f - G^b_\alpha f$ is in $W^{1,2}_0(D)$. By (3.43), for $g \in W^{1,2}_0(D)$,

$$\begin{aligned}&\mathscr{E}(G^b_\alpha(qG_\alpha f),\ g)\\&= \lim_{\beta\to\infty}\beta(G^b_\alpha(qG_\alpha f) - \beta G^b_\beta G^b_\alpha(qG_\alpha f),\ g)\\&= \lim_{\beta\to\infty}\frac{\beta}{\beta-\alpha}(\beta G^b_\beta(qG_\alpha f) - \alpha G^b_\alpha(qG_\alpha f),\ g)\\&= \lim_{\beta\to\infty}\frac{\beta}{\beta-\alpha}(qG^b_\alpha f, \beta\hat{G}^b_\beta g) - \alpha(G^b_\alpha(qG_\alpha f),\ g)\\&= (qG^b_\alpha f,\ g) - \alpha(G^b_\alpha(qG_\alpha f,\ g))\,.\end{aligned}$$

The last equality follows from (5.5) since $\beta\hat{G}^b_\beta g$ converges to g in $W^{1,2}_0(D)$ as $\beta \to \infty$ and $1_D q \in K_d$. Therefore

$$\mathscr{E}_\alpha(G^b_\alpha(qG_\alpha f), g) = (qG_\alpha f, g), \quad g \in W^{1,2}_0(D) \tag{5.14}$$

and (5.12) follows. □

Since

$$\mathscr{E}_\alpha(G_\alpha f, g) = \mathscr{E}^b_\alpha(G_\alpha f, g) - (qG_\alpha f, g) = (f, g), \quad g \in W^{1,2}_0(D)\,,$$

we have

Theorem 5.3 $(\Omega, X, \zeta, P^x, x \in D)$ *is the continuous Hunt process associated with the regular non-symmetric Dirichlet space* $(W^{1,2}_0(D), \mathscr{E})$ *on* $L^2(D, m)$. *Its infinitesimal generator is L with*

$$Domain(L) = \{f \in W^{1,2}_0(D) : Lf \in L^2(D, m)\}\,. \tag{5.15}$$

By [21, Lemma 3.2], we have

Corollary 5.4

$$W_0^{1,2}(D) = \{f \in L^2(D,m) : \sup_{\alpha > \alpha_1} \alpha(f - \alpha G_\alpha f, f) < \infty\},$$
$$\mathscr{E}(f,g) = \lim_{\alpha \to \infty} \alpha(f - \alpha G_\alpha f, g).$$

Theorem 5.5 *There exists a constant $\alpha_2 > \alpha_1$ such that for $\alpha > \alpha_2$ and $f \in L^2(D,m)$,*

$$\phi_n = G_\alpha^b (q G_\alpha^b)^n f, \quad n \geqq 1 \tag{5.16}$$

is in $W_0^{1,2}(D)$, $\sum_{n=1}^{\infty} \phi_n$ converges in $(W_0^{1,2}(D), \|\cdot\|_{1,2})$ and

$$G_\alpha f = \sum_{n=0}^{\infty} G_\alpha^b (q G_\alpha^b)^n f . \tag{5.17}$$

Proof. It suffices to consider nonnegative function $f \in L^2(D,m)$. By Lemma 5.1, there is a constant $\beta_0 > 0$ such that

$$\int_D |quv| m(dx) \leqq \frac{1}{2} \sqrt{\mathscr{E}_{\beta_0}^b(u,u)} \sqrt{\mathscr{E}_{\beta_0}^b(v,v)}, \quad u,v \in W_0^{1,2}(D). \tag{5.18}$$

Let $\alpha_2 = \alpha_1 \vee \beta_0$ and $\alpha > \alpha_2$. Set $q_n = q \vee (-n)$ and $g_n = G_\alpha^b(q_n G_\alpha^b f)$. Note that $g_n \in W_0^{1,2}(D)$ since $q_n G_\alpha^b f \in L^2(D,m)$. By (5.18) and (5.9)

$$\begin{aligned}
\|g_n\|_2^2 &= (q_n G_\alpha^b f, \hat{G}_\alpha^b g_n) \\
&\leqq \frac{1}{2} \sqrt{\mathscr{E}_\alpha^b(G_\alpha^b f, G_\alpha^b f)} \sqrt{\mathscr{E}_\alpha^b(\hat{G}_\alpha^b g_n, \hat{G}_\alpha^b g_n)} \\
&= \frac{1}{2} \sqrt{\mathscr{E}_\alpha^b(G_\alpha^b f, G_\alpha^b f)} \sqrt{(\hat{G}_\alpha^b g_n, g_n)} \\
&\leqq \frac{1}{2} \frac{1}{\sqrt{\alpha - \alpha_0}} \sqrt{\mathscr{E}_\alpha^b(G_\alpha^b f, G_\alpha^b f)}\, \|g_n\|_2 .
\end{aligned}$$

Thus $\|g_n\|_2 \leqq \frac{1}{2} \frac{1}{\sqrt{\alpha - \alpha_0}} \sqrt{\mathscr{E}_\alpha^b(\phi_0, \phi_0)}$, where $\phi_0 = G_\alpha^b f$. By monotone convergence theorem $\lim_{n \to \infty} g_n = G_\alpha^b(q G_\alpha^b f) = \phi_1$. Thus

$$\|\phi_1\|_2^2 \leqq \frac{1}{4(\alpha - \alpha_0)} \mathscr{E}_\alpha^b(\phi_0, \phi_0). \tag{5.19}$$

Since

$$\mathscr{E}_\alpha^b(g_n, g_n) = (q_n G_\alpha^b f, g_n) \leqq \frac{1}{2} \sqrt{\mathscr{E}_\alpha^b(G_\alpha^b f, G_\alpha^b f)} \sqrt{\mathscr{E}_\alpha^b(g_n, g_n)},$$

by (5.18)

$$\mathscr{E}_\alpha^b(g_n, g_n) \leqq \frac{1}{4} \mathscr{E}_\alpha^b(G_\alpha^b f, G_\alpha^b f), \quad n \geqq 1 . \tag{5.20}$$

Therefore

$$\mathscr{E}_\alpha^b(\phi_1, \phi_1) \leqq \frac{1}{4} \mathscr{E}_\alpha^b(\phi_0, \phi_0). \tag{5.21}$$

By induction, we have $\phi_n \in W_0^{1,2}(D)$ for $n \geqq 2$ and

$$\|\phi_n\|_2^2 \leqq \frac{1}{4(\alpha-\alpha_0)}\mathscr{E}_\alpha^b(\phi_{n-1},\phi_{n-1}), \tag{5.22}$$

$$\mathscr{E}_\alpha^b(\phi_n,\phi_n) \leqq \frac{1}{4}\mathscr{E}_\alpha^b(\phi_{n-1},\phi_{n-1}). \tag{5.23}$$

Thus $\sum_{n=1}^{\infty}\phi_n$ converges in $(W_0^{1,2}(D), \|\cdot\|_{1,2})$. By (5.10)

$$G_\alpha f = \sum_{k=0}^{n}\phi_n + G_\alpha(q\phi_n).$$

Since we assume $f \geqq 0$, $(-1)^n\phi_n \geqq 0$ and

$$\|G_\alpha(q\phi_n)\|_2^2 \leqq \|G_\alpha^b(q\phi_n)\|_2^2 = \|\phi_{n+1}\|_2^2$$

which converges to zero as $n \to \infty$. Thus

$$G_\alpha f = \sum_{k=0}^{\infty}\phi_k. \qquad \square$$

Lemma 5.6 *Let D_1 be a relatively compact open subset of D and let ϕ be a bounded function in $W_0^{1,2}(D)$. Set*

$$u(x) = E^x[\phi(X_{\tau(D_1)})], \quad x \in D_1, \tag{5.24}$$

$$u_1(x) = E_Q^x[\phi(X_{\tau(D_1)})], \quad x \in D_1. \tag{5.25}$$

Then u is the unique weak solution of $Lu = 0$ such that $u - \phi|_{D_1} \in W_0^{1,2}(D_1)$ and

$$u(x) = u_1(x) + G^{D_1}(qu_1)(x), \quad x \in D_1, \tag{5.26}$$

where G^{D_1} is the Green operator for the killed process of (X, P^x) upon leaving D_1.

Proof. By Theorem 4.2, u_1 is the unique weak solution of $L^b u_1 = 0$ in D_1 such that $u_1 - \phi|_{D_1} \in W_0^{1,2}(D_1)$. By setting $u_1 = \phi$ off D_1, $u_1 \in W_0^{1,2}(D)$.

$$\begin{aligned} u(x) &= E_Q^x[e^{\int_0^{\tau(D_1)} q(X_s)\,ds}\phi(X_{\tau(D_1)})] \\ &= E_Q^x\left[\left(1 + \int_0^{\tau(D_1)} q(X_s)e^{\int_0^s q(X_u)\,du}\,ds\right)u_1(X_{\tau(D_1)})\right] \\ &= u_1(x) + E_Q^x\left[\int_0^{\tau(D_1)} q(X_s)e^{\int_0^s q(X_u)\,du}u_1(X_s)\,ds\right] \\ &= u_1(x) + G^{D_1}(qu_1)(x). \end{aligned}$$

Let $I(\beta) = \beta(G^{D_1}(qu_1) - \beta G_\beta^{D_1}G^{D_1}(qu_1),\ G^{D_1}(qu_1))_{L^2(D_1)}$, where $G_\beta^{D_1}$ is the β-resolvent of the killed process of (X, Q^x) upon leaving D_1.

$$\begin{aligned} I(\beta) &= \beta(G_\beta^{D_1}(qu_1), G^{D_1}(qu_1))_{L^2(D_1)} \\ &= (qu_1, \beta\hat{G}_\beta^{D_1}(G^{D_1}(qu_1)))_{L^2(D_1)} \end{aligned}$$

which by Lemma 5.1 with $\delta = \frac{1}{2}$

$$\begin{aligned}
&\leqq C_1 \left(\int_D |\nabla u_1|^2 \, dx + \int_D |u_1|^2 \, dx \right) + C_2 \int_{D_1} |\beta \hat{G}_\beta^{D_1} G^{D_1}(qu_1)|^2 \, dx \\
&\quad + \frac{1}{2} \mathscr{E}(\beta \hat{G}_\beta^{D}(G^D(qu_1)), \beta \hat{G}_\beta^{D_1}(G^{D_1}(qu_1))) \\
&\leqq C_1 \left(\int_D |\nabla u_1|^2 \, dx + \int_D |u_1|^2 \, dx \right) + \frac{C_2 \beta}{\beta - \alpha_0} \int_{D_1} |G^{D_1}(qu_1)|^2 \, dx \\
&\quad + \frac{1}{2} I(\beta) \, .
\end{aligned}$$

The last inequality follows from [21, Lemma 3.1(i)]. Thus $\sup_{\beta > \alpha_1} I(\beta) < \infty$ and therefore $G^D(qu_1) \in W_0^{1,2}(D)$. By Corollary 3.6, for $g \in W_0^{1,2}(D_1)$

$$\begin{aligned}
&\mathscr{E}^b(G^{D_1}(qu_1), g) \\
&= \lim_{\beta \to \infty} \beta (G^D(qu_1) - \beta G_\beta^{D_1} G^{D_1}(qu_1), g)_{L^2(D_1)} \\
&= \lim_{\beta \to \infty} (qu_1, \beta \hat{G}_\beta^{D_1} g)_{L^2(D_1)} \\
&= (qu_1, g)_{L^2(D_1)} \, .
\end{aligned} \tag{5.27}$$

Thus for $g \in W_0^{1,2}(D)$,

$$\mathscr{E}^b(u, g) = \mathscr{E}^b(u_1, g) + \mathscr{E}^b(G^{D_1}(qu_1), g) = (qu_1, g)_{L^2(D_1)} \, .$$

This implies

$$\mathscr{E}(u, g) = 0, \quad g \in W_0^{1,2}(D_1) \, .$$

Hence u is a weak solution of $Lu = 0$ and $u - \phi|_{D_1} \in W_0^{1,2}(D_1)$. The uniqueness comes from [30]. □

Lemma 5.7 *For a bounded function f on D and $t > 0, x \to E^x[f(X_t)]$ is continuous in D and*

$$\lim_{\substack{x \to y \\ x \in D}} E^x[f(X_t)] = 0 \tag{5.28}$$

for $y \in \partial D$ which is regular for $\left(\frac{1}{2}\Delta, D\right)$.

Proof. Set $P_s f(x) = E^x[f(X_s)]$. For $0 < s < t$,

$$\begin{aligned}
E^x[f(X_t)] &= E^x[E^{X_s}[f(X_{t-s})]] \\
&= E^x[P_{t-s} f(X_s)] \\
&= E_Q^x[P_{t-s} f(X_s)] + E_Q^x[(e^{\int_0^s q(X_u) \, du} - 1) P_{t-s} f(X_s)] \, .
\end{aligned} \tag{5.29}$$

Since $1_D q \in K_d$, by (2.5) and [3, Theorem 4.5],

$$\lim_{s \downarrow 0} \sup_{x \in D} E_0^x \left[\int_0^s |q(X_t)| \, dt \right] = 0 \, . \tag{5.30}$$

Thus by Khasminskii's Lemma (see, e.g., [27, Lemma B.1.2]),

$$\lim_{s\downarrow 0}\sup_{x\in D} E^x[e^{k\int_0^s |q|(X_u)\,du}] = 1, \quad k > 0\,. \tag{5.31}$$

Therefore together with (3.6) and (5.29), we have

$$\begin{aligned}
&\lim_{t\downarrow 0}\sup_{x\in D}|E^x[f(X_t)] - E_Q^x[P_{t-s}f(X_s)]| \\
&= \lim_{t\downarrow 0}\sup_{x\in D}|E_Q^x[(e^{\int_0^s q(X_u)\,du} - 1)P_{t-s}f(X_s)]| \\
&\leqq \|f\|_\infty \lim_{t\downarrow 0}\sup_{x\in D}(E_0^x[M_s^2])^{\frac{1}{2}}\left(E_0^x\left[\left(1 - e^{\int_0^s q(X_u)\,du}\right)^2\right]\right)^{\frac{1}{2}} \\
&= 0\,.
\end{aligned} \tag{5.32}$$

Lemma 5.7 now follows from Proposition 5.6. □

Lemma 5.8 *The function u in Lemma 5.6 is continuous in D_1.*

Proof. By Theorem 4.2, $u_1 \in C(D_1)$. Let $g = G^{D_1}(qu_1)$, which is a bounded function on D_1 by (5.26). Denote by Y the process of X killed upon leaving D_1, i.e., $Y_t = X_t$ if $t < \tau(D_1)$ and $Y_t = \hat{\Delta}$ if $t \geqq \tau(D_1)$. Then by the Markov property of Y,

$$g(x) = E^x\left[\int_0^t (qu_1)(Y_s)\,ds\right] + E^x[g(Y_t)]\,.$$

But

$$\begin{aligned}
&\sup_{x\in D_1}\left|E^x\left[\int_0^t (qu_1)(Y_s)\,ds\right]\right| \\
&\leqq \|\phi\|_\infty \sup_{x\in D_1} E^x\left[-\int_0^t q(X_s)\,ds\right] \\
&= \|\phi\|_\infty \sup_{x\in D_1} E_Q^x\left[\int_0^t (-q(X_s))e^{\int_0^s q(X_u)\,du}ds\right] \\
&= \|\phi\|_\infty \sup_{x\in D_1} E_Q^x\left[1 - e^{\int_0^t q(X_u)\,du}\right] \\
&\leqq \|\phi\|_\infty \sup_{x\in D_1}(E_0^x[M_t^2])^{\frac{1}{2}}(E_0^x[(1 - e^{\int_0^t q(X_u)\,du})^2])^{\frac{1}{2}}\,.
\end{aligned}$$

Therefore $\lim_{t\downarrow 0}\sup_{x\in D_1}|g(x) - E^x[g(X_t)]| = 0$ by (3.6) and (5.31). Hence $g \in C(D_1)$ since $x \to E^x[g(Y_t)]$ is in $C(D_1)$ for $t > 0$ by Lemma 5.7 with D_1 in place of D. □

Corollary 5.9 *Any locally bounded weak solution of $Lu = 0$ in D has a continuous version.*

Proof. Same as that for Corollary 4.3. □

By a similar argument as that for Lemma 4.4, we have

Lemma 5.10 *For any weak solution u of $Lu = 0$ in D, the following inequality holds*

$$\int_{B(y,r)} |\nabla u|^2 \, dx \leqq C \int_{B(y,R)} u^2 \, dx \tag{5.33}$$

with $0 < r < R$ such that $\overline{B(y,R)} \subset D$, where $C > 0$ is a constant which only depends on λ in (1,2), the constants in (1.7) for $|b|^2$ and q, and on $R-r$.

Theorem 5.11 *Let D_1 be a relatively compact open subset of D and $\phi \in C(\partial D_1)$. Then*

$$u(x) = E^x[\phi(X_{\tau(D_1)})] \tag{5.34}$$

is the unique weak solution of $Lu = 0$ in D_1 such that

$$\lim_{\substack{x \to y \\ x \in D_1}} u(x) = \phi(y) \tag{5.35}$$

for $y \in \partial D_1$ which is regular for $\left(\frac{1}{2}\Delta, D_1\right)$. u is continuous in D_1.

Proof. Let $\phi_n \in C^2(\partial D_1)$ be such that ϕ_n converges uniformly to ϕ on ∂D_1. Define $u_n(x) = E^x[\phi_n(X_{\tau(D_1)})]$. By Lemma 5.6, 5.7, 5.8 and Theorem 4.5, $u_n \in W^{1,2}(D_1) \cap C_1(D_1)$ is the unique weak solution of $Lu = 0$ in D_1 such that (5.35) holds for (u_n, ϕ_n). u_n uniformly converges to u on D_1. By Lemma 5.10, u_n is $\|\cdot\|_{1,2}$-convergent to u in $W^{1,2}(D_2)$ for any relatively compact open subset D_2 in D_1. Thus u is the weak solution of $Lu = 0$ that satisfies (5.35). □

Remark 1. It can be shown that $y \in \partial D_1$ is regular for (L, D_1) if and only if it is regular for $\left(\frac{1}{2}\Delta, D_1\right)$.

Remark 2. Suppose that D_1 is a relatively compact subdomain of D and ϕ is a nonnegative continuous function on ∂D such that $\phi \not\equiv 0$. Since $u_0(x) = E_0^x[\phi(X_{\tau(D_1)})]$ is the weak solution of $L^0 u = 0$ such that

$$\lim_{\substack{x \to y \\ x \in D_1}} E_0^x[\phi(X_{\tau(D_1)})] = \phi(y)$$

for $y \in \partial D_1$ which is regular for $(\frac{1}{2}\Delta, D_1)$, by the Harnack principle for L^0 (cf. [22]), $u_0 > 0$ in D. Since for each $x \in D_1$, the probability measure P^x is absolutely continuous with respect to P_0^x, the weak solution of $Lu = 0$ in D with $u = \phi$ on ∂D_1 is strictly positive in D_1:

$$u(x) = E^x[\phi(X_{\tau(D_1)})] > 0, \qquad x \in D_1 .$$

Remark 3. Suppose D is a Euclidean domain. Then the diffusion process $(X, P_0^x, x \in D)$ associated with L^0 on D with Dirichlet boundary condition has a strictly positive transition density function $p_0(t,x,y)$ (see, e.g., [28]). Since P^x is absolutely continuous with respect to P_0^x, the diffusion process $(X, P^x, x \in D)$ of L on D with Dirichlet boundary condition has a transition density function $p(t,x,y)$ such that $p(t,x,\cdot) > 0$ a.e. on D for $(t,x) \in \mathbb{R}^+ \times D$.

We close this paper by a property concerning the lifetime ζ of the diffusion process $(X, \zeta, \{P^x, x \in D\})$.

Theorem 5.12

$$\lim_{\alpha \uparrow \infty} \sup_{x \in D} E^x[e^{-\alpha\zeta} 1_D(X_{\zeta-})] = 0 . \tag{5.36}$$

Proof. It follows from [26, p. 286] (by putting $m_t = e^{\int_0^t q(X_s)\,ds} 1_{[t<\zeta]}$ there and noticing that Q^x-a.e. $X_{\zeta-} = \hat{\Delta}$) that

$$\begin{aligned} &E^x[e^{-\alpha\zeta} 1_D(X_{\zeta-})] \\ &= E^x_Q\left[\int_0^\infty e^{-\alpha t}(-q(X_t)) e^{\int_0^t q(X_s)\,ds}\,dt\right] \\ &= 1 - \alpha E^x_Q\left[\int_0^\infty e^{-\alpha t} e^{\int_0^t q(X_s)\,ds}\,dt\right] \\ &= 1 - \alpha E^x_0\left[\int_0^\infty e^{-\alpha t} M_t e^{\int_0^t q(X_s)\,ds}\,dt\right] . \end{aligned} \tag{5.37}$$

We already see from (3.6) and (5.31) that

$$\lim_{t \downarrow 0} \sup_{x \in D} E^x_0[(M_t - 1)^2] = \lim_{t \downarrow 0} \sup_{x \in D} (E^x_0[M_t^2] - 1) = 0$$

and

$$\lim_{t \downarrow 0} \sup_{x \in D} E^x_0\left[(e^{\int_0^t q(X_s)\,ds} - 1)^2\right] = 0 .$$

Thus for any given $\varepsilon > 0$, there exists a constant $\delta = \delta(\varepsilon) > 0$ such that for $0 \leqq t \leqq \delta$,

$$\sup_{x \in D} E^x_0[(M_t - 1)^2] < \varepsilon^2 \quad \text{and} \quad \sup_{x \in D} E^x_0[(e^{\int_0^t q(X_s)\,ds} - 1)^2] < \varepsilon^2 . \tag{5.38}$$

Thus

$$\begin{aligned} &\left|\alpha E^x_0\left[\int_0^\delta e^{-\alpha t} M_t e^{\int_0^t q(X_s)\,ds}\,dt\right] - \alpha \int_0^\delta e^{-\alpha t}\,dt\right| \\ &\leqq \alpha E^x_0\left[\int_0^\infty e^{-\alpha t} M_t |e^{\int_0^t q(X_s)\,ds} - 1|\,dt\right] + \alpha E^x_0\left[\int_0^\delta e^{-\alpha t}|M_t - 1|\,dt\right] \\ &\leqq \alpha \int_0^\delta e^{-\alpha t} \left(E^x_0[M_t^2]\right)^{\frac{1}{2}} \left(E^x_0[(e^{\int_0^t q(X_s)\,ds} - 1)^2]\right)^{\frac{1}{2}} dt \\ &\quad + \alpha \int_0^\delta e^{-\alpha t} \left(E^x_0[(M_t - 1)^2]\right)^{\frac{1}{2}} dt \\ &\leqq \varepsilon(\sqrt{1 + \varepsilon^2} + 1)(1 - e^{-\alpha\delta}) \\ &\leqq (\sqrt{1 + \varepsilon^2} + 1)\varepsilon . \end{aligned} \tag{5.39}$$

Thus by (5.37)

$$\sup_{x\in D} E^x[e^{-\alpha\zeta} 1_D(X_{\zeta-})]$$
$$\leqq 1 - \alpha\int_0^\delta e^{-\alpha t}\,dt + \sup_{x\in D}\left|\alpha E_0^x\left[\int_0^\delta e^{-\alpha t} M_t e^{\int_0^t q(X_s)\,ds}\,dt\right] - \alpha\int_0^\delta e^{-\alpha t}\,dt\right|$$
$$\leqq e^{-\alpha\delta} + (\sqrt{1+\varepsilon^2}+1)\varepsilon\,.$$

Hence

$$\lim_{\alpha\uparrow\infty} E^x[e^{-\alpha\zeta} 1_D(X_{\zeta-})] \leqq (\sqrt{1+\varepsilon^2}+1)\varepsilon$$

and (5.36) follows. □

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