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Abelian local p-class field theory

Ivan B. Fesenko

Department of Mathematics and Mechanics, St. Petersburg State University, Staryj Petergof Bibliotechnaja pl. 2, 198904 Rossija,
and Max-Planck-Institut für Mathematik, Gottfried-Claren-Strasse, 26, D-53225 Bonn, Germany (e-mail: ites@mpim-bonn.mpg.de)

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The classical local class field theory deals with abelian extensions of the ground field with finite, quasi-finite, or algebraically closed residue field. In the case of arbitrary perfect residue field it seems there no general theories covering all types of extensions. Nevertheless, unramified, tamely ramified, and wildly ramified class field theories can be established separately. The latter theory ([7]) is the most essential; it describes abelian totally ramified p-extensions.

In the first section of this paper an abstract p-class field theory is developed. Assume that the ground field K possesses a distinguished galois extension $\tilde{K}/K$ with the galois group being an abelian pro-p-group (see C1, C2 in Sect. 1). We study galois p-extensions L/F linearly disjoint with $\tilde{F}/F, \tilde{F} = F\tilde{K}$ (these extensions are said to be totally ramified). Abelian totally ramified extensions are described by means of a suitable modulation A (see (1.1)). One can define a map $\Upsilon_{L/F}$ acting from the group $\mathrm{Gal}(L/F)^{\sim}$, of continuous $\mathbb{Z}_p$-homomorphisms from $\mathrm{Gal}(\tilde{F}/F)$ to the discrete group $\mathrm{Gal}(L/F)$, to the quotient group $U_F/N_{L/F}U_L$, where U_F is the kernel of the substantial epimorphism $v : A_F \to \mathbb{Z}$. $\Upsilon_{L/F}$ may be treated as a generalization of the Neukirch map ([19, 20]). One of the main condition on which the theory rests is Hilbert Satz 90 for A_F (C3 in Sect. 1). At the same time several other reasonable properties, such as $\mathrm{Gal}(L/F)$-stable elements of A_L are in one-to-one correspondence with elements of A_F, don't hold in general. Thereby, first of all we consider extensions for which a weaker condition: $v_{\tilde{L}}(A_{\tilde{L}}^{\mathrm{Gal}(\tilde{L}/\tilde{F})}) = |\tilde{L} : \tilde{F}|\mathbb{Z}$ holds for a cyclic p-extension $\tilde{L}/\tilde{F}$. The class of such extensions (they are called marked) is required to be sufficiently large (precise formulations in (1.5)). For a totally ramified galois extension L/F, such that any its intermediate cyclic subextension L_1/F_1 $(F \subset F_1 \subset L_1 \subset L)$ is marked, one can define a homomorphism $\Psi_{L/F}$ acting from $U_F/N_{L/F}U_L$ to $(\mathrm{Gal}(L/F)^{\mathrm{ab}})^{\sim}$; the composition $\Psi_{L/F} \circ \Upsilon_{L/F}$ is the identity map. $\Psi_{L/F}$ may be treated as a generalization of the Hazewinkel homomorphism ([9]). Then the last condition C5 (the surjectivity of $\Upsilon_{L/F}$ for a marked extension of degree p) permits one to show that $\Upsilon_{L/F}$ and $\Psi_{L/F}$ are inverse isomorphisms between $(\mathrm{Gal}(L/F)^{\mathrm{ab}})^{\sim}$ and $U_F/N_{L/F}U_L$. The general case of totally ramified galois extensions can be handled as well: the definition of $\Psi_{L/F}$ isn't extendable to arbitrary extensions, but $\Upsilon_{L/F}$ remains an isomorphism.

It should be noted that the first section with the abstract theory was written under the influence of Neukirch's work [22].

The second section briefly introduces a complete discrete valuation field F of rank n with perfect residue field k of positive characteristic p. Topological Milnor K-groups $K_m^{\mathrm{top}}(F)$ for such fields are defined and reviewed (for a leisurely exposition see [3,4,6]). In (2.6) we verify that these groups coincide with the quotient groups of $K_m(F)$ modulo the subgroup of divisible elements.

The third section contains p-class field theory for a complete discrete valuation field F of rank n with nonalgebraically-p-closed residue field k. Abelian totally ramified (with respect to the discrete valuation of rank n) p-extensions are described by means of the subgroup $VK_n^{\mathrm{top}}(F)$ in $K_n^{\mathrm{top}}(F)$ generated by principal units with respect to the discrete valuation. Here we choose for $n \geqq 2$ so-called $\wp$-extensions (a tower of subsequent Artin-Schreier extensions) as marked extensions. We establish the reciprocity map

$$\Psi_F \colon VK_n^{\mathrm{top}}(F) \to \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Gal}(\tilde{F}/F), \mathrm{Gal}(F_p^{\mathrm{ab}}/\tilde{F}))\,,$$

where $\tilde{F}/F$ is the maximal unramified with respect to the discrete valuation of rank n subextension in $F_p^{\mathrm{ab}}/F, F_p^{\mathrm{ab}}$ is the maximal abelian p-extension of F. The arguments of Sect. 1 and 3 provide, in particular, a new proof of the p-part of the main results of [3–6].

We touch ramification theory for the fields in the fourth section applying class field theory. A new Hasse-Herbrand function is defined and its properties are listed. This abelian ramification theory explains some known phenomena described in the works of Lomadze, Kato, and Hyodo ([18], [15–16], [10]) in the case of the imperfect residue field.

Finally, in the fifth section we clarify the behavior of the reciprocity map exposing the description of norm subgroups in $VK_n^{\mathrm{top}}(F)$ via the theory of decomposable additive polynomials from Sect. 2 of [7].

Higher dimensional local class field theory (with finite residue field) from different points of view has been treated by several authors, starting from Parshin ([23, 26, 27]) and Kato ([11–13]), then by Deninger and Wingberg ([2]), Koya ([17]), and the author. We hope that the approach of this paper could explain essence of the theory in a natural, explicit and simple way.

1 p-class field theory

1.1. Let K be a field and $G_p = \mathrm{Gal}(K_p/K)$ be the galois group of the fixed maximal separable p-extension K_p of K. Fields considered later are subfields in K_p over K. Let $Q(G_p)$ be the category, whose objects are finite subextensions F/K in K_p/K and the morphisms are the compositions of $\sigma\colon F \to \sigma F$ for $\sigma \in G_p$ and the inclusion $i_{\sigma F/L}\colon \sigma F \to L$.

Let A be a G_p-modulation (see S. 324 of [21], Sect. 4 of [22], Sect. 1 of [4]), or in other words, a double (Mackey) functor

$$A = (A^*, A_*) \colon Q(G_p) \to Ab ,$$

where A^* is a covariant functor, A_* is a contravariant functor, such that $A^*(F) = A_*(F) = A_F$ for all $F \in Q(G_p)$, and such that:
(1) $\sigma^*\sigma_* = \sigma_*\sigma^* = \mathrm{id}$ for $\sigma \in G_p, \sigma^* = A^*(\sigma), \sigma_* = A_*(\sigma)$;
(2) $N_{L/F} \circ i_{F/L} = |L : F|$ for $F, L \in Q(G_p)$, $F \subset L$ and $N_{L/F} = A_*(i_{F/L}), i_{F/L} = A^*(i_{F/L})$;
(3) for $L, M \in Q(G_p)$ and any system R of representatives of $\mathrm{Gal}(K_p/L) \setminus G_p/\mathrm{Gal}(K_p/M)$ the formula

$$i_{K/L} \circ N_{M/K} = \sum_{\sigma \in R} N_{L\sigma M/L} \circ i_{\sigma M/L\sigma M} \circ \sigma^*$$

holds.

Further we will write σ instead of σ^*.

1.2. Assume that the field K possesses the following properties:

C 1. *There exists a G_p-modulation A possessing a surjective homomorphism*

$$v \colon A_K \to \mathbb{Z} ,$$

such that for some galois subextension $\tilde{K}/K$ in K_p/K with $\mathrm{Gal}(\tilde{K}/K)$ *being an abelian pro-p-group the formula $v(N_{F/K}A_F) = |F \cap \tilde{K} : K|\mathbb{Z}$ holds for any $F \in Q(G_p)$.*

Let the cardinality of pro-p-basis of $\mathrm{Gal}(\tilde{K}/K)$ be equal to $\kappa \neq 0$. If $\mathrm{Gal}(\tilde{K}/K)$ has no nontrivial p-torsion, then $\mathrm{Gal}(\tilde{K}/K)$ is noncanonically isomorphic to $\prod_\kappa \mathbb{Z}_p$. Put $\tilde{F} = F\tilde{K}$ for $K \subset F \subset K_p$.

For a subextension F'/F in $\tilde{F}/F$ put

$$A_{F'} = \varinjlim A_{F_j}$$

where F_j runs all finite extensions of F in F' and the limit is taken with respect to $i_{F_j/F_{j'}}$. If $\tilde{L}/\tilde{F}$ is finite and $L' \cap \tilde{F} = F'$, then one can define $i_{F'/L'}, N_{L'/F'}$ as induced by $i_{F_j/L_j}, N_{L_j/F_j}$ where F_j runs all finite extensions of F in F' and $L_j = LF_j$. There is also $i_{F/F'} : A_F \to A_{F'}$ induced by i_{F/F_j} for finite subextensions F_j/F in F'/F. If L/F is galois, then similarly one can define $\sigma : A_L \to A_L$ for $\sigma \in \mathrm{Gal}(L/F)$.

Put $v_F = (1/|F \cap \tilde{K} : K|) v \circ N_{F/K}$, then $v_F(A_F) = \mathbb{Z}$. Introduce the homomorphism $v_{F'} : A_{F'} \to \mathbb{Z}$ as the natural extension of the homomorphisms $v_{F_j} : A_{F_j} \to \mathbb{Z}$ for finite subextensions F_j/F in F'/F.

An element $\pi_F \in A_F$ is called prime if $v_F(\pi_F) = 1$. Put

$$O_F = \{\alpha \in A_F : v_F(\alpha) \geqq 0\} , \quad V_F = 1 + \pi_F O_F, \quad U_F = O_F^*.$$

An extension L/F, where $\tilde{L}/\tilde{F}$ is finite, is called *totally ramified* (resp. *unramified*) if $L \cap \tilde{F} = F$ (resp. $L \subset \tilde{F}$).

The next property to be satisfied is

C 2. *If L/F is finite unramified, then $N_{L/F}U_L = U_F$; and if L/F is finite, then $N_{\tilde L/\tilde F}A_{\tilde L} = A_{\tilde F}$.*

1.3. For a p-group $\mathrm{Gal}(L/F)$ (of order of a power of p) let

$$\mathrm{Gal}(L/F)^{\sim} = \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Gal}(\tilde F/F), \mathrm{Gal}(L/F))$$

denote the group of continuous homomorphisms of $\mathbb{Z}_p$-module $\mathrm{Gal}(\tilde F/F)(a \circ \sigma = \sigma^a$ for $a \in \mathbb{Z}_p)$ to the discrete $\mathbb{Z}_p$-module $\mathrm{Gal}(L/F)$. If $\mathrm{Gal}(L/F)$ has no nontrivial p-torsion, this group is isomorphic (noncanonically) with $\oplus_\kappa \mathrm{Gal}(L/F)$, where κ was defined in (1.2).

Let L/F be a totally ramified galois extension, $\chi \in \mathrm{Gal}(L/F)^{\sim}$. Denote by Σ_χ the fixed field of all $\tau_\varphi \in \mathrm{Gal}(\tilde L/F)$ such that $\tau_\varphi|_{\tilde F} = \varphi$ and $\tau_\varphi|_L = \chi(\varphi)$, where φ runs $\mathrm{Gal}(\tilde F/F)$. Then Σ_χ/F is a totally ramified extension. Note that any totally ramified subextension Σ/F in $\tilde L/F$ of the same degree as L/F may be regarded as Σ_χ/F for a suitable $\chi \in \mathrm{Gal}(L/F)^{\sim}$.

If π_χ and π_L are prime elements in A_{Σ_χ} and A_L, then $N_{\Sigma_\chi/F}\pi_\chi$ and $N_{L/F}\pi_L$ are prime elements in F. Put

$$\Upsilon_{L/F}(\chi) = N_{\Sigma_\chi/F}\pi_\chi - N_{L/F}\pi_L \mod N_{L/F}U_L .$$

Lemma. *The map*

$$\Upsilon_{L/F} : \mathrm{Gal}(L/F)^{\sim} \to U_F/N_{L/F}U_L$$

is well defined.

Proof. $\Upsilon_{L/F}$ doesn't depend on the choice of π_L. Let M be the compositum of Σ_χ and L. Then M/Σ_χ is unramified and any prime element in Σ_χ can be written according to C2 as $\pi_\chi + N_{M/\Sigma_\chi}\varepsilon$ for a suitable $\varepsilon \in U_M$. Since $N_{M/F}\varepsilon = N_{L/F}(N_{M/L}\varepsilon) \in N_{L/F}U_L$, we complete the proof. □

1.4. Proposition. (1) *Let $L/F, L_1/F_1$ be totally ramified galois extensions, and $F_1 \cap \tilde F = F, L_1 \cap \tilde L = L$. Then the diagram*

$$\begin{array}{ccc} \mathrm{Gal}(L_1/F_1)^{\sim} & \xrightarrow{\Upsilon_{L_1/F_1}} & U_{F_1}/N_{L_1/F_1}U_{L_1} \\ \downarrow & & \downarrow N_{F_1/F} \\ \mathrm{Gal}(L/F)^{\sim} & \xrightarrow{\Upsilon_{L/F}} & U_F/N_{L/F}U_L \end{array}$$

is commutative, where the left vertical homomorphism is induced by the natural restrictions $\mathrm{Gal}(L_1/F_1) \to \mathrm{Gal}(L/F)$ and $\mathrm{Gal}(\tilde F_1/F_1) \xrightarrow{\sim} \mathrm{Gal}(\tilde F/F)$.

(2) *Let L/F be a totally ramified galois extension, and let σ be an automorphism. Then the diagram*

$$\begin{array}{ccc} \mathrm{Gal}(L/F)^{\sim} & \xrightarrow{\Upsilon_{L/F}} & U_F/N_{L/F}U_L \\ \sigma^{\sim}\downarrow & & \downarrow \\ \mathrm{Gal}(\sigma L/\sigma F)^{\sim} & \xrightarrow{\Upsilon_{\sigma L/\sigma F}} & U_{\sigma F}/N_{\sigma L/\sigma F}U_{\sigma L} \end{array}$$

is commutative, where $(\sigma^{\sim}\chi)(\sigma\phi\sigma^{-1}) = \sigma\chi(\phi)\sigma^{-1}$.

Proof. (1) Let $\chi_1 \in \mathrm{Gal}(L_1/F_1)^{\sim}$ and $\chi = \chi_1|_L \in \mathrm{Gal}(L/F)^{\sim}$. Put $\Sigma_1 = \Sigma_{\chi_1}$. Then $\Sigma_\chi = \Sigma_1 \cap \tilde{L}$. Therefore, $\pi = N_{\Sigma_1/\Sigma_\chi}\pi_{\Sigma_1}$ is a prime element in A_{Σ_χ} and $N_{\Sigma_\chi/F}\pi = N_{F_1/F}N_{\Sigma_1/F_1}\pi_{\Sigma_1}$.

(2) It follows from $\Sigma_{\sigma^{\sim}\chi} = \sigma\Sigma_\chi$. □

1.5. Now we assume that the following property also holds:

C 3. *Let $F \subset F' \subset \tilde{F}, L'/F'$ be a cyclic totally ramified extension of degree p, and σ be a generator of* $\mathrm{Gal}(L'/F')$. *Then the sequence*

$$A_{L'} \xrightarrow{\sigma-1} A_{L'} \xrightarrow{N_{L'/F'}} A_{F'}$$

is exact.

It immediately follows from C2 and C3 that for a cyclic extension $\tilde{L}/\tilde{F}$ with a generator σ the sequence

$$A_{\tilde{L}} \xrightarrow{\sigma-1} A_{\tilde{L}} \xrightarrow{N_{\tilde{L}/\tilde{F}}} A_{\tilde{F}} \longrightarrow 0$$

is exact.

In order to develop class field theory, one should add an additional condition C4. In general, the condition of C4 doesn't hold for all cyclic totally ramified extensions, see below (3.4). Nevertheless, for the theory it would be sufficient if this condition holds for a subset of marked extensions.

Definition. *Let $\mathfrak{m}$ be a set of totally ramified galois extensions $L/F, F, L \in Q(G_p)$, such that*

$\mathscr{P}1$. If $L/F \in \mathfrak{m}$, then $L/M, M/F \in \mathfrak{m}$ for some proper cyclic subextension M/F in L/F.

$\mathscr{P}2$. If L/F is a totally ramified galois extension, then there exists a totally ramified extension Q/F, such that $\tilde{L} \cap \tilde{Q} = \tilde{F}$ and any intermediate cyclic subextension L_1/Q_1 in $LQ/Q (Q \subset Q_1 \subset L_1 \subset LQ)$ belongs to $\mathfrak{m}$.

$\mathscr{P}3$. If L/F is a totally ramified galois extension of degree p, then there exists a totally ramified galois extension $E/F \in \mathfrak{m}, L \subset E$.

Extensions from $\mathfrak{m}$ will be called **marked**.

Note that the set $\mathfrak{m}$ depends on K, and it is more correctly to write $\mathfrak{m}_K$. Now the fourth condition may be stated as follows

C 4. *There exists a set $\mathfrak{m}$ of marked extensions, such that for any cyclic extension $L/F \in \mathfrak{m}$ with a generator $\sigma \in$* $\mathrm{Gal}(L/F)$

$$\sigma(\alpha) = \alpha \quad \text{for} \quad \alpha \in A_{\tilde{L}} \Longrightarrow v_{\tilde{L}}(\alpha) \in |L : F|\mathbb{Z}\,.$$

1.6. From now on we fix the set $\mathfrak{m}$ from C4. Denote by $\mathfrak{m}_0$ the set of totally ramified galois extensions L/F with the property: any intermediate cyclic subextension L_1/F_1 in L/F is marked. For $L/F \in \mathfrak{m}_0$ we introduce the map $\Psi_{L/F}$ inverse to $\Upsilon_{L/F}$. Put $G = \mathrm{Gal}(\tilde{L}/\tilde{F})$. Let $V(L|F)$ denote the subgroup in $U_{\tilde{L}}$ generated by the elements $\sigma(\alpha) - \alpha$ with $\sigma \in G, \alpha \in U_{\tilde{L}}$. For $\sigma \in G$ put

$$c(\sigma) = \sigma(\pi_{\tilde{L}}) - \pi_{\tilde{L}} \mod V(L|F),$$

where $\pi_{\tilde{L}}$ is a prime element in $\tilde{L}$ (e.g., $\pi_{\tilde{L}} = i_{L/\tilde{L}}\pi_L$). In fact, c induces the homomorphism

$$c : \mathrm{Gal}(\tilde{L}/\tilde{F})^{\mathrm{ab}} \to U_{\tilde{L}}/V(L|F),$$

since $\sigma(\pi_{\tilde{L}}) - \pi_{\tilde{L}} \in U_{\tilde{L}}$.

Proposition. *The sequence*

$$1 \longrightarrow \mathrm{Gal}(\tilde{L}/\tilde{F})^{\mathrm{ab}} \xrightarrow{c} U_{\tilde{L}}/V(L|F) \xrightarrow{N_{\tilde{L}\tilde{F}}} U_{\tilde{F}} \longrightarrow 0$$

is exact for $L/F \in \mathfrak{m}_0$.

Proof. (c.f. Sect. 4 of [9] or Sect. 2 of [28]). First assume that L/F is cyclic of degree p^n. Let σ be a generator of G. If $c(\sigma^m) \in V(L|F)$ for some m, then $\sigma(m\pi_{\tilde{L}}) - m\pi_{\tilde{L}} = \sigma(\varepsilon) - \varepsilon$ for a suitable $\varepsilon \in U_{\tilde{L}}$. Then $\sigma(m\pi_{\tilde{L}} - \varepsilon) = m\pi_{\tilde{L}} - \varepsilon$. By C4 we deduce that $p^n|m$ and c is injective. Further, let $N_{\tilde{L}/\tilde{F}}\alpha = 0$ for $\alpha \in U_{\tilde{L}}$. Then, according to $\mathscr{C}$, there is an element $\beta \in A_{\tilde{L}}$ such that $\alpha = (\sigma - 1)\beta$. Let $\beta = a\pi_{\tilde{L}} + \varepsilon$ with $\varepsilon \in U_{\tilde{L}}$. Therefore, $\alpha \equiv c(\sigma^a) \mathrm{mod}\ V(L|F)$.

Now let L/F be an arbitrary totally ramified galois extension, $L/F \in \mathfrak{m}_0$. Let M/F be a proper subextension in L/F, then $M/F \in \mathfrak{m}_0$. As $N_{\tilde{L}/\tilde{M}} U_{\tilde{L}} = U_{\tilde{M}}$ by $\mathscr{C}2$, we get $N_{\tilde{L}/\tilde{M}}(V(L|F)) = V(M|F)$. Argue by induction on $|L:F|$, one can show the exactness of the sequence of Proposition in the term $U_{\tilde{L}}/V(L|F)$. The injectivity of c follows as well, since the commutator group $\mathrm{Gal}(L/F)'$ coincides with the intersection of all $\mathrm{Gal}(L/M)$ for cyclic extensions M/F. □

Let $L/F \in \mathfrak{m}_0$. Let $\varepsilon \in U_F$. According to C2 there exists an element $\eta \in U_{\tilde{L}}$, such that $N_{\tilde{L}/\tilde{F}}\eta = i_{F/\tilde{F}}\varepsilon$. Let an extension of $\varphi \in \mathrm{Gal}(\tilde{F}/F)$ on $\tilde{L}$ be written by the same notation. As $N_{\tilde{L}/\tilde{F}}((\varphi - 1)\eta) = 0$, we deduce from Proposition that

$$(\varphi - 1)\eta \equiv (1 - \sigma)\pi_{\tilde{L}} \mod V(L|F)$$

with a suitable $\sigma \in \mathrm{Gal}(\tilde{L}/\tilde{F})^{\mathrm{ab}}$, where $\pi_{\tilde{L}}$ is a prime element in $A_{\tilde{L}}$. It is easy to verify that σ doesn't depend on the choice of an extension of φ and of η. Put $\chi(\varphi) = \sigma|_L$. One immediately obtains that $\chi(\varphi_1\varphi_2) = \sigma_1\sigma_2$, i.e. $\chi \in (\mathrm{Gal}(L/F)^{\mathrm{ab}})^{\check{}}$. Put $\Psi_{L/F}(\varepsilon) = \chi$.

Lemma. *The map* $\Psi_{L/F} : U_F/N_{L/F}U_L \to (\mathrm{Gal}(L/F)^{\mathrm{ab}})^{\check{}}$ *is a well-defined homomorphism.*

Proof. If $\varepsilon = \varepsilon_1\varepsilon_2$, then one can take $\eta = \eta_1\eta_2$. Therefore, $\sigma = \sigma_1\sigma_2$ and

$$\Psi_{L/F}(\varepsilon_1\varepsilon_2) = \Psi_{L/F}(\varepsilon_1)\Psi_{L/F}(\varepsilon_2). \quad \square$$

1.7. Proposition. *Let L/F be a totally ramified galois extension, $L/F \in \mathfrak{m}_0$. Then*

$$\Psi_{L/F} \circ \Upsilon_{L/F} : (\mathrm{Gal}(L/F)^{\mathrm{ab}})^{\check{}} \to (\mathrm{Gal}(L/F)^{\mathrm{ab}})^{\check{}}$$

is the identity map.

Proof. Let $\chi \in (\mathrm{Gal}(L/F)^{\mathrm{ab}})^{\sim}$, and let π_χ, π_L be prime elements in A_{Σ_χ}, A_L. Then

$$i_{\Sigma_\chi/\tilde{L}}\pi_\chi = i_{L/\tilde{L}}\pi_L + \eta \quad \text{with} \quad \eta \in U_{\tilde{L}}\,,$$

and $N_{\tilde{L}/\tilde{F}}\eta = i_{F/\tilde{F}}(N_{\Sigma_\chi/F}\pi_\chi - N_{L/F}\pi_L)$. Let $\varphi \in \mathrm{Gal}(\tilde{F}/F)$ and τ_φ be the element of $\mathrm{Gal}(\tilde{L}/F)$ determined by χ as above in (1.3). Then

$$(1-\tau_\varphi)i_{L/\tilde{L}}\pi_L \equiv (\tau_\varphi - 1)\eta \mod V(L|F)\,.$$

Therefore, $\chi = \Psi_{L/F}(\Upsilon_{L/F}\chi)$. □

Corollary. *For $L/F \in \mathfrak{m}_0$ the homomorphism*

$$\Psi_{L/F} : U_F/N_{L/F}U_L \to (\mathrm{Gal}(L/F)^{\mathrm{ab}})^{\sim}$$

is surjective; the map

$$\Upsilon_{L/F} : (\mathrm{Gal}(L/F)^{\mathrm{ab}})^{\sim} \to U_F/N_{L/F}U_L$$

is injective.

1.8. We are now in a position to indicate the last property to be satisfied.

C 5. *Let $L/F \in \mathfrak{m}$ be of degree p. Then the map $\Upsilon_{L/F}$ is surjective (or the homomorphism $\Psi_{L/F}$ is injective: if $\varepsilon \in U_F$ and $N_{\tilde{L}/\tilde{F}}\eta = i_{F/\tilde{F}}\varepsilon$ for $\eta \in U_{\tilde{L}}$ with $(\varphi-1)\eta \in V(L|F)$ for any $\varphi \in \mathrm{Gal}(\tilde{L}/F)$, then $\varepsilon \in N_{L/F}U_L$).*

If $\kappa = 1$, this property is equivalent to the first inequality $|U_F : N_{L/F}U_L| \leqq p$.

Theorem. *Let L/F be a totally ramified galois extension. Then the map*

$$\Upsilon_{L/F} : (\mathrm{Gal}(L/F)^{\mathrm{ab}})^{\sim} \to U_F/N_{L/F}U_L$$

is an isomorphism. The homomorphism

$$\Psi_{L/F} : U_F/N_{L/F}U_L \to (\mathrm{Gal}(L/F)^{\mathrm{ab}})^{\sim}$$

defined for $L/F \in \mathfrak{m}_0$ is the inverse one.

Proof. First let L/F be a cyclic extension of degree $p, L/F \in \mathfrak{m}$. Then it follows from C5 and (1.7) that $\Psi_{L/F}$ is an isomorphism and $\Upsilon_{L/F}$ is an isomorphism as well.

Let $L/F \in \mathfrak{m}_0, M/F$ be a galois subextension in $L/F, L/M, M/F \in \mathfrak{m}_0$. By Proposition (1.4) the following diagram is commutative:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Gal}(L/M)^{\sim} & \longrightarrow & \mathrm{Gal}(L/F)^{\sim} & \longrightarrow & \mathrm{Gal}(M/F)^{\sim} & \longrightarrow & 1 \\
 & & \downarrow \Upsilon_{L/M} & & \downarrow \Upsilon_{L/F} & & \downarrow \Upsilon_{M/F} & & \\
 & & U_M/N_{L/M}U_L & \xrightarrow{N^*_{M/F}} & U_F/N_{L/F}U_L & \longrightarrow & U_F/N_{M/F}U_M & \longrightarrow & 0
\end{array}$$

Hence $\Upsilon_{L/F}$ is surjective. Proposition (1.7) implies now that $\Psi_{L/F}$ is injective.

Now let L/F be a totally ramified galois extension, $L/F \notin \mathfrak{m}_0$. Then by $\mathscr{P}2$ in (1.5) there exists a finite totally ramified extension Q/F such that $LQ/Q \in \mathfrak{m}_0, \tilde{L} \cap \tilde{Q} = \tilde{F}$. By Proposition (1.4) we get the commutative diagram

$$\begin{array}{ccc} \mathrm{Gal}(LQ/Q)^{\sim} & \xrightarrow{\Upsilon_{LQ/Q}} & U_Q/N_{LQ/Q}U_{LQ} \\ \downarrow & & \downarrow N_{Q/F} \\ \mathrm{Gal}(L/F)^{\sim} & \xrightarrow{\Upsilon_{L/F}} & U_F/N_{L/F}U_L \end{array}$$

where the left vertical homomorphism is an isomorphism. It follows that $\Upsilon_{L/F}$ is a homomorphism. Note that if $\Upsilon_{L/M}$ is an isomorphism and M/F is cyclic, then the homomorphism $N^*_{M/F}$ in the first diagram is injective. Indeed, otherwise for some $x \notin N_{L/M}U_L$ there exists an element y in U_L such that $N_{M/F}z = 0$ for $z = x - N_{L/M}y$. Then, by C3 we obtain $z \in (1-\sigma)A_M$ for a generator σ of $\mathrm{Gal}(M/F)$. Proposition (1.4) shows that $z \in N_{L/M}U_L$, a contradiction.

Now we argue by induction on degree in order to show that $\Upsilon_{L/F}$ is an isomorphism for $L/F \in \mathfrak{m}$. According to property $\mathscr{P}1$ in (1.5) there exists a cyclic subextension M/F of L/F such that $L/M, M/F \in \mathfrak{m}$. Then $\Upsilon_{L/F}$ is surjective and injective by the first commutative diagram.

If L/F is of degree p, then by $\mathscr{P}3$ of (1.5) one deduces $\Upsilon_{L/F}$ is a homomorphism, surjective and injective. The general case follows immediately by induction arguements. □

Corollary. *Let $L_1/F, L_2/F, L_1L_2/F$ be abelian totally ramified extensions. Put $L_3 = L_1L_2, L_4 = L_1 \cap L_2$. Then*

$$\begin{aligned} N_{L_3/F}U_{L_3} &= N_{L_1/F}U_{L_1} \cap N_{L_2/F}U_{L_2}\,, \\ N_{L_4/F}U_{L_4} &= N_{L_1/F}U_{L_1} + N_{L_2/F}U_{L_2}\,. \end{aligned}$$

Moreover, $N_{L_1/F}U_{L_1} \subset N_{L_2/F}U_{L_2}$ if and only if $L_1 \supset L_2$. $N_{M/F}U_M = N_{L/F}U_L$ for the maximal abelian subextension M/F in L/F.

Proof. Put $H_i = \mathrm{Gal}(L_3/L_i)$. Then

$$\begin{aligned} N_{L_3/F}U_{L_3} &= \Upsilon_{L_3/F}(1) = \Upsilon_{L_3/F}((H_1 \cap H_2)^{\sim}) = \Upsilon_{L_3/F}(H_1^{\sim}) \cap \Upsilon_{L_3/F}(H_2^{\sim}) \\ &= N_{L_1/F}U_{L_1} \cap N_{L_2/F}U_{L_2}\,, \\ N_{L_4/F}U_{L_4} &= \Upsilon_{L_3/F}((H_1H_2)^{\sim}) = N_{L_1/F}U_{L_1} + N_{L_2/F}U_{L_2}\,. \end{aligned}$$

If $N_{L_1/F}U_{L_1} \subset N_{L_2/F}U_{L_2}$, then $N_{L_1/F}U_{L_1} = N_{L_3/F}U_{L_3}$ and $|L_1 : F| = |L_3 : F|$, i.e., $L_2 \subset L_1$. □

2 Complete discrete valuation fields of rank n

In this section we briefly treat the class of fields for which the theory of Sect. 1 will be applied later (for the case of a finite residue field see [3–6]).

2.1. Let F be a field and

$$\mathbf{v}_F : F^* \longrightarrow (\mathbb{Z})^n = \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{n \text{ times}}$$

be a surjective valuation, where the additive group $(\mathbb{Z})^n$ is considered to be lexicographically ordered, i.e., $(m_1, \ldots, m_n) < (m'_1, \ldots, m'_n)$ in $(\mathbb{Z})^n$ if $m_i < m'_i$ for the maximal i such that $m_i \neq m'_i$. The ring of integers of $\mathbf{v}_F$, its maximal ideal and the group of units will be denoted as $\mathcal{O}_F, \mathcal{M}_F, \mathcal{U}_F$. We will assume that the residue field $\mathcal{O}_F/\mathcal{M}_F = k$ is a perfect field of characteristic $p > 0$. The field F is said to be *complete* if it is complete with respect to the main component $v^{(n)}$ of $\mathbf{v}_F = (v^{(1)}, \ldots, v^{(n)})$ and the residue field $\overline{F}_{v^{(n)}}$ of F with respect to the discrete valuation $v^{(n)}$ of rank 1 is complete. The elements $t_n, \ldots, t_1$ of F such that

$$\mathbf{v}_F(t_i) = (0, \ldots, 1, \ldots, 0)\,,$$

with 1 at the ith position, are called *local parameters* of F. Denote also $\mathcal{V}_F = 1 + \mathcal{M}_F, \mathcal{U}_{m_1,\ldots,m_n} = 1 + t_n^{m_n} \ldots t_1^{m_1} \mathcal{O}_F$.

There is a chain of fields $k_n = F, k_{n-1} = \overline{F}_{v^{(n)}}, \ldots, k_0 = k$ where k_i is the residue field with respect to the discrete valuation of rank 1 on k_{i+1}.

If $\operatorname{char}(F) = p$, then it follows from the general theory that F is isomorphic (with respect to the discrete valuation of rank n) to the field of formal power series $k((X_1))\ldots((X_n))$ (c.f. section 5 Chapter II of [8]).

We are now going to introduce some special topology on F that takes into consideration the corresponding topology on $\overline{F}_{v^{(n)}}$. First assume that $\operatorname{char}(F) = p$. Let U_m, $m \in \mathbb{Z}$, be subgroups in k_{n-1}, which are neighborhoods of zero in this topology (the topology coincides with the induced topology by the discrete valuation of rank 1 if $n = 1$). Let $U_m = k_{n-1}$ for all sufficiently large m. Put $U = \{\sum_{m\in\mathbb{Z}} a_m t_n^m, a_m \in U_m\}$, where t_n is a prime element of F with respect to $v^{(n)}$. All such subgroups U in F form a fundamental system of neighborhoods of zero in the topology of F. The so-defined topology in the case of a finite residue field was introduced by Parshin, see [26].

Now let $\operatorname{char}(F) = 0$ and $\operatorname{char}(k_{n-1}) = p > 0$. According to the general theory there is a subfield F_0 in F which is a complete discrete valuation field of rank n under the induced valuation, and p is a prime element in F_0 with respect to the first component of the discrete valuation of rank n on F_0 (see section 5 Chapter II of [8]). In this case F is a finite extension of F_0. One may assume that the field of fractions F_{00} of the Witt ring $W(k)$ is contained in F_0. Let U_m, $m \in \mathbb{Z}$, be subgroups in k_{n-1} as above. Let $\tilde{U}_m$ be subgroups in F_0, such that the coefficients from k of elements of U_m are replaced by coefficients running the ring of integers of F_{00}. Then one can take $U = \{\sum_{m\in\mathbb{Z}} a_m t_n^m, a_m \in \tilde{U}_m\}$ as a fundamental system of neighborhoods of zero in the topology of F_0. Define the topology on F as of a finite-dimensional vector space over F_0.

The multiplicative group F^* is isomorphic to the product of the cyclic subgroups $\langle t_i \rangle$ generated by t_i, where $t_n, \ldots, t_1$ are local parameters in F, the group of multiplicative representatives $\mathcal{R}^*$ of k^* in F, and the group $\mathcal{V}_F$. If $\operatorname{char}(k_{n-1}) = p$, then introduce the topology on F^* as the product of the topology on $\mathcal{V}_F$ induced from F and the discrete topology on $\langle t_n \rangle \times \cdots \times$

$\langle t_1\rangle \times \mathcal{R}^*$. If char$(F)$ = char$(k_{m+1}) = 0$ and char$(k_m) = p, m < n-1$, then put $\mathcal{W}_F = 1 + t_{m+2}\mathcal{O}_F$. Note that the group $\mathcal{W}_F$ is uniquely divisible. The field F is isomorphic to the field $k_{m+1}((t_{m+1}))\dots((t_n))$, and k_{m+1} is a complete discrete valuation field of rank $m+1$ of the type considered above. We get the isomorphism

$$F^* \simeq k^*_{m+1} \times \mathcal{W}_F \times \langle t_{m+2}\rangle \times \dots \times \langle t_n\rangle \,.$$

Introduce the topology on F^* as the product of the trivial topology on $\mathcal{W}_F$, the discrete topology on $\langle t_{m+2}\rangle \times \dots \times \langle t_n\rangle$, and the above-defined topology on k^*_{m+1}.

The so-defined topology on F^* doesn't depend on the choice of local parameters and of imbeddings of the residue fields into the field. The multiplication is sequentially continuous with respect to this topology.

Any element $\alpha \in \mathcal{V}_F$ has precisely one expansion as the convergent product

$$\varepsilon = \varepsilon_1 \prod_{i_m>0} \prod_{i_{m-1}>I_{m-1}(i_m)} \dots \prod_{i_1>I_1(i_m,\dots,i_2)} (1+\theta_{i_m,\dots,i_1} t_m^{i_m}\dots t_1^{i_1})\,,$$

where ε_1 is a divisible element in $\mathcal{V}_F, \theta_{i_m,\dots,i_1} \in \mathcal{R} = \mathcal{R}^* \cup \{0\}$ and $I_{m-1}(0) > 0,\dots, I_1(0,\dots,0) > 0 (m = n$ if char$(k_{n-1}) = p)$.

2.2. Let $K_s(F)$ be the sth Milnor group of F. Introduce the topology on $K_s(F)$ as the strongest topology such that the map $F^* \times \dots \times F^* \to K_s(F)$ and the addition in $K_s(F)$ are sequentially continuous. Then the intersection $\Lambda_s(F)$ of all neighborhoods of zero in $K_s(F)$ is a subgroup.

Put

$$K_s^{\mathrm{top}}(F) = K_s(F)/\Lambda_s(F)\,.$$

Let $UK_s(F), VK_s(F), U_IK_s(F)$ denote the subgroups in $K_s(F)$ generated by $\mathcal{U}_F, \mathcal{V}_F, \mathcal{U}_I$ respectively, where $I = (i_1,\dots,i_n) \in (\mathbb{Z})^n$. Similarly the groups $UK_s^{\mathrm{top}}(F), VK_s^{\mathrm{top}}(F), U_IK_s^{\mathrm{top}}(F)$ are defined.

To study the structure of $K_s^{\mathrm{top}}(F)$ one can apply generalizations of the pairings in [26], or Sect. 2 of [4], and in Sect. 3 of [6], see (2.4)–(2.5).

2.3. Let $\tilde F$ be the maximal abelian unramified p-extension of F with respect to the discrete valuation of rank n, i.e., $\tilde F = F \otimes_{W(k)} W(k_p^{\mathrm{ab}})$. For $F \subset F' \subset \tilde F$ put

$$K_n^{\mathrm{top}}(F') = \varinjlim K_n^{\mathrm{top}}(F_j)$$

where F_j/F runs all finite extensions in F'/F and the limit is taken with respect to $i_{F_j/F_{j'}}$.

Let $\kappa = \dim_{\mathbb{F}_p} k/\wp(k)$ where $\wp(x) = x^p - x$. Further we will assume that $\kappa \neq 0$. Then the Witt theory implies that Gal$(\tilde F/F)$ is an abelian free pro-p-group without nontrivial p-torsion, and there is a noncanonical isomorphism Gal$(\tilde F/F) \simeq \prod_\kappa \mathbb{Z}_p$. The case $\kappa = 0$ requires special considerations taking into account the pro-quasi-algebraic structure of $VK_n^{\mathrm{top}}(F)$ over k as a generalization of Serre's theory ([28]) in the case of $n = 1$.

2.4. The first pairing we employ now in the case of $\mathrm{char}(F) = p$ is the Artin-Schreier-Witt pairing. Let $\alpha_1, \dots, \alpha_n \in F^*$, and let $(\beta_0, \dots, \beta_s) \in W_s(F)$ be a Witt vector. Let $\wp : W_s(F) \to W_s(F)$ be the operator defined as $\wp(\beta_0, \dots, \beta_s) = (\beta_0^p, \dots, \beta_s^p) - (\beta_0, \dots, \beta_s)$. For $\varphi \in \mathrm{Gal}(\tilde{F}/F)$ put

$$(\alpha_1, \dots, \alpha_n, (\beta_0, \dots, \beta_s)]_s(\varphi) = \varphi(\gamma_0, \dots, \gamma_s) - (\gamma_0, \dots, \gamma_s),$$

where $\wp(\gamma_0, \dots, \gamma_s) = (\lambda_0, \dots, \lambda_s)$ and the ith ghost component $\lambda^{(i)}$ of $(\lambda_0, \dots, \lambda_s)$ is defined as $\mathrm{res}_k(\beta^{(i)} \alpha_1^{-1} d\alpha_1 \wedge \dots \wedge \alpha_n^{-1} d\alpha_n)$. Then one can show similarly to section 2 of [4] and (1.11) of [7] that $(\,\cdot\,,\,\cdot\,]_s$ defines the nondegenerate pairing

$$\begin{aligned}(\,\cdot\,,\,\cdot\,]_s : VK_n^{\mathrm{top}}(F)/p^s \times W_s(F)/(\wp W_s(F) + W_s(\overline{F}))\\ \to \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Gal}(\tilde{F}/F), W_s(\mathbb{F}_p)).\end{aligned}$$

Applying this pairing in the same way as in [26] or section 2 of [4] one can prove

Proposition. *Let F be a complete discrete valuation field of rank n, $\mathrm{char}(F) = p$. Then any element $\alpha \in VK_n^{\mathrm{top}}(F)$ is uniquely expanded in the convergent series $\sum c_{I,\theta} x_{I,\theta}$ with $c_{I,\theta} \in \mathbb{Z}_p$,*

$$x_{I,\theta} = \{1 + \theta t_n^{i_n} \dots t_1^{i_1}, t_{j_1}, \dots, t_{j_{n-1}}\},$$

where θ belongs to the fixed basis of k over $\mathbb{F}_p, I = (i_1, \dots, i_n) > 0$, the set $\{j_1, \dots, j_{n-1}, j\}$ coincides with the set $\{1, \dots, n\}$, where j is the minimal integer such that $p \nmid i_j$. In particular, $K_n^{\mathrm{top}}(F)$ has no nontrivial p-torsion.

2.5. The second pairing is a generalization of the pairing introduced by Vostokov in the case of a finite k (see [29] and Appendix B of [8]). Assume that $\mathrm{char}(F) = 0, \mathrm{char}(k_{n-1}) = p$, and a primitive p^rth root of unity ζ is contained in F. Let

$$\alpha = t_n^{a_n} \dots t_1^{a_1} \theta \left(1 + \sum \theta_{i_n, \dots, i_1} t_n^{i_n} \dots t_1^{i_1}\right)$$

be an element of F^*, where $\theta \in \mathcal{R}^*, \theta_{i_n, \dots, i_1}$ belongs to the ring of integers $\mathcal{O}_{00}$ of the field F_{00} (see (2.1)). Put

$$\alpha(X) = X_n^{a_n} \dots X_1^{a_1} \theta \left(1 + \sum \theta_{i_n, \dots, i_1} X_n^{i_n} \dots X_1^{i_1}\right).$$

Let $z(X) = \zeta(X), s(X) = z(X)^{p^r} - 1$. Let the operator Δ act on elements of $\mathcal{O}_{00}$ as the Frobenius automorphism Fr and on X_i as raising to the pth power. For $\alpha \in F^*$ put

$$l(\alpha) = \frac{1}{p} \log \alpha(X)^{p - \Delta}, \quad \delta_i(\alpha) = \alpha^{-1} \frac{\partial \alpha}{\partial X_i}, \quad \eta_i(\alpha) = \delta_i(\alpha) - \frac{\partial l(\alpha)}{\partial X_i}.$$

For $\alpha_1, \dots, \alpha_{n+1} \in F^*$ put

$$\Phi(\alpha_1, \dots, \alpha_{n+1}) = l(\alpha_{n+1}) D_{n+1} - l(\alpha_n) D_n + \dots + (-1)^n l(\alpha_1) D_1,$$

where D_i is the determinant of the matrix

$$\begin{pmatrix} \delta_1(\alpha_1) & \cdots & \delta_n(\alpha_1) \\ \vdots & & \\ \delta_1(\alpha_{i-1}) & \cdots & \delta_n(\alpha_{i-1}) \\ \eta_1(\alpha_{i+1}) & \cdots & \eta_n(\alpha_{i+1}) \\ \vdots & & \\ \eta_1(\alpha_{n+1}) & \cdots & \eta_n(\alpha_{n+1}) \end{pmatrix}$$

Let $\mu = \mu_{p^r}$ denote the cyclic group generated by ζ. Define the map

$$\Gamma_r : (F^*)^{n+1} \to \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Gal}(\tilde{F}/F), \mu)$$

as

$$\Gamma_r(\alpha_1, \dots, \alpha_{n+1})(\varphi) = \zeta^{\gamma},$$

where $\gamma = (\varphi - 1)\delta$ and $\mathrm{Fr}(\delta) - \delta = \mathrm{res}_X \Phi(\alpha_1, \dots, \alpha_{n+1})/s(X)$.

Then one can show similarly to section 3 of [3] and [28] that Γ_r induces the nondegenerate pairing (for $p > 2$)

$$\Gamma_r : K_n^{\mathrm{top}}(F)/p^r \times F^*/F^{*p^r} \to \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Gal}(\tilde{F}/F), \mu) .$$

Applying this pairing in the same way as in section 3 of [3] (for $r = 1$), one can prove

Proposition. *Let F be a complete discrete valuation field of rank n, $\mathrm{char}(F) = 0$, $\mathrm{char}(k^{(n-1)}) = p$. Let $p = \theta_* t_n^{e_n} + \dots t_1^{e_1} + \dots$ with $\theta_* \in \mathcal{R}^*$. Then any element $\alpha \in VK_n^{\mathrm{top}}(F)/p$ is uniquely expanded as the convergent series $\sum c_{I,0} x_{I,0} + \sum c^*_{j,0} x^*_{j,0}$ where $c_{I,0}, c^*_{j,0} \in \mathbb{Z}/p$ and:*

(1) $x_{I,0} = \{1 + \theta t_n^{i_n} \dots t_1^{i_1}, t_{j_1}, \dots, t_{j_{n-1}}\}$ for $p \nmid I, 0 < I = (i_1, \dots, i_n) < p(e_1, \dots, e_n)/(p-1)$, where θ runs a fixed basis of $\mathcal{R}$ over $\mathbb{Z}/p, j_1 < \dots < j_{n-1}$, the set $\{j_1, \dots, j_{n-1}, j\}$ coincides with the set $\{1, \dots, n\}$, where j is the minimal index such that i_j is not divisible by p;

(2) $x^*_{j,0} = \{\omega_0, t_{j_1}, \dots, t_{j_{n-1}}\}$ where $\omega_0 = 1 + \theta t_n^{pe_n/(p-1)} \dots t_1^{pe_1/(p-1)} \in \mathcal{V}_F$, $1 \leqq j_1 < \dots < j_{n-1} \leqq n$, $\{j_1, \dots, j_{n-1}, j\} = \{1, \dots, n\}$, and $\bar{\theta}$ runs a $\mathbb{F}_p$-basis of $k/(-\bar{\theta}_*)^{p/(p-1)} \wp(k)$ if a primitive pth root of unity belongs to F; and $\omega_0 = 1, c^*_{j,0} = 0$ if a primitive pth root of unity doesn't belong to F.

In the case of $\mathrm{char}(k^{(m+1)}) = 0, \mathrm{char}(k^{(m)}) = p, m < n$ one can deduce similar assertions in the same way as in Sect. 5 of [6].

2.6. Finally, we concern with an explicit description of $\Lambda_s(F)$.

Proposition. *Let $t_n, \dots, t_1$ be a local parameters of F. Let $r \geqq 1, m \geqq 2$, and U be a neighborhood of 1 in $1 + t_m \mathcal{O}_F$ with respect to the discrete valuation $(v^{(m)}, \dots, v^{(n)})$ of rank $(n - m + 1)$ with k_{m-1} being endowed with the discrete topology if $m \leqq n + 1$, and $U = \{1\}$ if $m > n + 1$. Then for*

$\alpha_1 \in \mathscr{V}_F, \alpha_2, \ldots, \alpha_m \in F^*$ there exist elements $\beta_J \in \mathscr{V}_F$, which sequentially continuously depend on $\alpha_1, \ldots, \alpha_m$, such that

$$\{\alpha_1, \ldots, \alpha_m\} \equiv \sum \{\beta_J, t_{j_1}, \ldots, t_{j_{m-1}}\} \mod p^r VK_m(F) + \{U\} \cdot K_{m-1}(F),$$

where J consists of $j_1, \ldots, j_{m-1}$ and runs all $(m-1)$-elements subsets of $\{1, \ldots, n\}$.

Proof. The proof can be carried out essentially in the same way as the proof of Proposition (2.1) of [3], where in fact the case of $\mathrm{char}(F) = 0, m \geqq n$, has been considered. □

Corollary. (1) *The topology on $VK_m(F)/(p^r VK_m(F) + \{U\} \cdot K_{m-1}(F))$ induced from $(\mathscr{V}_F)^d, d = \binom{n}{m-1}$, by the surjective map $(\mathscr{V}_F)^d \to VK_m(F)/(p^r VK_m(F) + \{U\} \cdot K_{m-1}(F))$, is not stronger than induced from $VK_m(F)$;*

(2) *$\Lambda_m(F) = \bigcap_{l \geqq 1} lK_m(F)$ is a divisible group, and $\Lambda_m(F) \cap VK_m(F) = \bigcap_{r \geqq 1} p^r VK_m(F)$;*

(3) *There exists a writing of Proposition modulo $p^r VK_m(F)$ with β_J sequentially continuously depending on $\alpha_1, \ldots, \alpha_m$.*

Proof. (1) follows from the definition of the topology on $K_m(F)$.

(2) By (1) we obtain that $\Lambda_m(F)$ is contained in the intersection of all

$$p^r VK_m(F) + \{U\} \cdot K_{m-1}(F).$$

Evidently $\Lambda_m(F) \subset VK_m(F)$ and $\bigcap_{l \geqq 1} lK_m(F) \subset \Lambda_m(F)$. Hence we shall verify that for a fixed r the intersection of all $p^r VK_m(F) + \{U\} \cdot K_{m-1}(F)$ coincides with $p^r VK_m(F)$. If $\mathrm{char}(F) = \ldots = \mathrm{char}(k_s) = 0, \mathrm{char}(k_{s-1}) = p$, then $U_{i_1, \ldots, i_n} \subset F^{p^r}$ for all sufficiently large i_s. Consequently, applying the border homomorphism in K-theory, it remains to consider only the case of $\mathrm{char}(F) = p$. Then one can apply a differential symbol

$$d_F : K_m(F)/pK_m(F) \to \Omega_F^m, \quad d_F\{\alpha_1, \ldots, \alpha_m\} = \frac{d\alpha_1}{\alpha_1} \wedge \ldots \wedge \frac{d\alpha_m}{\alpha_m}.$$

The Bloch-Kato-Gabber Theorem asserts, in particular, that d_F is injective (c.f. Sect. 2 of [1]). Note that Ω_F^m is a finite-dimensional vector space over F/F^p, so the intersection of all neighborhoods of zero in Ω_F^m, with respect to the induced from F topology, is trivial. The homomorphism d_F is sequentially continuous, hence its injectivity implies $\Lambda_m(F) \subset pVK_m(F)$. Moreover, $\Lambda_m(F)$ is a divisible group. For let $x \in \Lambda_m(F), x = py$ with $y \in VK_m(F)$. The absence of nontrivial p-torsion in $K_m^{\mathrm{top}}(F)$ for the fields of positive characteristic (this can be shown in the same way as in Proposition (2.4)) implies now $y \in \Lambda_m(F)$.

Thus, $\Lambda_m(F) = \bigcap_{r \geqq 1} p^r VK_m(F)$ for $\mathrm{char}(F) = p$. If $\mathrm{char}(F) = 0$, then the description of p-torsion in $K_m^{\mathrm{top}}(F)$, see (3.3), shows that $\Lambda_m(F) = \bigcap_{r \geqq 1} p^r VK_m(F)$ is divisible as well.

(3) follows from (2) and existence of the required writing in $VK_m^{\mathrm{top}}(F)$. □

3 Multidimensional local p-class field theory

Let F be a complete discrete valuation field of rank n with the residue field k. Assume that k is a perfect field of characteristic p and $\kappa = \dim_{\mathbb{F}_p} k/\wp(k) \neq 0$. In this section we will show that F and $A_F = K_n^{\mathrm{top}}(F)$ satisfy conditions C1–C5 of Sect. 1 in the case of so-called $\wp$-extensions (see (3.4)) regarded as marked extensions. Thus, we establish class field theory for F. This theory may be regarded as a generalization of the known results in the case of a finite k ([11–13], [23, 26, 27], [3–6]).

3.1. It is well-known that $A_F = K_n^{\mathrm{top}}(F)$ is a G_p-modulation. C1 is satisfied with $\tilde{F}/F$ defined in (2.3): for the homomorphism $v_F : A_F \to \mathbb{Z}$ one can take the composition

$$K_n^{\mathrm{top}}(F) \xrightarrow{\partial} K_{n-1}^{\mathrm{top}}(k_{n-1}) \to \ldots \to K_0(k_0) \simeq \mathbb{Z}\,,$$

where ∂ is the border homomorphism in K-theory, c.f. section 2 Chapter IX of [8]. Then U_F of Sect. 1 coincides with $UK_n^{\mathrm{top}}(F)$ of Sect. 2. A prime element π_F of A_F can be written as $\{t_1, \ldots, t_n\} + \varepsilon$ with a suitable $\varepsilon \in UK_n^{\mathrm{top}}(F)$, where $t_n, \ldots, t_1$ are local parameters of F. The norm $N_{L/F}$ for K-groups maps $UK_n^{\mathrm{top}}(L)$ onto $UK_n^{\mathrm{top}}(F)$ for $L \subset \tilde{F}$ as it immediately follows.

3.2. In order to verify C3 we need the following description of the norm map (analogously to Proposition 4.1 of [3] and Proposition 3.1 of [6]):

Proposition. *Let L/F be a cyclic totally ramified extension of degree p, σ a generator of* Gal(L/F). *Let $L = F(t_{s,L})$ for some $s, 1 \leqq s \leqq n$. Take local parameters $t_n, \ldots, t_{s,F} = N_{L/F} t_{s,L}, \ldots, t_1$ in F and $t_n, \ldots, t_{s,L}, \ldots, t_1$ in L, and assume that*

$$\frac{\sigma t_{s,L}}{t_{s,L}} \equiv 1 + \theta_0 t_n^{r_n} \ldots t_{s,L}^{r_s} \ldots t_1^{r_1} \mod \mathcal{U}_{r_1+1,\ldots,r_n}$$

with $\theta_0 \in \mathcal{R}^*$.

Then for $\theta \in \mathcal{O}_F$

(1) *if $(i_1, \ldots, i_n) < (r_1, \ldots, r_n)$, then*

$$N_{L/F}\,(1 + \theta t_n^{i_n} \ldots t_{s,L}^{i_s} \ldots t_1^{i_1}) \equiv 1 + \theta^p t_n^{pi_n} \ldots t_{s,F}^{i_s} \ldots t_1^{pi_1} \mod \mathcal{U}_{pi_1+1,\ldots,pi_n,F};$$

(2) *if $(i_1, \ldots, i_n) = (r_1, \ldots, r_n)$, then*

$$N_{L/F}(1 + \theta t_n^{i_n} \ldots t_{s,L}^{i_s} \ldots t_1^{i_1}) \equiv 1 + (\theta^p - \theta\theta_0^{p-1}) t_n^{pi_n} \ldots t_{s,F}^{i_s} \ldots t_1^{pi_1} \mod \mathcal{U}_{pi_1+1,\ldots,pi_n,F};$$

(3) *if $(i_1, \ldots, i_n) > 0$, then*

$$N_{L/F}(1 + \theta t_n^{i_n+r_n} \ldots t_{s,L}^{pi_s+r_s} \ldots t_1^{i_1+r_1}) \equiv 1 - \theta\theta_0^{p-1} t_n^{i_n+pr_n} \ldots t_{s,F}^{i_s+r_s} \ldots t_1^{i_1+pr_1} \mod \mathcal{U}_{i_1+pr_1+1,\ldots,i_n+pr_n,F}\,.$$

The surjectivity of the norm map $N_{\tilde{L}/\tilde{F}} : K_n^{\text{top}}(\tilde{L}) \to K_n^{\text{top}}(\tilde{F})$ directly follows from Proposition.

3.3. Using the bijectivity of the norm residue symbol in the case of $\text{char}(F) = 0$ which can be verified on the base of the arguements of section 6 in [3] (employing Kato's theorem on the residue norm symbol for multidimensional complete fields, see Sect. 1.4 of [12]) one can deduce that the relation $px = 0$ for $x \in VK_n^{\text{top}}(F)$ in the case of $\text{char}(F) = 0, \mu_p \subset F$ implies $x = \{\zeta\} \cdot y$ for a generator ζ of μ_p and some $y \in K_{n-1}^{\text{top}}(F)$. From this fact and the absence of nontrivial p-torsion in the case of $\text{char}(F) = p$ (see the end of Sect. 2 of [6]) one can prove, using explicit calculations in $K_{n-1}^{\text{top}}(F)/p$ in the same way as in the proof of Theorem (4.2) of [3] and Theorem (3.2) of [6], the following assertion:

Theorem. *Let L/F be a cyclic totally ramified extension of degree p, and let σ be a generator of* Gal(L/F)*. Then the sequence*

$$K_n^{\text{top}}(L) \xrightarrow{\sigma - 1} K_n^{\text{top}}(L) \xrightarrow{N_{L/F}} K_n^{\text{top}}(F)$$

is exact.

3.4. At this point we note that C4 don't hold in the general case.

Example. Let f be a finite extension of $\mathbb{Q}_p$ such that the group of all pth roots of unity μ_p lies in f, and $\mu_{p^2} \not\subset f_1$, where f_1 is the unramified extension of degree p over f. Let ζ be a primitive pth root of unity and ω be a p-primary element in $f(\omega \in U_f$ and $f_1 = f(\sqrt[p]{\omega}))$. Then, according to well-known properties of K_2 of a local field (see, e.g., Chapter IX of [8]) there exist prime elements π and π' in f such that $\{\omega, \pi'\} - \{\zeta, \pi\} \in pK_2(f)$ and $\{\omega, \pi'\}$ is a generator of $K_2(f)/pK_2(f)$. Since $i_{f/f_1}\{\omega, \pi'\} \in pK_2(f_1)$, one can deduce from Moore's theorem (see, e.g., Sect. 4 Chapter IX of [8]) that $i_{f/f_1}\{\zeta, \pi\}$ is a divisible element in $K_2(f_1)$. Let t be a transcendental element over f and let $F = f\{\{t\}\}$ be a complete discrete valuation field of rank 2 with π, t as local parameters and with the residue field isomorphic to the residue field of f (see Sect. 1 of [3]). Put $L = F(\sqrt[p]{t})$: L/F is a totally ramified cyclic extension of degree p and $\pi_L = \{\sqrt[p]{t}, \pi\}$ is a prime element in $K_2^{\text{top}}(L)$. Then $\sigma i_{L/\tilde{L}}\pi_L = i_{L/\tilde{L}}\pi_L$ for a generator σ of Gal(L/F), since $i_{L/L_1}(\sigma\pi_L - \pi_L) = 0$ in $K_2^{\text{top}}(L_1)$ for $L_1 = L \otimes_f f_1$. We obtain also that $i_{F/\tilde{F}}\{\omega, \pi'\} = 0$ and $\{\omega, \pi'\} \notin N_{L/F}K_2^{\text{top}}(L)$.

Thus, at the first rate we will treat some special class of extensions of complete discrete valuation fields of rank $n \geqq 2$ to which the theory of Sect. 1 can be applied. Thereby class field theory in the general case of totally ramified p-extensions of the fields will be established.

Definition. *A totally ramified galois p-extension L/F will be called a $\wp$-extension if one of the following cases occurs:*

(1) $\text{char}(F) = p$,

(2) $\mathrm{char}(F) = 0$ *and* $\mu_p \not\subset F$,
(3) $\mathrm{char}(F) = 0, \mu_p \subset F$ *and there is a chain of subfields* $L = L_s - L_{s-1} - \dots - L_0 = F$ *such that* $L_i = L_{i-1}(\sqrt[p]{\varepsilon_i})$ *for some* $\varepsilon_i \in \mathscr{V}_{L_{i-1}}$, $1 \leqq i \leqq s$.

It is known that a $\wp$-extension L of F can be constructed as a tower of Artin-Schreier extensions: $L_i = L_{i-1}(\alpha_i')$ with suitable $\alpha_i = \wp(\alpha_i') \in L_{i-1}, 1 \leqq i \leqq s$ (for the case of a finite residue field see Sect. 1 of [30]), that justifies the notation. The following remark follows immediately: if $(r_1, \dots, r_n)$ for a totally ramified cyclic $\wp$-extension L/F of degree p is the same as in Proposition (3.2), then $p \nmid r_s$ (c.f. Corollary 2 of Proposition (4.1) of [3]).

3.5. Proposition. *$\wp$-extensions satisfy properties $\mathscr{P}1$–$\mathscr{P}3$ of (1.5) of marked extensions.*

Proof. The first property of marked extensions in (1.5) holds for $\wp$-extensions. For the rest one may assume that $\mathrm{char}(F) = 0, \mu_p \subset F$. At first we verify that for a finite extension L/F in which a local parameter $t_{s,F}$ ramifies, there exists an element $\varepsilon \in \mathscr{V}_F$ such that $\tilde{L}(\sqrt[p]{t_{s,F}\varepsilon})/\tilde{L}$ is of degree p. Indeed, if $\tilde{L}(\sqrt[p]{t_{s,F}}) \neq \tilde{L}$, then put $\varepsilon = 1$. Otherwise, let $t_{n,L}, \dots, t_{1,L}$ and $t_{n,F}, \dots, t_{1,F}$ be local parameters in L and F, and let $\mathbf{v}_L(p) = (e_1, \dots, e_n)$. Let $e_s = p^m v_F^{(s)}(p), m \geqq 1$. Put $\varepsilon = 1 + \sum \theta_I \varepsilon_I \in \mathscr{V}_F$ with $\theta_I \in \mathscr{R}$,

$$\varepsilon_I = t_{n,L}^{pe_n/(p-1)} \dots t_{s+1,L}^{pe_{s+1}/(p-1)} t_{s,L}^{pe_s/(p-1) - p^m + i_s} t_{s-1,L}^{i_{s-1}} \dots t_{1,L}^{i_1},$$

$I = (i_1, \dots, i_s) \geqq I_0, I_0 = (b_1, \dots, b_{s-1}, 0), \theta_{I_0} \neq 0$. Then

$$\varepsilon \equiv \left(1 + \sum_{p \nmid I} \theta_I \varepsilon_I\right)$$
$$\left(1 - p \sum_{I = pJ} \theta_{pJ}^{1/p} t_{n,L}^{e_n/(p-1)} \dots t_{s+1,L}^{e_{s+1}/(p-1)} t_{s,L}^{e_s/(p-1) - p^{m-1} + j_s} t_{s-1,L}^{j_{s-1}} \dots t_{1,L}^{j_1}\right)$$

modulo $\mathscr{V}_L^p$. If $\theta_I \neq 0$ for some $p \nmid I, I < (p^{-1}b_1, \dots, p^{-1}b_{s-1}, p^m - p^{m-1})$, or $m = 1$, then $\tilde{L}(\sqrt[p]{\varepsilon}) \neq \tilde{L}$. If $\theta_I = 0$ for all such I, and $m > 1$, then one can continue transformations for the second factor.

In particular, if L/F is a totally ramified galois extension of degree p, and $L = F(\sqrt[p]{t_{s,F}})$, then the same arguments show that there is $Q = F(\sqrt[p]{t_{s,F}\varepsilon})$ for a suitable $\varepsilon \in \mathscr{V}_F$, such that $\sqrt[p]{t_{s,F}}$ ramifies in LQ/L. If $t_{s,E}$ is a local parameter of $E = F(\sqrt[p]{\varepsilon})$, then $LQ = E(\sqrt[p]{t_{s,F}t_{s,E}^{-1}})$, hence LQ/E is a $\wp$-extension. Thus, LQ/F is a $\wp$-extension, so we have proved $\mathscr{P}3$.

In order to prove $\mathscr{P}2$ for $\wp$-extensions, it suffices to consider the case when L/L_1 is a subextension of degree p in L/F and to prove that there exists a finite totally ramified extension Q/F such that LQ/L_1Q is a $\wp$-extension and $\tilde{L} \cap \tilde{Q} = \tilde{F}$. Let $t_{n,L_1}, \dots, t_{1,L_1}$ be local parameters in L_1 and let $L = L_1(\sqrt[p]{t_{s,L_1}})$. We construct a tower of fields $F = Q_0 - Q_1 - \dots - Q_s = R$, such that in Q_i/Q_{i-1} only the local parameter $t_{i,F} = t_{i,Q_{i-1}}$ ramifies, and

$$t_{i,F} = t_{i,Q_i}^{e_{i,i}} t_{i-1,Q_{i-1}}^{e_{i,i-1}} \dots t_{1,Q_i}^{e_{i,1}} \theta_i \eta_i, \quad \text{with } \theta_i \in \mathscr{R}, \eta_i \in \mathscr{V}_{Q_i},$$

where

$$t_{i,F} = t_{i,L_1}^{e_{i,i}} t_{i-1,L_1}^{e_{i,i-1}} \dots t_{1,L_1}^{e_{i,1}} \theta_i \rho_i, \quad \text{with } \theta_i \in \mathscr{R}, \rho_i \in \mathscr{V}_{L_1} .$$

One can take Q_i/Q_{i-1} as subsequent extensions of the type considered at the beginning of the proof, such that $\widetilde{L_1}\tilde{R}/\widetilde{L_1}$ is of the same degree as R/F. We deduce that for any $1 \leqq i \leqq s$ the element $t_{i,L_1} t_{i,R}^{-1}$ belongs to $\mathscr{V}_{L_1R}$. If t_{s,L_1} ramifies in LR/L_1R, then take $Q = R(\sqrt[p]{t_{s,R}\varepsilon})$ with a suitable $\varepsilon \in \mathscr{V}_R$. Otherwise, $t_{s,L_1} \in (L_1R)^p$, and if LR/L_1R isn't a $\wp$-extension, then $LR = L_1R(\sqrt[p]{t_{s',L_1R}})$ for some $s' < s$, and one may repeat the previous constructions for $LR/L_1R/R$ instead of $L/L_1/F$. Finally we obtain the desired extension Q/F. □

Note that the quotient group $UK_n^{\text{top}}(F)/VK_n^{\text{top}}(F)$is isomorphic to $(k^*)^n$, and hence, is p-divisible. Thus, we get the map (according to section 1)

$$\Upsilon_{L/F} : (\mathrm{Gal}(L/F)^{\mathrm{ab}})^{\sim} \to VK_n^{\text{top}}(F)/N_{L/F} VK_n^{\text{top}}(L) .$$

3.6. We verify C4 for $\wp$-extensions.

Proposition. *Let L/F be a totally ramified cyclic $\wp$-extension of degree p^m, and let σ be a generator of* $\mathrm{Gal}(\tilde{L}/\tilde{F})$*. Then $\sigma\alpha = \alpha$ for $\alpha \in K_n^{\text{top}}(\tilde{L})$ implies $p^m | v_{\tilde{L}}(\alpha)$.*

Proof. Assume that L/F is of degree p. Let $(r_1, \dots, r_n)$ be as in Proposition (3.2). Then, as $p \nmid r_s$, Proposition (2.4) and Proposition (2.5) imply that $\sigma\pi - \pi$ belongs to $U_{r_1,\dots,r_n} K_n^{\text{top}}(\tilde{L})$ and doesn't belong to $U_{r_1+1,\dots,r_n} K_n^{\text{top}}(\tilde{L})$ for a prime element π in $K_n^{\text{top}}(\tilde{L})$. Therefore $p | v_{\tilde{L}}(\alpha)$.

Now let $\tilde{L}/\tilde{F}$ be of degree p^m. By the induction assumption, $\alpha = a p^{m-1} \pi + \varepsilon$, where π is a prime element in $K_n^{\text{top}}(\tilde{L}), \varepsilon \in UK_n^{\text{top}}(\tilde{L})$. Then

$$a(\tau\pi - \pi) \equiv 0 \mod V(L|F) ,$$

where $\tau = \sigma^{p^{m-1}}$. This means that $a(\tau\pi - \pi) = (\sigma - 1)\gamma$ for some $\gamma \in VK_n^{\text{top}}(\tilde{L})$. Assume that p doesn't divide a, then we may put $a = 1$.

Let $\tilde{M}$ be the fixed field of τ, and let $\varphi \neq 1$ be an element of $\mathrm{Gal}(\tilde{L}/L)$ with the fixed field L'. L' is a complete discrete valuation field of rank n with a pro-p-quasi-finite residue field k' such that k_p^{ab}/k' is a $\mathbb{Z}_p$-extension. Let M' be the fixed field of $\tilde{M}$ under the action of φ. The completeness of $\tilde{L}$ provides the relation $\mathscr{V}_{\tilde{L}} = \mathscr{V}_{\tilde{L}}^{\tau\varphi - 1}$ (c.f. Lemma (1.4) of [7]). Then for a prime element π of $K_n^{\text{top}}(L')$ there exist $\gamma_1, \delta \in VK_n^{\text{top}}(\tilde{L})$ such that

$$(\tau\varphi - 1) i_{L'/\tilde{L}} \pi = (\tau - 1) i_{L'/\tilde{L}} \pi = (\sigma - 1)\gamma = (1 - \tau\varphi)\delta, \quad \delta = (\sigma - 1)\gamma_1 .$$

Let Σ' be the fixed field of $\tau\varphi$. Then Propositions (2.4) and (2.5) imply that there is a prime element $\pi_{\Sigma'}$ in $K_n^{\text{top}}(\Sigma')$ such that $i_{L'/\tilde{L}}\pi + \delta = i_{\Sigma'/\tilde{L}}\pi_{\Sigma'} + p\nu, \nu \in K_n^{\text{top}}(\tilde{L})$. Therefore, the element

$$x = i_{M'/\tilde{M}}(N_{\Sigma'/M'}\pi_{\Sigma'} - N_{L'/M'}\pi) = N_{\tilde{L}/\tilde{M}}(\delta - p\nu)$$

belongs to $(\sigma-1)VK_n^{\rm top}(\tilde M)+pK_n^{\rm top}(\tilde M)$. Now we apply Propositions (2.4) and (2.5) again in order to deduce that (as L'/M' is a $\wp$-extension)

$$N_{\Sigma'/M'}\pi_{\Sigma'} - N_{L'/M'}\pi \in N_{L'/M'}UK_n^{\rm top}(L') .$$

This means that $i_{\Sigma'/\tilde L}\pi_{\Sigma'} - i_{L'/\tilde L}(\pi+\eta)$ for a suitable $\eta \in UK_n^{\rm top}(L')$ belongs to the kernel of $N_{\tilde L/\tilde M}$. By Theorem (3.3) we get $(\tau-1)i_{L'/\tilde L}\pi \in V(L|M)$ that contradicts the assumption $a=1$. □

Now according to Sect. 1 we obtain the homomorphism

$$\Psi_{L/F} : VK_n^{\rm top}(F)/N_{L/F}VK_n^{\rm top}(L) \to ({\rm Gal}(L/F)^{\rm ab})^{\sim}$$

for an extension $L/F \in \wp_0$ (any intermediate cyclic subextension in L/F is a $\wp$-extension).

3.7. At last we verify C5.

Proposition. *Let L/F be a totally ramified cyclic $\wp$-extension of degree p. Then $\Upsilon_{L/F}$ is surjective.*

Proof. Let $(r_1,\dots,r_n)$ be as in (3.2). By employing Proposition (3.2) it suffices to show that if

$$\varepsilon \equiv \{1+\theta t_n^{pr_n}\dots t_{s,F}^{r_s}\dots t_1^{pr_1}, t_{j_1},\dots,t_{j_{n-1}}\} \mod N_{L/F}VK_n^{\rm top}(L)$$

with $\bar\theta \notin \bar\theta_0^p\wp(k)$, where θ_0 is as in Proposition (3.2), then $\Psi_{L/F}(\varepsilon)\neq 1$. In terms of C5 we obtain that

$$\eta = \{1+\theta' t_n^{r_n}\dots t_{s,L}^{r_s}\dots t_1^{r_1}, t_{j_1},\dots,t_{j_{n-1}}\}+\dots ,$$

where $\overline{\theta'}^p - \bar\theta_0^{p-1}\overline{\theta'} = \bar\theta$. Now, if $\Psi_{L/F}(\varepsilon)=1$, then the element $(\varphi-1)\eta$ belongs to $U_{r_1+1,\dots,r_n}K_n^{\rm top}(\tilde L)$ for any $\varphi \in {\rm Gal}(\tilde L/F)$. This is impossible in view of (2.4) and (2.5). Thus, $\Psi_{L/F}$ is injective. □

3.8. According to Theorem (1.8) we obtain the following assertion:

Theorem. *Let F be a complete discrete valuation field of rank n with a perfect residue field k of characteristic $p, \kappa = \dim_{\mathbb{F}_p} k/\wp(k)\neq 0$. Then for a totally ramified galois extension $L/F, L\subset F_p$, the map*

$$\Upsilon_{L/F} : ({\rm Gal}(L/F)^{\rm ab})^{\sim} \to VK_n^{\rm top}(F)/N_{L/F}VK_n^{\rm top}(L)$$

is an isomorphism; and the map

$$\Psi_{L/F} : VK_n^{\rm top}(F)/N_{L/F}VK_n^{\rm top}(L) \to ({\rm Gal}(L/F)^{\rm ab})^{\sim}$$

is inverse if $L/F \in \wp_0$.

Remark 1. One can add to the functorial properties (1), (2) of Proposition (1.4) the following: Let L/F be a totally ramified galois extension and M/F be its subextension. Then the diagram

$$\begin{array}{ccc} (\mathrm{Gal}(L/F)^{\mathrm{ab}})^{\sim} & \xrightarrow{\Upsilon_{L/F}} & VK_n^{\mathrm{top}}(F)/N_{L/F}VK_n^{\mathrm{top}}(L) \\ {\scriptstyle \mathrm{Ver}^{\sim}}\downarrow & & \downarrow \\ (\mathrm{Gal}(L/M)^{\mathrm{ab}})^{\sim} & \xrightarrow{\Upsilon_{L/M}} & VK_n^{\mathrm{top}}(M)/N_{L/M}VK_n^{\mathrm{top}}(L) \end{array}$$

is commutative, where $\mathrm{Ver}^{\sim}$ is induced by $\mathrm{Ver} : \mathrm{Gal}(L/F)^{\mathrm{ab}} \to \mathrm{Gal}(L/M)^{\mathrm{ab}}$.

Indeed, assume first that $L/F \in \wp_0$. Let $\varepsilon = N_{\tilde{L}/\tilde{F}}\eta$ and $(\varphi - 1)\eta = (1 - \sigma)\pi + \gamma$ for a prime element π in $K_n^{\mathrm{top}}(\tilde{L}), \sigma \in \mathrm{Gal}(\tilde{L}/\tilde{F})$, $\gamma \in V(L|F)$. Then $\sigma|_L = \chi(\varphi), \chi = \Psi_{L/F}(\varepsilon)$. Let $\tau_i \in \mathrm{Gal}(\tilde{L}/\tilde{F})$ be a set of representatives of $\mathrm{Gal}(\tilde{L}/\tilde{F})$ over $\mathrm{Gal}(\tilde{L}/\tilde{M})$. Then $i_{F/\tilde{M}}\varepsilon = N_{\tilde{L}/\tilde{M}}\eta_1$ with $\eta_1 = \sum \tau_i\eta$ and $(\varphi - 1)\eta_1 = \sum(1 - \sigma)\tau_i\pi + \sum\tau_i\gamma$. Let $\sigma\tau_i = \tau_{i'}h_i(\sigma)$ with $h_i(\sigma) \in \mathrm{Gal}(\tilde{L}/\tilde{M})$. Now we deduce

$$\begin{aligned} \sum(1 - \sigma)\tau_i\pi &= \sum\tau_{i'}(1 - h_i(\sigma))\pi \equiv \prod(1 - h_i(\sigma))\pi \\ &= (1 - \mathrm{Ver}(\sigma))\pi \bmod V(L|M)\,. \end{aligned}$$

Since $\sum\tau_i\gamma \in V(L|M)$ we conclude that $(\varphi - 1)\eta_1 \equiv (1 - \mathrm{Ver}(\sigma))\pi \bmod V(L|M)$, as desired.

For arbitrary L/F one can apply $\mathscr{P}2$ of (1.5).

Remark 2. Let $\mathscr{F}/F$ be any subextension of F_p/F linearly disjoint with the maximal unramified p-extension F^{ur}/F and such that $\mathscr{F}F^{\mathrm{ur}} = F_p$. Passing to the projective limit for $\Upsilon_{L/F}^{-1}$ when L/F runs all finite galois subextensions in $\mathscr{F}/F$ we obtain the homomorphism $VK_n^{\mathrm{top}}(F) \to \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Gal}(\tilde{F}/F), \mathrm{Gal}(\mathscr{F}/F)^{\mathrm{ab}})$. It determines the *reciprocity map*

$$\Psi_F : VK_n^{\mathrm{top}}(F) \to \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Gal}(\tilde{F}/F), \mathrm{Gal}(F_p^{\mathrm{ab}}/\tilde{F}))\,,$$

where F_p^{ab}/F is the maximal abelian subextension in F_p/F, and elements of the right-hand group are continuous homomorphisms with respect to the discrete topology on $\mathrm{Gal}(F_p^{\mathrm{ab}}/\tilde{F})$. The kernel of Ψ_F coincides with the intersection of all norm groups $N_{L/F}VK_n^{\mathrm{top}}(L)$ where L/F runs finite galois subextensions in $\mathscr{F}/F$.

Remark 3. Let f be the residue field of F with respect to the discrete valuation $v^{(n)}$, then f is of rank $(n-1)$. Similarly with the proof of Theorem (5.3) of [6] one can verify that the diagram

$$\begin{array}{ccc} VK_n^{\mathrm{top}}(F) & \xrightarrow{\Psi_F} & \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Gal}(\tilde{F}/F), \mathrm{Gal}(F_p^{\mathrm{ab}}/\tilde{F})) \\ \downarrow & & \downarrow \\ VK_{n-1}^{\mathrm{top}}(f) & \xrightarrow{\Psi_f} & \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Gal}(\tilde{F}/F), \mathrm{Gal}(F_p^{\mathrm{ab}}/\tilde{F})) \end{array}$$

is commutative, where the left homomorphism is induced by the border homomorphism.

Let F' be an intermediate complete field between F and $\tilde{F}$. Then the diagram

$$\begin{array}{ccc} VK_n^{\mathrm{top}}(F) & \xrightarrow{\Psi_F} & \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Gal}(\tilde{F}/F), \mathrm{Gal}(F_p^{\mathrm{ab}}/\tilde{F})) \\ \downarrow & & \downarrow \\ VK_n^{\mathrm{top}}(F') & \xrightarrow{\Psi_{F'}} & \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Gal}(\tilde{F'}/F'), \mathrm{Gal}(F'^{\mathrm{ab}}_p/\tilde{F'})) \end{array}$$

is commutative.

Remark 4. One can show that if $\mathrm{char}(F) = p$, then for $\alpha \in VK_n^{\mathrm{top}}(F), \beta \in W_s(F)$ and the pairing of (2.4)

$$(\alpha, \beta]_s(\varphi) = \Psi_F(\alpha)(\varphi)(\gamma) - \gamma\,,$$

where $\varphi \in \mathrm{Gal}(\tilde{F}/F)$, and γ is the root of the polynomial $\wp(X) - \beta$.

If $\mathrm{char}(F) = 0$, then for $\alpha \in VK_n^{\mathrm{top}}(F), \beta \in F^*$ and the pairing of (2.5)

$$\Gamma_r(\alpha, \beta)(\varphi) = \gamma^{\Psi_F(\alpha)(\varphi)-1}\,,$$

where $\varphi \in \mathrm{Gal}(\tilde{F}/F)$ and $\gamma^{p^r} = \beta$.

4 Notes on ramification

Let F be a complete discrete valuation field of rank n as in Sect. 3. In this section we touch abelian ramification groups via p-class field theory.

4.1. There is a filtration (depends on the choice of the valuation $\mathbf{v}_F$ defined in (2.1))

$$U_{1,0,\dots,0}K_n^{\mathrm{top}}(F) \supset U_{2,0,\dots,0}K_n^{\mathrm{top}}(F) \supset \dots$$

on $VK_n^{\mathrm{top}}(F)$. Using the pairings of (2.4) and (2.5) in the context of Proposition (2.4) and Proposition (2.5), one can show that if $\mathrm{char}(F) = p$, then for $I > 0$

$$U_I K_n^{\mathrm{top}}(F)/U_{I+1}K_n^{\mathrm{top}}(F) \simeq k\,,$$

where $1 = (1, 0, \dots, 0)$. If $\mathrm{char}(F) = 0$, then for $I > 0$ and $E = (e_1, \dots, e_n)$ as in (2.5) one can obtain that

$$U_I K_n^{\mathrm{top}}(F) + pK_n^{\mathrm{top}}(F)/U_{I+1}K_n^{\mathrm{top}}(F) + pK_n^{\mathrm{top}}(F)$$

is equal to 0 if $p|I, I < pE/(p-1)$ or $I > pE/(p-1)$ or $I = pE/(p-1) \in (\mathbb{Z})^n$ and a primitive pth root of unity doesn't belong to F; is isomorphic to k if $p \nmid I, I < pE/(p-1)$; is isomorphic to $(k/\wp(k))^n$ if $I = pE/(p-1)$ and a primitive pth root of unity belongs to F.

Now let L/F be a totally ramified galois p-extension. The norm map

$$N_{L/F} : VK_n^{\mathrm{top}}(L) \to VK_n^{\mathrm{top}}(F)$$

can be described on the base of Proposition (3.2). However, the behavior of $N_{L/F}$ for $n > 1$ is more complicated than in the case of $n = 1$, because the norm map may act on different components of a symbol in $K_n^{\mathrm{top}}(L)$.

Example. Let L/F be of degree $p, n = 2$. Let $L = F(t_{2,L})$. We take $t_{2,F} = N_{L/F}t_{2,L}$ and t_1 as local parameters of F, and $t_{2,L}$ and t_1 as local parameters of L. Then $N_{L/F}\{1 + t_{2,L}t_1, t_1\} = \{1 + t_{2,F}t_1^p, t_1\}$ and $N_{L/F}\{1 + t_{2,F}t_1, t_{2,L}\} = \{1 + t_{2,F}t_1, t_{2,F}\}$, but $\mathbf{v}_L(t_{2,L}t_1) < \mathbf{v}_L(t_{2,F}t_1), \mathbf{v}_F(t_{2,F}t_1^p) > \mathbf{v}_F(t_{2,F}t_1)$.

4.2. Applying the construction of the reciprocity map of Sects. 1 and 3, one can describe the image of $U_I K_n^{\mathrm{top}}(F)$ in $(\mathrm{Gal}(L/F)^{\mathrm{ab}})^{\sim}$. Let π_L be a prime element in $K_n^{\mathrm{top}}(L)$. Assume first that L/F is an abelian totally ramified extension, $L/F \in \wp_0$. Then

$$\Upsilon_{L/F}^{-1}(U_I K_n^{\mathrm{top}}(F)) = \{\chi \in \mathrm{Gal}(L/F)^{\sim} :$$
$$i_{L/\tilde L}(\chi(\varphi)\pi_L - \pi_L) \in (\varphi - 1)N_{\tilde L/\tilde F}^{-1} i_{F/\tilde F} U_I K_n^{\mathrm{top}}(F) \quad \text{for any } \varphi \in \mathrm{Gal}(\tilde L/F)\}.$$

If $L/F \notin \wp_0$, then the description of the pre-image $\Upsilon_{L/F}^{-1}(U_I K_n^{\mathrm{top}}(F))$ follows from the commutative diagrams in (1.8).

In the case of $n = 1$ it is well-known that $(N_{\tilde L/\tilde F}^{-1} U_{i,F})^{\varphi-1}$ for an integer $i > 0$ can be replaced by $U_{h(i),L}V(L|F)$ and $\Upsilon_{L/F}^{-1}(U_{i,F})$ can be identified with the ramification group $\mathrm{Gal}(L/F)_{h(i)}$, where $h = \psi_{L/F}$ is the function defined as the maximal integer j such that (in the case of an infinite residue field)

$$N_{L/F}U_{j,L} \subset U_{i,F}, \not\subset U_{i+1,F} \quad N_{L/F}U_{j+1,L} \subset U_{i+1,F}$$

(c.f. Sect. 3 Chapter III of [7]). Moreover, h can be extended to galois extensions and all real $i \geqq 0$.

4.3. For $\alpha \in VK_n^{\mathrm{top}}(L)$ put

$$w_L(\alpha) = \min\{I : \alpha \in U_I K_n^{\mathrm{top}}(L)\} .$$

Let $L/F \in \wp_0$. We will assume that the surjective discrete valuation of rank n on F is induced by the surjective discrete valuation on L. Let

$$B = \{w_L(\sigma\pi_L - \pi_L) : \sigma \in \mathrm{Gal}(L/F)\} ,$$

where π_L is a prime element in $K_n^{\mathrm{top}}(L)$.

Define the modified Hasse-Herbrand function $h_n = h_{n,L/F} : (\mathbb{Z})_+^n \to (\mathbb{Z})_+^n$, where $(\mathbb{Z})_+^n$ is the subset of indices $I > 0$ in $(\mathbb{Z})^n$, as

$$h_n(I) = \min_{\varphi \in \mathrm{Gal}(\tilde L/F)} \{w_{\tilde L}(\alpha) \in B : \alpha \in (\varphi - 1)N_{\tilde L/\tilde F}^{-1} i_{F/\tilde F} U_I K_n^{\mathrm{top}}(F)\}$$

if the minimum exists, and $h_n(I) = +\infty$ otherwise. Then $h_{n,L/F}$ is an increasing function. This h_n doesn't coincide with the classical h. However, the equality $\Upsilon_{L/F}^{-1}(U_{i,F}) = \mathrm{Gal}(L/F)_{h_1(i)}$ holds for any integer $i > 0$. In fact, the function h_n reflects in the general case only properties of abelian extensions.

The equality $h_{n,L/F} = h_{n,L/M} \circ h_{n,M/F}$ holds for $n = 1$ and doesn't hold in the general case.

For $J \in (\mathbb{Z})^n_+$ put

$$\mathrm{Gal}(L/F)_J = \{\sigma \in \mathrm{Gal}(L/F) : \sigma\pi_L - \pi_L \in U_J K_n^{\mathrm{top}}(L)\} .$$

Then we deduce from (4.2) that $\Upsilon^{-1}_{L/F}(U_I K_n^{\mathrm{top}}(F)) \subset (\mathrm{Gal}(L/F)_{h_{n,L/F}(I)})^{\sim}$ for an abelian totally ramified $\wp$-extension L/F. There are examples of abelian extensions of degree p^3 in the case of $n > 1$ such that $\subset$ in the previous relation can't be replaced with $=$ (see, e.g., [18]). This means that in the general case the natural filtration on $\mathrm{Gal}(L/F)$ induced from the chosen filtration on $VK_n^{\mathrm{top}}(F)$ doesn't coincide with the induced filtration from $VK_n^{\mathrm{top}}(F)$ by the reciprocity map.

Let L/F be a finite totally ramified p-extension, and M/F be a galois subextension in L/F. Put $G = \mathrm{Gal}(L/F), H = \mathrm{Gal}(L/M)$. Then

$$(G/H)_{h_{n,M/F}(I)} = G_{h_{n,L/F}(I)}H/H \ , \quad I \in (\mathbb{Z})^n_+ \ .$$

This equality may be treated as an analog of the Herbrand theorem.

Problem. To extend the function h_n in a proper way on $(\mathbb{R})^n_+$.

5 Existence theorem

Let F be as in section 3. Recall that an additive polynomial over k is called k-decomposable if all its roots belong to k (see Sect. 2 of [7]).

5.1. A subgroup $\mathcal{N}$ in $VK_n^{\mathrm{top}}(F)$ is called *normic* if

(1) $\mathcal{N}$ is open;

(2) for any $I > 0$ there exists a polynomial $f_I(X) \in \mathcal{O}_F[X]$ such that the residue polynomial $\bar{f}_I \in k[X]$ is nonzero k-decomposable and

$$\{1 + f_I(\mathcal{O}_F)t_n^{i_n} \dots t_1^{i_1}\}K_{n-1}^{\mathrm{top}}(F) \subset \mathcal{N}$$

where $t_n, \dots, t_1$ are local parameters of $F, I = (i_1, \dots, i_n)$;

(3) for any $I > 0$ there exists a polynomial $g_I(X) \in \mathcal{O}_F[X]$ such that its residue $\bar{g}_I$ is nonzero k-decomposable, and

$$\mathcal{N} \cap U_I K_n^{\mathrm{top}}(F) + U_{I+1}K_n^{\mathrm{top}}(F)$$

$$= \{1 + g_I(\mathcal{O}_F)t_n^{i_n} \dots t_1^{i_1}\}K_{n-1}^{\mathrm{top}}(F) + U_{I+1}K_n^{\mathrm{top}}(F) \ ,$$

and for almost all I the polynomial $g_I(X)$ is equal to X.

We will show that the class of normic subgroups coincides with the class of norm groups $N_{L/F} VK_n^{\mathrm{top}}(L)$ of totally ramified galois extensions $L/F, L \subset F_p$.

5.2. It follows from the definition that the notion of a normic subgroup doesn't depend on the choice of local parameters in F.

Proposition. *Let L/F be a totally ramified galois p-extension, $L \subset F_p$. Then the group $N_{L/F} VK_n^{\mathrm{top}}(L)$ is normic in $VK_n^{\mathrm{top}}(F)$.*

Proof. The first property for $N_{L/F} VK_n^{\mathrm{top}}(L)$ is evident. To verify the second and third properties, one can proceed by induction on the degree of L/F. If L/F is of degree p, then all follows from Proposition (3.2). In the general case let M/F be a galois subextension in L/F of degree p. Let σ be a generator of $\mathrm{Gal}(M/F)$ and $M = F(t_{s,M})$. Let

$$t_{s,M}{}^{-1} \sigma t_{s,M} = 1 + \theta_0 t_n^{r_n} \dots t_{s,M}^{r_s} \dots t_1^{r_1} \mod \mathcal{U}_{r_1+1,\dots,r_n}$$

with $\theta_0 \in \mathcal{R}^*$. According to Proposition (3.2) the unique nontrivial polynomial arose from the norm map $N_{M/F}$ is $f_2(X) = \theta_0^p \wp(\theta_0^{-1} X)$. Now let π_L be a prime element in $K_n^{\mathrm{top}}(L)$, and $\sigma \in \mathrm{Gal}(L/F)$ be an extension of σ on L. Then

$$N_{L/M}(\sigma\pi_L - \pi_L) - \{1 + \theta_0 t_n^{r_n} \dots t_{s,M}^{r_s} \dots t_1^{r_1}, t_{j_1}, \dots, t_{j_{n-1}}\}$$

belongs to $U_{r_1+1,\dots,r_n} K_n^{\mathrm{top}}(M)$, where the set $\{j_1, \dots, j_{n-1}, s\}$ coincides with the set $\{1, \dots, n\}$ and $t_n, \dots, N_{M/F} t_{s,M}, \dots, t_1$ are local parameters of F. Therefore, by the induction assumption,

$$\begin{aligned} N_{L/M} \; & VK_n^{\mathrm{top}}(L) \cap U_R K_n^{\mathrm{top}}(M) + U_{R+1} K_n^{\mathrm{top}}(M) \\ & = \{1 + f_1(\mathcal{O}_F) t_n^{r_n} \dots t_{s,M}^{r_s} \dots t_1^{r_1}\} K_{n-1}^{\mathrm{top}}(M) + U_{R+1} K_n^{\mathrm{top}}(M) \end{aligned}$$

where $R = (r_1, \dots, r_n)$, and $\theta_0 \in f_1(\mathcal{O}_F), \bar{f}_1$ is k-decomposable. Thus, $\overline{\theta_0} \in \overline{f_1}(k)$ and the polynomial $\overline{f_2 \circ f_1}$ is nonzero k-decomposable. The second property for $N_{L/F} VK_n^{\mathrm{top}}(L)$ can be verified now similarly to the proof of Proposition 15 of [31]. □

5.3. Proposition. *Let L/F be an abelian totally ramified extension, $L \subset F_p$. Let $\mathcal{N}$ be a normic subgroup in $VK_n^{\mathrm{top}}(F)$. Then $N_{L/F}^{-1}(\mathcal{N})$ is a normic subgroup in $VK_n^{\mathrm{top}}(L)$.*

Proof. It suffices to verify the assertion for a cyclic totally ramified extension L/F of degree p. Then the first and second properties of $N_{L/F}^{-1}(\mathcal{N})$ can be established similarly with the proof of Lemma 5 in [31] employing Lemma (2.6) of [7]. The third property of $N_{L/F}^{-1}(\mathcal{N})$ follows immediately from Proposition (3.2) and Proposition (2.5), (2) of [7]. □

5.4. For a prime element π in $K_n^{\mathrm{top}}(F)$ let $\mathcal{E}_\pi$ denote the set of totally ramified abelian extensions $L/F, L \subset F_p$, such that $\pi \in N_{L/F} K_n^{\mathrm{top}}(L)$. Then, for $L_1/F, L_2/F \in \mathcal{E}_\pi$, one has $L_1 \cap L_2/F \in \mathcal{E}_\pi, L_1 L_2/F \in \mathcal{E}_\pi$. Indeed, let $M = L_1 \cap L_2$ and $N_{L_1/F}\pi_1 = N_{L_2/F}\pi_2 = \pi$. Then $N_{M/F}\varepsilon = 0$ for $\varepsilon = N_{L_1/M}\pi_1 - N_{L_2/M}\pi_2$. Now it follows from the first commutative diagram of Proposition (1.4) that $\varepsilon \in N_{L/M} VK_n^{\mathrm{top}}(L)$. Therefore, there is a prime element π_M in $K_n^{\mathrm{top}}(M)$ such that

$$N_{M/F}\pi_M = \pi, \quad \pi_M \in N_{L_1/M} K_n^{\mathrm{top}}(L_1) \cap N_{L_2/M} K_n^{\mathrm{top}}(L_2) \,.$$

Thus, it suffices to consider the case when $L_1 \cap L_2 = F$ and $L_1/F, L_2/F$ are cyclic extensions of degree p. Assume that L_1L_2/F is not totally ramified. Then there is an unramified cyclic extension E/F of degree $p, E \subset L_1L_2$. Let σ_1 and σ_2 be elements of $\mathrm{Gal}(L_1L_2/F)$ such that $\sigma_1|_{L_2}$, and $\sigma_2|_{L_1}$ are trivial, and $\sigma_1|_{L_1}$ and $\sigma_2|_{L_2}$ are generators of $\mathrm{Gal}(L_1/F)$ and $\mathrm{Gal}(L_2/F)$ respectively. One may assume that E is the fixed field of $\sigma_3 = \sigma_1\sigma_2$. Let $\pi = N_{L_1/F}\pi_1 = N_{L_2/F}\pi_2$ for some $\pi_1 \in K_n^{\mathrm{top}}(L_1), \pi_2 \in K_n^{\mathrm{top}}(L_2)$. Then $N_{L_1L_2/E}(i_{L_1/L_1L_2}\pi_1 - i_{L_2/L_1L_2}\pi_2) = 0$. Applying Theorem (3.3) we obtain that $i_{L_1/L_1L_2}\pi_1 - i_{L_2/L_1L_2}\pi_2 = \sigma_3(\gamma) - \gamma$ for some $\gamma \in K_n^{\mathrm{top}}(L_1L_2)$. Put $\beta = i_{L_1/L_1L_2}\pi_1 + \gamma - \sigma_1(\gamma)$. Then $\sigma_3(\beta) = \beta$ and $i_{F/L_1L_2}\pi = (1 + \sigma_1 + \ldots + \sigma_1^{p-1})\beta$. Therefore, $v_{L_1L_2}(\beta) = 1$. If L_1L_2/E is a $\wp$-extension, then according to Proposition (3.6) we obtain a contradiction. Otherwise, $L_1 = F(\sqrt[p]{t_s}), L_2 = F(\sqrt[p]{t_s\omega})$ for some local parameter t_s of F and $\omega = \omega_0$ as in Proposition (2.5). Then one can deduce equalities $N_{L_1/F}VK_n^{\mathrm{top}}(L_1) = N_{L_2/F}VK_n^{\mathrm{top}}(L_2)$ and $N_{L_1/F}VK_n^{\mathrm{top}}(L_1) + N_{L_2/F}VK_n^{\mathrm{top}}(L_2) = VK_n^{\mathrm{top}}(F)$, a contradiction. Thus, L_1L_2/F is totally ramified.

Let a prime element π' in $VK_n^{\mathrm{top}}(F)$ belong to $N_{L_1L_2/F}K_n^{\mathrm{top}}(L_1L_2)$. Then π' belongs to $N_{L_1/F}K_n^{\mathrm{top}}(L_1) \cap N_{L_2/F}K_n^{\mathrm{top}}(L_2)$, hence by Corollary (1.8) $\varepsilon = \pi' - \pi$ lies in the group $N_{L_1L_2/F}VK_n^{\mathrm{top}}(L_1L_2)$. This means that $L_1L_2/F \in \mathscr{E}_\pi$.

5.5. The following assertion can be verified similarly to the proof of Proposition (3.4) of [7].

Proposition. *Let π be a prime element in $K_n^{\mathrm{top}}(F)$. Let $\mathcal{N}$ be a normic subgroup in $VK_n^{\mathrm{top}}(F)$. Then there is precisely one abelian totally ramified p-extension L/F such that $\mathcal{N} = N_{L/F}VK_n^{\mathrm{top}}(L)$ and $\pi \in N_{L/F}K_n^{\mathrm{top}}(L)$.*

As a corollary, we obtain

Existence Theorem.. *There is an order reversing bijection between the lattice of normic subgroups in $VK_n^{\mathrm{top}}(F)$ with respect to the intersection and sum and the lattice of extensions $L/F \in \mathscr{E}_\pi$ with respect to the intersection and composition:*

$$\mathcal{N}_L = N_{L/F}VK_n^{\mathrm{top}}(L) \leftrightarrow L\,.$$

Finally, in the same way as in (3.4) of [7] one can show that for the compositum F_π of all fields L with $L/F \in \mathscr{E}_\pi$

$$F_\pi \cap \tilde{F} = F \quad \text{and} \quad F_\pi\tilde{F} = F_p^{\mathrm{ab}}\,.$$

Problem. To find an explicit description of the extension F_π/F as in the classical case Lubin-Tate formal groups provide.

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