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A general version of the fundamental theorem of asset pricing

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1 Introduction

A basic result in mathematical finance, sometimes called the *fundamental theorem of asset pricing* (see Dybvig 1987 e.g.), is that for a stochastic process $(S_t)_{t\in\mathbf{R}_+}$, the existence of an equivalent martingale measure is *essentially* equivalent to the absence of arbitrage opportunities. In finance the process $(S_t)_{t\in\mathbf{R}_+}$ describes the random evolution of the discounted price of one or several financial assets. The equivalence of no-arbitrage with the existence of an equivalent probability martingale measure is at the basis of the entire theory of 'pricing by arbitrage'. Starting from the economically meaningful assumption that S does not allow arbitrage profits (different variants of this concept will be defined below), the theorem allows the probability $\mathbf{P}$ on the underlying probability space $(\Omega, \mathscr{F}, \mathbf{P})$ to be replaced by an equivalent measure $\mathbf{Q}$ such that the process S becomes a martingale under the new measure. This makes it possible to use the rich machinery of martingale theory. In particular the problem of fair pricing of contingent claims is reduced to taking expected values with respect to the measure $\mathbf{Q}$. This method of pricing contingent claims is known to actuaries since the introduction of actuarial skills, centuries ago and known by the name of 'equivalence principle'.

The theory of martingale representation allows to characterise those assets that can be reproduced by buying and selling the basic assets. One might get the impression that martingale theory and the general theory of stochastic processes were tailor made for finance. (see Harrison and Pliska (1981)).

The change of measure from $\mathbf{P}$ to $\mathbf{Q}$ can also be seen as a result of risk aversion. By changing the physical probability measure from $\mathbf{P}$ to $\mathbf{Q}$, one can attribute more weight to unfavourable events and less weight to more favourable ones.

As an example that this technique has in fact a long history, we quote the use of mortality tables in insurance. The actual mortality table is replaced by a table reflecting more mortality if a life insurance premium is calculated but is replaced by a table reflecting a lower mortality rate if e.g. a lump sum buying a pension is calculated. Changing probabilities is common practice in actuarial sciences. It is therefore amazing to notice that today's actuaries are introducing these modern financial methods at such a slow pace.

The present paper focuses on the question: 'What is the precise meaning of the word *essentially* in the first paragraph of the paper?' The question has a twofold interest. From an economic point of view one wants to understand the precise relation between concepts of no-arbitrage type and the existence of an equivalent martingale measure in order to understand the exact limitations up to which the above sketched approach may be extended. From a purely mathematical point of view it is also of natural interest to get a better understanding of the question which stochastic processes are martingales after an appropriate change to an equivalent probability measure. We refer to the well known fact that a semi-martingale becomes a quasi-martingale under a well chosen equivalent law (see e.g. Protter (1990)); from here to the question whether we can obtain a martingale, or more generally a local martingale, is natural.

We believe that the main theorem (Theorem 1.1 below) of this paper contributes to both theories mathematics as well as economics. In economic terms the theorem contains essentially two messages. First that it is possible to characterise the existence of an equivalent martingale measure for a general class of processes in terms of the concept of no free lunch with vanishing risk, a concept to be defined below. In this notion the aspect of vanishing risk bears economic relevance. The second message is that – in a general setting – there is no way to avoid general stochastic integration theory. If the model builder accepts the possibility that the price process has jumps at all possible times, he needs a sophisticated integration theory, going beyond the theory for "simple integrands". In particular the integral of unbounded predictable processes of general nature has to be used. From a purely mathematical point of view we remark that the proof of the main Theorem 1.1 below, turns out to be surprisingly hard and requires heavy machinery from the theory of stochastic processes, from functional analysis and also requires some very technical estimates.

The process S, sometimes denoted $(S_t)_{t \in \mathbf{R}_+}$ is supposed to be $\mathbf{R}$-valued, although all proofs work with a d-dimensional process as well. We however prefer to avoid vector notation in d dimensions. If the reader is willing to accept the 1-dimensional notation for the d-dimensional case as well, nothing has to be changed. The theory of d-dimensional stochastic integration is a little more subtle than the one dimensional theory but no difficulties arise.

The general idea underlying the concept of no-arbitrage and its weakenings, stated in several variants of "no free lunch" conditions, is that there should be no trading strategy H for the process S, such that the final payoff described by the stochastic integral $(H \cdot S)_\infty$, is a non negative function, strictly positive with positive probability. The economic interpretation is that by betting on the process S and without bearing any risk, it should not be possible to make something out of nothing. If one wants to make this intuitive idea precise, several problems arise. First of all one has to restrict the choice of the integrands H to make sure that $(H \cdot S)_\infty$ exists. Besides the qualitative restrictions coming from the theory of stochastic integration, one has to avoid problems coming from so-called doubling strategies. This was already noted in the paper by Harrison and Pliska (1979). To explain this remark let us consider the classical doubling strategy. We draw a coin and when heads comes out the player is paid 2 times his bet. If tails comes up, the player loses his bet. The strategy is well known: the player doubles his bet until the first time he wins. If he starts with 1 ECU, his final gain (= last pay out – total sum of the preceding bets) is almost surely 1 ECU. He has an almost sure win. The probability that heads will eventually show up is indeed one, even if the coin is not fair. However, his accumulated losses are not bounded below. Everybody, especially the casino boss, knows that this is a very risky way of winning 1 ECU. This type of strategy has to be ruled out: there should be a lower bound on the player's loss. The described doubling strategy is known for centuries and in French it is still referred to as 'la martingale'.

One possible way to avoid these difficulties is to restrict oneself to simple predictable integrands. These are defined as linear combinations of buy and hold strategies. Mathematically such a buy and hold strategy is described as an integrand of the form $H = f\,\mathbf{1}_{]\!]T_1, T_2]\!]}$, where $T_1 \leqq T_2$ are finite stopping times and f is $\mathscr{F}_{T_1}$ measurable. The advantage of using such integrands is that they have a clear interpretation: when time $T_1(\omega)$ comes up, buy $f(\omega)$ units of the financial asset, keep them until time $T_2(\omega)$ and sell. A linear combination of such integrands is called a simple integrand. An elementary integrand is a linear combination of buy and hold strategies with stopping times that are deterministic. This terminology agrees with standard terminology of stochastic integration. (see Protter (1990), Dellacherie and Meyer (1980) and Chou et al. (1980)). Even if the process S is not a semi-martingale the stochastic integral $(H \cdot S)$ for $H = f\,\mathbf{1}_{]\!]T_1, T_2]\!]}$ can be defined as the process $(H \cdot S)_t = f.(S_{\min(t, T_2)} - S_{\min(t, T_1)})$. Also the definition of the limit $(H \cdot S)_\infty = \lim_{t \to \infty}(H \cdot S)_t = f.(S_{T_2} - S_{T_1})$ poses no problem. The net profit of the strategy is precisely $(H \cdot S)_\infty$. The use of stopping times is interpreted as the use of signals coming from available, observable information. This explains why in financial theories the filtration and the derived concepts such as predictable processes, are important. It is clear that the use of simple integrands rules out the introduction of doubling strategies. This led Harrison and Kreps (1979), Kreps (1981) and Harrison and Pliska (1981) to define no arbitrage and no free lunch in terms of simple integrands and to obtain theorems relating these notions to the existence of an equivalent martingale measure. In various directions these results were extended in Duffie and Huang (1986), Stricker (1990), Dalang et al. (1989), Ansel and Stricker (1993), Mc Beth (1992), Lakner (1992), Delbaen (1992), Schachermayer (1993), Kusuoka (1993).

To relate our work to earlier results, let us summarise the present state of the art. The case when the time set is finite is completely settled in Dalang et al. (1989) and the use of simple or even elementary integrands is no restriction at all (see Schachermayer (1992), Kabanov and Kramkov (1993) and Rogers (1993) for elementary proofs). For the case of discrete but infinite time sets, the problem is solved in Schachermayer (1993). The case of continuous and bounded processes in continuous time, is solved in Delbaen (1992). In these two cases the theorems are stated in terms of simple integrands and limits of sequences and by using the concept of no free lunch with bounded risk. We shall review these issues in Sect. 6.

In the general case, i.e. a time set of the form $[0, \infty[$ or $[0, 1]$ and with a possibility of random jumps, the situation is much more delicate. The existence of an equivalent martingale measure can be characterised in terms of 'no free lunch' involving the convergence of nets or generalised sequences, see e.g. Kreps (1981), Lakner (1992). Kusuoka (1993) used convergence in Orlicz spaces and Duffie and Huang (1986) and Stricker (1990) used L^p convergence for $1 \leqq p < \infty$. In the latter case the restrictions posed on S were such that the new measure has a density in $L^q\left(q = \frac{p}{p-1}\right)$. Contrary to the case of continuous processes or to the case of discrete time sets, no general solution was known in terms of "no free lunch" involving convergent sequences. Hence there remained the natural question whether for a general adapted process S, the existence of an equivalent martingale measure could be characterised in such terms.

The answer turns out to be no if one only uses simple integrands. In Sect. 7, we give an example of a process $S = M + A$ where M is a uniformly bounded martingale, A is a predictable process of finite variation, S admits no equivalent martingale measure but there is "no free lunch with bounded risk" if one only uses simple integrands. A closer look at the example shows that if one allows strategies of the form: 'sell before each rational number and buy back after it', then there is even a "free lunch with vanishing risk". Of course such a trading strategy is difficult to realise in practice but if we allow discontinuities for the price process at arbitrary times, then we should also allow strategies involving the same kind of pathology. The example shows that we should go beyond the simple integration theory to cover these cases as well. To back this assertion let us recall that the basis of the whole theory of asset pricing by arbitrage is, of course, the celebrated Black-Scholes formula (see Black and Scholes (1973) and Merton (1973)), widely used today by practitioners in option trading. Also in this case the trading strategy H, which perfectly replicates the payoff of the given option, is not a simple integrand. It is described as a smooth function of time and the underlying stock price. Being a smooth function of the stock price, its trajectories are in fact of unbounded variation. One can argue that in practice already this strategy is difficult to realise. In this case however one shows that the integrand can be approximated by simple integrands in a reasonable way; for details we refer the reader to books on stochastic integration theory with special emphasis on Brownian motion, e.g. Karatzas and Shreve (1988). In the case of the example of Sect. 7, this reduction is not possible and as already advocated, general integrands are really needed.

Summing up we are forced to leave the framework of simple integrands. However, we immediately face new problems. First the process S should be restricted in order to allow the definition of integrals $H \cdot S$ for more general trading strategies. S has to be a semi-martingale to realise this. This is precisely the content of the Bichteler-Dellacherie theorem (see Protter (1990)). It turns out that this is not really a restriction. From the work of Föllmer and Schweizer (1991) and Ansel and Stricker (1993), we know that no free lunch conditions stated with simple integrands, imply that a cadlag adapted process is a special semi-martingale. (A process is called cadlag, càdlàg would be better, if almost every trajectory admits left limits and is right continuous). We refer to Sect. 7 of this paper for a general version of this result, adapted to our framework. The second difficulty arises from the fact that doubling-like strategies have to be excluded. This may be done by using the concept of *admissible integrands* H, requiring that the process $H \cdot S$ is uniformly bounded from below, a concept going back to Harrison and Pliska (1981) and developed in Delbaen (1992), Mc Beth (1992) and Schachermayer (1993). The concept of admissible integrand is a mathematical formulation of the requirement that an economic agent's position cannot become too negative, a practice sometimes referred to as 'your friendly broker calls for extra margin'. The third problem is to make sure that $(H \cdot S)_\infty = \lim_{t \to \infty} (H \cdot S)_t$ has a meaning. We shall see that this problem has a very satisfactory solution if one restricts to admissible integrands.

The condition of **no free lunch with vanishing risk** (NFLVR) can now be described as follows. There should be no sequence of final payoffs of admissible integrands, $f_n = (H^n \cdot S)_\infty$ such that the negative parts f_n^- tend to 0 *uniformly* and such that f_n tends almost surely to a $[0, \infty]$-valued function f_0 satisfying $\mathbf{P}[f_0 > 0] > 0$. We will give a detailed discussion of this property below in Sect. 3. For the time being let us remark that the property (NFLVR) is different from the previously considered concept of no free lunch with bounded risk in the sense that we require that the risk taken, the lower bounds on the processes $(H^n \cdot S)$, tend to zero uniformly. In the property (NFLBR) one only requires that this risk is uniformly bounded below and that the variables f_n^- tend to zero in probability. The main theorem of the paper can now be stated as:

Theorem 1.1 *Let S be a bounded real valued semi-martingale. There is an equivalent martingale measure for S if and only if S satisfies* (NFLVR).

One implication in the above theorem is almost trivial: if there is an equivalent martingale measure for S then it is easy to see that S satisfies (NFLVR), see the first part of the proof in the beginning of Sect. 4. The interesting aspect of Theorem 1.1 lies in the reverse implication: the (economically meaningful) assumption (NFLVR) guarantees the existence of an equivalent martingale measure for S and thus opens the way to the wide range of applications from martingale theory.

If the process S is only a locally bounded semi-martingale we still obtain the following partial result:

Corollary 1.2 *Let S be a locally bounded real valued semi-martingale. There is an equivalent local martingale measure for S if and only if S satisfies* (NFLVR).

In Delbaen and Schachermayer (1993) counterexamples are given which show that in the above corollary one can only assert the existence of a measure $\mathbf{Q}$ under which S is a *local* martingale. Even if the variables S_t are uniformly bounded in L^p for some $p > 1$, this does not imply that S is a martingale. On the other hand we do not know whether the hypothesis of local boundedness is essential for the corollary to hold. There is some hope that the condition is superfluous but at present this remains an open question. In the discrete time case the local boundedness assumption is not needed as shown in Schachermayer (1993).

The proof of Theorem 1.1 is quite technical and will be the subject of Sect. 4. The rest of the paper is organised as follows. Section 2 deals with definitions, notation and results of general nature. In Sect. 3 we examine the property (NFLVR) and we prove that under this condition, the limit $(H \cdot S)_\infty = \lim_{t\to\infty}(H \cdot S)_t$ exists almost surely for admissible integrands. The fifth section is devoted to the study of the set of local martingale measures. Here we give a new characterisation of a complete market. It turns out that if each local martingale measure that is absolutely continuous with respect to the original measure, is already equivalent to the original measure, then the market is complete and there is only one equivalent (absolutely continuous) local martingale measure. These results are related to results from Ansel and Stricker (1992) and Jacka (1992). We also show that the framework of admissible integrands allows to formulate a general duality theorem (Theorem 5.7). In Sect. 6 we investigate the relation between the no free lunch with vanishing risk (NFLVR) property and the no free lunch with bounded risk (NFLBR) property. In the case of an infinite horizon the latter property permits to restrict to strategies that are of bounded support. They have a more intuitive interpretation since they only require 'planning' up to a bounded time. In Sect. 7 we introduce the no free lunch properties (NFLVR), (NFLBR) and (NFL) stated in terms of simple strategies. It is shown that in the case of continuous price processes one can avoid the use of general integrands and restrict oneself to simple integrands. The result generalises the main theorem of Delbaen (1992) in the case of a finite dimensional price process. The relation between the no free lunch with vanishing risk property for simple integrands and the semi-martingale property is also investigated in Sect. 7. We also give examples that show that the use of simple integrands is not enough to obtain a general theorem and relate the present results to previous ones, in particular to Delbaen (1992) and Schachermayer (1993). Appendix 1 contains some technical lemmas already used in Schachermayer (1993). We state versions which are more general and provide somewhat easier proofs.

2 Definitions and preliminary results

Throughout the paper we will work with random variables and stochastic processes which are defined on a fixed probability space $(\Omega, \mathscr{F}, \mathbf{P})$. We will without further notice identify variables that are equal almost everywhere. The space $L^0(\Omega, \mathscr{F}, \mathbf{P})$, sometimes written as L^0, is the space of equivalence classes of measurable functions, defined up to equality almost everywhere. The space L^0 is equipped with the topology of convergence in measure. It is a complete metrisable topological vector space, a Fréchet space, but it is not locally convex. The space $L^1(\Omega, \mathscr{F}, \mathbf{P})$ is the Banach space of all integrable $\mathscr{F}$-measurable functions. The dual space is identified with $L^\infty(\Omega, \mathscr{F}, \mathbf{P})$ the space of bounded measurable functions. The weak*-topology on L^∞ is the topology $\sigma(L^\infty, L^1)$.

The existence of an equivalent martingale measure is proved using Hahn-Banach type theorems. Central in this approach is the construction of a convex weak* closed subset of L^∞. To prove that a set is weak* closed we will use the following result. The proof essentially consists of a combination of the classical Krein-Smulian theorem and the fact that the unit ball of L^∞ under the weak* topology is an Eberlein compact. (see Diestel (1975) or Grothendieck (1954), p. 321, Exercise 1).

Theorem 2.1 *If C is a convex cone of L^∞ then C is weak* closed if and only if for each sequence $(f_n)_{n \geqq 1}$ in C that is uniformly bounded by 1 and converges in probability to a function f_0, we have that $f_0 \in C$.*

The properties of stochastic processes are always defined relative to a fixed filtration $(\mathscr{F}_t)_{t \in \mathbf{R}_+}$. This filtration is supposed to satisfy the usual conditions i.e. the filtration is right continuous and contains all negligible sets: if $B \subset A \in \mathscr{F}$ and $\mathbf{P}[A] = 0$ then $B \in \mathscr{F}_0$. We also suppose that the sigma algebra $\mathscr{F}$ is generated by $\bigcup_{t \geqq 0} \mathscr{F}_t$. Stochastic intervals are denoted as $[\![T, S]\!]$ where $S \leqq T$ are stopping times and $[\![T, S]\!] = \{(t, \omega) | t \in \mathbf{R}_+, \omega \in \Omega, T(\omega) \leqq t \leqq S(\omega)\}$. Stochastic intervals of the form $]\!]T, S]\!]$ etc. are defined in the same way. The interval $[\![T, T]\!]$ is denoted by $[\![T]\!]$ and it is the graph of the stopping time T, $\{(T(\omega), \omega) | T(\omega) < \infty\}$. We note that according to this definition the set $[\![0, \infty]\!]$ equals $\mathbf{R}_+ \times \Omega$. Stochastic processes are indexed by a time set. In this paper the time set will be $\mathbf{R}_+$. This will cover the case of infinite horizon and indeed represents the general case since bounded time sets $[0, t]$ can of course be imbedded by requiring the processes to be constant after time t. It also contains the case of discrete time sets, by requiring the processes and the filtration to be constant between two consecutive natural numbers. A mapping $X : \mathbf{R}_+ \times \Omega \to \mathbf{R}$ is called an adapted stochastic process if for each $t \in \mathbf{R}_+$ the mapping $\omega \to X(t, \omega) = X_t(\omega)$ is $\mathscr{F}_t$ measurable. X is called continuous (right continuous, left continuous), if for almost all $\omega \in \Omega$, the mapping $t \to X_t(\omega)$ is continuous (right continuous, left continuous). Stochastic processes that are indistinguishable are always identified. Other concepts such as optional and predictable processes are also used in this paper and we refer the reader to Protter (1990) for the details. The predictable

σ-algebra $\mathscr{P}$ on $\mathbf{R}_+ \times \Omega$ is the σ-algebra generated by the stochastic intervals $[\![0, T]\!]$, where T runs through all the stopping times. A predictable process H is a process that is measurable for the σ-algebra $\mathscr{P}$. For the theory of stochastic integration we refer to Protter (1990) and to Chou et al. (1980). If X is a real valued stochastic process the variable X^* is defined as $X^* = \sup_{t \geqq 0} |X_t(\omega)|$. This variable is measurable if X is right or left continuous. Sometimes we will use X_t^* which is defined as $\sup_{t \geqq u \geqq 0} |X_u(\omega)|$. X^* is called the maximum function and it plays a central role in martingale theory. If X is a cadlag process, i.e. a right continuous process possessing left limits for each $t > 0$, then ΔX denotes the process that describes the jumps of X. More precisely $(\Delta X)_t = X_t - X_{t-}$ and $(\Delta X)_0 = X_0$.

If X is a semi-martingale then X defines a continuous operator on the space of bounded predictable processes of bounded support into the space L^0. The space of semi-martingales can therefore be considered as a space of linear operators. The semi-martingale topology is precisely induced by the topology of linear operators. It is therefore metrisable by a translation invariant metric given by the distance of X to the zero semi-martingale:

$$\mathbf{D}(X) = \sup \left\{ \sum_{n \geqq 1} 2^{-n} \mathbf{E}[\min(|(H \cdot X)_n|, 1)] \mid H \text{ predictable } |H| \leqq 1 \right\}.$$

For this metric, the space of semi-martingales is complete, see Émery (1979).

A semi-martingale X is called special if it can be decomposed as $X = M + A$ where M is a local martingale and A is a *predictable* process of finite variation. In this case such a decomposition is unique and it is called the canonical decomposition. It is well known (see Chou et al. (1980)) that a semi-martingale is special if and only if X is locally integrable, i.e. there is an increasing sequence of stopping times T_n, tending to ∞ such that $X_{T_n}^*$ is integrable. The following theorem on special semi-martingales will be used on several occasions, for a proof we refer to Chou et al. (1980).

Theorem 2.2 *If X is a special semi-martingale with canonical decomposition $X = M + A$ and if H is X-integrable then the semi-martingale $H \cdot X$ is special if and only if*

(1) *H is M-integrable in the sense of stochastic integrals of local martingales and*

(2) *H is A-integrable in the usual sense of Stieltjes-Lebesgue integrals.*

In this case the canonical decomposition of $H \cdot X$ is given by $H \cdot X = H \cdot M + H \cdot A$.

The following theorem seems to be folklore. Essentially it may be deduced from (the proof of) an inequality of Stein (Stein (1970)), see also Lépingle (1978) and Yor (1978b). For a survey of these results and related inequalities see Delbaen and Schachermayer (1994). For convenience of the reader we include the easy proof, suggested by Stricker, of Theorem 2.3.

The theorem and more precisely its Corollary 2.4, will be used in Sect. 4. It allows to control the jumps of the martingale part in the canonical decomposition of a special semi-martingale.

Theorem 2.3 *If X is a semi-martingale satisfying $\|(\Delta X)^*\|_p < \infty$, where $1 < p \leqq \infty$, then*

(a) *X is special and has a canonical decomposition $X = M + A$*

(b) *A satisfies $\|(\Delta A)^*\|_p \leqq \dfrac{p}{p-1}\|(\Delta X)^*\|_p$;*

(c) *M satisfies $\|(\Delta M)^*\|_p \leqq \dfrac{2p-1}{p-1}\|(\Delta X)^*\|_p$.*

Proof. Since X is locally p-integrable it is certainly locally integrable and hence is special. (a) is therefore proved. Let $X = M + A$ be the canonical decomposition where A is the predictable process of finite variation and M is the local martingale part. Let Y be the cadlag martingale defined as $Y_t = E[(\Delta X)^* | \mathscr{F}_t]$.

Since A is predictable the set $\{\Delta A \neq 0\}$ is the union of a sequence of sets of the form $[\![T_n]\!]$ where T_n are predictable stopping times. For each predictable stopping time T we have that $\Delta A_T = E[\Delta X_T | \mathscr{F}_{T-}]$ and hence

$$|\Delta A_T| \leqq E[|\Delta X_T| | \mathscr{F}_{T-}] \leqq E[(\Delta X)^* | \mathscr{F}_{T-}] = Y_{T-} \leqq Y^* .$$

This implies that $(\Delta A)^* \leqq Y^*$. From Doob's maximal inequality, see Dellacherie and Meyer (1980), it now follows that

$$\|Y^*\|_p \leqq \frac{p}{p-1}\|(\Delta X)^*\|_p \quad \text{and therefore}$$

$$\|(\Delta A)^*\|_p \leqq \frac{p}{p-1}\|(\Delta X)^*\|_p \quad \text{and} \quad \|(\Delta M)^*\|_p \leqq \frac{2p-1}{p-1}\|(\Delta X)^*\|_p . \quad \square$$

Corollary 2.4 If *T is a stopping time then*:

$$\|(\Delta A)_T\|_p \leqq \frac{p}{p-1}\|(\Delta X)^*\|_p ;$$

$$\|(\Delta M)_T\|_p \leqq \frac{2p-1}{p-1}\|(\Delta X)^*\|_p .$$

Corollary 2.5 *If $1 < p \leqq \infty$ and the semi-martingale X satisfies $\sup\{\|(\Delta X)_T\|_p | T$ stopping time$\} = N < \infty$, then for $p' < p$ there is constant $k(p, p')$ depending only on p and p' such that*

$$\|(\Delta A)^*\|_{p'} \leqq k(p, p')N .$$

Proof. Let the stopping time T be defined as $T = \inf\{t | |(\Delta X)_t| \geqq c\}$. From the hypothesis we deduce that $\int |\Delta X_T| \leqq N^p$ and this implies, by the Markov-Tchebychev inequality, that $c^p \mathbf{P}[(\Delta X)^* > c] \leqq N^p$. The rest follows easily. $\square$

Remark 2.6 In Corollary 2.4 we cannot replace $(\Delta X)^*$ by $(\Delta X)_T$. The following example illustrates this. We construct a bounded semi-martingale X such that for each $\varepsilon > 0$ there is a stopping time T with $|\Delta A_T| = 1$ and $|\Delta X_T| \leqq \varepsilon$. This clearly shows that there is no constant K such that $\|(\Delta A)_T\|_p \leqq K\|(\Delta X)_T\|_p$. The construction is as follows: For $0 \leqq t < 1$ put $X_t = 0$. We now proceed by recursion. For n a natural number we suppose the process X is already constructed for $t < n$. The filtration $\mathscr{F}_s$ is defined as $\mathscr{F}_s = \sigma(X_u;\, u \leqq s)$ and $\mathscr{F}_{s-} = \sigma(X_u;\, u < s)$. At $t = n$ we put a jump $(\Delta X)_n$ such that $|(\Delta X)_n|$ is uniformly distributed over the interval $[0, 2]$ and is independent of the past $\mathscr{F}_{n-}$ of the process. This means that $|(\Delta X)_n|$ is independent of the variables $(\Delta X)_1, \ldots, (\Delta X)_{n-1}$. If $(X)_{n-} \geqq 0$ then $(\Delta X)_n$ is uniformly distributed over the interval $[-2, 0]$, otherwise if $(X)_{n-} < 0$ then $(\Delta X)_n$ is uniformly distributed over $[0, 2]$. For $n \leqq t < n+1$ we put $X_t = X_n$. The filtration $\mathscr{F}_s$ is clearly right continuous and if we augment it with the null sets we obtain that the natural filtration of X satisfies the usual conditions. For $\varepsilon > 0$ we now define $T = \inf\{t \mid |(\Delta X)_t| \leqq \varepsilon\}$. Clearly $T < \infty$ almost surely and satisfies the desired properties.

If A is a predictable process of finite variation with $A_0 = 0$, we can associate with it a (random) measure on $\mathbf{R}_+$. The variation of A, a process denoted by V, is given by

$$V_t = \sup\left\{ \sum_{k=1}^{n} |A_{s_k} - A_{s_{k-1}}| \,\middle|\, 0 = s_0 < s_1 \ldots < s_n = t \right\}.$$

The process V is predictable and it also defines a (random) measure on $\mathbf{R}_+$. The process V defines a σ-finite measure μ_V on the predictable σ-algebra on $\mathbf{R}_+ \times \Omega$. The definition of μ_V is, for K a predictable subset of $\mathbf{R}_+ \times \Omega$:

$$\mu_V(K) = E\left[\int_0^\infty K_u \, dV_u \right].$$

The measure μ_A is defined in a similar way, but its definition is restricted to a σ-ring to avoid expressions like $\infty - \infty$. It is well known, see Meyer (1976, chap. I), that the measure μ_V is precisely the variation measure of μ_A. From the Hahn decomposition theorem we deduce that there is a partition of $\mathbf{R}_+ \times \Omega$, in two sets, B_+ and B_-, both predictable, such that $(\mathbf{1}_{B_+} \cdot A)$ and $(-\mathbf{1}_{B_-} \cdot A)$ are increasing. Moreover $V = ((\mathbf{1}_{B_+} - \mathbf{1}_{B_-}) \cdot A)$. For almost all ω the measure dA on $\mathbf{R}_+$ is absolutely continuous with respect to dV and the Radon-Nikodym derivative is precisely $1_{F_+} - 1_{F_-}$ where $F_\pm = \{t \mid (t, \omega) \in B_\pm\}$. We will refer to this decomposition as the **Hahn decomposition of A**. Note that the difficulty in the definition of the pathwise decomposition of the measures $dA(\omega)$ comes from the fact that the sets F_+ and F_- have to be glued together in order to form the predictable sets B_+ and B_-. See Meyer (1976, chap. I), for the details of this result which is due to Cathérine Doléans-Dade.

Throughout the paper, with the exception of Sect. 7, S will be a fixed semi-martingale. As mentioned in the introduction S represents the discounted price of a financial asset.

Definition 2.7 *Let a be a positive real number. An S-integrable predictable process H is called* **a-admissible** *if $H_0 = 0$ and $(H \cdot S) \geqq -a$ (i.e. for all $t \geqq 0$: $(H \cdot S)_t \geqq -a$ almost everywhere). H is called admissible if it is admissible for some $a \in \mathbf{R}_+$.*

Given the semi-martingale S we denote, in a similar way as in Stricker (1990), by K_0 the convex cone in L^0, formed by the functions

$$K_0 = \left\{ (H\cdot S)_\infty \,|\, H \text{ admissible and } (H\cdot S)_\infty = \lim_{t\to\infty} (H\cdot S)_t \text{ exists a.s.} \right\}.$$

By C_0 we denote the cone of functions dominated by elements of K_0 i.e. $C_0 = K_0 - L^0_+$. With C and K we denote the corresponding intersections with the space L^∞ of bounded functions $K = K_0 \cap L^\infty$ and $C = C_0 \cap L^\infty$. By $\bar{C}$ we denote the closure of C with respect to the norm topology of L^∞ and by $\bar{C}^*$ we denote the weak* closure of C.

Definition 2.8 *We say that the semi-martingale S satisfies the condition of*
(i) **no arbitrage** (NA) *if* $C \cap L^\infty_+ = \{0\}$
(ii) **no free lunch with vanishing risk** (NFLVR) *if* $\bar{C} \cap L^\infty_+ = \{0\}$.

It is clear that (ii) implies (i). The no-arbitrage property (NA) is equivalent to $K_0 \cap L^0_+ = \{0\}$ and has an obvious interpretation: there should be no possibility of obtaining a positive profit by trading alone (according to an admissible strategy): it is impossible to make something out of nothing without risk. It is well known that in general the notion (NA) is too restrictive to imply the existence of an equivalent martingale measure for S, see Sect. 7. Compare also to the results in Dalang et al. (1989) and Schachermayer (1993, Remark 4.11).

The notion (NFLVR) is a slight generalisation of (NA). If (NFLVR) is not satisfied then there is a f_0 in L^∞_+, not identically 0, as well as a sequence $(f_n)_{n\geq 1}$ of elements in C, tending almost surely to f_0, such that for all n we have that $f_n \geqq f_0 - \frac{1}{n}$. In particular we have $f_n \geqq -\frac{1}{n}$. In economic terms this amounts to almost the same thing as (NA), as the risk of the trading strategies becomes arbitrarily small. See also Proposition 3.6 below.

We emphasise that the set C and hence the properties (NA) and (NFLVR) are defined using *general admissible predictable processes H*. This is a more general definition than the one usually taken in the literature and used by the authors in previous papers. (See e.g. Schachermayer (1993) and Delbaen (1992)). These classical concepts were defined using simple integrands or/and integrands with bounded support. In these cases we will say that S satisfies (NA) *for simple integrands*, (NFLVR) *for integrands with bounded support*, etc. These notions will reappear in Sect. 7, where we will emphasise on the differences between these notions.

We close this section by quoting a result due to Émery and Ansel and Stricker. The result states that under suitable conditions the stochastic integral of a local martingale is again a local martingale. A counterexample due to Émery (1980) shows that in general a stochastic integral of a local martingale need not be a local martingale. From Theorem 2.2 it follows that if M is a local martingale with respect to a measure $\mathbf{P}$, then $H\cdot M$ is a local martingale if and only if it is a special semi-martingale, i.e. if it is locally integrable. The next theorem gives us a criterion that is related to admissibility of H.

Theorem 2.9 *If M is a local martingale and if H is an admissible integrand for M, then $H \cdot M$ is a local martingale. Consequently $H \cdot M$ is a supermartingale.*

Proof. We refer to Émery (1980) and Corollaire 3.5 in Ansel and Stricker (1992). It is an easy consequence of Fatou's lemma that if $H \cdot M$ is a local martingale uniformly bounded from below, then it is a supermartingale. □

3 No free lunch with vanishing risk

The main result of this section states that for a semi-martingale S, under the condition of no free lunch with vanishing risk (NFLVR), the limit $(H \cdot S)_\infty = \lim_{t \to \infty} (H \cdot S)_t$ exists and is finite whenever the integrand H is admissible. To get a motivation for this result, consider the case where we already know that there is an equivalent local martingale measure $\mathbf{Q}$. In this case, by Theorem 2.9, the stochastic integral $H \cdot S$ is a $\mathbf{Q}$-local martingale if H is admissible. This implies that it is a supermartingale and the classical convergence theorem shows that the limit $(H \cdot S)_\infty = \lim_{t \to \infty}(H \cdot S)_t$ exists and is finite almost everywhere. But of course we do not know yet that there is an equivalent martingale measure $\mathbf{Q}$ and the art of the game is to derive the convergence result simply from the property (NFLVR). We start with two preparatory results.

Proposition 3.1 *If S is a semi-martingale with the property* (NFLVR), *then the set*

$$\{(H \cdot S)_\infty \mid H \text{ is 1-admissible and of bounded support}\}$$

is bounded in L^0.

Proof. H 1-admissible means that H is S-integrable and $(H \cdot S)_t \geqq -1$. Being of bounded support means that H is 0 outside $[\![0, T]\!]$ where T is a positive real number. The limit $(H \cdot S)_\infty = \lim_{t \to \infty}(H \cdot S)_t$ exists without difficulty because $(H \cdot S)_t$ becomes eventually constant. Suppose that the set $\{(H \cdot S)_\infty \mid H$ is 1-admissible and of bounded support$\}$ is not bounded in L^0. This implies the existence of a sequence H^n of 1-admissible integrands of bounded support and the existence of $\alpha > 0$ such that $\mathbf{P}[(H^n \cdot S)_\infty \geqq n] > \alpha > 0$. The sequence $f_n = \min\left(\frac{1}{n}(H^n \cdot S)_\infty, 1\right)$ is in C, $\mathbf{P}[f_n = 1] > \alpha > 0$ and $\| f_n^- \|_\infty \leqq \frac{1}{n}$. By taking convex combinations we may take $g_n \in \text{conv}(f_n, f_{n+1}, \ldots)$ that converge a.s. to $g \colon \Omega \to [0, 1]$. (we can use Lemma A1.1 but a simpler argument in L^∞ can do the job, compare to Remark 3.4 in Schachermayer 1993). Clearly $E[g] \geqq \alpha$ and therefore $\mathbf{P}[g > 0] = \beta \geqq \alpha > 0$. By Egorov's theorem $g_n \to g$ uniformly on a set Ω' of measure at least $1 - \beta/2$. The functions $h_n = \min(g_n, \mathbf{1}_{\Omega'})$ are still in the set C and $h_n \to g\, \mathbf{1}_{\Omega'}$ in the norm topology of L^∞. Since $\mathbf{P}[g \mathbf{1}_{\Omega'} > 0] \geqq \beta/2 > 0$ we obtain a contradiction to (NFLVR). □

Proposition 3.2 *If S is a semi-martingale satisfying* (NFLVR), *then for each admissible H the function $(H \cdot S)^* = \sup_{0 \leqq t} |(H \cdot S)_t|$ is finite almost everywhere and the set $\{(H \cdot S)^* \mid H$ 1-admissible$\}$ is bounded in L^0.*

Proof. If the set is not bounded, we can find a sequence of 1-admissible integrands H^n, stopping times T_n and $\alpha > 0$ such that $\mathbf{P}[T_n < \infty] > \alpha > 0$ and $(H^n \cdot S)_{T_n} > n$ on $\{T_n < \infty\}$. For each natural number n take t_n large enough so that $\alpha < \mathbf{P}[T_n \leqq t_n]$ and observe that for $K^n = H^n \, \mathbf{1}_{[\![0, \min(T_n, t_n)]\!]}$ we have that K^n is of bounded support and $\mathbf{P}[(K^n \cdot S)_\infty > n] > \alpha > 0$, a contradiction to Proposition 3.1. □

We now prove the main result of this section. It extends Proposition 4.2 from Schachermayer (1993) to the present case of a general semi-martingale S.

Theorem 3.3 *If S is a semi-martingale satisfying* (NFLVR), *then for H admissible the limit $(H \cdot S)_\infty = \lim_{t \to \infty} (H \cdot S)_t$ exists and is finite almost everywhere.*

Proof. We will mimic the proof of the martingale convergence theorem of Doob. The classical idea of considering upcrossings through an interval $[\beta, \gamma]$ may in mathematical finance be interpreted as the well known procedure: "Buy low, sell high". We may suppose that H is 1-admissible and hence $(H \cdot S)^* = \sup_{0 \leqq t} |(H \cdot S)_t| < \infty$ almost surely by Proposition 3.2. We therefore only have to show that $\liminf_{t \to \infty} (H \cdot S)_t = \limsup_{t \to \infty} (H \cdot S)_t$ a.s. Suppose this were not the case and that $\mathbf{P}[\liminf_{t \to \infty} (H \cdot S)_t < \limsup_{t \to \infty} (H \cdot S)_t] > 0$. Take $\beta < \gamma$ and $\alpha > 0$ so that $\mathbf{P}[\liminf_{t \to \infty} (H \cdot S)_t < \beta < \gamma < \limsup_{t \to \infty} (H \cdot S)_t] > \alpha$. We will construct finite stopping times $(U_n, V_n)_{n \geqq 1}$ such that

(1) $U_1 \leqq V_1 \leqq U_2 \leqq V_2 \leqq \ldots \leqq U_n \leqq V_n \leqq U_{n+1} \leqq \ldots$

(2) $L^n = \sum_{k=1}^n H \, \mathbf{1}_{]\!]U_k, V_k]\!]}$ is $(1 + \beta)$-admissible

(3) $\mathbf{P}[(L^n \cdot S)_\infty > n(\gamma - \beta)] > \alpha/2$.

The existence of such a sequence clearly violates the conclusion of Proposition 3.2 and this will prove the theorem.

The stopping times are constructed by induction. Take $(\varepsilon_n)_{n \geqq 1}$ strictly positive and such that the sum $\sum_{n \geqq 1} \varepsilon_n < \alpha/100$. Let A be the set defined as $A = \{\liminf_{t \to \infty} (H \cdot S)_t < \gamma < \beta < \limsup_{t \to \infty} (H \cdot S)_t\}$. Since the Boolean algebra $\bigcup_{0 \leqq t} \mathscr{F}_t$ is dense in the sigma-algebra $\mathscr{F}$ we have that there is t_1 and $A_1 \in \mathscr{F}_{t_1}$ such that $\mathbf{P}[A \Delta A_1] < \varepsilon_1$. For $\omega \notin A_1$ we put $U_1 = V_1 = t_1$ and we concentrate on $\omega \in A_1$.

Define first

$$U'_1 = \inf\{t \mid t \geqq t_1 \text{ and } (H \cdot S)_t < \beta\} \text{ for } \omega \text{ in } A_1$$

$$V'_1 = \inf\{t \mid t \geqq U'_1 \text{ and } (H \cdot S)_t > \gamma\} \text{ for } \omega \text{ in } A_1 .$$

The variables U'_1 and V'_1 are clearly stopping times and take values in $[0, \infty]$. By construction of A_1 we have that

$$\mathbf{P}[V'_1 < \infty] \geqq \mathbf{P}[A \cap A_1] > \alpha - \varepsilon_1 .$$

Take $s_1 > t_1$ so that $\mathbf{P}[V'_1 \leqq s_1] > \alpha - \varepsilon_1$ and define

$$U_1 = \min(U'_1, s_1) ,$$

$$V_1 = \min(V'_1, s_1) .$$

The set $B_1 = \{(H \cdot S)_{U_1} \leqq \beta < \gamma \leqq (H \cdot S)_{V_1}\}$ is in $\mathscr{F}_{s_1}$ and $\mathbf{P}[B_1 \cap A] > \alpha - \varepsilon_1$. Put $K^1 = H \mathbf{1}_{]\!]U_1, V_1]\!]}$. We claim that K^1 is $(1 + \beta)$-admissible. Indeed on A_1^c clearly $(K^1 \cdot S)_t = 0$ for all t. For $\omega \in A_1$ and $t \leqq U_1$ we also have $(K^1 \cdot S)_t(\omega) = 0$. For $\omega \in A_1$ and $U_1 < t \leqq V_1$ we have

$$(K^1 \cdot S)_t = (H \cdot S)_t - (H \cdot S)_{U_1} \geqq -1 - \beta = -(1 + \beta) .$$

Let us put $L^1 = K^1$. We now apply the same reasoning on the set $(B_1 \cap A)$ i.e. we take $t_2 \geqq s_1$, $A_2 \in \mathscr{F}_{t_2}$ such that $A_2 \subset B_1$, $\mathbf{P}[A_2 \Delta (B_1 \cap A)] > \alpha - \varepsilon_1 - \varepsilon_2$. On the set A_2 we define

$$U'_2 = \inf\{t \mid t \geqq t_2 \text{ and } (H \cdot S)_t < \beta\}$$

$$V'_2 = \inf\{t \mid t \geqq U'_2 \text{ and } (H \cdot S)_t > \gamma\}$$

$\mathbf{P}[V'_2 < \infty] > \alpha - \varepsilon_1 - \varepsilon_2$ and we select $s_2 > t_2$ so that $\mathbf{P}[V'_2 \leqq s_2] > \alpha - \varepsilon_1 - \varepsilon_2$. Take

$$U_2 = \min(U'_2, s_2)$$

$$V_2 = \min(V'_2, s_2)$$

$$K^2 = H \mathbf{1}_{]\!]U_2, V_2]\!]} .$$

The integrand is $(1 + \beta)$-admissible, but outside the set B_1 the process $(K^2 \cdot S)$ is zero. On the set B_1 however, $(L^1 \cdot S)_{t_2} = (L^1 \cdot S)_{s_2} \geqq \gamma - \beta > 0$. The integrand $L^2 = L^1 + K^2$ remains therefore $(1 + \beta)$-admissible. Furthermore $\mathbf{P}[(L^2 \cdot S)_{t_2} \geqq 2(\gamma - \beta)] > \alpha - \varepsilon_1 - \varepsilon_2$. This permits us to continue the construction and to define L^n by induction. □

The rest of this section is devoted to some results giving a better understanding of the property (NFLVR) of no free lunch with vanishing risk and relating this property to previous results of Delbaen (1992) and Schachermayer (1993).

Corollary 3.4 *If the semi-martingale S satisfies* (NFLVR) *then the set*

$$\{(H \cdot S)_\infty \mid H \text{ is } 1\text{-admissible}\}$$

is bounded in L^0.

Proof. This follows immediately from the existence of the limit $(H \cdot S)_\infty$ and from Proposition 3.1. □

Remark. The convergence theorem shows in particular that in the definition of K_0 the requirement that the limit exists is superfluous. We also want to point out that to derive the above results 3.1 to 3.4, we only used the condition (NFLVR) for integrands with bounded support, i.e. for integrands that are zero outside a stochastic interval $[\![0, k]\!]$ for some real number k.

The next result only uses the (very weak) assumption of no arbitrage. We emphasize that the property (NA), as we defined it, refers to general integrands.

Proposition 3.5 (compare Schachermayer (1993), Proposition 4.2) *If the semi-martingale S satisfies* (NA) *then for every admissible integrand H, such that*

$(H \cdot S)_\infty = \lim_{t \to \infty} (H \cdot S)_t$ *exists, we have for each* $t \in \mathbf{R}_+$:

$$\|(H \cdot S)_t^- \|_\infty \leqq \|(H \cdot S)_\infty^- \|_\infty .$$

Proof. If $\|(H \cdot S)_t^- \|_\infty > \|(H \cdot S)_\infty^- \|_\infty$ then we define the set $A \in \mathscr{F}_t$ as

$$A = \{(H \cdot S)_t < - \|(H \cdot S)_\infty^- \|_\infty\} .$$

The integrand $K = 1_A 1_{]\!]t, \infty[\![}$ is admissible, the random variable $(K \cdot S)_\infty$ exists, is non negative and $\mathbf{P}[(K \cdot S)_\infty > 0] > 0$. This violates (NA). □

The next result may be seen as a sharpening of Proposition 1.5 of Schachermayer (1993). It combines the property (NA) with the conclusion of Proposition 3.1.

Proposition 3.6 *If the semi-martingale S fails the property* (NFLVR) *then either S fails* (NA) *or there exists* $f_0 : \Omega \to [0, \infty]$ *not identically* 0, *a sequence of variables* $(f_n)_{n \geqq 1} = ((H^n \cdot S)_\infty)_{n \geqq 1}$ *in* K_0 *with* H^n *a* $\frac{1}{n}$*-admissible integrand and such that* $\lim_{n \to \infty} f_n = f_0$ *in probability.*

Proof. It is clear that the existence of such sequences violates (NFLVR). Indeed the set $\{n(H^n \cdot S)_\infty ; n \geqq 1\}$ is unbounded in L^0, whereas the integrands $(nH^n)_{n \geqq 1}$ are 1-admissible. This contradicts Proposition 3.1.

The converse is less obvious. Suppose that S satisfies (NA) and suppose that $(g_n)_{n \geqq 1}$ is a sequence in C such that $g_0 = \lim_{n \to \infty} g_n$ in L^∞, $g_0 \geqq 0$, $\mathbf{P}[g_0 > \alpha] > \alpha > 0$. From the hypothesis on the sequence $(g_n)_{n \geqq 1}$ we deduce that $\|g_n^-\|_\infty$ tends to 0. By passing to a subsequence, if necessary, we may suppose that $\|g_n^-\|_\infty \leqq \frac{1}{n}$. For each n we take a function h_n in K_0 such that $h_n \geqq g_n$. If $h_n = (L^n \cdot S)_\infty$ then $\|h_n^-\|_\infty \leqq \frac{1}{n}$ and hence L^n is $\frac{1}{n}$-admissible by Proposition 3.5 and the property (NA) of S. Lemma A1.1 allows us to replace h_n by $f_n \in \text{conv}(h_n, h_{n+1} \ldots)$ such that f_n converges to $f_0 : \Omega \to [0, \infty]$ in probability. Let H^n be the corresponding convex combination of the integrands $(L^k)_{k \geqq n}$. Obviously H^n is still $\frac{1}{n}$-admissible and f_n^- tends to 0 in L^∞. For n large enough we have $\|g_n - g_0\|_\infty \leqq \alpha/2$ and hence $\mathbf{P}[h_n > 0] \geqq \mathbf{P}[g_n > \alpha/2] > \alpha/2$. Lemma A1.1 now shows that $\mathbf{P}[f_0 > 0] > 0$. □

The following corollary relates the condition (NFLVR) with the condition (d) in Delbaen (1992) (which in turn is just reformulating the concept of (NFLBR) to be defined in Sect. 6 below).

Corollary 3.7 *The semi-martingale S satisfies the condition* (NFLVR) *if and only if for a sequence* $(g_n)_{n \geqq 1}$ *in* K_0, *the condition* $\|g_n^-\|_\infty \to 0$ *implies that* g_n *tends to* 0 *in probability.*

Proof. We first observe that the condition stated in the corollary implies (NA). The corollary is now a direct consequence of the Proposition 3.6 and the Lemma A1.1. □

Corollary 3.8 *Under the assumption* (NA), *the semi-martingale S satisfies the condition* (NFLVR) *if and only if the set*

$$\{(H\cdot S)_\infty \mid H \text{ 1-admissible and of bounded support}\}$$

is bounded in L^0.

Proof. From the proof of Proposition 3.2, it follows that the set $\{\sup_{0\leqq t}(H\cdot S)_t \mid H \text{ 1-admissible}\}$ is also bounded in L^0.

If the sequence $(g_n)_{n\geqq 1}$ in K_0, satisfies $\|g_n^-\|_\infty \to 0$ then by the (NA) property and Proposition 3.5, $g_n = (H^n\cdot S)_\infty$ where H^n is ε_n-admissible with $\varepsilon_n = \|g_n^-\|_\infty$. The sequence $\frac{1}{\varepsilon_n} g_n$ has to be bounded which is only possible when g_n tends to 0 in probability. The conclusion now follows from the preceding corollary. □

4 Proof of the main theorem

In this section we prove the main theorem of the paper. The proof follows the following plan: prove that the set C, introduced in Sect. 2, is weak*-closed in L^∞ and apply the separation theorem of Kreps and Yan (see Schachermayer (1993)), which in turn is a consequence of the Hahn–Banach theorem. We use similar arguments as in Delbaen (1992) and Schachermayer (1993). The technicalities are however different and more complicated.

Definition 4.1 (compare Mc Beth (1992) and Schachermayer (1993), Definition 3.4) *A subset D of L^0 is* **Fatou closed** *if for every sequence $(f_n)_{n\geqq 1}$ uniformly bounded from below and such that $f_n \to f$ almost surely, we have $f\in D$.*

We remark that if D is a cone then D is **Fatou closed** if for every sequence $(f_n)_{n\geqq 1}$ in D with $f_n \geqq -1$ and $f_n \to f$ almost surely, we have $f\in D$.

The next result is the technical version of the main theorem.

Theorem 4.2 *If S is a bounded semi-martingale satisfying* (NFLVR), *then*
(i) *C_0 is Fatou closed and hence*
(ii) *$C = C_0 \cap L^\infty$ is $\sigma(L^\infty, L^1)$ closed.*

Proof. We will not prove the first part of Theorem 4.2 immediately, its proof is quite complicated and will fill the rest of this section.

The second assertion is proved using Theorem 2.1. If C_0 is Fatou closed then we have to prove that $C = C_0\cap L^\infty$ is closed for the topology $\sigma(L^\infty, L^1)$. Take a sequence $(f_n)_{n\geqq 1}$ in C, uniformly bounded in absolute value by 1 and such that $f_n \to f$ almost surely. Since C_0 is Fatou closed the element f belongs to C_0 and hence also $f\in C$. □

We now show how Theorem 4.2 implies the main theorem of the paper. For convenience of the reader we restate the main Theorem 1.1.

Theorem 1.1 (Main theorem) *Let S be a bounded real valued semi-martingale. There is an equivalent martingale measure* $\mathbf{Q}$ *for S if and only if S satisfies* (NFLVR).

Proof. We proceed on a well known path (Delbaen (1992), Mc Beth (1992), Schachermayer (1992), Stricker (1990), Lakner (1992), Kreps (1978)). Since S satisfies (NA) we have $C \cap L^\infty_+ = \{0\}$. Because C is weak* closed in L^∞ we know that there is an equivalent probability measure $\mathbf{Q}$ such that $\mathbf{E}_\mathbf{Q}[f] \leqq 0$ for each f in C. This is precisely the Kreps-Yan separation theorem, for a proof of which we refer to Schachermayer (1993, Theorem 3.1). For each $s < t$, $B \in \mathscr{F}_s$, $\alpha \in \mathbf{R}$ we have $\alpha(S_t - S_s)\mathbf{1}_B \in C$ (S is bounded!). Therefore $\mathbf{E}_\mathbf{Q}[(S_t - S_s)\mathbf{1}_B] = 0$ and $\mathbf{Q}$ is a martingale measure for S.

The condition (NFLVR) is not altered if we replace the original probability measure by an equivalent one. In the proof that condition (NFLVR) is also necessary, we may therefore suppose that $\mathbf{P}$ is already a martingale measure for the bounded semi-martingale S. If H is an admissible integrand then by Theorem 2.9 we know that the process $(H \cdot S)$ is a supermartingale. Therefore $E[(H \cdot S)_\infty] \leqq E[(H \cdot S)_0] = 0$. Every function f in C therefore satisfies $\mathbf{E}[f] \leqq 0$. The same applies for elements in the norm closure $\bar{C}$ of C. Therefore $\bar{C} \cap L^\infty_+ = \{0\}$. □

We now show how the main theorem implies Corollary 1.2 pertaining to the locally bounded case. We refer to Delbaen and Schachermayer (1992) for examples that show that we can only obtain an equivalent local martingale measure for the process S. The proof of Corollary 1.2 is similar to Schachermayer (1993, Theorem 5.1).

Corollary 1.2 *Let S be a locally bounded real valued semi-martingale. There is an equivalent local martingale measure* $\mathbf{Q}$ *for S if and only if S satisfies* (NFLVR).

Proof. Since S is locally bounded, there is a sequence $\alpha_n \to +\infty$ and an increasing sequence of stopping times $T_n \to \infty$ so that on $[\![0, T_n]\!]$ the process S is bounded by α_n. We replace S by

$$\tilde{S} = S\mathbf{1}_{[\![0, T_1]\!]} + \sum_{n \geqq 1} 2^{-n} \frac{1}{\alpha_n + \alpha_{n+1}} (\mathbf{1}_{]\!]T_n, T_{n+1}]\!]} \cdot S),$$

$\tilde{S}$ is bounded and satisfies (NFLVR) since the outcomes of admissible integrands are the same for S and $\tilde{S}$. A martingale measure for $\tilde{S}$ is a local martingale measure for S and therefore the corollary follows from the main theorem. The proof of the necessity of the condition (NFLVR) is proved in the same way as in the Theorem 1.1. □

Remark. The necessity of the condition (NFLVR) and Theorem 4.2 show that if S is a locally bounded local martingale then the set C_0 is Fatou closed.

We now proceed with the proof of Theorem 4.2. The bounded semi-martingale S will be assumed to satisfy the property (NFLVR). We take a sequence $h_n \in C_0$, $h_n \geqq -1$ and $h_n \to h$ a.s.; we have to show $h \in C_0$. This is the same as showing that there is a $f_0 \in K_0$ with $f_0 \geqq h$. For each n we take $g_n \in K_0$ such that $g_n \geqq h_n$. The sequence g_n is not necessarily convergent and even if it were, this does not give good information about the sequence of integrands used to construct g_n. To overcome this difficulty we introduce a maximal element (compare Remark 4.4 below). Define $\mathfrak{D}$ as the set $\mathfrak{D} = \{f \mid \text{there is a sequence } K^n$ of 1-admissible integrands such that $(K^n \cdot S)_\infty \to f$ a.s. and $f \geqq h\}$.

Lemma 4.3 *The set $\mathfrak{D}$ is not empty and contains a maximal element f_0.*

Proof. $\mathfrak{D}$ is not empty. Indeed $\mathfrak{D}$ contains an element g that dominates h. To see this we take g_n as above and apply Lemma A1.1. Next observe that the set $\mathfrak{D}$ is bounded in L^0 since it is contained in the closure of the set $\{(H \cdot S)_\infty \mid H$ 1-admissible$\}$ which is bounded by Corollary 3.4. The set $\mathfrak{D}$ is clearly closed for the convergence in probability. We now apply the well known fact that a bounded closed set of L^0 contains a maximal element. For completeness we give a proof. We will use transfinite induction. For $\alpha = 1$ take an arbitrary element f_1 of $\mathfrak{D}$. If α is of the form $\alpha = \beta + 1$ and if f_β is not maximal then choose $f_\alpha \geqq f_\beta$; $\mathbf{P}[f_\alpha > f_\beta] > 0$ and $f_\alpha \in \mathfrak{D}$. If α is a countable limit ordinal then $\alpha = \lim \beta_n$ where β_n is increasing to α. The sequence f_{β_n} is increasing and converges to a function f_α finite a.s. ($\mathfrak{D}$ is bounded!). In this way we construct for each countable ordinal the variable f_α. Since $\mathbf{E}[\exp(-f_\alpha)]$ is well defined and form a decreasing "long sequence", this sequence has to become eventually stationary, say at a countable ordinal α_0. By construction $f_0 = f_{\alpha_0}$ is maximal. □

Remark 4.4 Let us motivate why we introduced the maximal element f_0 in the above lemma. As already observed the sequence g_n introduced before Lemma 4.3 is not of immediate use. Our goal is, of course, to find a 1-admissible integrand H_0 which is, in some sense, a limit of the sequence H_n of the 1-admissible integrands used to construct the sequence g_n. But the convergence of $(g_n)_{n \geqq 1}$ (which we may assume by Lemma A1.1) does not imply the convergence of the sequence $(H_n)_n$ in any reasonable sense. We illustrate this with the following example in discrete time. Let $(r_m)_{m \geqq 1}$ be a sequence of Rademacher functions i.e. a sequence of independent identically distributed variables with $\mathbf{P}[r_m = +1] = \mathbf{P}[r_m = -1] = \frac{1}{2}$. Let $S_m = \sum_{k=1}^m r_k$ and $S_0 = 0$. For each n, an odd natural number we take for the strategy H^n the so called doubling strategy. This strategy is defined as

$$H_t^n = 2^{t-1} \quad \text{if} \quad r_1 = \ldots = r_{t-1} = 1$$

$$= 0 \text{ elsewhere} .$$

Clearly $(H^n \cdot S)_t = H_1^n r_1 + \ldots + H_{t-1}^n r_t$ hence we obtain

$$(H^n \cdot S)_t = 2^t - 1 \quad \text{with probability } 2^{-t}$$

$$= -1 \quad \text{with probability } 1 - 2^{-t}.$$

For odd n the final outcome g_n satisfies $g_n = \lim_{t \to \infty} (H^n \cdot S)_t = -1$ almost surely.

For each n, an even natural number, we introduce a "doubling strategy" H^n starting at time n. More precisely

$$
\begin{aligned}
H_t^n &= 0 \quad \text{for} \quad t \leqq n \\
&= 2^{m-1} \text{ if } t = n + m \quad \text{and} \quad r_{n+1} = \ldots r_{n+m-1} = 1 \\
&= 0 \text{ elsewhere} .
\end{aligned}
$$

Clearly for $t \leqq n$: $(H^n \cdot S)_t = 0$ and for $t > n$: $(H^n \cdot S)_t = H_n^n(r_{n+1}) + \ldots + H_{t-1}^n(r_t)$ hence for $t > n$:

$$
\begin{aligned}
(H^n \cdot S)_t &= 2^{(t-n)} - 1 \quad \text{with probability } 2^{-(t-n)} \\
&= -1 \quad \text{with probability } 1 - 2^{-(t-n)} .
\end{aligned}
$$

Again, for each even number n, the final outcome g_n satisfies $g_n = \lim_{t \to \infty} (H^n \cdot S)_t = -1$ almost surely. Hence all the variables g_n, for odd as well as for even n, are equal to -1 almost surely and hence trivially $g = \lim g_n = -1$ a.s. On the other hand the sequence H^n, along the even numbers, tends to zero on $\mathbf{R}_+ \times \Omega$. Along the odd numbers the sequence H^n is constant and equal to the same doubling strategy. The sequence H^n is therefore not converging. Note however that the limit function g is not maximal in the sense of Lemma 4.3. If we take limits along the even numbers then the pointwise limit H of H^n is zero and hence $(H \cdot S)_\infty = 0$. The example suggest that the outcome 0, which is larger than g, can be obtained by looking at limits of the strategies H^n. So the remedy is to replace the function g by the larger outcome 0. Replacing g by a maximal element is in this sense a "best try".

Of course, this is only a very simple example and the reader may construct examples where even more pathological phenomena occur. But the present example shows in a convincing way, that the convergence of the final outcome g_n does not imply any kind of convergence of the corresponding integrands H^n.

The difficulties arising from the above introduced "suicide strategies" H^n were already addressed in Harrison and Pliska (1981).

We finish this remark by giving an example of a process $(S_t)_{t \geqq 0}$ such that $K = K_0 \cap L^\infty$ is not $\sigma(L^\infty, L^1)$ closed. This underlines again the importance of considering the cone C_0 of elements dominated by elements of K_0, a phenomenon already encountered in the Kreps-Yan theorem (see Schachermayer (1993), Theorem 3.1). The example is in discrete time. We consider a sequence Y_n of independent variables taking 3 possible values $\{a, b, c\}$. The probability is defined as $\mathbf{P}[Y_n = a] = \frac{1}{2}$; $\mathbf{P}[Y_n = b] = \frac{1}{2} - 4^{-n}$; $\mathbf{P}[Y_n = c] = 4^{-n}$. We again use the sequence of Rademacher functions defined this time as $r_n = 1$ if $Y_n = a$, and $r_n = -1$ if $Y_n = b$ or c. Let T be defined as the first n so that $Y_n = c$. It is clear that $\mathbf{P}[\text{there is } n \text{ such that } Y_n = c] \leqq \frac{1}{3}$. We define the process S as $S_m = \sum_{n=1}^{\min(m, T)} r_n$. More precisely we take the sum of the first m Rademacher functions but we stop the process at T. The original measure is clearly a martingale measure for S. Let us now define B_n as the set $\{T > n\}$ and let H^n be the doubling strategy starting at time n. From the definition of T it follows that the final outcome $g_n = (H^n \cdot S)_\infty = -\mathbf{1}_{B_n}$. The sequence g_n tends weak* to $g = -\mathbf{1}_{\{T=\infty\}}$. This random variable g however is not in the set K. Suppose on the contrary that H is a predictable integrand such that $(H \cdot S)_\infty = -\mathbf{1}_{\{T=\infty\}}$. On the set $\{T \leqq n-1\}$ we can without disturbing the final outcome, replace $H_1, \ldots, H_n$ by 0. This new integrand is still denoted by H. Let now n be the first integer such that H_n is not identically 0. On the set $\{T = n\}$ the product $H_n r_n$ is also the final outcome. Since this set is disjoint from the set $\{T = \infty\}$ we find that $H_n = 0$ on the set $\{T = n\}$. The variable H_n is $\mathscr{F}_{n-1}$ measurable and by independence of $\mathscr{F}_{n-1}$ and Y_n we therefore have $H_n = 0$ on the set $\{T > n-1\}$. This contradicts the assumption on n. □

For the rest of the proof we will denote by f_0 a maximal element of $\mathfrak{D}$, $(f_n)_{n \geqq 1}$ is a sequence of elements, obtained as $f_n = (H^n \cdot S)_\infty$ where H^n are 1-admissible strategies H^n, and the sequence f_n converges to f_0 almost surely. Remark that if we can prove that $f_0 \in K_0$, we finish the proof of Theorem 4.2.

Lemma 4.5 *With the notation introduced above we have that the random variables*

$$F_{n,m} = ((H^n - H^m) \cdot S)^* = \sup_{t \in \mathbf{R}_+} |(H^n \cdot S)_t - (H^m \cdot S)_t|$$

tend to zero in probability as $n, m \to \infty$.

Proof. Suppose to the contrary that there is $\alpha > 0$, sequences $(n_k, m_k)_{k \geqq 1}$ tending to ∞ and for each k: $\mathbf{P}[\sup_{t \geqq 0}((H^{n_k} \cdot S)_t - (H^{m_k} \cdot S)_t) > \alpha] \geqq \alpha$.

Define the stopping times T_k as

$$T_k = \inf\{t \mid (H^{n_k} \cdot S)_t - (H^{m_k} \cdot S)_t \geqq \alpha\}$$

so that we have $\mathbf{P}[T_k < \infty] \geqq \alpha$.

Define L^k as $L^k = H^{n_k} \mathbf{1}_{[\![0, T_k]\!]} + H^{m_k} \mathbf{1}_{]\!]T_k, \infty[\![}$. L^k is predictable and it is 1-admissible. Indeed for $t \leqq T_k$ we have $(L^k \cdot S)_t = (H^{n_k} \cdot S)_t \geqq -1$ since H^{n_k} is 1-admissible. For $t \geqq T_k$ we have

$$\begin{aligned}(L^k \cdot S)_t &= (H^{n_k} \cdot S)_{T_k} + (H^{m_k} \cdot S)_t - (H^{m_k} \cdot S)_{T_k} \\ &\geqq (H^{m_k} \cdot S)_t + \alpha \geqq -1 + \alpha\,.\end{aligned}$$

Denote $\lim_{t \to \infty}(L^k \cdot S)_t$ by ρ_k. From the preceding inequalities we deduce that ρ_k can be written as $\rho_k = \varphi_k + \psi_k$ where

$$\varphi_k = f_{n_k} \mathbf{1}_{\{T_k = \infty\}} + f_{m_k} \mathbf{1}_{\{T_k < \infty\}} \quad \text{and} \quad \mathbf{P}[\psi_k \geqq \alpha] \geqq \alpha\,.$$

By assumption $\varphi_k \to f_0$ and by taking convex combination as in Lemma A1.1 we may suppose that $\psi_k \to \psi_0$ where $\mathbf{P}[\psi_0 > 0] > 0$. Therefore convex combinations of ρ_k converge almost surely to an element $f_0 + \psi_0$, a contradiction to the maximality of f_0. □

Remark 4.6 Let us give an economic interpretation of the argument of the proof. At time T_k we know that the trading strategy H^{n_k} has obtained the result $(H^{n_k} \cdot S)_{T_k}$ which is at least α better than $(H^{m_k} \cdot S)_{T_k}$ on a set of measure bigger than α. On the other hand we know that, for k big enough, both strategies yield at time ∞ a result close to f_0. Having this information the economic agent will switch from the strategy H^{n_k} to H^{m_k} since, starting from a lower level, H^{m_k} yields almost the same final result, i.e. the gain on the interval $]\!]T_k, \infty[\![$ is better for H^{m_k} than for H^{n_k}. The strategy L^k precisely describes this attitude.

The proof used convergence in probability. In the rest of the proof we will make use of decomposition theorems, estimation of maximal functions etc. These methods are easier when applied in an "L^2-environment". We therefore replace the original measure **P** by a new equivalent measure **Q** we will now construct.

First we observe that $(H^n \cdot S)_t$ converges uniformly in t. The variable $q = \sup_n \sup_t |(H^n \cdot S)_t|$ is therefore finite almost surely. For **Q** we now take a probability measure equivalent with **P** and such that $q \in L^2(\mathbf{Q})$ e.g. we can take **Q** with density $\frac{d\mathbf{Q}}{d\mathbf{P}} = \frac{\exp(-q)}{\mathbf{E_P}[\exp(-q)]}$. From the dominated convergence theorem we then easily deduce that

$$\lim_{n,m\to\infty} \left\| \sup_t |(H^n \cdot S)_t - (H^m \cdot S)_t| \right\|_{L^2(\mathbf{Q})} = 0 .$$

From now on **Q** will be fixed. Since S is bounded it is a special semi-martingale and its canonical decomposition (with respect to **Q**) will be denoted as $S = M + A$, where M is the local martingale part and A is of finite variation and predictable. The symbols M and A are from now on reserved for this decomposition.

The next lemma is crucial in the proof of the main theorem. It is used to obtain bounds on $H^n \cdot M$. Because we shall need such an estimate also for other integrands we state it in a more abstract way. For $\lambda > 0$, let $\mathscr{H}_\lambda$ be the convex set of 1-admissible integrands H with the extra property $\|(H \cdot S)^*\|_{L^2(\mathbf{Q})} \leqq \lambda$.

Lemma 4.7 *For* $\lambda > 0$ *the set of maximal functions* $\{(H \cdot M)^* | H \in \mathscr{H}_\lambda\}$ *is bounded in* $L^0(\mathbf{Q})$.

Proof. Fix $\lambda > 0$ and abbreviate the set $\mathscr{H}_\lambda$ by $\mathscr{H}$. The semi-martingales $H \cdot S$ where H is in $\mathscr{H}$, are special (with respect to **Q**) because their maximal functions are in $L^2(\mathbf{Q})$. Therefore, by Theorem 2.2, the canonical decomposition of $H \cdot S$ comes from the decomposition $S = M + A$ i.e., $H \cdot M$ is the local martingale part of $H \cdot S$ and $H \cdot A$ is the predictable part of finite variation.

Because the proof of the lemma is rather lengthy let us roughly sketch the idea, which is quite simple. If K^n is a sequence in $\mathscr{H}$ such that $(K^n \cdot M)^*$ is unbounded in probability, then $K^n \cdot A$ is also unbounded and – keeping in mind that $K^n \cdot A$ is predictable – using good strategies we might take advantage of positive gains. This turns out to be possible as the calculations will show that the gains coming from the predictable part A in the long run overwhelm the possible losses coming from the martingale part M. This will contradict the property (NFLVR). Very roughly speaking, the gains coming from the predictable part A add up proportionally in time, whereas the expected losses from the martingale part only add up proportionally to

$\sqrt{\text{time}}$. These phenomena are due to the orthogonality of martingale differences, whereas the variation of the predictable part over the union of two intervals is the sum of the variations over each interval.

Let us now turn to the technicalities. If $\{(H\cdot M)^* \mid H \in \mathscr{H}\}$ in not bounded in L^0, there is a sequence $(K^n)_{n\geqq 1}$ in $\mathscr{H}$, as well as $\alpha > 0$, such that for all $n \geqq 1$ we have $\mathbf{Q}[(K^n\cdot M)^* > n^3] > 8\alpha$. From the L^2 bound on $(H\cdot S)^*$ and Tchebychev's inequality we deduce that $\mathbf{Q}[\sup_t |(K^n\cdot S)_t| > n] \leqq \frac{\lambda^2}{n^2}$ and for n large enough (say $n \geqq N$) this expression is smaller than $\alpha/3$. For each n we now define T_n as

$$T_n = \inf\left\{t \,\middle|\, |(K^n\cdot M)_t| \geqq n^3 \text{ or } |(K^n\cdot S)_t| \geqq n\right\}.$$

If we now define the integrand $L^n = \frac{1}{n^2} K^n \mathbf{1}_{[\![0,\,T_n]\!]}$ we obtain that

(i) $L^n\cdot M$ are local martingales

(ii) $\mathbf{Q}[(L^n\cdot M)^* \geqq n] \geqq \mathbf{Q}[(K^n\cdot M)^* \geqq n^3] - \mathbf{Q}[(K^n\cdot S)^* \geqq n] \geqq 8\alpha - \frac{\lambda^2}{n^2} \geqq 7\alpha$ for all $n \geqq N$.

(iii) $L^n\cdot M$ is constant after T_n.

(iv) The jumps of $L^n\cdot S$ are bounded from below by $-\frac{n+1}{n^2}$. Indeed the process $(K^n\cdot S)^{T_n}$ are bounded above by n on $[\![0, T_n[\![$. Its value is always bigger than -1 and hence jumps of $(K^n\cdot S)^{T_n}$ are bounded from below by $-(n+1)$.

(v) $\|(L^n\cdot M)^*\|_{L^2(\mathbf{Q})} \leqq n + \|\Delta(L^n\cdot M)_{T_n}\|_{L^2(\mathbf{Q})} \leqq n + \frac{3\lambda}{n^2}$. The last inequality follows from Corollary 2.4 and the inequality $\|(L^n\cdot S)^*\|_{L^2(\mathbf{Q})} \leqq \frac{\lambda}{n^2}$.

The local martingale $L^n\cdot M$ is therefore an $L^2(\mathbf{Q})$ martingale. For each n we define a sequence of stopping times $(T_{n,i})_{i\geqq 0}$. We start with $T_{n,0} = 0$ and put (eventually the value is $+\infty$)

$$T_{n,i} = \inf\{t \mid t \geqq T_{n,i-1} \text{ and } |(L^n\cdot M)_t - (L^n\cdot M)_{T_{n,i-1}}| \geqq 1\}\ .$$

We then may estimate

$$\begin{aligned}
&\|(L^n\cdot M)_{T_{n,i}} - (L^n\cdot M)_{T_{n,i-1}}\|_{L^2(\mathbf{Q})} \\
&\leqq 1 + \|\Delta(L^n\cdot M)_{T_{n,i}}\|_{L^2(\mathbf{Q})} \\
&\leqq 1 + \frac{3\lambda}{n^2} \leqq 1 + \alpha \leqq 2 \quad \text{for all} \quad n \geqq N\ .
\end{aligned}$$

Let k_n be the integer part of $\frac{n\alpha}{4}$. We claim that for $i = 1, \ldots, k_n$ and all $n \geqq N$, we have $\mathbf{Q}[T_{n,i} < \infty] > 6\alpha$. An inequality of this type is suggested by the fact that the variables $f_{n,i} = (L^n\cdot M)_{T_{n,i}} - (L^n\cdot M)_{T_{n,i-1}}$ are bounded by 2 in $L^2(\mathbf{Q})$ but their sum has to be large, so we need many of them. To prove that for each $i \leqq k_n$ we have $\mathbf{Q}[T_{n,i} < \infty] > 6\alpha$, it is of course sufficient to prove that

$$\mathbf{Q}[T_{n,k_n} < \infty] = \mathbf{Q}[|(L^n \cdot M)_{T_{n,k_n}} - (L^n \cdot M)_{T_{n,k_n-1}}| \geqq 1] > 6\alpha .$$

Put $B = \{T_{n,k_n} < \infty\}$ and estimate, for $n \geqq N$, the $L^2(\mathbf{Q})$ norm of $(L^n \cdot M)^* \mathbf{1}_{B^c}$:

$$\begin{aligned}
&\|(L^n \cdot M)^* \mathbf{1}_{B^c}\|_{L^2(\mathbf{Q})} \\
&\leqq \left\| \sum_{i=1}^{k_n} (L^n \mathbf{1}_{]\!]T_{n,i-1}, T_{n,i}]\!]} \cdot M)^* \mathbf{1}_{B^c} \right\|_{L^2(\mathbf{Q})} \\
&\leqq \sum_{i=1}^{k_n} \|(L^n \mathbf{1}_{]\!]T_{n,i-1}, T_{n,i}]\!]} \cdot M)^* \mathbf{1}_{B^c}\|_{L^2(\mathbf{Q})} \\
&\leqq \sum_{i=1}^{k_n} \|(L^n \mathbf{1}_{]\!]T_{n,i-1}, T_{n,i}]\!]} \cdot M)^*\|_{L^2(\mathbf{Q})} \\
&\leqq 2 \sum_{i=1}^{k_n} \|(L^n \mathbf{1}_{]\!]T_{n,i-1}, T_{n,i}]\!]} \cdot M)_\infty\|_{L^2(\mathbf{Q})} \quad \text{(by Doob's inequality)} \\
&\leqq 4 k_n \\
&\leqq n\alpha .
\end{aligned}$$

Tchebychev's inequality now yields $\mathbf{Q}[(L^n \cdot M)^* \mathbf{1}_{B^c} \geqq n] \leqq \alpha^2$ which implies $\mathbf{Q}[B^c \cap \{(L^n \cdot M)^* \geqq n\}] \leqq \alpha^2 \leqq \alpha$ and hence

$$\mathbf{Q}[B] \geqq \mathbf{Q}[(L^n \cdot M)^* \geqq n] - \mathbf{Q}[B^c \cap \{(L^n \cdot M)^* \geqq n\}] > 7\alpha - \alpha = 6\alpha .$$

For $n \geqq N$ and $i = 1, \ldots, k_n$, the random variables $f_{n,i}$ are bounded in $L^2(\mathbf{Q})$-norm by 2 but in $L^0(\mathbf{Q})$ they satisfy the lower bound $\mathbf{Q}[|f_{n,i}| \geqq 1] > 6\alpha$. This will allow us to obtain a lower $L^0(\mathbf{Q})$ estimate for $f_{n,i}^-$. Let $\beta = \alpha^2$ and $B_{n,i} = \{f_{n,i}^- \geqq \alpha\}$. We will show that $\mathbf{Q}[B_{n,i}] > \beta$.

The martingale property implies that

$$\mathbf{E_Q}[f_{n,i}^-] = \mathbf{E_Q}[f_{n,i}^+] = \frac{\mathbf{E_Q}[|f_{n,i}|]}{2} > 3\alpha .$$

Therefore as $f_{n,i}^-$ is bounded by α outside $B_{n,i}$:

$$\mathbf{E_Q}[f_{n,i}^- \mathbf{1}_{B_{n,i}}] \geqq \mathbf{E_Q}[f_{n,i}^-] - \alpha > 2\alpha .$$

On the other hand the Cauchy-Schwarz inequality gives

$$\mathbf{E_Q}[f_{n,i}^- \mathbf{1}_{B_{n,i}}] \leqq \|f_{n,i}\|_{L^2(\mathbf{Q})} \mathbf{Q}[B_{n,i}]^{1/2} \leqq 2\mathbf{Q}[B_{n,i}]^{1/2} .$$

Both inequalities show that $\mathbf{Q}[B_{n,i}] > \alpha^2 = \beta$.

We now turn to $L^n \cdot A$. Because $L^n \cdot S = L^n \cdot M + L^n \cdot A$ and we know that $L^n \cdot S$ is small and the negative parts of $L^n \cdot M$ are big, we can deduce that positive parts in $L^n \cdot A$ are also big. Let us formalise this idea: from the definition of λ we infer that for all i

$$\|(L^n \cdot S)_{T_{n,i}} - (L^n \cdot S)_{T_{n,i-1}}\|_{L^2(\mathbf{Q})} \leqq \frac{2\lambda}{n^2} .$$

Tchebychev's inequality implies

$$\mathbf{Q}\left[\,|(L^n\cdot S)_{T_{n,i}} - (L^n\cdot S)_{T_{n,i-1}}| \geqq \frac{2\lambda}{n}\right] \leqq \left(\frac{2\lambda}{n^2}\right)^2 \frac{n^2}{4\lambda^2} = n^{-2}\,.$$

Because $\mathbf{Q}[((L^n\cdot M)_{T_{n,i}} - (L^n\cdot M)_{T_{n,i-1}})^- \geqq \alpha] > \beta$ we necessarily have $\mathbf{Q}\left[\,(L^n\cdot A)_{T_{n,i}} - (L^n\cdot A)_{T_{n,i-1}} \geqq \alpha - \frac{2\lambda}{n}\right] > \beta - n^{-2}$ and this holds for all $i \leqq k_n$ and $n \geqq N$.

We will now construct a strategy that allows us to take profit of these k_n positive differences. The process $L^n\cdot A$ is of bounded variation. The Hahn decomposition of this measure, see the discussion preceding Definition 2.7, produces a partition of $\mathbf{R}_+ \times \Omega$ in two predictable sets B^n_+ and B^n_- on which this measure is respectively positive and negative. The processes $(L^n\mathbf{1}_{B^n_+}\cdot A)$ and $(-L^n\mathbf{1}_{B^n_-}\cdot A)$ are therefore increasing. Let R^n be the process $L^n\mathbf{1}_{B^n_+\cap[\![0,T_{n,k_n}]\!]}$.

The process $(R^n\cdot A) = (L^n\mathbf{1}_{B^n_+\cap[\![0,T_{n,k_n}]\!]}\cdot A)$ satisfies

$$(R^n\cdot A)_{T_{n,i}} - (R^n\cdot A)_{T_{n,i-1}} \geqq (L^n\cdot A)_{T_{n,i}} - (L^n\cdot A)_{T_{n,i-1}}$$

and we therefore obtain

$$\mathbf{Q}\left[\,(R^n\cdot A)_{T_{n,i}} - (R^n\cdot A)_{T_{n,i-1}} \geqq \alpha - \frac{2\lambda}{n}\right] > \beta - n^{-2}$$

$$\text{for } i = 1, \ldots, k_n \text{ and all } n \geqq N\,.$$

Unfortunately we do not know that R^n is 1-admissible or even admissible. A final stopping time argument and some estimates will allow us to control the "admissibility" of R^n. The jumps of $R^n\cdot S$ are part of the jumps of $L^n\cdot S$ and hence

$$\Delta(R^n\cdot S) \geqq \Delta(L^n\cdot S) \geqq -\frac{n+1}{n^2} \geqq -\frac{2}{n}\,.$$

An upper bound for $(R^n\cdot M)$ is obtained by

$$\begin{aligned}
&\|(R^n\cdot M)_{T_{n,k_n}}\|^2_{L^2(\mathbf{Q})}\\
&\leqq \|(L^n\cdot M)_{T_{n,k_n}}\|^2_{L^2(\mathbf{Q})}\\
&\leqq \sum_{i=1}^{k_n} \|f_{n,i}\|^2_{L^2(\mathbf{Q})}\,.
\end{aligned}$$

For $n \geqq N$ this is smaller than $4k_n$. Doob's maximal inequality applied on the $L^2(\mathbf{Q})$-martingale $(R^n\cdot M)^{T_{n,k_n}}$ yields

$$\left\|\sup_{t\geqq 0}|(R^n\cdot M)_t|\right\|_{L^2(\mathbf{Q})} \leqq 4\sqrt{k_n}\,.$$

This inequality will show that $R^n\cdot S$ will not become too negative on big sets. First note that we may estimate $(R^n\cdot S)$ from below by $R^n\cdot M$. Indeed, $R^n\cdot S =$

$R^n \cdot M + R^n \cdot A \geqq R^n \cdot M$ since $R^n \cdot A$ is increasing and hence positive. The following estimates hold

$$\mathbf{Q}\left[\inf_{t \geqq 0} (R^n \cdot S)_t \leqq -k_n n^{-1/4}\right]$$

$$\leqq \mathbf{Q}\left[\sup_{t \geqq 0} |(R^n \cdot M)_t| \geqq k_n n^{-1/4}\right]$$

$$\leqq 16 \frac{\sqrt{n}}{k_n} \text{ by Tchebychev's inequality and the above estimate}$$

$$\leqq 64\,\alpha \frac{1}{\sqrt{n}}.$$

Let now $U_n = \inf\{t \mid (R^n \cdot S)_t < -k_n n^{-1/4}\}$. The preceding inequality says that $\mathbf{Q}[U_n < \infty] \leqq 64\,\alpha \frac{1}{\sqrt{n}}$. We define yet another integrand: let $V^n = \frac{1}{k_n} R^n \mathbf{1}_{[\![0,\, U_n]\!]}$. The jumps of $V^n \cdot S$ are then bounded from below by $\frac{-2}{nk_n}$ and the process $(V^n \cdot S)$ is therefore bounded below by $-n^{-1/4} - \frac{2}{nk_n}$. The integrands V^n are therefore admissible and their uniform lower bound tend to zero. We now claim that $(V^n \cdot S)_\infty$ is positive with high probability.

From $\mathbf{Q}\left[(R^n \cdot A)_{T_{n,i}} - (R^n \cdot A)_{T_{n,i-1}} \geqq \alpha - \frac{2\lambda}{n}\right] > \beta - n^{-2}$ and from the Corollary A1.3 we deduce that

$$\mathbf{Q}\left[(R^n \cdot A)_{T_{n,k_n}} \geqq \frac{k_n}{2}\left(\alpha - \frac{2\lambda}{n}\right)(\beta - n^{-2})\right] > \frac{\beta - n^{-2}}{2}.$$

It follows that

$$\mathbf{Q}\left[(V^n \cdot A)_{T_{n,k_n}} \geqq \frac{1}{2}\left(\alpha - \frac{2\lambda}{n}\right)(\beta - n^{-2})\right] > \frac{\beta - n^{-2}}{2} - \mathbf{Q}[U_n < \infty] \quad \text{or}$$

$$\mathbf{Q}\left[(V^n \cdot A)_\infty \geqq \left(\frac{\alpha}{2} - \frac{\lambda}{n}\right)(\beta - n^{-2})\right] > \frac{\beta - n^{-2}}{2} - 64\,\alpha \frac{1}{\sqrt{n}}.$$

Since $\left(\frac{\alpha}{2} - \frac{\lambda}{n}\right)(\beta - n^{-2})$ tends to $\gamma = \frac{\alpha\beta}{2}$ we obtain that for n large enough, say $n \geqq N'$

$$\mathbf{Q}\left[(V^n \cdot A)_\infty \geqq \frac{\gamma}{2}\right] > \frac{\beta}{4}.$$

Let us now look at $(V^n \cdot S)_\infty = (V^n \cdot M)_\infty + (V^n \cdot A)_\infty$. The first term $(V^n \cdot M)_\infty$ tends to zero in $L^2(\mathbf{Q})$. Indeed

$$\|(V^n \cdot M)_\infty\|_{L^2(\mathbf{Q})} \leqq \frac{1}{k_n} \|(R^n \cdot M)_{T_{n,k_n}}\|_{L^2(\mathbf{Q})} \leqq 2 \frac{1}{\sqrt{k_n}} \to 0 .$$

The second term satisfies $\mathbf{Q}\left[(V^n \cdot A)_\infty > \frac{\gamma}{2}\right] > \frac{\beta}{4}$.

Tchebychev's inequality therefore implies that for n large enough, say $n \geqq N''$ we have

$$\mathbf{Q}\left[(V^n \cdot S)_\infty > \frac{\gamma}{4}\right] \geqq \frac{\beta}{4} - \mathbf{Q}\left[(R^n \cdot M)_{T_{n,k_n}} > \frac{\gamma}{4}\right] \geqq \frac{\beta}{8} .$$

The functions $g_n = (V^n \cdot S)_\infty$ have their negative parts going to zero in the norm of L^∞. This is a contradiction to Corollary 3.7. □

The next step in the proof is to obtain convex combinations $L^n \in \operatorname{conv}(H^n; n \geqq 1)$ so that the local martingales $L^n \cdot M$ converge in the semi-martingale topology. If we knew that the elements $H^n \cdot M$ were bounded in $L^2(\mathbf{Q})$ then we could proceed as follows: by taking convex combinations the elements H^n can be replaced by elements L^n such that $L^n \cdot M$ converge in the $L^2(\mathbf{Q})$ topology, whence in the semi-martingale topology. Afterwards we then should concentrate on the processes $L^n \cdot A$. Unfortunately we do not dispose of such an $L^2(\mathbf{Q})$-bound but only a L^0-bound and a slightly more precise information given by the preceding lemma. It suggests that we should stop the local martingales $H^n \cdot M$ when they cross the level $c > 0$, apply Corollary 2.4 to control the final jumps in $L^2(\mathbf{Q})$ and apply some L^2-argument on the so obtained L^2-bounded martingales. Afterwards we should take care of the remaining parts and let c tend to ∞. Again the idea is simpler than the technique. Let us introduce the following sequence of stopping times (c is supposed to be > 0).

$T_c^n = \inf\{t \mid |(H^n \cdot M)_t| \geqq c\}$. The local martingales $(H^n \cdot M)$ will be stopped at T_c^n causing an error $K_c^n \cdot M$ where $K_c^n = H^n \mathbf{1}_{]\!]T_c^n, \infty[\![}$.

Lemma 4.8. *For all $\varepsilon > 0$, there is $c_0 > 0$ such that for arbitrary n, for all convex weights $(\lambda_1, \ldots, \lambda_n)$ and all $c \geqq c_0$, we have*

$$\mathbf{Q}\left[\left(\sum_{i=1}^n \lambda_i K_c^i \cdot M\right)^* > \varepsilon\right] < \varepsilon .$$

Proof. Suppose on the contrary that there is $\alpha > 0$ such that for all c_0 there are convex weights $(\lambda_1, \ldots, \lambda_n)$ and $c \geqq c_0$, such that

$$\mathbf{Q}\left[\left(\sum_{i=1}^n \lambda_i K_c^i \cdot M\right)^* > \alpha\right] > \alpha .$$

From this we will deduce the existence of a sequence of 1-admissible integrands L^n such that $\sup_n \|(L^n \cdot S)^*\|_{L^2(\mathbf{Q})}$ is bounded and such that $(L^n \cdot M)^*$ is unbounded in $L^0(\mathbf{Q})$. This will contradict Lemma 4.7.

Let N be large enough so that $\mathbf{Q}[q > N] < \frac{\alpha}{4}$ (remember $q = \sup_n \sup_t |(H^n \cdot S)_t|$). This is easy since q is finite a.s. If we define τ as the stopping time

$$\tau = \inf\{t \mid \text{for some } n \geqq 1 : |(H^n \cdot S)_t| > N\}$$

we trivially have $\mathbf{Q}[\tau < \infty] < \frac{\alpha}{4}$. From Lemma 4.7, applied with $\lambda = \sup_n \|(H^n \cdot S)^*\|_{L^2(\mathbf{Q})}$, we deduce that $\lim_{c \to \infty} \sup_n \mathbf{Q}[T_c^n < \infty] \leqq \lim_{c \to \infty} \sup_n \mathbf{Q}[(H^n \cdot M)^* \geqq c] = 0$. For $0 < \delta < \frac{\alpha}{4}$, let c_1 be chosen so that for all n and all $c \geqq c_1$ we have $\mathbf{Q}[T_c^n < \infty] < \delta^2$. For each n we have

$$\begin{aligned} \|(K_c^n \cdot S)^*\|_{L^2(\mathbf{Q})} &\leqq \|2(H^n \cdot S)^* \mathbf{1}_{\{T_c^n < \infty\}}\|_{L^2(\mathbf{Q})} \\ &\leqq 2\|q\|_{L^2(\mathbf{Q})} \mathbf{Q}[T_c^n < \infty]^{1/2} . \end{aligned}$$

If follows that there is c_2 so that for all n and all $c \geqq c_2$

$$\|(K_c^n \cdot S)^*\|_{L^2(\mathbf{Q})} \leqq \delta .$$

For $c \geqq \max(c_1, c_2)$ take $\lambda_1 \ldots \lambda_n$ a convex combination that guarantees $\mathbf{Q}\left[\left(\sum_{i=1}^n \lambda_i K_c^i \cdot M\right)^* > \alpha\right] > \alpha$ and let $\sigma = \inf\left\{t \,\middle|\, \left|\left(\sum_{i=1}^n \lambda_i K_c^i \cdot M\right)_t\right| \geqq \alpha\right\}$.

Put $K = \left(\sum_{i=1}^n \lambda_i K_c^i\right) \mathbf{1}_{[\![0, \min(\tau, \sigma)]\!]}$.

Clearly $\mathbf{Q}[(K \cdot M)^* \geqq \alpha] > \alpha - \mathbf{Q}[\tau < \infty] = \frac{3\alpha}{4}$ and the inequality $(K \cdot S)^* \leqq \sum_{i=1}^n \lambda_i (K_c^i \cdot S)^*$ implies $\|(K \cdot S)^*\|_{L^2(\mathbf{Q})} \leqq \delta$. Let us now investigate whether K is admissible.

$$\begin{aligned} (K \cdot S)_t &= \sum_{i=1}^n \lambda_i \mathbf{1}_{\{t > T_c^i\}} \left((H^i \cdot S)_{\min(t, \tau, \sigma)} - (H^i \cdot S)_{\min(T_c^i, \tau, \sigma)}\right) \\ &\geqq \sum_{i=1}^n \lambda_i \mathbf{1}_{\{t > T_c^i\}} (-1 - N) \\ &\geqq -(N+1) \sum_{i=1}^n \lambda_i \mathbf{1}_{\{t > T_c^i\}} \\ &\geqq -(N+1) F_t \end{aligned}$$

where F is the process $F = \sum_{i=1}^n \lambda_1 \mathbf{1}_{]\!]T_c^i, \infty[\![}$. F is an increasing adapted left continuous process, it is therefore predictable. By construction $\mathbf{E}_{\mathbf{Q}}[F_\infty] \leqq \delta^2$ and therefore $\mathbf{Q}[F_\infty > \delta] \leqq \delta$. This implies that the stopping time v defined as

$$v = \inf\{t \mid F_t > \delta\} \text{ satisfies}$$

$$\mathbf{Q}[v < \infty] < \delta < \alpha/4 .$$

This implies that $K' = K\mathbf{1}_{[\![0,\nu]\!]}$ satisfies

$$\|(K'\cdot S)^*\|_{L^2(\mathbf{Q})} \leqq \delta \quad \text{and}$$

$$\mathbf{Q}[(K'\cdot M)^* > \alpha] > \alpha - \mathbf{Q}[\tau < \infty] - \mathbf{Q}(\nu < \infty) \geqq \frac{\alpha}{2} \text{ as well as}$$

$$(K'\cdot S) \geqq -(N+1)\delta\,.$$

The integrand $L^\delta = \dfrac{K'}{(N+1)\delta}$ therefore is 1-admissible and

$$\|(L^\delta\cdot S)^*\|_{L^2(\mathbf{Q})} \leqq \left(\frac{1}{N+1}\right).$$

Furthermore $\mathbf{Q}\left[(L^\delta\cdot M)^* > \dfrac{\alpha}{(N+1)\delta}\right] > \dfrac{\alpha}{2}$.

For δ tending to zero this produces the desired contradiction to Lemma 4.7. □

The following lemma relates, in the L^0 topology, the maximal function of a local martingale with the maximal function of a stochastic integral for an integrand that is bounded by 1. The proof uses the fact that the sequence $(H^n\cdot M)_{n\geqq 1}$ is a sequence of local L^2-martingales with uniform L^2-control of the jumps.

Lemma 4.9 *With the same notation as in Lemma* 4.8, *for all* $\varepsilon > 0$ *there is* $c_0 > 0$ *such that for all* h *predictable* $|h| \leqq 1$, *all convex weights* $(\lambda_1 \dots \lambda_n)$ *and all* $c \geqq c_0$

$$\mathbf{Q}\left[\left(\left(h\sum_{i=1}^n \lambda_i K_c^i\right)\cdot M\right)^* > \varepsilon\right] < \varepsilon\,.$$

In particular $\mathbf{D}(\sum \lambda_i K_c^i\cdot M) < 2\varepsilon$ *where* $\mathbf{D}$ *is the quasi-norm introduced in Sect.* 2 *and inducing the semi-martingale topology.*

Proof. Let $\varepsilon > 0$ and take c_0 as in Lemma 4.8 i.e.

$$\mathbf{Q}[(\sum \lambda_i K_c^i\cdot M)^* > \varepsilon] < \varepsilon$$

for all $(\lambda_1 \dots \lambda_n)$ convex combination and all $c \geqq c_0$. By enlarging c_0 we also may suppose that $\sup_n \|(K_c^n\cdot S)^*\|_{L^2(\mathbf{Q})} \leqq \frac{\varepsilon}{3}$ (see the proof of the Lemma 4.8). Corollary 2.4 now implies that for all n and every stopping time σ

$$\|\Delta(K_c^n\cdot M)_\sigma\|_{L^2(\mathbf{Q})} \leqq \varepsilon\,.$$

Take now h predictable and bounded by 1, take $c \geqq c_0$, $\lambda_1 \dots \lambda_n$ a convex combination. Define σ as

$$\sigma = \inf\left\{t \,\middle|\, \left|\left(\sum_{i=1}^n \lambda_i (K_c^i\cdot M)_t\right)\right| > \varepsilon\right\}.$$

The following estimate holds:

$$\sup_{t \leqq \sigma} \left| \left(\sum_{i=i}^{n} \lambda_i K_c^i \right) \cdot M \right|_t \leqq \varepsilon + \sum \lambda_i | \Delta (K_c^i \cdot M)_\sigma | .$$

The L^2-norm of the left hand side is therefore smaller than 2ε and we have an L^2-martingale. This implies that the martingale $(h \sum \lambda_i K_c^i) \mathbf{1}_{[\![0,\sigma]\!]} \cdot M$ is in L^2 and its norm is smaller than 2ε. Hence

$$\begin{aligned} &\mathbf{Q}[((h \sum \lambda_i K_c^i) \cdot M)^* > \sqrt{\varepsilon}] \\ &\leqq \mathbf{Q}[((h \sum \lambda_i K_c^i) \mathbf{1}_{[\![0,\sigma]\!]} \cdot M)^* > \sqrt{\varepsilon}] + \mathbf{Q}[\sigma < \infty] \\ &\leqq \frac{4\varepsilon^2}{\varepsilon} + \varepsilon = 5\varepsilon . \end{aligned}$$

□

Lemma 4.10 *There is a sequence of convex combinations* $L^n \in \operatorname{conv}(H^k, k \geqq n)$ *such that* $(L^n \cdot M)$ *converges in the semi-martingale topology.*

Proof. We use the notation introduced before Lemma 4.8. For $\varepsilon = \frac{1}{n}$ we apply Lemma 4.9 to find c_n such that

$$\mathbf{D}\left(\left(\sum_{i=1}^{m} \lambda_i K_{c_n}^i \right) \cdot M \right) \leqq \frac{1}{n} \quad \text{for all convex weights } \lambda_1 \ldots \lambda_m .$$

For each n and each k we have $(H^k \mathbf{1}_{[\![0, T_{c_n}^k]\!]} \cdot M)^* \leqq c_n + |\Delta(H^k \cdot M)_{T_{c_n}^k}|$ and an application of Corollary 2.4 yields that each $H^k \mathbf{1}_{[\![0, T_{c_n}^k]\!]} \cdot M$ is an $L^2(\mathbf{Q})$-martingale with bound $c_n + 3\|q\|_{L^2(\mathbf{Q})}$. A standard diagonalisation argument shows that existence of convex weights $\lambda_0^k, \lambda_1^k, \ldots \lambda_{N_k}^k$ such that

$$Y_n^k = \sum_{j=0}^{N_k} \lambda_j^k H^{k+j} \mathbf{1}_{[\![0, T_{c_n}^k]\!]} \cdot M$$

is, for each n, converging in the space of $L^2(\mathbf{Q})$ martingales. An easy way to prove this assertion, is via the following reasoning in Hilbert spaces.

Let $\mathscr{M}^2$ be the Hilbert space of $L^2(\mathbf{Q})$ martingales and let $\mathfrak{H} = (\sum \oplus \mathscr{M}^2)_{\ell^2}$ be its ℓ^2-sum (see Diestel (1975)). An element of this space is a sequence $X = (X_n)_n$ where each X_n is in $\mathscr{M}^2$. This space is also a Hilbert space when equipped with the norm $\|X\|^2 = \sum_{n \geqq 1} \|X_n\|_2^2$. The sequence X^k, defined by the co-ordinates

$$X_n^k = \frac{1}{2^n(c_n + 3\|q\|_{L^2(Q)})} (H^k \mathbf{1}_{[\![0, T_{c_n}^k]\!]} \cdot M)$$

is bounded in the Hilbert space $\mathfrak{H}$ and hence there are convex combinations $Y^k \in \operatorname{conv}(X^k, X^{k+1}, \ldots)$ that converge with respect to the norm of $\mathfrak{H}$. It follows that each "co-ordinate" converges in $\mathcal{M}^2$. This implies the existence of convex weights $\lambda_0^k, \lambda_1^k, \ldots \lambda_{N_k}^k$ such that

$$Y_n^k = \sum_{j=0}^{N_k} \lambda_j^k H^{k+j} \mathbf{1}_{[\![0, T_{c_n}^k]\!]} \cdot M$$

is, for each n, converging in the space of $L^2(\mathbf{Q})$ martingales.

The sequence $L^k = \sum_{j=0}^{N_k} \lambda_j^k H^{k+j} \cdot M$ is now a Cauchy sequence in the space of semi-martingales. Indeed for given $\varepsilon > 0$ take N such that $\frac{1}{N} < \varepsilon$. We find that for k, l:

$$\begin{aligned}
&\mathbf{D}((L^k - L^l) \cdot M) \\
&\leqq \mathbf{D}(Y_N^k - Y_N^l) + \mathbf{D}\left(\sum_{j=1}^{n} \lambda_j^k K_{c_N}^{k+j} \cdot M \right) + \mathbf{D}\left(\sum_{j=1}^{n} \lambda_j^l K_{c_N}^{l+j} \cdot M \right) \\
&\leqq \mathbf{D}(Y_N^k - Y_N^l) + 2\varepsilon .
\end{aligned}$$

For k and l large enough this is smaller than 3ε. □

Lemma 4.11 *The sequence* $(L^k)_{k \geqq 1}$ *of Lemma* 4.10 *is such that* $(L^k \cdot A)$ *converges in the semi-martingale topology.*

Proof. We know that $L^k \cdot S \geqq -1$ and that $(L^k \cdot M)$ converges in the semi-martingale topology. To show that $(L^k \cdot A)$ converges in the semi-martingale topology we have to prove that for each $t \geqq 0$ the total variation $\int_0^t |d((L^k - L^m) \cdot A)|$ converges to 0 in probability as k and m tend to ∞. We will show the stronger statement that $\int_0^\infty |d((L^k - L^m) \cdot A)|$ tend to 0 in probability as k and m tend to ∞. If this were not the case then by the Hahn decomposition, described in Sect. 2, we could find h^k predictable with values in $\{+1, -1\}$, $\alpha > 0$ and two increasing sequences $(i_k, j_k)_{k \geqq 1}$ such that $\mathbf{Q}[\varphi_k > \alpha] > \alpha$ where

$$\begin{aligned}
\varphi_k &= \int_{[0,\infty[} h_u^k d((L^{i_k} - L^{j_k}) \cdot A)_u \\
&= \int_{[0,\infty]} h_u^k (L_u^{i_k} - L_u^{j_k}) dA_u \\
&= \int_{[0,\infty[} |L_u^{i_k} - L_u^{j_k}| |dA_u| .
\end{aligned}$$

We now define the integrand R^k as

$$\begin{aligned}
R^k &= (L^{j_k} + \tfrac{1}{2}(1 + h^k)(L^{i_k} - L^{j_k})) \\
&= \tfrac{1}{2}(L^{i_k} + L^{j_k} + h^k(L^{i_k} - L^{j_k})) .
\end{aligned}$$

The idea is simple if $h^k = 1$ i.e. if $(L^{i_k} - L^{j_k}) \cdot dA \geqq 0$ we take L^{i_k}, if $h^k = -1$ i.e. if $(L^{i_k} - L^{j_k}) dA \leqq 0$ we take L^{j_k}. In some sense R^k takes the best of both. The processes $(R^k - L^{i_k}) \cdot A$ and $(R^k - L^{j_k}) \cdot A$ define positive measures and are therefore increasing. Indeed

$$(R^k - L^{i_k}) \cdot A = ((L^{j_k} - L^{i_k}) + \tfrac{1}{2}(1 + h^k)(L^{i_k} - L^{j_k})) \cdot A$$
$$= \tfrac{1}{2}((h^k - 1)(L^{i_k} - L^{j_k})) \cdot A \quad \text{and}$$
$$(R^k - L^{j_k}) \cdot A = \tfrac{1}{2}((h^k + 1)(L^{i_k} - L^{j_k})) \cdot A \,.$$

Both measures are positive by the construction of h^k. Also

$$\varphi_k = ((R^k - L^{i_k}) \cdot A)_\infty + ((R^k - L^{j_k}) \cdot A)_\infty .$$

We may therefore suppose that $\mathbf{Q}\left[((R^k - L^{i_k}) \cdot A)_\infty > \frac{\alpha}{2}\right] > \frac{\alpha}{2}$ (if necessary we interchange i_k and j_k and take subsequences to keep them increasing). Because $(R^k - L^{i_k}) \cdot M = \frac{1}{2}((h^k - 1)(L^{i_k} - L^{j_k}) \cdot M)$ and because $(L^{i_k} - L^{j_k}) \cdot M$ tend to zero in the semi-martingale topology on $[0, \infty)$ we deduce that the maximal functions $((R^k - L^{i_k}) \cdot M)^*$ tend to zero in probability. The same holds for $((R^k - L^{j_k}) \cdot M)^*$. Let now $(\delta_k)_{k \geqq 1}$ be a sequence of strictly positive numbers tending to 0. By taking subsequences and by the above observation we may suppose that $\mathbf{Q}[((R^k - L^{i_k}) \cdot M)^* > \delta_k \text{ or } ((R^k - L^{j_k}) \cdot M)^* > \delta_k] < \delta_k$ holds for all k. This implies that the stopping time τ_k defined as $\tau_k = \inf\{t \mid (R^k \cdot M)_t \leqq \max((L^{i_k} \cdot M)_t, (L^{j_k} \cdot M)_t) - \delta_k\}$ satisfies $\mathbf{Q}[\tau_k < \infty] < \delta_k$. Define now $\tilde{R}^k = R^k \mathbf{1}_{[\![0, \tau_k]\!]}$. We claim that the integrands $\tilde{R}^k$ are $(1 + \delta_k)$ admissible!

For $t < \tau_k$ we have

$$\begin{aligned}
(\tilde{R}^k \cdot S)_t &= (R^k \cdot S)_t \\
&= (R^k \cdot A)_t + (R^k \cdot M)_t \\
&\geqq \max((L^{i_k} \cdot A)_t, (L^{j_k} \cdot A)_t) + (R^k \cdot M)_t \\
&\geqq \max((L^{i_k} \cdot A)_t, (L^{j_k} \cdot A)_t) + \max((L^{i_k} \cdot M)_t, (L^{j_k} \cdot M)_t) - \delta_k \\
&\geqq \max((L^{i_k} \cdot S)_t, (L^{j_k} \cdot S)_t) - \delta_k \\
&\geqq -1 - \delta_k \,.
\end{aligned}$$

At time τ_k the jump $\Delta(\tilde{R}^k \cdot S)$ is either $\Delta(L^{i_k} \cdot S)$ or $\Delta(L^{j_k} \cdot S)$ and hence $(R^k \cdot S)_{\tau_k} \geqq -1 - \delta_k$ because the left limit of $(\tilde{R}^k \cdot S)$ at τ_k is at least $\max((L^{i_k} \cdot S)_{\tau_k -}, (L^{j_k} \cdot S)_{\tau_k -}) - \delta_k$.

The integrands $(1 + \delta_k)^{-1} \tilde{R}^k$ are 1-admissible. We will use them to construct a contradiction to the maximal property of $f_0 = \lim_{k \to \infty} (L^{j_k} \cdot S)_\infty = \lim_{m \to \infty} (H^m \cdot S)_\infty$.

$$\left(\frac{\tilde{R}^k}{1 + \delta_k} \cdot S - L^{i_k} \cdot S\right)_\infty$$
$$= \frac{1}{1 + \delta_k} ((\tilde{R}^k - L^{i_k}) \cdot S)_\infty - \frac{\delta_k}{1 + \delta_k} (L^{i_k} \cdot S)_\infty$$

$$= \left(\frac{1}{1+\delta_k}\right)((\tilde{R}^k - L^{i_k})\cdot A)_\infty +$$

$$\frac{1}{1+\delta_k}((\tilde{R}^k - L^{i_k})\cdot M)_\infty - \frac{\delta_k}{1+\delta_k}(L^{i_k}\cdot S)_\infty .$$

This first term is estimated from below

$$\mathbf{Q}\left[((\tilde{R}^k - L^{i_k})\cdot A)_\infty > \frac{\alpha}{2}\right] > \frac{\alpha}{2} \quad \text{and} \quad ((\tilde{R}^k - L^{i_k})\cdot A)_\infty \geqq 0 .$$

The second term is estimated from above

$$\mathbf{Q}[((\tilde{R}^k - L^{i_k})\cdot M)_\infty \leqq -\delta_k] < \delta_k \quad \text{and} \quad ((\tilde{R}^k - L^{i_k})\cdot M)_\infty \to 0 .$$

The third term tends to zero since $\delta_k \to 0$. From Lemma A1.1 we know that there are convex combinations $V^k \in \operatorname{conv}(\tilde{R}^k, \tilde{R}^{k+1} \dots)$ such that $(V^k \cdot S)_\infty$ will converge to a function g. Because $(L^{i_k}\cdot S)_\infty \to f_0$ and because $\mathbf{Q}\left[((\tilde{R}^k - L^{i_k})\cdot S)_\infty > \frac{\alpha}{2} - \delta_k\right] > \frac{\alpha}{2} - \delta_k$ we deduce from Lemma A3.1 that $\mathbf{Q}[g > f_0] > 0$. Also $g \geqq f_0$, a contradiction to the construction of f_0. □

Final part of the proof of Theorem 4.2 From Lemmas 4.11 and 4.12 we deduce the existence of 1-admissible integrands $L^k \in \operatorname{conv}(H^k, H^{k+1}, \dots)$ such that $L^k \cdot M$ and $L^k \cdot A$ both converge in the semi-martingale topology. The sequence $(L^k \cdot S)_{k \geqq 1}$ is therefore convergent in the semi-martingale topology. Memin's theorem (see Memin (1980)) now implies the existence of a predictable process L such that $L^k \cdot S \to L \cdot S$ in the semi-martingale topology. In particular L is 1-admissible and the final value satisfies

$$(L\cdot S)_\infty = \lim_{t\to\infty} (L\cdot S)_t = \lim_{t\to\infty} \lim_{n\to\infty} (L^n \cdot S)_t$$

$$= \lim_{n\to\infty} \lim_{t\to\infty} (L^n\cdot S)_t = \lim_{n\to\infty} (L^n \cdot S)_\infty = f_0 .$$

The interchange of the limits is allowed because almost surely $(L^n \cdot S)_t \to (L\cdot S)_t$ uniformly in t, by Lemma 4.5. Indeed $(H^n \cdot S)_t$ converge uniformly on $\mathbf{R}_+$ and the convex combinations $L^k \in \operatorname{conv}(H^k, H^{k+1}, \dots)$ preserve this uniform convergence. This shows that $f_0 \in K_0$ and as remarked before Lemma 4.5 this implies Theorem 4.2. □

Remark. The topology of semi-martingales was defined in Sect. 2. It was defined using the open end interval $[0, \infty)$. A similar but stronger topology could have been defined using the time interval $[0, \infty]$. This amounts to using the distance function:

$$\mathbf{D}(X) = \sup\{\mathbf{E}[\min(|(H\cdot X)_\infty|, 1)] \,|\, H \text{ predictable}, |H| \leqq 1\} .$$

The difference between the two topologies is comparable to the difference between uniform convergence on compact sets of $[0, \infty)$ and uniform convergence on $[0, \infty]$. A careful inspection of the proofs, mainly devoted to checking the existence of the limits at ∞, shows that the semi-martingales $(L^n \cdot S)$ tend to $(L \cdot S)$ in the semi-martingale topology on $[0, \infty]$ and not only on $[0, \infty)$. We preferred not to use this approach in order to keep the proofs easier.

5 The set of representing measures

In this section we use the results obtained in Ansel and Stricker (1992) "Couverture des actifs contingents" and we give a new criterion under which the market is complete. Throughout this paragraph the process S is supposed to be locally bounded and to be a local martingale under the measure $\mathbf{P}$. This will facilitate the notation. We will study the following sets of "representing measures" defined on the sigma algebra $\mathscr{F}$, (see e.g. Delbaen (1992) for an explanation concerning the name "representing measures"):

$$M(\mathbf{P}) = \{\mathbf{Q} \mid \mathbf{Q} \ll \mathbf{P}, \mathbf{Q} \text{ is } \sigma\text{-additive and } S \text{ is a } \mathbf{Q}\text{-local martingale}\}$$

$$M^e(\mathbf{P}) = \{\mathbf{Q} \mid \mathbf{Q} \sim \mathbf{P}, \mathbf{Q} \text{ is } \sigma\text{-additive and } S \text{ is a } \mathbf{Q}\text{-local martingale}\} .$$

The space $M(\mathbf{P})$ consists of all absolutely continuous local martingale measures and it can happen that some of the elements will give a measure zero to events that under the original measure are supposed to have a strictly positive probability to occur. This phenomenon was studied in detail in Delbaen (1992). We will show that $M^e(\mathbf{P}) = M(\mathbf{P})$ implies $M(\mathbf{P}) = \{\mathbf{P}\}$.

We will need the following set of attainable assets:

$$W^0 = \{f \mid \text{there is an } S\text{-integrable } H,\ H \cdot S \text{ bounded and } (H \cdot S)_\infty = f\} .$$

The set W^0 is a subspace of L^∞. There is no problem in this notation since if $H \cdot S$ is bounded, then H as well as $(-H)$ is admissible and therefore $f = (H \cdot S)_\infty$ exists and is a bounded random variable. From Proposition 3.5 it follows that $W^0 = K \cap (-K)$. The same notation for a space related to W^0 is already used in Delbaen (1992). The set W is simply $\{\alpha + f \mid \alpha \in \mathbf{R}$ and $f \in W^0\}$. Because S is supposed to be locally bounded these vector spaces are quite big. The following lemma seems to be obvious but, because unbounded S-integrable processes are used, it is not so trivial as one might suspect. The proof we give uses rather heavy material but it saves place.

Lemma 5.1 *If H is S-integrable and $H \cdot S$ is bounded, then $H \cdot S$ is a $\mathbf{Q}$-martingale for all $\mathbf{Q} \in M(\mathbf{P})$.*

Proof. Take $\mathbf{Q} \in M(\mathbf{P})$. Clearly S is a special semi-martingale under the measure $\mathbf{Q}$. Since it is a local martingale it decomposes as $S = S + 0$. The stochastic integral $H \cdot S$ is bounded and hence is a special martingale under $\mathbf{Q}$. Its decomposition is, according to Theorem 2.2, $H \cdot S = H \cdot S + H \cdot 0$, i.e. $H \cdot S$ is a $\mathbf{Q}$-local martingale. Being bounded it is a martingale under $\mathbf{Q}$. □

It follows from the martingale property that if H and G are two S-integrable processes such that $H \cdot S$ and $G \cdot S$ are bounded and such that

$(H \cdot S)_\infty = (G \cdot S)_\infty$ then necessarily $(H \cdot S) = (G \cdot S)$. (This also follows from arbitrage considerations.).

The following theorem is due to Ansel and Stricker (1992) and Jacka (1992). Earlier versions can be found in Pagès (1987) and in Karatzas et al. (1991). The theorem is particularly important in the setting of incomplete markets (e.g. semi-martingales with more than one equivalent martingale measure). It shows exactly what elements can be constructed or hedged, using admissible strategies.

Theorem 5.2 *If* $f \in L^0(\Omega, \mathscr{F}, \mathbf{P})$ *with* $f^- \in L^\infty(\Omega, \mathscr{F}, \mathbf{P})$ *then the following are equivalent*

(i) *there is* H *predictable,* S*-integrable,* $\mathbf{Q} \in M^e(\mathbf{P})$ *and* $\alpha \in \mathbf{R}$ *such that* $H \cdot S$ *is a* $\mathbf{Q}$*-uniformly integrable martingale with* $f = \alpha + (H \cdot S)_\infty$

(ii) *there is* $\mathbf{Q} \in M^e(\mathbf{P})$ *such that* $\mathbf{E}_\mathbf{R}[f] \leqq \mathbf{E}_\mathbf{Q}[f]$ *for all* $\mathbf{R} \in M^e(\mathbf{P})$.

For f *bounded these two properties are also equivalent to*

(iii) $\mathbf{E}_\mathbf{R}[f]$ *is constant as a function of* $\mathbf{R} \in M(\mathbf{P})$.

Proof. We refer to Ansel and Stricker (1992, Théorème 3.2). For (iii) we remark that $M^e(\mathbf{P})$ is $L^1(\mathbf{P})$ dense in $M(\mathbf{P})$ and hence $\mathbf{E}_\mathbf{R}[f]$ is constant on $M^e(\mathbf{P})$ if and only if it is constant on $M(\mathbf{P})$. □

Corollary 5.3 W *is* $\sigma(L^\infty, L^1)$ *closed in* L^∞.

Proof. This follows immediately from (iii) of the theorem. W is the subspace of these elements in L^∞ that are constant on a subset of L^1. □

Remark. The corollary was known long before Theorem 5.2 was known. The earliest versions of it are due to Yor (1978). Contrary to intuition, the boundedness condition needed in (iii) of Theorem 5.2 cannot be relaxed to f being a member of $L^1_+(\mathbf{R})$ for each $\mathbf{R}$ in $M^e(\mathbf{P})$. A counterexample can be found in Schachermayer (1993b). □

The next theorem is a new criterion for the completeness of the market.

Theorem 5.4 *If* S *is locally bounded and* $\mathbf{P}$ *is a local martingale measure for* S, *then*

(i) $M(\mathbf{P})$ *is a closed convex bounded set of* $L^1(\Omega, \mathscr{F}, \mathbf{P})$

(ii) $M(\mathbf{P}) = M^e(\mathbf{P})$ *implies that* $M(\mathbf{P}) = \{\mathbf{P}\}$.

Proof. (i) We only have to show that $M(\mathbf{P})$ is closed. Take $\mathbf{Q}_n$ a sequence in $M(\mathbf{P})$ and suppose that $\mathbf{Q}_n$ converges to $\mathbf{Q}$. Take T a stopping time such that S^T is bounded. If $t < s$ and $A \in \mathscr{F}_t$ then we can see that: $\mathbf{E}_\mathbf{Q}[S_t^T \mathbf{1}_A] = \lim \mathbf{E}_{\mathbf{Q}_n}[S_t^T \mathbf{1}_A] = \lim \mathbf{E}_{\mathbf{Q}_n}[S_s^T \mathbf{1}_A] = \mathbf{E}_\mathbf{Q}[S_s^T \mathbf{1}_A]$. This proves $\mathbf{Q} \in M(\mathbf{P})$.

(ii) If $M(\mathbf{P}) = M^e(\mathbf{P})$ then $M^e(\mathbf{P})$ is a closed, bounded, convex set. The Bishop-Phelps theorem, see Diestel (1975), states that the set G of elements f of $L^\infty(\Omega, \mathscr{F}, \mathbf{P})$ that attain their supremum on $M^e(\mathbf{P})$, is a norm dense set in $L^\infty(\Omega, \mathscr{F}, \mathbf{P})$. The preceding theorem, part (ii), states that G is a subset of W. Since W is weak* closed it is certainly norm closed. Since W is closed and G is dense for the norm topology we obtain $W = L^\infty(\Omega, \mathscr{F}, \mathbf{P})$. By the Hahn-Banach theorem, two distinct elements of L^1 can be separated by an element of L^∞ i.e. by an element of W. However elements of W are constant on $M(\mathbf{P})$. This implies that $M(\mathbf{P}) = \{\mathbf{P}\}$. □

As we remarked in the introduction our results remain true for $\mathbf{R}^d$-valued processes. The same holds for Theorem 5.4. As the example of Artzner and Heath (1990) shows, Theorem 5.4 is no longer true for an infinite number of assets. The example uses the set $\{0, 1\}$ as time set, but as easily seen and stated in Artzner and Heath (1990) it is easy to transform the example into a setting with continuous time.

In Delbaen (1992) the following identity was proved for a continuous process S. For every $f \in L^\infty$:

$$\sup_{\mathbf{Q} \in M(P)} \mathbf{E}_\mathbf{Q}[f] = \inf\{x \mid \text{there is } h \in W^0 \text{ with } x + h \geqq f\} .$$

In the general case this equality becomes false as the following example in discrete time shows. The left hand side of the equality is always dominated by the right hand side. The example shows that a "gap" is possible. Some further properties displayed by Example 5.5 are: W^0 is weak*-closed but the set $W^0 - L^\infty_+$ is not even norm closed. We will also see that the norm closure and the weak* closure of $W^0 - L^\infty_+$ are different.

Example 5.5 The set Ω is the set $N = \{1, 2, 3, \ldots\}$ of natural numbers. The σ-algebra $\mathscr{F}_n$ is the σ-algebra generated by the atoms $\{k\}$ for $k \leqq 3n$ and the atom $(3n + 1, 3n + 2, \ldots\}$. $S_0 = 0$ and $S_n - S_{n-1}$ is defined as the variable $g_n(3(n-1) + 1) = n$; $g_n(3(n-1) + 2) = 1$; $g_n(3n) = -1$. The process S is not bounded but a normalisation of the functions g_n allows us to replace S by a bounded process. To keep the notation simple we prefer to continue with the locally bounded process S given above. For the measure $\mathbf{P}$ we choose any measure that gives a strictly positive mass to all natural numbers and such that for all n we have $\mathbf{E}_\mathbf{P}[g_n] = 0$. The space W^0 is precisely the set:

$$\left\{ \sum_{n \geqq 1} a_n g_n \mid (n a_n)_{n \geqq 1} \text{ is bounded} \right\} .$$

Take now for f the function defined as

$$\text{for all } n \geqq 1 \colon f(3(n-1) + 1) = 0; \quad f(3(n-1) + 2) = 1 \quad \text{and} \quad f(3n) = 0 .$$

From the description of W^0 it follows that for h in W^0 and $x \in \mathbf{R}$ the random variable $x + h$ can only dominate f if $x \geqq 1$. The constant function 1 clearly dominates f. This shows that

$$\inf\{x \mid \text{there is } h \in W^0 \text{ with } x + h \geqq f\} = 1 .$$

On the other hand if $\mathbf{Q}$ is a local martingale measure for S then $n\mathbf{Q}[3(n-1) + 1] + \mathbf{Q}[3(n-1) + 2] = \mathbf{Q}[3n]$ implies that $\mathbf{Q}[3(n-1) + 2] \leqq 1/2\,\mathbf{Q}[\{3(n-1) + 1, 3(n-1) + 2, 3n\}]$, with strict inequality if $\mathbf{Q}$ is equivalent to $\mathbf{P}$. Therefore $\mathbf{E}_\mathbf{Q}[f] \leqq 1/2$ with strict inequality for $\mathbf{Q}$ in $M^e(\mathbf{P})$. If we take any measure $\mathbf{Q}$ such that $\mathbf{Q}[3(n-1)+1] = 0$ and $\mathbf{Q}[3(n-1)+2] = \mathbf{Q}[3n]$

then $\mathbf{Q}$ is in $M(\mathbf{P})$ and $\mathbf{E_Q}[f] = 1/2$. It is now clear that $\max_{\mathbf{Q} \in M(\mathbf{P})} \mathbf{E_Q}[f] = 1/2$.

This example also shows that in Theorem 5.2(ii), the condition $\mathbf{Q} \in M^e(\mathbf{P})$ may not be replaced by the condition $\mathbf{Q} \in M(\mathbf{P})$. Referring to the proof in Delbaen (1992, Lemma 5.7), we remark that in this example the function f is not in $1/2 + W^0 - L^\infty_+$ but it is in the weak* closure of it. To see this let f_n be the function defined as $f_n(3(k-1)+2) = 1$ for all $k \leqq n$ and 0 elsewhere. The functions f_n are smaller than $1/2 + \sum_{k=1}^n (\frac{1}{2} g_k)$ and therefore are in $1/2 + W^0 - L^\infty_+$, they converge weak* to f. The set $W^0 - L^\infty_+$ is not even norm closed as the following reasoning shows. An element h in $W^0 - L^\infty_+$ is of the form $\sum_{n \geqq 1} a_n g_n - k$ where k is in L^∞_+ and $|n.a_n|$ is bounded, say by m. If a_n is positive then $h(3(n-1)+2) \leqq a_n g_n \leqq \frac{m}{n}$ and if a_n is negative then $h(3(n-1)+2) \leqq 0$. In any case $h(3(n-1)+2) \leqq \frac{m}{n}$. Take now the function p defined as $p(3(n-1)+1) = 0$, $p(3(n-1)+2) = \frac{1}{\sqrt{n}}$ and $p(3n) = \frac{-1}{\sqrt{n}}$. It is easy to see that p is in the norm closure of $W^0 - L^\infty_+$ but it cannot be in $W^0 - L^\infty_+$ since the converge of $p(3(n-1)+2)$ to 0 is too slow. This reasoning also shows that the element f, described above, cannot be in the norm closure of the set $x + W^0 - L^\infty_+$ for any $x < 1$. □

To remedy this "gap" phenomenon, well known in *infinite* dimensional linear programming, we will use another set to calculate the infimum. The set we will use is precisely the set C introduced in Sects. 2, 3 and 4. In Sect. 4, Theorem 4.2, it is proved that C is weak* closed in L^∞. In the case of processes which are not necessarily continuous, C is the exact substitute for the set $W^0 - L^\infty_+$, so useful in the continuous case. The polar C° of the cone C is by definition

$$C^\circ = \{g \mid g \in L^1\, E[gh] \leqq 0 \text{ for all } h \text{ in } C\}\ .$$

Theorem 5.6

$$M(\mathbf{P}) = \{\mathbf{Q} \mid \mathbf{Q} \in L^1\, \mathbf{Q}[\Omega] = 1 \textit{ and } \mathbf{Q} \in C^\circ\}\ .$$

Proof. If $\mathbf{Q}$ is in $M(\mathbf{P})$ then for H admissible we know by Theorem 2.9 that $H \cdot S$ is a $\mathbf{Q}$-supermartingale. Therefore $\mathbf{E_Q}[h] \leqq 0$ for every h in C. Conversely let $\mathbf{Q}$ be in L^1, of norm 1 and $\mathbf{Q} \in C^\circ$. The set $-L^\infty_+$ is a subset of C and hence every element of C° is in L^1_+. Therefore $\mathbf{Q}$ is a probability measure. If T is a stopping time and S^T is bounded then the random variables $\alpha(S^T_u - S^T_t)\mathbf{1}_A$ for $u \geqq t$, α real and A in $\mathscr{F}_t$, are in C and hence $\mathbf{Q}$ is a local martingale measure for S. □

The following theorem is the precise form of the duality equality stated above. We will prove it for bounded functions, referring to a forthcoming paper of Ansel for the case of measurable functions with bounded negative parts.

Theorem 5.7 *For every f in L^∞ we have*

$$\sup_{\mathbf{Q} \in M^e(\mathbf{P})} \mathbf{E}_\mathbf{Q}[f] = \sup_{\mathbf{Q} \in M(\mathbf{P})} \mathbf{E}_\mathbf{Q}[f]$$

$$= \inf\{x \mid \textit{there is } h \in C \textit{ with } x + h \geqq f\}$$

$$= \inf\{x \mid \textit{there is } h \in C \textit{ with } x + h = f\} .$$

Proof. From the definition of C it follows that $x + h \geqq f$ for h in C if and only if there is h in C with $f = x + h$. The second equality is therefore obvious. From the preceding theorem it follows that

$$\sup_{\mathbf{Q} \in M(\mathbf{P})} \mathbf{E}_\mathbf{Q}[f] \leqq \inf\{x \mid \text{there is } h \in C \text{ with } x + h \geqq f\} .$$

If $z < \inf\{x \mid \text{there is } h \in C \text{ with } x + h \geqq f\}$ then $f - z$ is not an element of the weak* closed cone C. By the Hahn-Banach theorem there is a signed measure $\mathbf{Q} \in L^1$, $\mathbf{E}_\mathbf{Q}[h] \leqq 0$ for all h in C and $\mathbf{E}_\mathbf{Q}[f - z] > 0$. The preceding theorem shows that $\mathbf{Q}$ can normalise as $\mathbf{Q}[\Omega] = 1$ and then it is in $M(\mathbf{P})$. It follows that $z < \mathbf{E}_\mathbf{Q}[f] \leqq \sup_{\mathbf{R} \in M(\mathbf{P})} \mathbf{E}_R[f]$. This shows that

$$\sup_{\mathbf{Q} \in M(\mathbf{P})} \mathbf{E}_\mathbf{Q}[f] \geqq \inf\{x \mid \text{there is } h \in C \text{ with } x + h = f\} . \qquad \square$$

Remark 5.9 The infimum is a minimum since C is weak* and hence norm closed.

Remark 5.10 Let us recall that the dual of L^∞ is $ba(\Omega, \mathscr{F}, \mathbf{P})$, the space of all bounded, finitely additive measures on the sigma-algebra $\mathscr{F}$, absolutely continuous with respect to $\mathbf{P}$. We can try to define the set of all finitely additive measures that can be considered as local martingale measures for S. It is not immediately clear how this can be done in a canonical way. But, if we define $M_{ba}(\mathbf{P}) = \{\mathbf{Q} \mid \mathbf{Q} \in ba, \mathbf{Q}[\Omega] = 1, \mathbf{E}_\mathbf{Q}[h] \leqq 0 \text{ for all } h \text{ in } C\}$, then it is easy to see, via the equality in Theorem 5.8, that $M(\mathbf{P})$ is $\sigma(ba, L^\infty)$ dense in $M_{ba}(\mathbf{P})$. In other words $M_{ba}(\mathbf{P})$ is the $\sigma(L^\infty, L^1)$ closure of $M(\mathbf{P})$ in the space $ba(\Omega, \mathscr{F}, \mathbf{P})$, the dual of L^∞. This is of course the good definition of $M_{ba}(\mathbf{P})$. We remark that the set C has to be used and not just the set W^0. Indeed the Example 5.5 shows that the set $M(\mathbf{P})$ is not necessarily $\sigma(ba, L^\infty)$ dense in the set $\{\mathbf{Q} \mid \mathbf{Q}$ finitely additive, positive, $\mathbf{Q}[\Omega] = 1$ and $\mathbf{E}_\mathbf{Q}[h] = 0$ for all h in $W^0\}$. To see this, we observe that the function f defined in Example 5.5 is not in the norm closure of $x + W^0 - L^\infty_+$ for any $x < 1$. By the Hahn-Banach theorem there is a finitely additive positive probability $\mathbf{Q}$ such that $\mathbf{E}_\mathbf{Q}[f] = 1$ and $\mathbf{E}_\mathbf{Q}[h] = 0$ for all h in W^0. Because $\sup_{\mathbf{Q} \in M(\mathbf{P})} \mathbf{E}_\mathbf{Q}[f] = 1/2$ this element $\mathbf{Q}$ cannot be in the $\sigma(ba, L^\infty)$ closure of the set $M(\mathbf{P})$. This suggests that the "good" definition of such finitely additive measures should use the inequality $\mathbf{E}_\mathbf{Q}[h] \leqq 0$ for all h in C and not only for all h in $W^0 - L^\infty_+$.

6 No free lunch with bounded risk

In this section we will compare the property of no free lunch with vanishing risk (NFLVR) with the previously used property of no free lunch with bounded risk (NFLBR). This property was used in a series of papers: Mc Beth (1992), Delbaen (1992) and Schachermayer (1993). The property (NFLBR) is a generalisation of the property (NFLVR). To define this property we need some more notation. By $\bar{C}$ we denoted the closure of C with respect to the norm topology of L^∞, by $\bar{C}^*$ we will denote the weak* closure of C. The set $\tilde{\bar{C}}$ is the set of all limits of weak* converging *sequences* of elements of C. Although the fact that a convex set in L^∞ is weak* closed if and only if it is sequentially closed for the weak* topology, the closure of a convex set cannot necessarily be obtained by taking all limits of sequences. (In Banach's book (1932) (Annexe théorème 1), one can find for each k, examples of convex sets such that after k iterations of taking weak* limits of sequences, the weak* closure is not obtained but after $k+1$ iterations the closure is found.) Therefore in general, there is a difference between $\bar{C}^*$ and $\tilde{\bar{C}}$ and the use of nets is essential to find the weak* closure of C.

Definition 6.1 *If S is a semi-martingale then we say that S satisfies the property*
(i) **no free lunch with bounded risk** (NFLBR) *if* $\tilde{\bar{C}} \cap L^\infty_+ = \{0\}$
(ii) **no free lunch** (NFL) *if* $\bar{C}^* \cap L^\infty_+ = \{0\}$.

From the definitions and the results of Sect. 3 it follows that (NFL) implies (NFLBR) implies (NFLVR) implies (NA). As regards the notion of no free lunch (NFL), this was introduced by Kreps (1981) and is at the basis of subsequent work on the topic. It requires that there should not exist f_0 in L^∞_+, not identically 0, as well as a net $(f_\alpha)_\alpha$ of elements in C such that f_α converges to f_0 in the weak* topology of L^∞. Because nets are used, there is no bound on the negative part f_α^- of f_α. It is not excluded that e.g. $\| f_\alpha^- \|_\infty$ tends to ∞, reflecting the enormous amount of risk taken by the agent. It is well known that for bounded cadlag adapted processes S, (NFL) (even when defined by simple strategies) is equivalent to the existence of an equivalent martingale measure. ii) See Schachermayer (1993) for a proof of this theorem which is essentially due to Kreps (1981) and Yan (1980). The drawback of this theorem is twofold. First it is stated in terms of nets, a highly non intuitive concept. Second it involves the use of very risky positions. The main theorem of the present paper remedies this drawback. We therefore focus attention on variants of the properties (NFLVR). The following characterisation, the proof of which is almost the same as the proof of Proposition 3.6 and Corollary 3.7, was proved in Schachermayer (1993). The proof makes essential use of the Banach-Steinhaus theorem on the boundedness of weak* convergent *sequences*.

Proposition 6.2 *The semi-martingale S satisfies the condition* (NFLBR) *if and only if for a sequence of* 1*-admissible integrands $(H^n)_{n \geqq 1}$ with final values $g_n = (H^n \cdot S)_\infty$, the condition $g_n^- \to 0$ in probability implies that g_n tends to* 0 *in probability.*

Proof. Suppose that S satisfies the property (NFLBR) and let $(H^n)_{n \geqq 1}$ be a sequence of 1-admissible integrands $(H^n)_{n \geqq 1}$ with final values $g_n = (H^n \cdot S)_\infty$ such that $g_n^- \to 0$ in probability. Suppose that this sequence does not tend to 0 in probability. By selecting a subsequence, still denoted by $(g_n)_{n \geqq 1}$ we may suppose that there is $\alpha > 0$ such that $\mathbf{P}[g_n > \alpha] > \alpha$ for all n. By Lemma A1.1 we may take convex combinations $f_n \in \operatorname{conv}(g_k; k \geqq n)$ that converge in probability to a function f. The negative parts f_n^- still tend to 0 in probability and hence $f: \Omega \to [0, \infty]$. The function f satisfies $\mathbf{P}[f > 0] > 0$. The functions $h_n = \min(f_n, 1)$ are in the convex set C and converge in probability to $h = \min(f, 1)$. The functions h_n are uniformly bounded by 1 and therefore the convergence in probability implies the convergence in the weak* topology of L^∞. The function h is therefore in $\tilde{C}$ and the property (NFLBR) now implies that $h = 0$ almost everywhere. This however is a contradiction to $\mathbf{P}[f > 0] > 0$.

Suppose conversely that S satisfies the announced property. It is clear that S satisfies the no arbitrage property (NA). Suppose now that h_n is a sequence in C that converge weak* to h. We have to prove that $h = 0$ almost everywhere. By the Banach-Steinhaus property on weak* bounded sets, the sequence h_n is uniformly bounded. Without loss of generality we may suppose that it is uniformly bounded by 1 and hence $h_n \geqq -1$ almost surely. Since the sequence h_n tends to h weak* in L^∞ it certainly converges weakly to h in L^2. Therefore there is a sequence of convex combinations $g_n \in \operatorname{conv}(h_k; k \geqq n)$ that converges to h in L^2 and therefore in probability. The sequence g_n is bigger than -1 and by the no arbitrage property g_n is the final value of 1-admissible integrands H_n (see Proposition 3.5). The property of S now says that $h = 0$. □

The difference between (NFLVR) and (NFLBR) is now clear. In the no free lunch with vanishing risk property we deal with sequences such that the negative parts tend to 0 uniformly. In the no free lunch with bounded risk property we only require these negative parts to tend to 0 in probability and remain uniformly bounded!

If the case of an infinite time horizon the set K_0 was defined using general admissible integrands. The infinite time horizon and especially strategies that require action until the very end, are not easy to interpret. It would be more acceptable if we could limit the properties (NFLBR) and (NFLVR) to be defined with integrands having bounded support. The following proposition remedies this. (We recall as already stated in the remark following Corollary 3.4 that an integrand H is of bounded support if H is zero outside a stochastic interval $[\![0, k]\!]$ for some real number k.

Proposition 6.3 (1) *If the semi-martingale S satisfies* (NFLBR) *for integrands with bounded support, then it satisfies* (NA) *for general admissible integrands.*

(2) *If the semi-martingale S satisfies* (NFLVR) *for integrands with bounded support and* (NA) *for general integrands, then it satisfies* (NFLVR) *for general integrands.*

Proof. We start with the remark that if S satisfies (NFLVR) for integrands with bounded support then from Theorem 3.3 it follows that for each admissible H, the limit $(H \cdot S)_\infty = \lim_{t\to\infty} (H \cdot S)_t$ exists and is finite almost everywhere. We now show (1) of the proposition. Let $g = (H \cdot S)_\infty$ for H 1-admissible and suppose that $g \geqq 0$ almost everywhere. Let $g_n = (H \cdot S)_n$. Clearly g_n^- tends to 0 in probability and each g_n is the result of a 1-admissible integrand with bounded support. The property (NFLBR) for integrands with bounded support shows that $g \geqq 0$ implies that $g = 0$. The semi-martingale therefore satisfies (NA) for admissible integrands.

We now turn to (2) of the proposition. Let $g_n = (H^n \cdot S)_\infty$ with H^n admissible, be a sequence such that the sequence g_n^- tends to 0 in L^∞-norm. Because the process S satisfies (NA) it follows from Proposition 3.5 that each H^n is $\|g_n^-\|_\infty$-admissible. For each n we take t_n big enough so that $h_n = (H^n \cdot S)_{t_n}$ is close to g_n in probability, e.g. such that $\mathbf{E}[\min(|h_n - g_n|, 1)] \leqq \frac{1}{n}$. Since each h_n is the result of a $\|g_n^-\|_\infty$-admissible integrand with bounded support, the property (NFLVR) for integrands with bounded support implies that h_n tends to 0 in probability. As a result we obtain that also g_n tends to 0 in probability. □

The Proposition 6.3 allows us to obtain a sharpening of the main theorem of Schachermayer (1993, Theorem 1.6). We leave the economic interpretation to the reader.

Proposition 6.4 *Let $(S_n)_n$ be a locally bounded adapted stochastic process for the discrete time filtration $(\mathscr{F}_n)_n$. If there does not exist an equivalent local martingale measure for S then at least one of the following two conditions must hold*:

(1) *S fails* (NA) *for general admissible integrands, i.e. there is an admissible integrand H such that* $(H \cdot S)_\infty \geqq 0$ a.s. *and* $\mathbf{P}[(H \cdot S)_\infty > 0] > 0$.

(2) *S fails* (NFLVR) *for elementary integrals, i.e. there is a sequence $(H_n)_n$ of elementary integrals such that $(H_n \cdot S) \geqq -n^{-1}$ and $(H_n \cdot S)_\infty$ tends almost surely to a function $f\colon W \to [0, \infty]$ with* $\mathbf{P}[f > 0] > 0$.

Proof. For discrete time processes, elementary integrands and general integrands with bounded support are the same. Therefore if S satisfies both conditions (1) and (2), then by Proposition 6.3, S also satisfies (NFLVR) for general integrands. The main Theorem 1.1 now asserts that S admits an equivalent local martingale measure. The proposition is the contraposition of this statement. □

The following example shows that in general the no free lunch with vanishing risk property for admissible integrands with bounded support does not imply the no free lunch with vanishing risk property for general admissible integrands! As Proposition 6.3 indicates there should be arbitrage for general integrands.

Example 6.5 We give the example in discrete time. The extension to continuous time processes is obvious. The set Ω is the compact space of all sequences of -1 or $+1$: $\{-1, +1\}^{\mathbb{N}}$. The sigma-algebra's $\mathscr{G}_n$ of the filtration are defined as the smallest sigma-algebra's making the first n co-ordinates measurable. On Ω we put two measures $\mathbf{P}$ and $\mathbf{Q}$. The measure $\mathbf{P}$ is defined as the Haar measure, this is the only measure such that the co-ordinates r_n are a sequence of independent, identically distributed variables with $\mathbf{P}[r_n = \pm 1] = 1/2$. The measure $\mathbf{Q}$ is defined as $\frac{1}{2}(\mathbf{P} + \delta_{\mathbf{a}})$, where $\delta_{\mathbf{a}}$ is the Dirac measure giving all its mass to the element $\mathbf{a}$, the sequence identically 1. Define f as the variable $f = -\mathbf{1}_{\{\mathbf{a}\}} + \mathbf{1}_{\Omega\setminus\{\mathbf{a}\}}$. Clearly $\mathbf{E}_{\mathbf{Q}}[f] = 0$. Define now the process S_n by $S_n = \mathbf{E}_{\mathbf{Q}}[f \mid \mathscr{G}_n]$. The sigma-algebra's $\mathscr{F}_n$ of the filtration are defined as the smallest sigma-algebra's making the $S_1, \ldots, S_n$ measurable, i.e. the natural filtration of S. The sigma-algebra $\mathscr{F}$ is generated by the sequence $(S_n)_n$. It is easy to see that on $\mathscr{F}$, S admits only one equivalent martingale measure, namely $\mathbf{Q}$. We will now consider the process S under the measure $\mathbf{P}$. On each sigma-algebra $\mathscr{F}_n$ the two measures, $\mathbf{P}$ and $\mathbf{Q}$, are equivalent. Suppose now that H^n is a sequence of boundedly supported predictable integrands such that $g_n = (H^n \cdot S)_\infty \geqq \dfrac{-1}{n}$ almost everywhere for the measure $\mathbf{P}$. For each n there is k_n big enough such that g_n is measurable for $\mathscr{F}_n$. Therefore also $\mathbf{Q}\left[g_n \geqq \dfrac{-1}{n}\right] = 1$ for each n. Since $\mathbf{Q}$ is a martingale measure for S it follows that $\mathbf{E}_{\mathbf{Q}}[g_n] = 0$ and that the sequence $(g_n)_{n \geqq 1}$ tends to 0 in $L^1(\mathbf{Q})$. Therefore the sequence $(g_n)_{n \geqq 1}$ tends to 0 in probability for the measure $\mathbf{Q}$. Because $\mathbf{P}$ is absolutely continuous with respect to $\mathbf{Q}$ we deduce that g_n tends to 0 in probability for the probability $\mathbf{P}$. This implies that S satisfies the (NFLVR)-property for integrands with bounded support. Because $\mathbf{Q}$ is the only martingale measure for S and because $\mathbf{Q}$ is not absolutely continuous with respect to $\mathbf{P}$, the process S cannot satisfy the no free lunch with vanishing risk property for general integrands (Theorem 1.1). In fact, precisely as predicted in Proposition 6.4, there is already arbitrage if general integrands are allowed! Take e.g. H the predictable process identically one. Because $S_0 = 0$, we have $H \cdot S = S$ and H is therefore admissible. Now S_n tends to f for the probability $\mathbf{Q}$ and hence S_n tends to $\mathbf{1}_{\Omega\setminus\{\mathbf{a}\}}$ for the measure $\mathbf{P}$, i.e. tends to the constant function 1 for the probability $\mathbf{P}$. The process S does not satisfy (NA) for general integrands.

7 Simple integrands

In this section we investigate the consequences of the no-free-lunch like properties when defined with simple integrands. It turns out that there is a relation between the semi-martingale property and the no free lunch with vanishing risk (NFLVR) property for simple integrands. For continuous processes we are able to strengthen Theorem 1.1 and the main theorem of Delbaen (1992).

Definition 7.1 *A* ***simple predictable integrand*** *is a linear combination of processes of the form* $H = f\, \mathbf{1}_{]\!]T_1, T_2]\!]}$ *where* f *is* $\mathscr{F}_{T_1}$ *measurable and* T_1 *and* T_2 *are finite stopping times with respect to the filtration* $((\mathscr{F}_t)_{t \in \mathbf{R}_+})$. *(see also* Protter (1990).*) The expression 'elementary predictable integrand' is reserved* *for processes of the same kind but with the restriction that the stopping times are deterministic times.*

Simple predictable integrands seem to be the easiest strategies an investor can use. The integrand $H = f\ \mathbf{1}_{]\!]T_1, T_2]\!]}$ corresponds to buying f units at time T_1 and selling them at time T_2. The requirement that only stopping times and predictable integrands are used reflects the fact that only information available from the past can be used. The interpretation of simple integrands is therefore straightforward. The use of general integrands however seems more difficult to interpret and their use can be questioned in economic models. It is therefore reasonable to investigate how far one can go in requiring the integrands to be simple.

As pointed out in Sect. 2, we can define the concepts such as no arbitrage, ... with the extra restriction that the integrands are simple. In the case of simple integrands, stochastic integrals are defined for adapted processes. In this section we therefore suppose that S is a cadlag adapted process. The following theorem shows that the condition of no free lunch with vanishing risk for simple integrands, already implies that S is a semi-martingale. In particular this theorem shows that in the main Theorem 1.1, the hypothesis that the price process is a semi-martingale is not a big restriction. The theorem is a version of Theorem 8 in Ansel and Stricker (1993). The proof follows the same lines but control in L^2 norm is replaced by other means. The theorem only uses conditions that are invariant under the equivalent changes of measure. The context of the following theorem is therefore more natural than the same theorem stated in an L^2-environment. We however pay a price by requiring the process S to be locally bounded. A counterexample will show that the local boundedness cannot be dropped.

Theorem 7.2 *Let $S: \mathbf{R}_+ \times \Omega: \to \mathbf{R}$ be an adapted cadlag process. If S is locally bounded and satisfies the no free lunch with vanishing risk property for simple integrands, then S is a semi-martingale.*

The proof requires some intermediate results that have their own merit. Since S is locally bounded there is an increasing sequence of stopping times $(\tau_n)_{n \geqq 1}$ such that each stopped process S^{τ_n} is bounded and $\tau_n \to \infty$ a.s. To prove that S is a semi-martingale it is sufficient to prove that each S^{τ_n} is a semi-martingale. We therefore may and do suppose that S is bounded. To simplify notation we suppose that $|S| \leqq 1$. In the following lemmas it is always assumed that S satisfies (NFLVR) for simple integrands.

Lemma 7.3 *Under the assumptions of Theorem 7.2, let $\mathscr{H}$ be a family of simple predictable integrands each bounded by 1, i.e. $|H_t(\omega)| \leqq 1$ for all t and $\omega \in \Omega$. If*

$$\left\{ \sup_{0 \leqq t} (H \cdot S)_t^- \mid H \in \mathscr{H} \right\} \text{ is bounded in } L^0 \text{, then}$$

$$\left\{ \sup_{0 \leqq t} (H \cdot S)_t^+ \mid H \in \mathscr{H} \right\} \text{ is also bounded in } L^0 .$$

Proof. Suppose that the set $\{\sup_{0\leqq t}(H\cdot S)_t^+ \mid H\in\mathscr{H}\}$ is not bounded in L^0. This implies the existence of a sequence $c_n\to\infty$, $H^n\in\mathscr{H}$ and $\varepsilon>0$ such that $\mathbf{P}[\sup_{0\leqq t}(H\cdot S)_t^+ > c_n]>\varepsilon$. Take K such that $\mathbf{P}[\sup_{0\leqq t}(H\cdot S)_t^- < -K] < \varepsilon/2$ for all n and all $H\in\mathscr{H}$ and define the stopping times

$$T_n' = \inf\{t \mid (H^n\cdot S)_t \geqq c_n\}\,,$$

$$U_n = \inf\{t \mid (H^n\cdot S)_t < -K\}\,.$$

Clearly $(H^n\cdot S)_t \geqq -K-2$ on $[\![0, U_n]\!]$ since each H^n is bounded by 1 and $|S|\leqq 1$. Take $T_n=\min(T_n', U_n)$ and observe that

$$\mathbf{P}[(H^n\cdot S)_{T_n}\geqq c_n,\ \sup_{0\leqq t\leqq T_n}(H^n\cdot S)_t \leqq K+2] \geqq \varepsilon/2\,.$$

Take now $\delta_n\to 0$ so that $\delta_n c_n\to\infty$ and remark that

(a) $(\delta_n H^n \mathbf{1}_{[\![0,T_n]\!]}\cdot S)_\infty^- \leqq \delta_n(K+2)$

(b) $f_n=(\delta_n H^n\mathbf{1}_{[\![0,T_n]\!]}\cdot S)_\infty$ satisfies $\mathbf{P}[f_n\geqq\delta_n c_n]\geqq\varepsilon/2$.

By Lemma A1.1 there are $g_n\in\operatorname{conv}(f_n, f_{n+1}\ldots)$ such that $g_n\to g$ a.e. where $g:\Omega\to[0,\infty]$. Also $\mathbf{P}[g>0]>0$. If $g_n=\lambda_0^n f_n+\ldots+\lambda_k^n f_{n+k}$ is the convex combination, let us put $K^n=\lambda_0^n H^n+\ldots+\lambda_k^n H^{n+k}$. Clearly

(a) $\|(K^n\cdot S)_\infty^-\|\to 0$ and

(b) $(K^n\cdot S)_\infty\to g:\Omega\to[0,\infty]$.

Since $\mathbf{P}[g>0]>0$, this is a contradiction to (NFLVR) with simple integrands. □

Lemma 7.4 *The set*

$$\mathscr{G}=\left\{\sum_{k=0}^{n}(S_{T_{k+1}}-S_{T_k})^2 \mid 0\leqq T_0\leqq T_1\leqq\ldots\leqq T_{n+1}<\infty \text{ stopping times}\right\}$$

is bounded in L^0.

Proof. For $0\leqq T_0\leqq T_1\leqq\ldots\leqq T_{n+1}<\infty$ stopping times put:

$$H=-2\sum_{k=0}^{n} S_{T_k}\mathbf{1}_{]\!]T_k,T_{k+1}]\!]}\,.$$

Because $|S|\leqq 1$ we have that H is bounded by 2. Also

$$(H\cdot S)_\infty=\sum_{k=0}^{n}(S_{T_{k+1}}-S_{T_k})^2 - S_{T_{n+1}}^2+S_{T_0}^2 \quad\text{and hence}$$

$(H\cdot S)_\infty\geqq -1$. The same calculation applied to the sequence of stopping times $\min(T_1,t),\ldots,\min(T_n,t)$ yields $(H\cdot S)_t\geqq -1$ and therefore $\sup_{0\leqq t}(H\cdot S)_t^-\leqq 1$. The preceding lemma now implies that $\mathscr{G}$ is bounded in L^0. □

Proof of Theorem 7.2 We have to show that if H^n is a sequence of simple predictable processes such that $H^n\to 0$ uniformly over $\mathbf{R}_+\times\Omega$, then $(H^n\cdot S)_\infty\to 0$ in probability. By the Bichteler-Dellacherie theorem this implies the classical definition of a semi-martingale. (In Protter (1990) this property is used as the definition of a semi-martingale). It is of course, sufficient to show that the sequence $(H^n \cdot S)_\infty$ is bounded in L^0. If this were not true then there would exist a subsequence of simple integrands, still denoted by $(H^n)_{n \geqq 1}$, such that

(a) $H^n \to 0$ uniformly over $\mathbf{R}_+ \times \Omega$;
(b) $\mathbf{P}[(H^n \cdot S)_\infty \geqq n] \geqq \varepsilon > 0$.
(c) Each H^n can be written as

$$H^n = \sum_{k=0}^{N_n} f_k^n \mathbf{1}_{]\!]T_k^n, T_{k+1}^n]\!]}$$

where $0 \leqq T_0 \leqq \ldots \leqq T_{N_n+1} < \infty$ are stopping times and the functions f_k^n are $\mathscr{F}_{T_k^n}$ measurable functions, bounded by 1.

For each n we put ζ_t^n the process defined as

$$\zeta_t^n = \sum (S_{T_{k+1}^n} - S_{T_k^n})^2 ,$$

where the summation is done over the set of indices $k = 0, \ldots, N_n$ such that $T_{k+1}^n \leqq t$.

Since by the preceding lemma $\mathscr{G}$ is bounded in L^0, there is $c > 0$ such that $\mathbf{P}[\zeta_\infty^n \geqq c] \leqq \varepsilon/2$. Let for each n the stopping time T_n' be defined as $T_n' = \inf\{t \mid \zeta_t^n \geqq c\}$. This definition implies that T_n' takes values in the set $\{T_0, \ldots, T_{n+1}, \infty\}$ and is a stopping time with respect to the discrete time filtration $(\mathscr{F}_{T_k^n})_{k=0,\ldots,N_n+1}$. The bound $\zeta_{T_n'}^n \leqq c + 4$ (since $|S| \leqq 1$) and $\mathbf{P}[T_n' < \infty] \leqq \varepsilon/2$ are straightforward. Take now $K^n = H^n \mathbf{1}_{[\![0, T_n']\!]}$ and observe that $\mathbf{P}[(K^n \cdot S)_\infty \geqq n] \geqq \varepsilon/2$.

Each discrete time, stopped, process $(S_{\min(T_k^n, T_n')})_{k=0,\ldots,N_n+1}$ is now decomposed according to the discrete time Doob decomposition:

$$A_{T_{k+1}^n}^n - A_{T_k^n}^n = \mathbf{E}[S_{\min(T_{k+1}^n, T_n')} - S_{\min(T_k^n, T_n')} \mid \mathscr{F}_{T_k^n}]$$

$$M_{T_{k+1}^n}^n - M_{T_k^n}^n = (S_{\min(T_{k+1}^n, T_n')} - S_{\min(T_k^n, T_n')})$$
$$- \mathbf{E}[(S_{\min(T_{k+1}^n, T_n')} - S_{\min(T_k^n, T_n')}) \mid \mathscr{F}_{T_k^n}]$$

$(M_{T_k^n}^n)_{k=0,\ldots,N_n+1}$ is now a martingale bounded in L^2. Indeed

$$\mathbf{E}[(M_{T_{N_n+1}^n}^n)^2] = \sum_{k=0}^{N_n} \mathbf{E}[(M_{T_{k+1}^n}^n - M_{T_k^n}^n)^2] + \mathbf{E}[(M_{T_0^n}^n)^2]$$

$$\leqq \sum_{k=0}^{N_n} \mathbf{E}[(S_{\min(T_{k+1}^n, T_n')} - S_{\min(T_k^n, T_n')})^2] + \mathbf{E}[(S_{\min(T_0^n, T_n')})^2]$$

$$\leqq \mathbf{E}[(\zeta_{T_n'}^n)^2] + 1 \leqq c + 5 .$$

For each t we put $M_t^n = \mathbf{E}[M_{T_{N_n+1}^n}^n \mid \mathscr{F}_t]$ and we take a cadlag version of this martingale. Because of the optional sampling theorem this definition coincides with the previously given construction of M_t^n for times $t = T_{N_k}^n$. In the definition of H^n we now replace each f_k^n by $\tilde{f}_k^n = f_k^n \operatorname{sign}(A_{T_{k+1}^n}^n - A_{T_k^n}^n)$. The functions

$\tilde{f}_k^n$ are still measurable with respect to the sigma-algebra $\mathscr{F}_{T_k^n}$. The resulting process is denoted by $\tilde{K}^n$ i.e.

$$\tilde{K}^n = \sum_{k=0}^{N_n} \tilde{f}_k^n \mathbf{1}_{]\!]T_k^n, T_{k+1}^n]\!]} .$$

Since $|\tilde{K}^n| \leqq 1$ we still have

$$\mathbf{E}[((\tilde{K}^n \cdot M^n)_{T_{N_n+1}^n})^2] \leqq c + 5 .$$

On the other hand

$$\begin{aligned}(\tilde{K}^n \cdot S)_\infty &= (\tilde{K}^n \cdot S)_{T_{N_n+1}^n} \\ &= (\tilde{K}^n \cdot M)_{T_{N_n+1}^n} + \sum_{k=0}^{N_n} |f_k^n| |A_{T_{k+1}^n}^n - A_{T_k^n}^n| \\ &\geqq (\tilde{K}^n \cdot M)_{T_{N_n+1}^n} + \sum_{k=0}^{N_n} (f_k^n)(A_{T_{k+1}^n}^n - A_{T_k^n}^n) \\ &\geqq (\tilde{K}^n \cdot M)_{T_{N_n+1}^n} + (K^n \cdot S)_\infty - (K^n \cdot M)_{T_{N_n+1}^n} .\end{aligned}$$

Because the sequences $(\tilde{K}^n \cdot M)_{T_{N_n+1}^n}$ and $(K^n \cdot M)_{T_{N_n+1}^n}$ are bounded in L^2 and the sequence $(K^n \cdot S)_\infty^+$ is unbounded in L^0, the sequence $(\tilde{K}^n \cdot S)_\infty$ is necessarily unbounded in L^0. On the other hand $\sup_{0 \leqq t} (\tilde{K}^n \cdot S)_t^-$ is a bounded sequence in L^0. Indeed for $t = T_k^n$ we have

$$(\tilde{K}^n \cdot S)_{T_k^n}^- \leqq (\tilde{K}^n \cdot M)_{T_k^n} \leqq \sup_{0 \leqq t} |(\tilde{K}^n \cdot M)_t| .$$

And for $T_k^n \leqq t \leqq T_{k+1}^n$ we find:

$$(\tilde{K}^n \cdot S)_t^- \leqq (\tilde{K}^n \cdot S)_{T_k^n}^- + |f_k^n| |S_t - S_{T_k^n}| \leqq 2 + \sup_{0 \leqq t} |(\tilde{K}^n \cdot M)_t|$$

and hence

$$\begin{aligned}\left\| \sup_{0 \leqq t} (\tilde{K}^n \cdot S)_t^- \right\|_2 &\leqq 2 + \left\| \sup_{0 \leqq t} (\tilde{K}^n \cdot M)_t \right\|_2 \\ &\leqq 2 + 2\|(\tilde{K}^n \cdot M)_t\|_2 \\ &\leqq 2 + 2\sqrt{c+5} \quad \text{(by Doob's maximum inequality)} .\end{aligned}$$

This proves the boundedness in L^0. From Lemma 2.3 it now follows that $(\tilde{K}^n \cdot S)_t^+$ is bounded in L^0. This contradicts the choice of the sequence $\tilde{K}^n$. □

The following example shows that the requirement that S is locally bounded cannot be dropped. The same notation will also be used in a later example.

Example 7.5 We suppose that on a probability space $(\Omega, \mathscr{F}, \mathbf{P})$ the following sequences of variables are defined: a sequence $(\gamma_n)_{n \geqq 1}$ of Gaussian normalised

$\mathcal{N}(0,1)$ variables, a sequence $(\phi_n)_{n \geqq 1}$ of random variables with distribution $\mathbf{P}[\phi_n = 1] = 2^{-n}$ and $\mathbf{P}[\phi_n = 0] = 1 - 2^{-n}$. All these variables are supposed to be independent. The countable set of rationals in the interval $]0,1[$ is enumerated as $(q_n)_{n \geqq 1}$. Because $\sum \mathbf{P}[\phi_n = 1] < \infty$ the Borel-Cantelli lemma tells us that for almost all $\omega \in \Omega$ there are only a finite number of natural numbers n such that $\phi_n = 1$.

The stochastic process X defined as

$$X_t = \sum_{n; q_n \leqq t} \phi_n \gamma_n$$

is therefore right continuous, even piecewise constant (by the above Borel-Cantelli argument). The natural filtration generated by this process is therefore right continuous (see Protter (1990), p. 16, Theorem 25 for a proof that can be adapted to this case) and so is the filtration augmented with the negligible sets. The filtration so constructed therefore satisfies the usual conditions.

Take now $F:[0,1] \to \mathbf{R}$ a continuous function of unbounded variation, e.g. $F(t) = t \sin\left(\frac{1}{t}\right)$. Let now $S_t = X_t + F(t)$. It is easy to verify that X is an L^2-martingale and hence it is a semi-martingale. If S were a semi-martingale then F would also be a semi-martingale. This however implies that F is of bounded variation. We conclude that S is not a semi-martingale. We will now show that S satisfies the (NFL) property for simple integrands. This certainly implies that S satisfies the (NFLVR) property for simple integrands and it shows that the local boundedness condition in Theorem 7.2 is not superfluous. To verify the (NFL) property with simple integrands let us start with an integrand $H = f \mathbf{1}_{]\!]T;T']\!]}$ where $T \leqq T'$ are two stopping times and f is $\mathscr{F}_T$ measurable. We will show that $H \cdot S$ is not uniformly bounded from below unless $H = 0$. Suppose on the contrary that $\mathbf{P}[\{T < T'\} \cap \{f > 0\}] > 0$ (the case $\{f > 0\}$ is similar). Take t real and q_n rational such that $t < q_n$ and $\mathbf{P}[\{f > 0\} \cap \{T \leqq t\} \cap \{q_n \leqq T'\}] > 0$. Because f is $\mathscr{F}_T$ measurable, $t < q_n$ and T' is a stopping time we obtain that $A = \{f > 0\} \cap \{T \leqq t\} \cap \{q_n \leqq T'\} \in \mathscr{F}_{q_n -}$ and hence is independent of $\phi_n \gamma_n$. Because $\phi_n \gamma_n$ is unbounded from below (and from above for the other similar case) we obtain that $\mathbf{P}[A \cap \{\phi_n \gamma_n < -K\}] > 0$ for all $K > 0$. It is now easy to see that this implies that $H \cdot S$ is unbounded from below. It also follows that the only simple integrand H for which $H \cdot S$ is bounded from below is the zero integrand. Since there are no admissible simple integrands, the (NFL) property with simple integrands is trivially satisfied! □

Theorem 7.2 and the main Theorem 1.1 allow us to strengthen the main theorem in Delbaen (1992). The theorem shows that in the case of continuous price processes and finite horizon, the condition (d) in Delbaen (1992), an equivalent form of the no free lunch with bounded risk for simple integrands, can be relaxed. The case of infinite horizon is already treated in Example 6.5. By using the techniques developed in Schachermayer (1993b) one can translate this example into an example where S is a continuous process.

Theorem 7.6 (a) *If* $S:[0,1]\times\Omega:\to\mathbf{R}$ *is an adapted continuous process, then the condition no free lunch with vanishing risk* (NFLVR) *for simple integrands implies the existence of an equivalent local martingale measure.*

(b) *If* $S:\mathbf{R}_+\times\Omega:\to\mathbf{R}$ *is an adapted continuous process then the condition no free lunch with bounded risk* (NFLBR) *for simple integrands implies the existence of an equivalent local martingale measure.*

Proof. Because the process S satisfies the condition no free lunch with vanishing risk (NFLVR) for simple integrands, it follows from Theorem 7.2 that S is a semi-martingale. General stochastic integrals can now be used. Let H^n be a sequence of general ε_n-admissible integrands where ε_n tends to 0 and let $g_n=(H^n\cdot S)_1$. We have to prove that g_n tends to 0 in probability, which will prove part (a) in view of the main Theorem 1.1. From the theory of stochastic integration (see Chou et al. 1980) we deduce that there are simple integrands L^n such that $\mathbf{P}[\sup_{0\leqq t\leqq 1}|(L^n\cdot S)_t-(H^n\cdot S)_t|\geqq\varepsilon_n]\leqq\varepsilon_n$. For each n we define the stopping time T_n as $\inf\{t|(L^n\cdot S)_t<-2\varepsilon_n\}$. Clearly $\mathbf{P}[T_n<1]\leqq\varepsilon_n$. Since the process S is continuous, the random variables $h_n=(L^n\cdot S)_{T_n}$ are bounded below by $-2\varepsilon_n$ and are therefore results of $2\varepsilon_n$-admissible simple integrands. Because $\mathbf{P}[T_n<1]\leqq\varepsilon_n$ and $\mathbf{P}[\sup_{0\leqq t\leqq 1}|(L^n\cdot S)_t-(H^n\cdot S)_t|\geqq\varepsilon_n]\leqq\varepsilon_n$, the sequence h_n-g_n tends to 0 in probability. From the property no free lunch with vanishing risk (NFLVR) for simple integrands we deduce that h_n and hence g_n tend to 0 in probability. Therefore the property no free lunch with vanishing risk property (NFLVR) is satisfied and by the main Theorem 1.1 there is an equivalent martingale measure.

For the second part we refer to Schachermayer ((1993), Sect. 5, Proposition 5.1). □

The above theorem seems to indicate that for continuous processes simple integrands are sufficient to describe no arbitrage conditions. This is not true in general. The $\mathrm{Bes}^3(1)$ process, $(R_t)_{0\leqq t\leqq 1}$, gives a counterexample. This process can be seen as the Euclidean norm of a three dimensional Brownian motion starting at the point $(1,0,0)$ of $\mathbf{R}^3$. It plays a major role in the theory of continuous martingales and Brownian motion, see Revuz and Yor (1991) for details. The process R satisfies the no arbitrage (NA) property for simple integrands but fails the no arbitrage (NA) property for general integrands. We refer to our forthcoming paper Delbaen and Schachermayer (1993b) for the details. The inverse of this process, $L=R^{-1}$, a local martingale, has been used by Delbaen and Schachermayer (1993).

As a general question one might ask whether for continuous processes the no arbitrage (NA) property for general integrands is sufficient for the existence of an equivalent local martingale measure. The following example shows that this is not true.

Example 7.7 We take a standard Wiener process W with its natural filtration $(\mathscr{G}_t)_{0\leqq t\leqq 1}$. Before we define the price process S, we first define a local martingale of exponential type by:

$$L_t = \exp\left(-(f\cdot W)_t - \frac{1}{2}\left(\int_0^t f^2(u)\,du\right)\right) \quad \text{if } t < 1 \quad \text{and}$$

$$L_1 = 0$$

where f is the deterministic function defined as $f(t) = \frac{1}{\sqrt{1-t}}$.

We define the stopping time T as $T = \inf\{t \mid L_t \geqq 2\}$. The stopped process L^T is a bounded martingale starting at zero. Clearly $L_T = 2$ if $T < 1$ and equals 0 if $T = 0$. Therefore $\mathbf{P}[T < 2] = 1/2$. We now define the price process by its differential

$$dS_t = dW_t + \frac{1}{\sqrt{1-t}}\,dt \quad \text{if } t \leqq T \quad \text{and}$$

$$dS_t = 0\,, \quad \text{if } t \geqq T\,.$$

The filtration is now defined as $(\mathscr{F}_t)_{0 \leqq t \leqq 1} = (\mathscr{G}_{\min(t,T)})_{0 \leqq t \leqq 1}$. Except for sets of measure zero, this is also the natural filtration of the process S and of the stopped Wiener process W^T. All local martingales with respect to this filtration are stochastic integrals with respect to the Wiener process (stopped at T) (see Revuz and Yor (1991), p. 187, Theorem 4.2 and stop all the local martingales at the stopping time T). Girsanov's formula therefore implies that the only probability measure $\mathbf{Q}$, absolutely continuous with respect to $\mathbf{P}$ and for which S is a local martingale, is precisely the measure $\mathbf{Q}$ defined through its density on $\mathscr{F}_1$ as $d\mathbf{Q} = L_T\,d\mathbf{P}$. As we shall see, S satisfies the property of no arbitrage (NA). Important in the proof of this, is the fact that for $t < 1$, the measures $\mathbf{Q}$ and $\mathbf{P}$ are equivalent on $\mathscr{F}_t$ (the density L_t^T is strictly positive). Because the process S is continuous the proof that S satisfies (NA) reduces to verifying the statement that for H admissible, $(H\cdot S)_1$ cannot be almost everywhere positive without being zero a.s. Take H admissible and suppose that $(H\cdot S)_1 \geqq 0$, $\mathbf{P}$ a.s. This certainly implies that $(H\cdot S)_1 \geqq 0$, $\mathbf{Q}$ a.s. Because S is a continuous $\mathbf{Q}$-local martingale, we know that $H\cdot S$ is a continuous $\mathbf{Q}$-local martingale and because H is admissible for $\mathbf{Q}$, $\mathbf{Q}$ being absolutely continuous with respect to $\mathbf{P}$, $H\cdot S$ is a $\mathbf{Q}$-supermartingale. From this it follows that $\mathbf{E}_{\mathbf{Q}}[(H\cdot S)_1] \leqq 0$ and by positivity of $(H\cdot S)_1$, this in turn implies that $(H\cdot S)_1 = 0$, $\mathbf{Q}$ a.s. Under the probability $\mathbf{Q}$, the process S is a local martingale and hence satisfies (NA) with respect to $\mathbf{Q}$! For each $\varepsilon > 0$, let now V be the stopping time defined as $\inf\{t \mid (H\cdot S)_t \geqq \varepsilon\}$. The integrand $K = (1_{[\![0,V]\!]}H)$ is clearly admissible and $(K\cdot S)_1 = 0$ on $\{V = 1\}$, whereas on $\{V < 1\}$ the outcome is ε, i.e. strictly positive. The (NA) property for S (under $\mathbf{Q}$!) implies that $\mathbf{Q}[V < 1] = 0$. In other words the process $H\cdot S$ never exceeds ε $\mathbf{Q}$ a.s. This implies $(H\cdot S)_t \leqq 0$, $\mathbf{Q}$ a.s. for all $t < 1$. Because $\mathbf{Q}$ and $\mathbf{P}$ are equivalent on $\mathscr{F}_t$ for $t < 1$, this is the same as $(H\cdot S)_t \leqq 0$, $\mathbf{P}$ a.s. for all $t < 1$. From this and the continuity of the process $(H\cdot S)$ we deduce that $(H\cdot S)_1 \leqq 0$,

P a.s. This in turn implies that $(H \cdot S)_1 = 0$, **P** a.s. The process S therefore satisfies (NA) under the probability **P**. □

We now give some more examples motivating the introduction of general integrands. As seen in the above theorems and examples, the case of continuous processes can essentially be reduced to simple integrands. The following examples show that for general semi-martingales the no free lunch with bounded risk (NFLBR) property for simple integrands is not sufficient to imply the existence of an equivalent local martingale measure.

The examples are very similar in nature; the problems arise from the fact that the jumps do not occur at an increasing sequence $(T_n)_{n \geqq 1}$ of *predictable* stopping times (a case already solved in Schachermayer (1993)). In our examples the jumps occur at an *increasing* sequence of *accessible* stopping times, similarly as in Example 7.5. The first example of this kind is an unbounded process but it contains all the ingredients and the general idea. The second example of this kind gives a bounded process. Of course the price to pay is the use of more technique.

Example 7.8 The first example uses the process X introduced in Example 7.5. The semi-martingale S we will need is defined as $S_t = X_t + t$. The process S is now a special semi-martingale and again if H is simple predictable with $H \cdot S$ bounded from below then $H = 0$. Therefore S trivially satisfies the no free lunch (NFL) property with simple integrands. If however we put $H = \mathbf{1}_{([0,1] \setminus \mathbf{Q}) \times \Omega}$ (sell before each rational and buy back immediately after it) we have $(H \cdot S)_t = t$ (for $0 \leqq t \leqq 1$) and this violates (NA) for general integrands. If **Q** were an equivalent local martingale measure for the process S, then because $H = \mathbf{1}_{([0,1] \setminus \mathbf{Q}) \times \Omega}$ is bounded, $H \cdot S$ is also a local martingale (see Protter 1990, p. 142, Theorem 2.9). This is absurd. □

The previous example has at least one disadvantage: the process S is unbounded. The next example overcomes this problem. This time we will work with a doubly indexed sequence of Rademacher variables $(r_{n,m})_{n \geqq 1, m \geqq 1}$ i.e. variables with distribution $\mathbf{P}[r_{n,m} = 1] = \mathbf{P}[r_{n,m} = -1] = 1/2$, and with a doubly indexed sequence of variables $(\phi_{n,m})_{n \geqq 1, m \geqq 1}$ with the property $\mathbf{P}[\phi_{n,m} = 1] = 2^{-(n+m)}$ and $\mathbf{P}[\phi_{n,m} = 0] = 1 - 2^{-(n+m)}$. We also need a sequence of Brownian motions W^n starting at 0. All these variables and processes are supposed to be independent. The rationals in $]0,1[$ are again enumerated as $(q_n)_{n \geqq 1}$. We first define the L^2 martingales Y^m as:

$$Y_t^m = \sum_{n; q_n \leqq t} \phi_{n,m} r_{n,m} .$$

The Borel-Cantelli implies, as in Example 7.5, that each Y^m is piecewise constant. We define the stopping time T_m as:

$$T_m = \min(\inf\{t \mid |W_t^m| = m \text{ or } Y_t^m \neq 0\}, 1) .$$

We make the crucial observation that

$$\mathbf{P}[T_m < 1] \leqq \mathbf{P}\left[\sup_{0 \leqq t \leqq 1} |W_t^m| > m\right] + \sum_{m \geqq 1} 2^{-(n+m)}$$

$$\leqq \sqrt{\frac{2}{\pi}\frac{1}{m}} e^{-m^2/2} + 2^{-m}$$

and hence $\sum \mathbf{P}[T_m < 1] < \infty$. This implies, via the Borel-Cantelli lemma, that for almost all $\omega \in \Omega$, $T_m(\omega)$ becomes eventually 1.

The process Z^m is now defined as

$$Z_t^m = Y_t^m + \alpha_m(W_t^m + m^2 t) \quad \text{for } t \leqq T_m$$

$$= Y_{T_m}^m + \alpha_m(W_{T_m}^m + m^2 T_m) \quad \text{for } T_m \leqq t \leqq 1 .$$

The sequence α_m will be chosen later, but will satisfy $0 < \alpha_m \leqq 1$.

The process Z^m is clearly bounded by $1 + (m + m^2)\alpha_m \leqq 1 + m + m^2$.

Finally we define

$$S_t = \tfrac{1}{2} Z_t^1 \quad \text{for } 0 \leqq t \leqq 1$$

$$= S_{m-1} + 2^{-m} Z_{t-(m-1)}^m \quad \text{for } m - 1 \leqq t \leqq m .$$

The process S is cadlag and $|S| \leqq \sum_{m \geqq 1} 2^{-m}(1 + m + m^2)\alpha_m \leqq 24$. It is a semi-martingale with decomposition $S = M + A$, where A is given by the recurrence relations

$$A_{m-1+t} - A_{m-1} = 2^{-m}\alpha_m m^2 t \quad \text{for } t \leqq T_m \quad \text{and}$$

$$A_{m-1+t} - A_{m-1} = 2^{-m}\alpha_m m^2 T_m \quad \text{for } T_m \leqq t \leqq 1 .$$

The martingale M is uniformly bounded on each interval $[\![0, m]\!]$.

With respect to its natural filtration, augmented with the zero sets, S is a special semi-martingale and the filtration satisfies the usual assumptions. The last statement is not trivial to verify but it follows from the same property of the filtration of the Brownian motion.

Lemma 7.9 *For each sequence* $(\alpha_m)_{m \geqq 1}$ *in* $]0, 1]$, *the process* S *fails the equivalent (local) martingale property.*

Proof. Consider the sequence $(H^m)_{m \geqq 1}$ defined as

$$H^m = \alpha_m^{-1} m^{-2} 2^m \mathbf{1}_{(]m-1, m]\setminus \mathbf{Q}) \times \Omega} .$$

Each H^m is a deterministic process, hence predictable. The process $(H^m \cdot S)$ is uniformly bounded from below by -1 and $((H^m \cdot S)_\infty)_{m \geqq 1}$ equals $\frac{1}{m^2} W_{T_m}^m + T_m \geqq T_m - \frac{1}{m}$.

Because $T_m = 1$ for m big enough we see that $(H^m \cdot S)_\infty$ tends to 1 for m tending to ∞. This clearly violates (NFLVR). Because of the main Theorem 1.1 we see that S cannot have an equivalent martingale measure. □

Lemma 7.10 *If* $(\alpha_m)_{m \geqq 1}$ *is a sequence in* $]0, 1]$ *such that* $\alpha_m \to 0$ *fast enough, then* S *satisfies* (NFLBR) *for simple integrands.*

(By fast enough we mean that for all m_0 *we have:*

$$\sum_{m > m_0} 2^{m+1} m^2 \alpha_m < \frac{\beta_{m_0}}{2m_0} \quad \textit{where } \beta_{m_0} = \exp(-3\, m_0^5)) \,.$$

Proof. For each m natural number, we know that the process S^m, i.e. S stopped at m, admits an equivalent martingale measure $\mathbf{Q}_m$. Indeed we can use a Girsanow transformation to find an equivalent martingale measure such that for k fixed, the process $(W_t^k + k^2 t)_{0 \leqq t \leqq 1}$ stopped at T_k is a martingale. The density of this measure is given by $\exp(\delta W^k_{T_k} - \frac{1}{2}\delta^2 T_k)$ where $\delta = -k^2$. This density is bounded above by $\exp(k^3)$ and below by $\exp(-k^3 - \frac{1}{2}k^4)$. The density of $\mathbf{Q}_m$ on $\mathscr{F}_m$ is therefore bounded below by $\exp(-\sum_{k=1}^m (k^3 + \frac{1}{2}k^4)) \geqq \exp(-2m^5)$ and bounded above by $\exp(\sum_{k=1}^m k^3) \leqq \exp(m^4)$. Under the measure $\mathbf{Q}_m$ the process S^m is a martingale and hence for each H that is 1-admissible, $H \cdot S^m$ is a $\mathbf{Q}_m$-supermartingale (by Theorem 2.9) and hence for each 1-admissible integrand we find

$$\mathbf{E}_{\mathbf{Q}_m}[(H \cdot S)_m^+] \leqq \mathbf{E}_{\mathbf{Q}_m}[(H \cdot S)_m^-] \quad \text{and hence}$$

$$\exp(-2m^5)\,\mathbf{E}_{\mathbf{P}}[(H \cdot S)_m^+] \leqq \exp(m^4)\,\mathbf{E}_{\mathbf{P}}[(H \cdot S)_m^-] \quad \text{and}$$

$$\mathbf{E}_{\mathbf{P}}[(H \cdot S)_m^-] \geqq \beta_m \mathbf{E}_{\mathbf{P}}[(H \cdot S)_m^+] \quad \text{for } \beta_m = \exp(-3m^5)\,.$$

We will show that if $\alpha_m \to 0$ as announced, the process S satisfies (NFLBR) with simple integrands.

Suppose on the contrary that S does not satisfy the (NFLBR) property for simple integrands. We then choose H^j simple, predictable, 1-admissible such that $(H^j \cdot S)_\infty$ tends to $f_0 \geqq 0$ where $\mathbf{P}[f_0 > 0] > 0$. Find m_0 so that $\mathbf{E}_{\mathbf{P}}[\min(f_0, 1)] > \frac{2}{m_0}$. For each j we define the stopping time U_j as $\inf\{t \mid (H^j \cdot S)_t \geqq 1\}$ and let $L^j = H^j \mathbf{1}_{[\![0, U_j]\!]}$. For each j the simple predictable process L^j is still 1-admissible and $(L^j \cdot S)_\infty \geqq \min((H^j \cdot S)_\infty, 1)$, therefore $\liminf_{j \to \infty} (L^j \cdot S)_\infty \geqq \min(f_0, 1)$. Each L^j is of the form $\sum_{k=1}^n f_k \mathbf{1}_{]\!]V_{k-1}, V_k]\!]}$ where f_k is $\mathscr{F}_{T_k}$ measurable and $V_0 \leqq V_1 \leqq \ldots \leqq V_n < \infty$ are stopping times.

If $]\!]V_{k-1}, V_k]\!] \cap]\!]m-1, m-1+T_m]\!]$ is not equivalent to the zero process, then the probability of a jump between V_{k-1} and V_k is strictly positive by the same arguments as in Example 7.5. Because the jumps of S are positive or negative with the same probability we conclude that the downward jump of $(L^j \cdot S)$ cannot be smaller than -2. (Indeed the process is always bigger than -1 and is stopped when it hits the level 1). We conclude that also the positive jump is bounded by 2. Therefore $|L^j_{T_m} \Delta S_{T_m}| \leqq 2$. We conclude that $|L^j| \leqq 2^{m+1}$ on $]\!]m-1, m-1+T_m]\!]$. Because we stopped the process $(L^j \cdot S)$ when it exceeds the level 1, we see that:

$$\min((L^j \cdot S)_m, 1) - \min((L^j \cdot S)_{m-1}, 1) \leqq (L^j \cdot S)_m - (L^j \cdot S)_{m-1}\,.$$

The process L^j is bounded in intervals $[0, m]$ and because S is also uniformly bounded with only one jump in each interval $[k, k+1]$, the semimartingale $L^j \cdot S$ is locally bounded, therefore special and decomposed as $L^j \cdot S = L^j \cdot M + L^j \cdot A$. The local martingale part is a square integrable martingale and hence:

$$\mathbf{E_P}[(L^j \cdot M)_m - (L^j \cdot M)_{m-1}] = 0 .$$

This yields the following estimates:

$$\begin{aligned}
&\mathbf{E_P}[\min((L^j \cdot S)_m, 1) - \min((L^j \cdot S)_{m-1}, 1)] \\
&\quad \leqq \mathbf{E_P}[(L^j \cdot S)_m - (L^j \cdot S)_{m-1}] \\
&\quad \leqq \mathbf{E_P}[(L^j \cdot A)_m - (L^j \cdot A)_{m-1}] \\
&\quad \leqq \mathbf{E_P}\left[\int_{]m-1, m]} L_u^j \alpha_m m^2 \, du \right] \\
&\quad \leqq 2^{m+1} m^2 \alpha_m .
\end{aligned}$$

This implies that

$$\begin{aligned}
\mathbf{E_P}[\min((L^j \cdot S)_{m_0}, 1)] &\geqq \mathbf{E_P}[\min((L^j \cdot S)_\infty, 1)] - \sum_{m > m_0} 2^{m+1} m^2 \alpha_m \\
&\geqq \mathbf{E_P}[\min((L^j \cdot S)_\infty, 1)] - \frac{\beta_{m_0}}{2m_0} \text{(by the choice of } \alpha_m) .
\end{aligned}$$

Because

$$\liminf_{j \to \infty} \mathbf{E_P}[\min((L^j \cdot S)_\infty, 1)] > \frac{2}{m_0} \quad \text{we can deduce that}$$

$$\liminf_{j \to \infty} \mathbf{E_P}[\min((L^j \cdot S)_{m_0}, 1)] > \frac{2}{m_0} - \frac{\beta_{m_0}}{2m_0} > \frac{1}{m_0} .$$

We may now suppose that $\mathbf{E_P}[\min((L^j \cdot S)_{m_0}, 1)] > \frac{1}{m_0}$ for all j. Because of the choice of β_m we also see that

$$\begin{aligned}
&\mathbf{E_P}[\min((L^j \cdot S)_{m_0}^-, 1)] \\
&\quad \geqq \beta_m \mathbf{E_P}[\min((L^j \cdot S)_{m_0}^+, 1)] \geqq \beta_m \mathbf{E_P}[\min((L^j \cdot S)_{m_0}, 1)] > \frac{\beta_m}{m_0} .
\end{aligned}$$

Let the set A_j be defined as $A_j = \{(L^j \cdot S)_{m_0} < 0\}$.

Because $\liminf_{j \to \infty} \min((L^j \cdot S)_\infty, 1) \geqq \min(f_0, 1)$ we also have that

$$\liminf_{j \to \infty} (1_{A_j} \min((L^j \cdot S)_\infty, 1)) \geqq \liminf_{j \to \infty} (1_{A_j} \min(f_0, 1)) .$$

An application of Fatou's lemma yields that

$$\begin{aligned}
&\mathbf{E}_{\mathbf{P}}\left[\liminf_{j\to\infty}(1_{A_j}\min(f_0,1))\right]\\
&\leqq \mathbf{E}_{\mathbf{P}}\left[\liminf_{j\to\infty} 1_{A_j}\min((L^j\cdot S)_\infty,1)\right]\\
&\leqq \liminf_{j\to\infty}\mathbf{E}_{\mathbf{P}}[1_{A_j}\min((L^j\cdot S)_\infty,1)]\\
&\leqq \liminf_{j\to\infty}\mathbf{E}_{\mathbf{P}}[1_{A_j}\min((L^j\cdot S)_{m_0},1)]\\
&\quad+\sum_{m>m_0}\mathbf{E}_{\mathbf{P}}[\min((L^j\cdot S)_m,1)-\min((L^j\cdot S)_{m-1},1)]\\
&\leqq -\frac{\beta_{m_0}}{m_0}+\sum_{m>m_0}2^{m+1}m^2\alpha_m\\
&\leqq -\frac{\beta_{m_0}}{2m_0}.
\end{aligned}$$

This is clearly a contradiction to $f_0 \geqq 0$. □

Appendix 1: Some measure theoretical lemmas

In this appendix we prove two lemmas we used at several places. We assume that, especially regarding the second lemma, the results are known, but we could not find a reference. We therefore give full proofs and we also add some remarks that are of independent interest but are not used elsewhere in this paper. The first lemma was already proved in Schachermayer (1992, Lemma 3.5). We give a similar but simpler proof.

Lemma A1.1 *Let* $(f_n)_{n\geqq 1}$ *be a sequence of* $[0,\infty[$ *valued measurable functions on a probability space* $(\Omega,\mathcal{F},\mathbf{P})$. *There is a sequence* $g_n\in\operatorname{conv}(f_n,f_{n+1},\dots)$ *such that* $(g_n)_{n\geqq 1}$ *converges almost surely to a* $[0,\infty]$ *valued function* g.

If $\operatorname{conv}(f_n;\, n\geqq 1)$ *is bounded in* L^0, *then* g *is finite almost surely. If there are* $\alpha>0$ *and* $\delta>0$ *such that for all* n: $\mathbf{P}[f_n>\alpha]>\delta$, *then* $\mathbf{P}[g>0]>0$.

Proof. Let $u:\mathbf{R}_+\cup\{+\infty\}\to[0,1]$ be defined as $u(x)=1-e^{-x}$. Economists may see u as a utility function but there is no need to. Define s_n as

$$s_n=\sup\{\mathbf{E}[u(g)]\,|\,g\in\operatorname{conv}(f_n,f_{n+1},\dots)\}$$

and choose $g_n\in\operatorname{conv}(f_n,f_{n+1},\dots)$ so that

$$\mathbf{E}[u(g_n)]\geqq s_n-\frac{1}{n}.$$

Clearly s_n decreases to $s_0\geqq 0$ and $\lim_{n\to\infty}E[u(g_n)]=s_0$. We shall show that the sequence $(g_n)_{n\geqq 1}$ converges in probability to a function g. We will work with the compact (metrisable) space $[0, \infty]$. A sequence $(x_n)_{n \geqq 1}$ of elements of $[0, \infty]$ is a Cauchy sequence in $[0, \infty]$ if and only if for each $\alpha > 0$ there is n_0 so that for all $n, m \geqq n_0$ we have $|x_n - x_m| \leqq \alpha$ or $\min(x_n, x_m) \geqq \alpha^{-1}$. From the properties of u it also follows that for $\alpha > 0$ there is $\beta > 0$, so that $|x - y| > \alpha$ and $\min(x, y) \leqq \alpha^{-1}$, implies $u\left(\frac{x+y}{2}\right) > \frac{1}{2}(u(x) + u(y)) + \beta$.

We can now easily proceed with the proof of the lemma. By the observation on the topology of $[0, \infty]$ we have to show that $\lim_{n,m\to\infty} \mathbf{P}[|g_n - g_m| > \alpha$ and $\min(g_n, g_m) \leqq \alpha^{-1}] = 0$.

For given $\alpha > 0$ we take β as above and we obtain

$$\mathbf{E}\left[u\left(\frac{g_n + g_m}{2}\right)\right] \geqq \tfrac{1}{2}\mathbf{E}[u(g_n)] + \tfrac{1}{2}\mathbf{E}[g_m]$$

$$+ \beta\mathbf{P}[|g_n - g_m| > \alpha \text{ and } \min(g_n, g_m) < \alpha^{-1}]\,.$$

By construction $\mathbf{E}\left[u\left(\frac{g_n + g_m}{2}\right)\right] \leqq s_n$, but by concavity of u we have

$$\mathbf{E}\left[u\left(\frac{g_n + g_m}{2}\right)\right] \geqq \tfrac{1}{2}(\mathbf{E}[u(g_n)] + \mathbf{E}[u(g_m)])\,.$$

From this it follows

$$\beta\mathbf{P}[|g_n - g_m| > \alpha \text{ and } \min(g_n, g_m) < \alpha^{-1}]$$

$$\leqq \mathbf{E}\left[u\left(\frac{g_n + g_m}{2}\right)\right] - \tfrac{1}{2}(\mathbf{E}[u(g_n)] + \mathbf{E}[u(g_m)])\,.$$

The choice of the sequence $(g_n)_{n \geqq 1}$ implies that the right hand side tends to 0. We therefore proved that $(g_n)_{n \geqq 1}$ is a Cauchy sequence in probability and hence there is a function $g: \Omega \to [0, \infty]$ so that g_n converges to g in probability. If one wants a sequence converging almost surely one can pass to a subsequence.

If $\operatorname{conv}(f_n; n \geqq 1)$ is bounded in L^0 then for each $\varepsilon > 0$ there is N so that $\mathbf{P}[h > N] < \varepsilon$ for all $h \in \operatorname{conv}(f_n; n \geqq 1)$. In particular this implies that $\mathbf{P}[g_n > N] < \varepsilon$ and hence $\mathbf{P}[g > N] \leqq \varepsilon$. The function g so obtained is therefore finite almost surely.

If $\mathbf{P}[f_n > \alpha] > \delta > 0$ for each n and fixed $\alpha > 0$, we obtain that $\mathbf{E}[u(g_n)] \geqq \delta u(\alpha) > 0$. Since g_n tends to g we find $u(g_n) \to u(g)$ and by the bounded convergence theorem we obtain $\mathbf{E}[u(g)] \geqq \delta u(\alpha) > 0$ and therefore $\mathbf{P}[g > 0] > 0$. □

Remark 1 If $(f_n)_{n \geqq 1}$ is a sequence of $[0, \infty]$ valued measurable functions then the same conclusion can be obtained. The proof is the same up to minor changes in the notation. The reader can convince himself that there is almost no gain in generality.

Remark 2 If $(f_n)_{n \geqq 1}$ is a sequence of $\mathbf{R}$-valued measurable functions such that $\operatorname{conv}(f_n^-; n \geqq 1)$ is bounded in L^0, then there are $g_n \in \operatorname{conv}(f_n; n \geqq 1)$ so that g_n converges almost surely to a $]-\infty, +\infty]$ valued measurable function g.

Proof. We first take convex combinations of $\{f_n^-; n \geqq 1\}$ that converge almost surely. Since conv $\{f_n^-; n \geqq 1\}$ is bounded in L^0, the limit is finite almost surely. We now apply the lemma to the same convex combination of f_n^+. This procedure yields convex combinations of the original sequence $(f_n)_{n \geqq 1}$, converging almost surely to a $]-\infty, +\infty]$ valued function. □

Remark 3 If in Remark 2 we only require that $\{f_n^-; n \geqq 1\}$ is bounded in L^0 then the conclusion breaks down. Indeed take $(f_n)_{n \geqq 1}$ a sequence of 1-stable (see Loève 1978 for a definition) independent random variables. If there were convex combinations converging a.s. we could make the convex combinations so that $g_k \in \operatorname{conv}(f_{n_k+1}, \ldots, f_{n_{k+1}})$ where $n_1 < n_2 < \ldots$. This implies that $(g_k)_{k \geqq 1}$ is an independent sequence. Since convex combinations of independent 1-stable variables are 1-stable this would produce an iid sequence converging almost surely, a contradiction.

Remark 4 If in the setting of Lemma A1.1 the sequence $\{f_n, n \geqq 1\}$ is bounded in L^0, but $\operatorname{conv}(f_n; n \geqq 1)$ is not bounded in L^0, then the procedure used in the proof does not necessarily yield a function g that is finite almost surely. The next example shows that there is a sequence $\{f_n; n \geqq 1\}$ bounded in L^0 and such that *every* g that is a limit of functions $g_n \in \operatorname{conv}(f_n, f_{n+1}, \ldots)$, is identically $+\infty$. Before we give the construction let us recall some results from the theory of Brownian motion (see Revuz and Yor (1991) for details). If $(B_t)_{0 \leqq t}$ is a standard 1-dimensional Brownian motion, let us denote by T_β the stopping time defined as $T_\beta = \inf\{t \mid B_t = \beta\}$. It is known (see Revuz and Yor, (1991), p. 67) that for $\beta > 0$, $T_\beta < \infty$ a.s. and for each $u \geqq 0$: $\mathbf{E}[\exp(-uT_\beta)] = \exp(-\beta\sqrt{2u})$. It follows that if f has the same distribution as T_β, then for $\lambda > 0$, λf has the same distributions as $T_{(\lambda)^{1/2}\beta}$. If $f_1 \ldots f_N$ are independent and have the same distribution as $T_{\beta_1}, \ldots, T_{\beta_N}$ then $f_1 + \quad + f_N$ has the same distribution as $T_{\beta_1 + \ldots + \beta_N}$ (this follows easily from the interpretation of f_n as the hitting time of β_n). Take now $(f_n)_{n \geqq 1}$ a sequence of independent identically distributed variables, each having the same distribution as T_1. Suppose that $g_n \in \operatorname{conv}(f_n, f_{n+1}, \ldots)$ and $g_n \to g$ a.e. We will show $g = +\infty$ a.e. We can assume that the functions g_n are independent, eventually we take subsequences. Each g_n has a distribution of the form

$$\lambda_1^n f_1 + \ldots + \lambda_{N_n}^n f_{N_n}$$

where $(\lambda_1^n, \ldots, \lambda_{N_n}^n)$ is a convex combination. From preceding considerations it follows that the distribution of g_n is T_{α_n} where $\alpha_n = \sum_1^{N_n} \sqrt{\lambda_i^n} \geqq 1$. The 0–1 law gives us that either $g = +\infty$ or that $\mathbf{P}[g < \infty] = 1$. In this case we conclude that there is a real number α such that $\alpha_n \to \alpha \geqq 1$ and g has the same distribution as T_α. From the 0–1 law it follows again that the distribution of g is degenerate, impossible if $\alpha \geqq 1$. Therefore $g = +\infty$ identically. □

The following lemma is quite simple, it was used in the proof of Lemma 4.7.

Lemma A1.2 *Let $(g_k)_{1 \leqq k \leqq n}$ be non negative functions defined on the probability space $(\Omega, F, \mathbf{P})$. Suppose that there are positive numbers $(a_k)_{1 \leqq k \leqq n}$ as well* *as* $\delta > 0$ *so that for every* k: $\mathbf{P}[g_k \geqq a_k] \geqq \delta > 0$. *If* $g = \sum_{j=1}^n g_j$ *then for all* $0 < \eta < 1$ *we have* $\mathbf{P}[g \geqq \eta(\sum_{j=1}^n a_j)\delta] \geqq \frac{\delta(1-\eta)}{1-\eta\delta}$.

Proof. Let $A = \{g \geqq (\sum_{j=1}^n a_j)\delta\eta\}$. Clearly

$$\mathbf{E}[g\mathbf{1}_{A^c}] \leqq \left(\sum_{j=1}^n a_j\right)\delta\eta\,\mathbf{P}[A^c] = \left(\sum_{j=1}^n a_j\right)\delta\eta(1-\mathbf{P}[A])$$

On the other hand

$$\begin{aligned}\mathbf{E}[g\mathbf{1}_{A^c}] &= \left(\sum_{j=1}^n \mathbf{E}[g_j\mathbf{1}_{A^c}]\right)\\ &\geqq \left(\sum_{j=1}^n a_j\mathbf{P}[A^c \cap \{g_j \geqq a_j\}]\right)\\ &\geqq \left(\sum_{j=1}^n a_j(\mathbf{P}[g_j \geqq a_j] - \mathbf{P}[A])\right)\\ &\geqq \left(\sum_{j=1}^n a_j\right)\delta - \left(\sum_{j=1}^n a_j\right)\mathbf{P}[A]\,.\end{aligned}$$

Both inequalities imply

$$\left(\sum_{j=1}^n a_j\right)\mathbf{P}[A](1-\delta\eta) \geqq \left(\sum_{j=1}^n a_j\right)\delta(1-\eta)\,.$$

We may of course suppose that $(\sum_{j=1}^n a_j) > 0$ and this yields the desired result $\mathbf{P}[A] \geqq \frac{\delta(1-\eta)}{1-\delta\eta}$ □

Corollary A1.3 *If* $(g_j)_{1 \leqq j \leqq n}$ *are non negative functions defined on the probability space* $(\Omega, \mathscr{F}, \mathbf{P})$ *and if for* $j = 1, \ldots, n$ *we have* $\mathbf{P}[g_j \geqq a] \geqq b$ *where* a, $b > 0$, *then for* $g = (\sum_{j=1}^n g_j)$ *we have* $\mathbf{P}\left[g \geqq \frac{nab}{2}\right] \geqq \frac{b}{2}$.

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