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Hölder continuity of normalized solutions of the Schrödinger equation

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1 Introduction and statement of results

We investigate the behaviour of local weak solutions u of the Schrödinger equation

$$\left(-\frac{1}{2}\Delta + V\right)u = 0 \tag{1.1}$$

with a potential $V \geqq 0$ on $\mathbb{R}^d$, $d \geqq 2$. There exist potentials V for which the solutions are discontinuous but satisfy Harnack's inequality. Nevertheless, the quotients of any two positive solutions are Hölder continuous.

We aim to give lower bounds for the Hölder exponents of quotients of positive solutions of the Schrödinger equation. Our main result states that the quotients are Hölder continuous of order arbitrarily close to 1 if the solutions of the original equation are continuous.

Definition 1. Assume that $V \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$. A function u is called *local weak solution* of the Schrödinger equation $(-\frac{1}{2}\Delta + V)u = 0$ on an open subset $D \subset \mathbb{R}^d$ if $u \in L_1^{\mathrm{loc}}(D)$, $V \cdot u \in L_1^{\mathrm{loc}}(D)$, and

$$-\frac{1}{2}\int_D u \cdot \Delta\psi\, dx + \int_D V \cdot u \cdot \psi\, dx = 0 \quad \forall \psi \in \mathscr{C}^\infty_{\mathrm{comp}}(D)\,.$$

A function u is called *V-harmonic on D* if it is a local weak solution of the Schrödinger equation $(-\frac{1}{2}\Delta + V)u$ on D and, in addition, finely continuous in D. The class of V-harmonic functions on D will be denoted by $\mathscr{H}^V(D)$.

Since we are interested in pointwise estimates, we quote the following result (see [5, Proposition 2]): *For every local weak solution u of the Schrödinger equation*

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$(-\frac{1}{2}\Delta + V)u = 0$ *on* D *there exists a (pointwise) uniquely defined* V*-harmonic function* $\tilde{u}$ *on* D *that coincides with* u *a.e. on* D.

Moreover, every V*-harmonic function on* D *is locally bounded in* D.

We say that a potential $V \geqq 0$ on $\mathbb{R}^d$, $d \geqq 3$, belongs to the *Dynkin class* ($V \in \mathscr{K}_\infty(\mathbb{R}^d)$) if and only if the Kato norm of V is finite, i.e.,

$$\sup_{x \in \mathbb{R}^d} \int_{B_r(x)} \frac{V(y)}{\|x-y\|^{d-2}} dy < \infty \tag{1.2}$$

for some (and hence all) $r > 0$, and that it belongs to the *Kato class* ($V \in \mathscr{K}_0(\mathbb{R}^d)$) if and only if

$$\lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{B_r(x)} \frac{V(y)}{\|x-y\|^{d-2}} dy = 0 . \tag{1.3}$$

In dimension $d = 2$ the kernel $\dfrac{1}{\|x-y\|^{d-2}}$ will be replaced by $\left(\ln \dfrac{1}{\|x-y\|}\right)_+$.

We write $V \in \mathscr{K}_\infty^{\mathrm{loc}}(D)$ if $1_C \cdot V \in \mathscr{K}_\infty(\mathbb{R}^d)$ for every compact set $C \Subset D$. The class $\mathscr{K}_0^{\mathrm{loc}}(D)$ is defined analogously.

It is easy to see that a non-negative function V belongs to the class $\mathscr{K}_\infty^{\mathrm{loc}}(D)$ (resp., $\mathscr{K}_0^{\mathrm{loc}}(D)$) if and only if every point $x \in D$ has a neighborhood B on which the Green potential of V

$$y \mapsto U^B V(y) \equiv \mathbb{E}^y \left[\int_0^{\tau(B)} V(X_s) ds \right] \tag{1.4}$$

is bounded (resp., continuous) (cf. [1, 3, 7]). Here $\mathbb{E}^y$ denotes expectation w.r.t. Brownian motion starting in $y \in \mathbb{R}^d$ and $\tau(B)$ denotes the first exit time of B, i.e., $\tau(B) \equiv \inf\{t > 0 : X_t \notin B\}$.

According to the following propositions, the classes $\mathscr{K}_\infty^{\mathrm{loc}}(D)$ and $\mathscr{K}_0^{\mathrm{loc}}(D)$ can also be characterized by the fact that a potential $V \geqq 0$ belongs to the class $\mathscr{K}_\infty^{\mathrm{loc}}(D)$ (or $\mathscr{K}_0^{\mathrm{loc}}(D)$ resp.) if and only if for every point $x \in D$ there exist a neighborhood of x and a strictly positive function which is V-harmonic in that neighborhood and satisfies Harnack's inequality (or is continuous resp.). Moreover, $V \in \mathscr{K}_\infty^{\mathrm{loc}}(D)$ implies that all V-harmonic functions satisfy Harnack's inequality and $V \in \mathscr{K}_0^{\mathrm{loc}}(D)$ implies that all V-harmonic functions are continuous (cf. [1, 5]). Following [8] we obtain explicitly the dependence of the constant in Harnack's inequality.

Proposition 2. a) *If* $d \geqq 3$ *and* $1_{B_R(z)} \cdot V \in \mathscr{K}_\infty(\mathbb{R}^d)$ *(for some* $R > 0$ *and* $z \in \mathbb{R}^d$*) then for every* $r \in]0, R[$ *the following holds:*

$$\sup_{x \in B_r(z)} u(x) \leqq K \cdot \inf_{x \in B_r(z)} u(x) \quad \forall u \in \mathscr{H}^v(B_R(z)), \quad u \geqq 0 ,$$

with $K = K(r, R, d, \beta) \equiv \left(\dfrac{R+r}{R-r}\right)^d \cdot \exp \beta$ *and* $\beta \equiv 3 \cdot (d-2) \cdot 2^{d+1} \cdot \|U(1_{B_R(z)} \cdot V)\|_\infty$.

b) $V \in \mathscr{K}_\infty^{\mathrm{loc}}(D)$ *if and only if for every point* $x \in D$ *there exist a neighborhood* B *of* x *and a* V*-harmonic function* u *on* B *with* $\inf_{y \in B} u(y) > 0$.

Proposition 3. a) *If* $V \in \mathscr{K}_0^{\mathrm{loc}}(D)$ *then all* V*-harmonic functions on open subsets of* D *are continuous.*

b) $V \in \mathscr{K}_0^{\mathrm{loc}}(D)$ *if and only if for every point* $x \in D$ *there exists at least one strictly positive continuous function which is* V*-harmonic in an appropriate neighborhood of* x.

The proofs of these results (as well as that of the following theorem) will be postponed to the end of the paper.

We emphasize that in order to obtain that the V-harmonic functions are continuous, we need a more restrictive condition on V than for the validity of Harnack's inequality. In fact, $\mathscr{K}_0^{\mathrm{loc}}(D)$ is a proper subset of $\mathscr{K}_\infty^{\mathrm{loc}}(D)$ according to the well-known fact that there exist functions V whose Green potentials $U^B V$ (for suitable $B \subset D$) are bounded but not continuous (recall that $d \geqq 2$). An explicit example of such a function V is given in [1, Example 1].

For those potentials V the positive V-harmonic functions satisfy Harnack's inequality (cf. Proposition 3), but they are not continuous in D (cf. Proposition 2). According to Proposition 2, the continuity of the V-harmonic functions on D is equivalent to $V \in \mathscr{K}_0^{\mathrm{loc}}(D)$.

Note, however, that even in the case $V \in \mathscr{K}_0^{\mathrm{loc}}(D)$ the V-harmonic functions are not necessarily Hölder continuous of any order; cf. the example below.

We recall that there is a classical argument due to Moser [6] in order to prove Hölder continuity of solutions using Harnack's inequality (cf. also the proof of our Theorem). This argument does obviously *not* apply directly to our situation. The reason is that the constants are in general not solutions of the Schrödinger equation.

The situation is quite different if we consider quotients of solutions. We set

$$R^V(D) \equiv \left\{ \frac{u}{v} \,|\, u, v \in \mathscr{H}^V(D), v(x) > 0 \text{ for every } x \in D \right\}$$

for every open set $D \subseteq \mathbb{R}^d$. Obviously, $1 + R^V(D) = R^V(D)$ and $\lambda R^V(D) = R^V(D)$ for every real number λ with $\lambda \neq 0$.

Our main result is the following theorem:

Theorem 4. *Suppose that $1_D V \in \mathscr{K}_\infty(\mathbb{R}^d)$. Then there exists a positive number γ which depends only on $\|U(1_D V)\|_\infty$ and the dimension d such that every function u from $R^V(D)$ is locally γ-Hölder continuous.*

If $1_D V \in \mathscr{K}_0^{\mathrm{loc}}(\mathbb{R}^d)$ then every function u from $R^V(D)$ is locally γ-Hölder continuous for every $\gamma < 1$.

We notice that Moser's original argument applied to the equation $\nabla\left(v^2 \nabla \frac{u}{v}\right) = 0$ yields that $\frac{u}{v}$ is Hölder continuous. However, it seems to be difficult to prove γ-Hölder continuity for every $\gamma < 1$ by arguments of this type.

Combining the above propositions with the theorem, we obtain the following corollary:

Corollary 5. *Let V be an arbitrary non-negative potential on $\mathbb{R}^d$, $d \geqq 2$, and let u and v be any two local weak solutions of the Schrödinger equation $(-\frac{1}{2}\Delta + V)u = 0$ on an open set $D \subset \mathbb{R}^d$.*

a) *If $\operatorname{ess\,inf}_{x \in D} v(x) > 0$ then $\frac{u}{v}$ coincides a.e. with a locally Hölder continuous function in D.*

b) *If in addition v is continuous, then $\frac{u}{v}$ coincides a.e. with a function which is locally γ-Hölder continuous in D for every $\gamma < 1$.*

Example 6. Let us consider the potential

$$V(x) = \frac{1}{(\|x\| \cdot \ln\|x\|)^2}$$

on $D \equiv B_1(0) \subset \mathbb{R}^d$, $d \geqq 2$. Elementary calculations show that $V \in \mathscr{K}_0^{\mathrm{loc}}(D)$ (cf. [1, Theorem 4.11 and the subsequent Remark 2.2]). On the other hand, it is easy to see that the (strictly positive) function

$$u(x) = \frac{d-2}{2} - \frac{1}{\ln\|x\|}$$

– which is *not* Hölder continuous of any order – is a local weak solution of the Schrödinger equation $(-\frac{1}{2}\Delta + V)u = 0$ in D. However, if v is another local weak solution of that equation, then, according to our Theorem from above, the quotient $\frac{v}{u}$ is γ-Hölder continuous in D for every $\gamma < 1$.

Remark 7. In this article we have restricted ourselves to the investigation of (quotients of) local weak solutions u of the Schrödinger equation $(-\frac{1}{2}\Delta + V)u = 0$ with non-negative potentials $V \in \mathscr{K}_1^{\mathrm{loc}}(\mathbb{R}^d)$.

Essentially the same results hold true if one replaces the Schrödinger operator $-\frac{1}{2}\Delta + V$ by a generalized Schrödinger operator $-\frac{1}{2}\Delta + \mu$ with a measure $\mu \geqq 0$ on $\mathbb{R}^d$ charging no polar sets (cf. [3, 2, 5]).

A measure μ belongs to the generalized class $\bar{\mathscr{K}}_\infty^{\mathrm{loc}}(D)$ (resp., $\bar{\mathscr{K}}_0^{\mathrm{loc}}(D)$) if there exist open sets B_i with $\bigcup_{i \in I} B_i = D$ and bounded (resp., continuous) superharmonic functions u_i on B_i with $-\frac{1}{2}\Delta u_i = \mu$.

In the case $d = 1$, we obviously have $\bar{\mathscr{K}}_0^{\mathrm{loc}}(D) = \bar{\mathscr{K}}_\infty^{\mathrm{loc}}(D)$. For dimension $d \geqq 2$, however, it is easy to construct superharmonic functions u which are bounded but not continuous. Any such function u leads to a measure $\mu = -\frac{1}{2}\Delta u \in \bar{\mathscr{K}}_\infty^{\mathrm{loc}}(D) \backslash \bar{\mathscr{K}}_0^{\mathrm{loc}}(D)$, and finally to a generalized Schrödinger equation $(-\frac{1}{2}\Delta + \mu)v = 0$ with discontinuous solutions v whose quotients are (Hölder) continuous.

Similar results also hold true for Schrödinger operators $-\frac{1}{2}\Delta + V$ with arbitrary potentials $V \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$ (or, more generally, for operators $-\frac{1}{2}\Delta + \mu$ with signed measures μ charging no polar sets). The main difference to our situation occurs in the proof of Harnack's inequality for the V-harmonic functions (cf. Proposition 2). In order to estimate the conditional gauge

$$\mathbb{E}^{x,z}\left[\exp\left(-\int_0^{\tau(B)} V(X_s)ds\right)\right]$$

one uses Jensens' inequality together with Khas'minskii's Lemma. This leads to

$$\sup_{x,y \in B_r(z)} \frac{u(x)}{u(y)} \leqq \left(\frac{R+r}{R-r}\right)^d \cdot \frac{e^\beta}{1-\beta}$$

provided that $\beta \equiv 3 \cdot (d-2) \cdot 2^{d+1} \cdot \|U(1_{B_R(z)} \cdot |V|)\|_\infty < 1$ (cf. also [8, 4]).

2 Proofs

Proof of Proposition 2

a) Let $B \equiv B_{R'}(z)$ for an arbitrary R' with $r < R' < R$. Then, according to [5], every $u \in \mathscr{H}^V(B_R(z))$ admits the representation

$$u(x) = \mathbb{E}^x\left[\exp\left(-\int_0^{\tau(B)} V(X_s)ds\right)\cdot u(X_{\tau(B)})\right]$$

for $x \in B$.

Since V is non-negative, we obviously have

$$h(x) \equiv \mathbb{E}^x[u(X_{\tau(B)})] \geqq u(x)$$

on B; here h is harmonic (w.r.t. the Laplacian) in B. On the other hand, for $x \in B$,

$$\begin{aligned} u(x) &= \mathbb{E}^x\left[\exp\left(-\int_0^{\tau(B)} V(X_s)ds\right)\cdot u(X_{\tau(B)})\right] \\ &= \mathbb{E}^x\left[\mathbb{E}^{x,X_{\tau(B)}}\left[\exp\left(-\int_0^{\tau(B)} V(X_s)ds\right)\right]\cdot u(X_{\tau(B)})\right] \\ &\geqq \inf_{\substack{x\in B\\ y\in\partial B}} \mathbb{E}^{x,y}\left[\exp\left(-\int_0^{\tau(B)} V(X_s)ds\right)\right]\cdot \mathbb{E}^x[u(X_{\tau(B)})]\,; \end{aligned}$$

here $\mathbb{E}^{x,y}$ denotes expectation with respect to the Brownian motion restricted to B, starting in x, and conditioned to converge to y. By Jensen's inequality

$$\inf_{\substack{x\in B\\ y\in\partial B}} \mathbb{E}^{x,y}\left[\exp\left(-\int_0^{\tau(B)} V(X_s)ds\right)\right] \geqq \exp\left(-\sup_{\substack{x\in B\\ y\in\partial B}} \mathbb{E}^{x,y}\left[\int_0^{\tau(B)} V(X_s)ds\right]\right)$$

and by the 3G-Theorem (see [8] or [4])

$$\sup_{\substack{x\in B\\ y\in\partial B}} \mathbb{E}^{x,y}\left[\int_0^{\tau(B)} V(X_s)ds\right] \leqq 3(d-2)2^{d+1}\cdot \|U(1_{B_R(z)}V)\|_\infty\,.$$

The right-hand side of the last inequality is by definition equal to β.

We have shown that $u \leqq h$ and $u \geqq e^{-\beta}\cdot h$ in B. Therefore

$$\sup_{x_1,x_2\in B_r(z)} \frac{u(x_1)}{u(x_2)} \leqq \sup_{x_1,x_2\in B_r(z)} \frac{h(x_1)}{h(x_2)}\cdot e^\beta \leqq \left(\frac{R'+r}{R'-r}\right)^d\cdot e^\beta\,;$$

here we have applied the classical Harnack's inequality to the harmonic function h. Now we let R' tend to R. This establishes part a) of the assertion.

b) Let $V \in \mathscr{K}_\infty^{\mathrm{loc}}(D)$ and let B be any relatively compact open subset of D. Then the function

$$y \mapsto \mathbb{E}^y\left[\exp\left(-\int_0^{\tau(B)} V(X_s)ds\right)\right]$$

is V-harmonic in B and bounded below by

$$\exp\left(-\sup_{y\in B} \mathbb{E}^y\left[\int_0^{\tau(B)} V(X_s)ds\right]\right) > 0\,.$$

Conversely, let $u \in \mathscr{H}^V(B)$ with $\varepsilon \equiv \inf_{y\in B} u(y) > 0$.

For every relatively compact open subset $B' \subset B$ the function

$$h \equiv u + U^{B'}(V \cdot u) \tag{2.5}$$

is harmonic (w.r.t. the Laplacian) in B'. In particular, h is locally bounded in B'. According to

$$U^{B'} V \leqq \frac{1}{\varepsilon} \cdot U^{B'}(V \cdot u) \leqq \frac{1}{\varepsilon} \cdot h$$

this implies $V \in \mathscr{K}_{\infty}^{\mathrm{loc}}(B')$. □

Proof of Proposition 3
a) See [1], Theorem 1.5.
b) If $V \in \mathscr{K}_0^{\mathrm{loc}}(D)$, then the existence of suitable strictly positive V-harmonic functions follows from Proposition 2.b. According to Proposition 3.a, these functions are continuous.

Let us conversely assume that there exists a strictly positive continuous V-harmonic function u on a suitable neighborhood B' of x. For any bounded open set $B \Subset B'$ we obtain $\varepsilon \equiv \inf_{y \in B} u(y) > 0$. Since u is V-harmonic on a neighborhood of the closure of B, the function

$$h \equiv u + U^B(V \cdot u)$$

is harmonic on B and hence continuous on B. The continuity of u therefore implies the continuity of $U^B(V \cdot u)$ which in turn implies that

$$V \cdot u \in \mathscr{K}_0^{\mathrm{loc}}(B) .$$

But $V \leqq \frac{1}{\varepsilon} \cdot V \cdot u$ on B, hence we have also $V \in \mathscr{K}_0^{\mathrm{loc}}(B)$. □

Proof of the Theorem
Our proof is based on the explicit version of Harnack's inequality given in Proposition 2.a and a simple iteration argument due to Moser (see [6], Section 2). The proof of the Theorem for $d = 2$ can be reduced to the case $d = 3$ by consideration of functions which are constant with respect to the variable x_3. We therefore restrict ourselves to the case $d \geqq 3$. Let $z \in D$ be given. We introduce the notation

$$\operatorname*{osc}_M w \equiv \sup_M w - \inf_M w$$

for every set M and every bounded real function w on M. We consider positive numbers r and R with $r < R$ and $B_R(z) \Subset D$. If we apply Harnack's inequality (Proposition 2.a) to pairs of non-negative functions in $\mathscr{H}^V(B_R(z))$, then we obtain a Harnack inequality for non-negative functions in $R^V(B_R(z))$. More specifically, for every non-negative function $v \in R^V(B_R(z))$ the following holds:

$$\sup_{B_r(z)} v \leqq K(r, R, d, \beta)^2 \cdot \inf_{B_r(z)} v$$

where $K(r, R, d, \beta) = \left(\frac{R+r}{R-r}\right)^d \cdot \exp \beta$ and $\beta = 3(d-2)2^{d+1} \cdot \|U(1_{B_R(z)} V)\|_\infty$.

Hence,

$$\operatorname*{osc}_{B_r(z)} v \leqq (K(r, R, d, \beta)^2 - 1) \cdot \inf_{B_r(z)} v .$$

Now we set $v \equiv \sup_{B_R(z)} u - u$. Thus,

$$\begin{aligned}\operatorname*{osc}_{B_r(z)} u &= \operatorname*{osc}_{B_r(z)} v \\ &\leqq (K(r, R, d, \beta)^2 - 1) \cdot \inf_{B_r(z)} v \\ &\leqq (K(r, R, d, \beta)^2 - 1) \cdot (\operatorname*{osc}_{B_R(z)} u - \operatorname*{osc}_{B_r(z)} u) .\end{aligned}$$

We can conclude that

$$\operatorname*{osc}_{B_r(z)} u \leqq \left(1 - \frac{1}{K(r, R, d, \beta)^2}\right) \cdot \operatorname*{osc}_{B_R(z)} u .$$

Let δ be a positive number with $r = \delta R$. Since $K(r, R, d, \beta) = \left(\frac{1+\delta}{1-\delta}\right)^d \cdot \exp \beta$, we obtain by iteration that u is γ-Hölder continuous at z for

$$\gamma \equiv \frac{\ln\left(1 - \left(\frac{1-\delta}{1+\delta}\right)^{2d} \exp(-2\beta)\right)}{\ln \delta} .$$

This establishes the first part of the assertion since δ can be chosen arbitrarily in the interval (0, 1).

In order to prove the second part, notice that there exists a positive constant C_d such that

$$1 - \left(\frac{1-\beta}{1+\beta}\right)^{2d} \cdot \exp(-2\beta) \leqq C_d \beta$$

for every positive β with $\beta < 1$. We choose $\delta \equiv \beta$. Then,

$$\gamma \geqq 1 + \frac{\ln C_d}{\ln \beta} .$$

Since the right-hand side of the last inequality tends to 1 if $\|U(1_{B_R(z)} V)\|_\infty$ tends to 0, the proof of the theorem is now complete.

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