Werk

Titel: Symmetric spaces and strongly isotropy irreducibel spaces.

Autor: Wang, McKenzei; Ziller, Wolfgang

Jahr: 1993

PURL: https://resolver.sub.uni-goettingen.de/purl?235181684_0296|log20

Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

Math. Ann. 296, 285–326 (1993)

Mathematische Annalen
© Springer-Verlag 1993

Symmetric spaces and strongly isotropy irreducible spaces

McKenzie Wang[1] **and Wolfgang Ziller**[2]

[1] Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada L8S 4K1
[2] Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104, USA

Received September 24, 1992

Mathematics Subject Classification (1991): 53C35, 53C30, 53C25

A strongly isotropy irreducible homogeneous space is a connected effective homogeneous space L/H where H is a compact subgroup of a Lie group L such that the identity component H_0 of H acts irreducibly on $T_{eH}(L/H)$. If we assume instead that H, but not necessarily H_0, acts irreducibly on $T_{eH}(L/H)$, then L/H is called an isotropy irreducible homogeneous space. These spaces are interesting because by Schur's Lemma they all admit an L-invariant Einstein metric, and because they include the irreducible Riemannian symmetric spaces as a sub-family. The non-symmetric strongly isotropy irreducible homogeneous spaces were classified independently by Manturov [M1,2,3], Wolf [Wo1], and later by Krämer [Kr], with some minor omissions in the first two references. In addition, in [Wo1] a detailed study of the geometric properties of strongly isotropy irreducible spaces was carried out. The isotropy irreducible spaces which are not strongly isotropy irreducible have recently been classified in [WZ3].

In a different direction, we present in this paper a direct proof of a beautiful observation of Wall concerning a correspondence between the compact simply connected irreducible symmetric spaces on the one hand and the compact strongly isotropy irreducible quotients of the classical groups on the other hand (cf. the remarks added in proof in [Wo1, pp. 147–148]). This correspondence has a number of exceptions: certain Grassmannians and the isotropy irreducible space $SO(7)/G_2$, which is diffeomorphic to $\mathbb{R}P^7$. These exceptions appear at first sight to be even more mysterious than the correspondence. But our main result is that there is an explanation, using general principles, of this correspondence as well as all its exceptions. In particular, this means that the classification of the non-symmetric strongly isotropy irreducible quotients of the classical groups may be read off from that of the irreducible symmetric spaces.

Consider a compact simply connected irreducible symmetric space G/K, where G is the identity component of the isometry group, and K is the isotropy group,

The first author is partially supported by the Natural Sciences and Engineering Research Council of Canada. The second author is partially supported by a grant from the National Science Foundation

which is necessarily connected. Then K is locally of the form $H \cdot \mathrm{U}(1)$ or $H \cdot \mathrm{Sp}(1)$, or it is semisimple. In the first case the symmetric space is hermitian and the isotropy representation is of the form $[\pi \hat{\otimes} \phi]_{\mathbb{R}}$, where ϕ is a non-trivial 1-dimensional representation of $\mathrm{U}(1)$. In the second case the symmetric space is quaternionic and the isotropy representation is of the form $\pi \hat{\otimes} \nu_2$, where ν_2 is the two-dimensional representation of $\mathrm{Sp}(1)$. In the third case the isotropy group is simple, unless G/K is a real or quaternionic Grassmannian. We hasten to point out that these cases are not mutually exclusive. The precise correspondence can now be formulated as follows:

Theorem A. *Let G/K be a compact, simply connected, effective, irreducible symmetric space.*

(a) *If $G/K = G/H \cdot \mathrm{U}(1)$ is Hermitian symmetric of real dimension $2n$ with isotropy representation $[\pi \hat{\otimes} \phi]_{\mathbb{R}}$, then $\pi: H \to \mathrm{SU}(n)$ is a faithful unitary representation and the homogeneous manifold $\mathrm{SU}(n)/\pi(H)$ is strongly isotropy irreducible. Conversely, if $\mathrm{SU}(n)/\pi(H)$ is strongly isotropy irreducible and non-symmetric with H compact and connected, then $[\pi \hat{\otimes} \phi]_{\mathbb{R}}$ is the isotropy representation of a simply connected, irreducible Hermitian symmetric space.*

(b) *If $G/K = G/H \cdot \mathrm{Sp}(1)$ is quaternionic symmetric of real dimension $4n$ with isotropy representation $\pi \hat{\otimes} \nu_2$, then $\pi: H \to \mathrm{Sp}(n)$ is a faithful irreducible quaternionic representation and the homogeneous manifold $\mathrm{Sp}(n)/\pi(H)$ is strongly isotropy irreducible. Conversely, if $\mathrm{Sp}(n)/\pi(H)$ is strongly isotropy irreducible and nonsymmetric with H compact and connected, then $\pi \hat{\otimes} \nu_2$ is the isotropy representation of a simply connected, irreducible quaternionic symmetric space.*

(c) *If K is simple or G/K is a quaternionic projective space $\mathrm{Sp}(n+1)/\mathrm{Sp}(n)\mathrm{Sp}(1)$, then the isotropy representation $\pi: K \to \mathrm{SO}(n)$ is a faithful irreducible orthogonal representation and the homogeneous manifold $\mathrm{SO}(n)/\pi(K)$ is strongly isotropy irreducible. Conversely, if $\mathrm{SO}(n)/\pi(K)$ is strongly isotropy irreducible, nonsymmetric and not equal to $\mathrm{SO}(7)/\mathrm{G}_2$ and with K compact and connected, then π is the isotropy representation of a simply connected, irreducible symmetric space.*

The exceptions preventing the above correspondence from being one-to-one are described in detail in the comments after Theorem 3.1.

The forward correspondence can also be uniformly formulated as follows. If G/K is a compact simply connected irreducible symmetric space of dimension n, then either K is maximal in $\mathrm{SO}(n)$ and then $\mathrm{SO}(n)/K$ is strongly isotropy irreducible, or $K \subset L \subset \mathrm{SO}(n)$ and L/K is strongly isotropy irreducible. Here L can be either $\mathrm{U}(n)$ if G/K is hermitian symmetric or $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$ if G/K is quaternionic symmetric and hence L/K, after being made effective, becomes a quotient of $\mathrm{SU}(n)$ or $\mathrm{Sp}(n)$. The only exception to this rule are the Grassmannians $\mathrm{SO}(p+q)/\mathrm{SO}(p)\mathrm{SO}(q)$, $p \geqq q \geqq 3$, $p \neq 4$, $q \neq 4$, and $\mathrm{Sp}(p+q)/\mathrm{Sp}(p)\mathrm{Sp}(q)$, $p \geqq q \geqq 2$, in which case K is maximal in $\mathrm{SO}(n)$, but the isotropy representation of $\mathrm{SO}(n)/K$ has as many irreducible subrepresentations as there are simple factors in K.

These stronlgy isotropy irreducible spaces also have the following geometric interpretation. If G/K is a symmetric space as in part (c) of Theorem A and of dimension n, then it gives rise to a curvature tensor on $\mathbb{R}^n$ which uniquely characterises the symmetric space. The set of all curvature tensors on $\mathbb{R}^n$ which give rise to the given symmetric space is then equal to $\mathrm{SO}(n)/\pi(K)$ since any two of them are orthogonally equivalent and since the isotropy group of a given one is precisely equal to K. Similarly in the case of $\mathrm{SU}(n)/\pi(H)$ or $\mathrm{Sp}(n)/\pi(H)$ if we consider curvature tensors compatible with the complex or the quaternionic structure.

The statement of the forward correspondence can be found in Theorem 3.1 whose proof involves only general principles. The irreducible symmetric spaces which do not satisfy the hypothesis of Theorem 3.1 give the exceptions in the forward correspondence. To enumerate these we use results in 2.10. In proving the forward correspondence, we need to study carefully some properties of the isotropy representation of a symmetric space. It turns out that the dominant weight of the isotropy representation satisfies a very simple, remarkable condition, which admits further refinements in the hermitian and quaternionic cases. It can be formulated as follows:

Theorem B. *Let G/K be a compact, irreducible symmetric space and B the negative of the Killing form of G.*

(a) *If G/K is not a sphere $\mathrm{SO}(n+1)/\mathrm{SO}(n)$ or a quaternionic projective space $\mathrm{Sp}(n+1)/\mathrm{Sp}(n)\mathrm{Sp}(1)$ and has isotropy representation $\pi \otimes \mathbb{C} = \pi_\lambda$ or $\pi_\lambda \oplus \pi_\lambda^*$, where λ is the dominant weight, then $B(\lambda,\lambda) = 2B(\lambda,\alpha)$, where α is any simple root of K with $B(\lambda,\alpha) \neq 0$.*

(b) *If $G/H \cdot \mathrm{U}(1)$ is Hermitian symmetric other than the complex projective spaces, with isotropy representation $[\pi_\lambda]_{\mathbb{R}} = [\pi_{\lambda'} \hat{\otimes} \phi]_{\mathbb{R}}$, then $B(\lambda,\lambda^*) = 0$ and if α is any simple root of H such that $B(\lambda',\alpha) \neq 0$, we also have $B(\lambda',\lambda') + B(\lambda',\lambda'^*) = 2B(\lambda',\alpha)$.*

(c) *If $G/H \cdot \mathrm{Sp}(1)$ is quaternionic symmetric other than the quaternionic projective spaces, with isotropy representation $\pi \otimes \mathbb{C} = \pi_\lambda = \pi_{\lambda'} \hat{\otimes} \nu_2$ or $\pi \otimes \mathbb{C} = \pi_\lambda \oplus \pi_\lambda^* = [\pi_{\lambda'} \oplus \pi_{\lambda'}^*] \hat{\otimes} \nu_2$, then $B(\lambda',\lambda') = \frac{3}{2} B(\lambda',\alpha)$, where α is any simple root of H such that $B(\lambda',\alpha) \neq 0$.*

Theorem B is proved in Sect. 2 from general principles, see 2.5 and 2.7. It turns out that the identities in Theorem B actually characterize the isotropy representations of symmetric spaces, among all orthogonal representations of compact Lie groups. The precise formulation is as follows:

Theorem C. *Let $\pi_\lambda : H \to Gl(V)$ be a faithful irreducible complex representation of a semisimple, compact, connected Lie group H with dominant weight λ, and let $\langle\,,\rangle$ be a bi-invariant metric on $\mathfrak{h}$.*

(a) *If π_λ is orthogonal and $\langle\lambda,\lambda\rangle = 2\langle\lambda,\alpha\rangle$ for every simple root α such that $\langle\lambda,\alpha\rangle \neq 0$, then (H,π_λ) is the isotropy representation of a compact, simply connected, irreducible symmetric space, except in the case (G_2,ϕ_7), where ϕ_7 is the 7-dimensional representation of G_2.*

(b) *If $\pi_\lambda \neq \pi_\lambda^*$, and $\langle\lambda,\lambda\rangle + \langle\lambda,\lambda^*\rangle = 2\langle\lambda,\alpha\rangle$ for every simple root α such that $\langle\lambda,\alpha\rangle \neq 0$, then $(H \cdot \mathrm{U}(1), [\pi_\lambda \hat{\otimes} \phi]_{\mathbb{R}})$ is the isotropy representation of a compact, simply connected, hermitian symmetric space.*

(c) *If π_λ is symplectic and $\langle\lambda,\lambda\rangle = \frac{3}{2}\langle\lambda,\alpha\rangle$ for every simple root α such that $\langle\lambda,\alpha\rangle \neq 0$, then $(H \cdot \mathrm{Sp}(1), \pi_\lambda \hat{\otimes} \nu_2)$ is the isotropy representation of a compact, simply connected, quaternionic symmetric space.*

These identities turn out to be very strong conditions on the dominant weights. Hence, as a byproduct of our study, we obtain a short classification of the simply connected irreducible symmetric spaces in terms of their isotropy representation (which coincides with the holonomy representation). This classification is presented in Sect. 2, and may be regarded as a modern version of Cartan's first method of classifying symmetric spaces – that of analysing the holonomy representation (cf. pp. 216–217, 233–263

of [Ca]). In fact, Cartan also used an identity for the weights of the isotropy representation, but it did not involve the dominant weight alone.

The construction of irreducible symmetric spaces from the strongly isotropy irreducible quotients of the classical groups occupies Sect. 4. The general idea is to analyse the space of invariant curvature tensors on the candidate for the isotropy representation of the corresponding symmetric space. This candidate is constructed from the given strongly isotropy irreducible space. We show the existence of a non-trivial invariant curvature operator taking values in the appropriate holonomy algebra, and using this curvature operator to define Lie brackets, we recover the corresponding irreducible symmetric pair.

At the end of this paper we compiled a table giving the correspondence and exceptions.

As an application of our results, we mention a proof of the fact that the second symmetric power of the $(1,0)$-part of the isotropy representation of an irreducible hermitian symmetric space other than complex projective space contains exactly two irreducible subrepresentations (see Theorem 5.2). This fact was proved by invoking the classification theory of symmetric spaces in [CV; Bo, Lemma 3.3], and was used in the computations of [CV; Bo; Mo, p. 184]. Other consequences of this paper were used in [WZ2]. Lemma II.2.1 in [WZ2] is just (2.2), and the facts about symmetric spaces on pp. 585–589 in [WZ2] are proved in detail here.

Lastly, we would like to mention that similar techniques allow us to give a proof, along general principles, of Dynkin's classification of the maximal subalgebras of the classical Lie algebras [Dy]. This will be the content of a forth-coming paper.

1 Preliminaries

Recall that a connected effective homogeneous space L/H is said to be isotropy irreducible if H is compact and acts irreducibly on $T_{eH}(L/H)$ via the isotropy representation χ. L/H is said to be strongly isotropy irreducible if the identity component H^0 of H also acts irreducibly on $T_{eH}(L/H)$. If L/H is non-compact, one easily shows (see [WZ3, Theorem 2.1]) that L/H is a symmetric space. Clearly, L/H is strongly isotropy irreducible iff $L^0/(H \cap L^0)$ is strongly isotropy irreducible, and hence we can assume that L is connected. If L/H is not simply connected but L is connected, then a covering group L^* of L acts transitively on the universal cover of L/H with connected isotropy group H^* and L/H is strongly isotropy irreducible iff L^*/H^* is strongly isotropy irreducible. Hence we can assume, whenever necessary, that L/H is a compact, simply connected, effective homogeneous space with L and H connected.

Let L/H be an arbitrary homogeneous space with isotropy representation χ. If we choose a reductive decomposition $\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{p}$, we can identify $\mathfrak{p}$ with $T_{eH}(L/H)$ in the usual manner, sending $X \in \mathfrak{p}$ to $X^*(eH) = \left.\dfrac{d}{dt}\right|_{t=0} [\exp(tX)H]$. Under this identification the isotropy representation χ becomes the adjoint representation of H on $\mathfrak{p}$ and hence is given by

$$\mathrm{Ad}_L\,|H = \mathrm{Ad}_H \oplus \chi\,.$$

Recall that for $L = \mathrm{U}(n)$, $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$, $\mathrm{SO}(n)$ respectively, $\mathrm{Ad}_L \otimes \mathbb{C}$ is given by $\mu_n \otimes \mu_n^*$, $\mu_n \otimes \mu_n^* - \mathbf{1}$, $S^2\nu_{2n}$, $\Lambda^2\varrho_n$, where μ_n, ν_{2n}, ϱ_n are respectively the standard representations of L on $\mathbb{C}^n$, $\mathbb{C}^{2n}$, and $\mathbb{C}^n$. Hence, if H is a subgroup of

such an L, and if the inclusion is given via a representation π, which must be unitary, symplectic, or orthogonal respectively, then the isotropy representation is determined by the equations $\pi \otimes \pi^* = \mathrm{Ad}_H \oplus \chi$, $\pi \otimes \pi^* = \mathbf{1} \oplus \mathrm{Ad}_H \oplus \chi$, $S^2\pi = \mathrm{Ad}_H \oplus \chi$, or $\Lambda^2\pi = \mathrm{Ad}_H \oplus \chi$ respectively.

Now let (M, g) be a compact, irreducible Riemannian symmetric space. Then from the general theory of symmetric spaces (see, e.g. [Wo2, pp. 231–247, 257–263]) we can write $M = G/K$, where G is the identity component $\mathrm{Isom}^0(M, g)$ of the isometry group of g and K is the isotropy group of a point. Furthermore, K^0 coincides with the identity component of the linear holonomy group. G is simple, except when $M = (K \times K)/\Delta K$, where K is simple, and ΔK is the diagonal subgroup of $K \times K$. Therefore, we may assume that g is the normal homogeneous metric induced by B, the negative of the Killing form of G. The curvature tensor of g is given by $R_{X^*,Y^*}(Z^*) = -[[X, Y], Z]$ where $X, Y, Z \in \mathfrak{p}$, $\mathfrak{g} = \mathfrak{k} \perp \mathfrak{p}$. Notice that $[X, Y] \in \mathfrak{k}$, so that $[[X, Y], Z] = \pi([X, Y])(Z)$ is the infinitesimal isotropy action. Conversely, if $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, then there exists a symmetric space G/K, where $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{k}$ is the Lie algebra of K. Here K is given by the group which makes the infinitesimal isotropy action of $\mathfrak{k}$ on $\mathfrak{p}$ given by $\pi(X)a = [X, a]$ into a faithful representation.

Since (M, g) is irreducible, the isotropy representation π of G/K is irreducible. Furthermore, $\pi \otimes \mathbb{C}$ is reducible (as a complex representation) iff (M, g) is Hermitian symmetric, in which case $K = H \cdot \mathrm{U}(1)$ with H semisimple. Then $\pi \otimes \mathbb{C} = \pi_\lambda \oplus \pi_\lambda^*$, where π_λ is an irreducible unitary representation of K with dominant weight λ and $\pi_\lambda \ncong \pi_\lambda^*$. Furthermore, $\pi_\lambda = \pi_{\lambda'} \hat{\otimes} \phi$, where ϕ is a non-trivial 1-dimensional representation of $\mathrm{U}(1)$ and $\pi_{\lambda'} : H \to \mathrm{SU}(n)$ is a unitary representation of H with dominant weight λ'.

G/K is called a quaternionic symmetric space if $K = H \cdot \mathrm{Sp}(1)$, and if the isotropy representation satisfies $\pi = \sigma \hat{\otimes} \nu_2$, where σ is a quaternionic representation of H. A quaternionic symmetric space is automatically irreducible and hence π is irreducible, which means that $\sigma \otimes \mathbb{C} = \pi_\lambda$ or $\sigma \otimes \mathbb{C} = \pi_\lambda \oplus \pi_\lambda^*$. Notice that the second case can indeed occur, since the symmetric space $\mathrm{SU}(n+2)/S(\mathrm{U}(n)\mathrm{U}(2))$ is both hermitian symmetric and quaternionic symmetric.

Furthermore, the (infinitesimal) isotropy action determines the symmetric space in the following sense (see e.g. [He, p. 427, Theorem 5.11]): If G_i/K_i are two simply connected irreducible symmetric spaces with isotropy representations π_i then any isomorphism ϕ of $\mathfrak{k}_1$ onto $\mathfrak{k}_2$ such that $\pi_2 \circ \phi$ is equivalent to π_1 can be extended to an isomorphism of $\mathfrak{g}_1$ onto $\mathfrak{g}_2$ and hence induces an isometry of G_1/K_1 onto G_2/K_2.

Also notice that if (K, π) is the isotropy representation of a symmetric space G/K, then so is $(K, \pi \circ \sigma)$ where σ is any automorphism of K. Of course, the two symmetric spaces are isometric by the previous result. Indeed, on the level of Lie algebras we can define $\mathfrak{g}' = \mathfrak{k} \oplus \mathfrak{p}$ with Lie brackets $[X, Y]' = [X, Y]$, $[X, a]' = [\sigma(X), a]$ and $[a, b]' = \sigma^{-1}([a, b])$ where $X, Y \in \mathfrak{k}$ and $a, b \in \mathfrak{p}$. $\mathfrak{g}'$ is of course isomorphic to $\mathfrak{g}$ via the isomorphism which sends $X \in \mathfrak{k}$ to $\sigma^{-1}(X) \in \mathfrak{k}$ and is the identity on $\mathfrak{p}$. The isotropy representation of the corresponding symmetric space G'/K is by construction equal to $\pi \circ \sigma$. For example, the spin representation $\Delta_8^{\pm}$ can be viewed as the isotropy representation of the sphere SO(9)/SO(8).

Although we do not want to assume the classification of symmetric spaces, we will need the isotropy representation of the Grassmannians (and in particular the corresponding rank 1 symmetric spaces) which are easily determined to be as follows: $(G/K, \pi) = (\mathrm{SO}(p+q)/\mathrm{SO}(p)\mathrm{SO}(q), \varrho_p \hat{\otimes} \varrho_q)$, $(\mathrm{SU}(p+q)/\mathrm{S}(\mathrm{U}(p)\mathrm{U}(q)),$

$[\mu_p \hat{\otimes} \mu_q^* \hat{\otimes} \phi]_{\mathbb{R}})$, $(\mathrm{Sp}(p+q)/\mathrm{Sp}(p)\mathrm{Sp}(q), \nu_p \hat{\otimes} \nu_q)$ where ϕ is a 1-dimensional representation of a 1-dimensional torus. We will use the fact that any of these representations π is the isotropy representation of a symmetric space, and by the above uniqueness statement, must be the isotropy representation of one of the Grassmannians.

2 On the isotropy representation of a symmetric space

In this section we characterize the isotropy representation of a symmetric space in terms of their dominant weights and apply it to give a quick classification of the irreducible symmetric spaces.

Our proofs will use some consequences of the classification of simple Lie algebras, which we collect in the following Lemma. The first part consists of standard facts about the adjoint representation. For the second part see e.g. [He, p. 523, Exercise 3]. We will use the convention that for each simple root α of a semisimple Lie algebra $\mathfrak{h}$ we have $\tau_\alpha = \dfrac{2}{\langle \alpha, \alpha \rangle} H_\alpha$, where H_α is given by $\alpha(H) = \langle H_\alpha, H \rangle$ for all H in the chosen Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{h} \otimes \mathbb{C}$. If λ_α is the basis of $\mathfrak{t}^*$ dual to the basis τ_α of $\mathfrak{t}$, then λ_α are the dominant weights of the basic representations of $\mathfrak{h}$. Furthermore, for the exceptional Lie groups we use the notation ϕ_k for an irreducible representation of dimension k.

(2.1) Lemma. *Let $\mathfrak{h}$ be a complex simple Lie algebra of finite dimension.*

(a) *The dominant weight of the adjoint representation, i.e. the maximal root μ, satifies one of the following:*

(i) $\mathfrak{h} = \mathfrak{su}(n)$, $n \geqq 3$, *with* $\mu = \lambda_\beta + \lambda_\beta^*$ *where β is one of the simple roots at the end of the Dynkin diagram.*

(ii) $\mathfrak{h} = \mathfrak{sp}(n)$, $n \geqq 1$, *with* $\mu = 2\lambda_\beta$ *where β is the short root at the end of the Dynkin diagram.*

(iii) $\mathfrak{h} = \mathfrak{so}(n)$, $n \geqq 7$, *with* $\mu = \lambda_\beta$ *where β is a long simple root not at the end of the Dynkin diagram.*

(iv) $\mathfrak{h}$ *is an exceptional Lie algebra and* $\mu = \lambda_\beta$ *where β is a long simple root at the end of the Dynkin diagram.*

(b) *The root system consists of at most 2 orbits under the Weyl group since all roots of the same length are equivalent under the Weyl group, and there are two orbits when $\mathfrak{h}$ is of type B_n, C_n, G_2 or F_4. In each Weyl orbit there is a unique dominant form. Hence the only roots that are also dominant are the maximal root and, in the case of the above Lie aglebras, there exists a unique short root. In the latter case the dominant form is a basic weight and corresponds to the representations* $(\mathfrak{h}, \pi) = (\mathfrak{so}(2n+1), \varrho_{2n+1})$, $n \geqq 2$; $(\mathfrak{sp}(n), \Lambda^2 \nu_{2n} - \mathbf{1})$, $n \geqq 3$; $(\mathfrak{g}_2, \phi_7)$; *and* $(\mathfrak{f}_4, \phi_{26})$. □

Next we collect some facts from representation theory.

(2.2) Lemma. *Let $\mathfrak{h}$ be a compact Lie algebra and V a finite dimensional complex vector space. Let $\langle\,,\rangle$ be a bi-invariant metric on $\mathfrak{h}$.*

(a) *Let $\pi_\lambda : \mathfrak{h} \to \mathfrak{gl}(V)$ be a faithful irreducible complex representation of $\mathfrak{h}$ with dominant weight λ. If $\mathfrak{h}$ is not semisimple and hence has a 1-dimensional center, we have $\pi_\lambda \otimes \pi_\lambda^* = \mathrm{ad}(\mathfrak{h})_{\mathbb{C}} \oplus \chi$ with $\chi = 0$ iff $\mathfrak{h} = \mathfrak{u}(n)$, $n \geqq 1$, and $\pi_\lambda = \mu_n$ or μ_n^*. If $\mathfrak{h}$ is semisimple, then $\pi_\lambda \otimes \pi_\lambda^* = [\mathrm{ad}(\mathfrak{h})_{\mathbb{C}} + \mathbf{1}] \oplus \chi$ with $\chi = 0$ iff $\mathfrak{h} = \mathfrak{su}(n)$, $n \geqq 2$, and $\pi_\lambda = \mu_n$ or μ_n^*. In both cases $\pi_{\lambda+\lambda^*}$ occurs with multiplicity 1 in $\pi_\lambda \otimes \pi_\lambda^*$ and lies in χ unless* (i) $\chi = 0$, (ii) $\mathfrak{h} = \mathfrak{sp}(n)$, $n \geqq 2$, *with* $\pi_\lambda = \nu_{2n}$, *or* (iii) $\mathfrak{h} = \mathfrak{sp}(n) \oplus \mathbb{R}$, $n \geqq 2$, *with* $\pi_\lambda = \nu_{2n} \hat{\otimes} \phi$.

(b) *Let* $\pi_\lambda : \mathfrak{h} \to \mathfrak{sp}(V, \omega)$ *be a faithful irreducible complex representation of* $\mathfrak{h}$ *on* V *with invariant symplectic form* ω. *Then* $S^2\pi_\lambda = \mathrm{ad}(\mathfrak{h})_{\mathbb{C}} \oplus \chi$ *with* $\chi = 0$ iff $\mathfrak{h} = \mathfrak{sp}(n)$ *and* $\pi_\lambda = \nu_{2n}$. $\pi_{2\lambda}$ *occurs with multiplicity* 1 *in* $S^2\pi_\lambda$ *and if* $\chi \neq 0$ *then* $\pi_{2\lambda} \subset \chi$.

(c) *Let* $\pi_\lambda : \mathfrak{h} \to \mathfrak{so}(V, b)$ *be a faithful irreducible complex representation of* $\mathfrak{h}$ *on* V *with invariant symmetric bilinear form* b. *Then* $\Lambda^2\pi_\lambda = \mathrm{ad}(\mathfrak{h})_{\mathbb{C}} \oplus \chi$ *with* $\chi = 0$ iff $\mathfrak{h} = \mathfrak{so}(n)$ *and* $\pi_\lambda = \varrho_n$ *or* $\mathfrak{h} = \mathfrak{so}(8)$ *and* π_λ *is one of the two half-spin representations* $\Delta_8^{\pm}$. *Let* α *be a simple root of* $\mathfrak{h}$ *such that* $\langle \lambda, \alpha \rangle \neq 0$. *Then* $\pi_{2\lambda-\alpha}$ *occurs with multiplicity* 1 *in* $\Lambda^2\pi_\lambda$. *Furthermore,* $\pi_{2\lambda-\alpha} \subset \chi$ *except when* (i) $\chi = 0$, (ii) $\mathfrak{h} = \mathrm{G}_2$, $\pi_\lambda = \phi_7$, (iii) $\mathfrak{h} = \mathfrak{so}(7)$, $\pi_\lambda =$ *spin representation* Δ_7, *or* (iv) $\mathfrak{h} = \mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$, $\pi_\lambda = \nu_{2n} \otimes \nu_2$ *and* α *is the simple root of* $\mathfrak{sp}(1)$.

Proof. (a) If $\mathfrak{h}$ is not semisimple $\pi_\lambda(\mathfrak{h}) \subset \mathfrak{u}(n)$ and hence $\pi_\lambda \otimes \pi_\lambda^* = \mathrm{ad}(\mathfrak{h})_{\mathbb{C}} \oplus \chi$ where χ is the isotropy representation of the corresponding homogeneous space $\mathrm{U}(n)/\pi_\lambda(H)$. If $\chi = 0$ this means that $\pi_\lambda(\mathfrak{h}) = \mathfrak{u}(n)$ and hence $\mathfrak{h} = \mathfrak{u}(n)$ with $\pi_\lambda = \mu_n$ or μ_n^*. Similarly if $\mathfrak{h}$ is semisimple, since in this case $\pi_\lambda(\mathfrak{h}) \subset \mathfrak{su}(n)$. If v_λ is a weight vector of π_λ with weight λ, then v_λ^* is a weight vector of π_λ^* with weight λ^*, and hence $v_\lambda \otimes v_\lambda^*$ is a dominant weight vector in $\pi_\lambda \otimes \pi_\lambda^*$ with weight $\lambda + \lambda^*$, which clearly has multiplicity one. Hence $\pi_{\lambda+\lambda^*}$ occurs in $\pi_\lambda \otimes \pi_\lambda^*$ with multiplicity one. Notice that $\pi_{\lambda+\lambda^*}$ is a representation of $\mathfrak{h}_s$, the semisimple part of $\mathfrak{h}$. Now if $\pi_{\lambda+\lambda^*} \subset \mathrm{ad}(\mathfrak{h})_{\mathbb{C}}$, then $\lambda + \lambda^*$ is the maximal root of a simple ideal of $\mathfrak{h}$. Since π_λ is faithful, $\mathfrak{h}$ has only one simple ideal. (If $\mathfrak{h}$ has no simple ideal, then $\mathfrak{h} = \mathfrak{u}(1)$ and our assertions are trivial.) Since both λ and λ^* are dominant (2.1.a) implies that $\mathfrak{h}_s = \mathfrak{su}(n)$ or $\mathfrak{sp}(n)$ with $\pi_\lambda|_{\mathfrak{h}_s} = \mu_n$, μ_n^* or ν_{2n}. If $\mathfrak{h}$ is semisimple, we obtain (i) and (ii) and if $\mathfrak{h}$ is not semisimple we obtain (i) and (iii). Notice that in the case of $\pi_\lambda = \nu_{2n}$ or $\pi_\lambda = \nu_{2n} \hat{\otimes} \phi$ we have $\chi = 0$ iff $n = 1$.

(b) By assumption $\mathfrak{h}$ must be semisimple. As in (a) it follows that $S^2\pi_\lambda = \mathrm{ad}(\mathfrak{h})_{\mathbb{C}} \oplus \chi$ where χ is the isotropy representation of $\mathrm{Sp}(n)/\pi_\lambda(H)$ and hence if $\chi = 0$ we have $\mathfrak{h} = \mathfrak{sp}(n)$ and $\pi_\lambda = \nu_{2n}$. If v_λ is a weight vector in π_λ, then $v_\lambda \circ v_\lambda$ is a dominant weight vector in $S^2\pi_\lambda$ with weight 2λ which clearly has multiplicity one. Suppose $\pi_{2\lambda} \subset \mathrm{ad}(\mathfrak{h})_{\mathbb{C}}$, then 2λ is the maximal root of a simple ideal of $\mathfrak{h}$. By faithfulness, $\mathfrak{h}$ is simple, and by (2.1.a) $\mathfrak{h} = \mathfrak{sp}(n)$, $n \geqq 1$, with $\pi_\lambda = \nu_{2n}$.

(c) $\mathfrak{h}$ must also be semisimple in this case. $\Lambda^2\pi_\lambda = \mathrm{ad}(\mathfrak{h})_{\mathbb{C}} \oplus \chi$ follows as before. But this time $\chi = 0$ can also occur if $\mathfrak{h} = \mathfrak{so}(8)$ with $\pi_\lambda = \Delta_8^{\pm}$ since in this case π_λ gives rise to an isomorphism of $\mathfrak{so}(8)$ onto itself. If $\langle \lambda, \alpha \rangle \neq 0$, then $\lambda - \alpha$ is a weight in π_λ with multiplicity one and if v_λ and $v_{\lambda-\alpha}$ are corresponding weight vectors, then $v_\lambda \wedge v_{\lambda-\alpha}$ is a dominant weight vector in $\Lambda^2\pi_\lambda$ with weight $2\lambda - \alpha$ which clearly has multiplicity one. Suppose that $\pi_{2\lambda-\alpha} \subset \mathrm{ad}(\mathfrak{h})_{\mathbb{C}}$. Then $2\lambda - \alpha$ is the maximal root of a simple ideal of $\mathfrak{h}$. If $\mathfrak{h}$ is not simple, faithfulness implies that α is a root of a different simple ideal of $\mathfrak{h}$ and that these are the only simple ideals in $\mathfrak{h}$. We can therefore assume that $\lambda = \lambda_1 + \lambda_2$ where λ_i is a dominant weight of $\mathfrak{h}_i$, $i = 1, 2$, α is a root of $\mathfrak{h}_2$ and $2\lambda - \alpha$ is the maximal root μ of $\mathfrak{h}_1$. Then $\alpha = 2\lambda_2$ and $\mu = 2\lambda_1$. Hence α is a dominant weight of $\mathfrak{h}_2$ and also a simple root, which is only possible if $\mathfrak{h}_2 = \mathfrak{sp}(1)$ and hence $\lambda_2 = \nu_2$. The condition $\mu = 2\lambda_1$ implies by (2.1.a) that $\mathfrak{h}_1 = \mathfrak{sp}(n)$ and $\lambda_1 = \nu_{2n}$. This gives case (iv).

Assume next that $\mathfrak{h}$ is simple and hence $2\lambda - \alpha = \mu$. If $\mathfrak{h} = \mathfrak{su}(n)$, $n \geqq 3$, then $\mu = \lambda_\beta + \lambda_\beta^*$ where λ_β is the basic weight corresponding to the simple root β at one end of the Dynkin diagram of $\mathfrak{h}$. Notice that we must have $\alpha = \alpha^*$ since $\lambda^* = \lambda$ and $\mu^* = \mu$. This means that n is even and α is the middle vertex of the Dynkin diagram. Since $2\lambda = \mu + \alpha$ is dominant, this is only possible if $n = 3$. Furthermore

$2\lambda(\tau_\beta) = (\lambda_\beta + \lambda_{\beta^*})(\tau_\beta) + \alpha(\tau_\beta) = 0$ and $2\lambda(\tau_\alpha) = (\lambda_\beta + \lambda_{\beta^*})(\tau_\alpha) + \alpha(\tau_\alpha) = 2$. Similarly $\lambda(\tau_{\beta^*}) = 0$ and hence $\lambda = \lambda_\alpha$ or $\mathfrak{h} = \mathfrak{so}(6)$ and $\pi_\lambda = \varrho_6$, which implies $\chi = 0$.

If $\mathfrak{h} = \mathfrak{sp}(n)$, $n \geqq 1$, then $\mu = 2\lambda_\beta$, where β is the short root at the end of the Dynkin diagram. Since $2\lambda = \mu + \alpha$ is dominant, we must have $n = 2$ if $\alpha \neq \beta$ and $n = 1$ if $\alpha = \beta$. In the first case $2\lambda(\tau_\beta) = 2\lambda_\beta(\tau_\beta) + \alpha(\tau_\beta) = 0$ and $2\lambda(\tau_\alpha) = 2\lambda_\beta(\tau_\alpha) + \alpha(\tau_\alpha) = 2$. Hence $\lambda = \lambda_\alpha$ or $\mathfrak{h} = \mathfrak{sp}(2) = \mathfrak{so}(5)$, $\pi_\lambda = \varrho_5$. If $n = 1$, $2\lambda(\tau_\beta) = 2\lambda_\beta(\tau_\beta) + \beta(\tau_\beta) = 4$ and hence $\lambda = 2\lambda_\beta$ or $\mathfrak{h} = \mathfrak{so}(3)$, $\pi_\lambda = \varrho_3$. In both cases $\chi = 0$.

In the remaining cases, $\mu = \lambda_\beta$ for some simple root β. The fact that $2\lambda = \mu + \alpha$ is dominant implies that α cannot have any adjacent simple roots except for β, in particular $\alpha \neq \beta$. Hence α must be at the end of the Dynkin diagram and β right next to it. (2.1.a) now implies that $\mathfrak{h} = \mathfrak{so}(n)$ or $\mathfrak{g}_2$. If $\mathfrak{h} = \mathfrak{so}(n)$ it follows that $2\lambda(\tau_\beta) = \lambda_\beta(\tau_\beta) + \alpha(\tau_\beta) = 0$ and $2\lambda(\tau_\alpha) = 2\lambda_\beta(\tau_\alpha) + \alpha(\tau_\alpha) = 2$. Hence $\lambda = \lambda_\alpha$ which implies $\pi_\lambda = \varrho_n$ or $n = 7$ with $\pi_\lambda = \Delta_7$ or $n = 8$ with $\pi_\lambda = \Delta_8^\pm$. Similarly if $\mathfrak{h} = \mathfrak{g}_2$ it follows that $\lambda = \lambda_\alpha$ and hence $\pi_\lambda = \phi_7$. □

Remark. (2.2) proves Lemma II.2.1 of [WZ2]. Note, however, the correction for the statement of the unitary case. This omission does not effect the rest of [WZ2].

Later we will also need a somewhat more general version of (2.2.c).

(2.3) Lemma. *Let $\pi_\lambda : \mathfrak{h} \to \mathfrak{so}(V, b)$ be a faithful irreducible complex representation of $\mathfrak{h}$ on V with invariant symmetric bilinear form b and let α be a simple root of $\mathfrak{h}$ with $\langle \lambda, \alpha \rangle \neq 0$. Then $2\lambda - \alpha$ is not a root of $\mathfrak{h}$ except in the following cases:*

(a) $\mathfrak{h} = \mathfrak{so}(n)$, $\pi_\lambda = \varrho_n$
(b) $\mathfrak{h} = \mathfrak{so}(8)$, $\pi_\lambda = \Delta_8^\pm$
(c) $\mathfrak{h} = \mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$, $\pi_\lambda = \nu_{2n} \hat{\otimes} \nu_2$ *and α is the simple root of* $\mathfrak{sp}(1)$.
(d) $\mathfrak{h} = \mathrm{G}_2$, $\pi_\lambda = \phi_7$
(e) $\mathfrak{h} = \mathfrak{so}(7)$, $\pi_\lambda = \Delta_7$.

Proof. If $\mathfrak{h}$ is not simple we can argue as in (2.2.c). If $\mathfrak{h}$ is simple, we observe that $2\lambda - \alpha$ is a dominant form and hence if $2\lambda - \alpha$ is equal to a root β, β must be dominant also. By (2.1.b) there are only two possibilities. If β is a long root it must be equal to the maximal root, and in this case the proof of (2.2.c) implies the possibilities (a)–(e). If β is a short root, (2.1.b) implies that $\beta = \lambda_\gamma$ for some simple root γ. The fact that $2\lambda = \lambda_\gamma + \alpha$ is dominant, again implies that α must be at the end of the Dynkin diagram and γ right next to it. By (2.1.b) this is only possible if $(\mathfrak{h}, \pi_{\lambda_\gamma}) = (\mathfrak{sp}(n), \Lambda^2 \nu_{2n} - \mathbf{1})$, $n \geqq 2$ or $(\mathfrak{g}_2, \phi_7)$. In the first case $2\lambda(\tau_\gamma) = \lambda_\gamma(\tau_\gamma) + \alpha(\tau_\gamma) = 0$ and $2\lambda(\tau_\alpha) = \lambda_\gamma(\tau_\alpha) + \alpha(\tau_\alpha) = 2$ and hence $\lambda = \lambda_\alpha = \nu_{2n}$. But this representation is excluded since ν_{2n} is not orthogonal. In the second case $2\lambda(\tau_\gamma) = \lambda_\gamma(\tau_\gamma) + \alpha(\tau_\gamma) = -2$ implies that λ is not dominant. □

(2.4) Lemma. *The following cannot be the isotropy representation of a symmetric space:*

(a) $\mathfrak{k} = \mathfrak{sp}(n) \oplus \mathfrak{u}(1)$, $\pi = [\nu_{2n} \hat{\otimes} \phi]_{\mathbb{R}}$, $n \geqq 2$
(b) $\mathfrak{k} = \mathrm{G}_2$, $\pi = \phi_7$
(c) $\mathfrak{k} = \mathfrak{so}(7)$, $\pi = \Delta_7$.

Proof. (a) In this case the corresponding simply connected irreducible symmetric space G/K would have holonomy group $K = \mathrm{Sp}(n) \cdot \mathrm{U}(1)$. But in general the holonomy group can be either $\mathrm{Sp}(n)$ or $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$, but not $\mathrm{Sp}(n) \cdot \mathrm{U}(1)$, see [Be, 10.66 and 14.46].

In the case of (b) and (c) one easily sees that the representation π of $\mathfrak{k}$ acts transitively on the set of 2-frames. Hence if this were the isotropy representation of a symmetric space M, M would have to have constant curvature. But in this case the holonomy group of M would be bigger than K. □

We now come to the main result of this chapter.

(2.5) Theorem. *Let $(M = G/K, g)$ be a compact symmetric space with $G = \mathrm{Isom}^0(M, g)$ and without Euclidean factors. Let B be the negative of the Killing form of G and λ be a dominant weight of its (complex) isotropy representation π. Suppose that α is any root of $\mathfrak{k}$ such that $B(\lambda, \alpha) \neq 0$ and that $2\lambda - \alpha$ is not a root of $\mathfrak{k}$. Then $B(\lambda, \lambda) = 2B(\lambda, \alpha)$.*

Proof. We write $\mathfrak{g} = \mathfrak{k} \perp \mathfrak{p}$ and select a maximal abelian subalgebra $\mathfrak{t} \subset \mathfrak{k}_s$, as well as a system of simple roots $\{\alpha_i\}$ for $(\mathfrak{k}_s \otimes \mathbb{C}, \mathfrak{t} \otimes \mathbb{C})$. As is well-known, there is a basis of $\mathfrak{k} \otimes \mathbb{C}$ of the form $\{z_i, H_j, X_\alpha, X_{-\alpha}\}$ where $\{z_i\}$ spans the center of $\sqrt{-1}\mathfrak{k}$, $\{H_j = H_{\alpha_i}\}$ spans $\sqrt{-1}\mathfrak{t}$, and $[X_\alpha, X_{-\alpha}] = H_\alpha \in \sqrt{-1}\mathfrak{t}$, where H_α is dual to α with respect to $(B|_\mathfrak{t}) \otimes \mathbb{C}$. Next let $\{v_\varrho\}$ be a basis of weight vectors for $\pi \otimes \mathbb{C}$ such that $B(v_\lambda, v_{-\lambda}) = 1$ and $\pi(X_{-\alpha})v_\lambda = v_{\lambda-\alpha}$. Notice that $v_{\lambda-\alpha} \neq 0$ since $B(\lambda, \alpha) \neq 0$. Since $B([v_\lambda, v_{-\lambda}], H) = -B([v_\lambda, H], v_{-\lambda}) = B(\lambda(H)v_\lambda, v_{-\lambda}) = \lambda(H)$ it follows that $[v_\lambda, v_{-\lambda}] = H_\lambda \in \sqrt{-1}\mathfrak{t}$, the element dual to λ.

By the Jacobi identity (or equivalently the Bianchi identity for the curvature tensor), we have

$$[v_\lambda, [v_{-\lambda}, v_{\lambda-\alpha}]] + [v_{-\lambda}, [v_{\lambda-\alpha}, v_\lambda]] + [v_{\lambda-\alpha}, [v_\lambda, v_{-\lambda}]] = 0\,.$$

$[v_{\lambda-\alpha}, v_\lambda] = 0$ since in general $[v_{\varrho_1}, v_{\varrho_2}] = 0$ unless $\varrho_1 + \varrho_2$ is a root of $\mathfrak{k} \otimes \mathbb{C}$. Therefore,

$$[v_\lambda, [v_{-\lambda}, [X_{-\alpha}, v_\lambda]]] + [v_{\lambda-\alpha}, H_\lambda] = 0\,.$$

By the Jacobi identity again,

$$\begin{aligned} B(\lambda, \lambda - \alpha)v_{\lambda-\alpha} &= (\lambda - \alpha)(H_\lambda)v_{\lambda-\alpha} = [H_\lambda, v_{\lambda-\alpha}] = [v_\lambda, [v_{-\lambda}, [X_{-\alpha}, v_\lambda]]] \\ &= -[v_\lambda, [X_{-\alpha}, H_\lambda]] + [v_\lambda, [v_\lambda, \pi(X_{-\alpha})v_{-\lambda}]] = -[v_\lambda, [X_{-\alpha}, H_\lambda]] \\ &= B(\lambda, \alpha)v_{\lambda-\alpha} \end{aligned}$$

which yields the identity. □

Remark. Cartan first classified the symmetric spaces by classifying their holonomy representation, which coincides with their isotropy representation. Geometrically this approach is especially pleasing in a general discussion of holonomy, in view of the Berger-Simons theorem. However, Cartan never published all the technical details of this classification. The general method is described in [Ca, see 216–217, 233–263]. The key lemma seems to be the following: If λ and μ are two weights of the isotropy representation of a symmetric space such that $\lambda + \mu$ and $\lambda - \mu$ are not roots of $\mathfrak{k}$, then $B(\lambda, \mu) = 0$. This condition can be deduced by applying the Jacobi identity to v_λ, $v_{-\lambda}$, and v_μ as in the proof of (2.5). It implies the condition in (2.5) only if $B(\lambda, \alpha) \geqq B(\alpha, \alpha)$ (which by (2.11) is almost never the case) since then $\lambda - 2\alpha$ is also a weight of π_λ. In the following we shall see that (2.5) can be used quite easily to give a shorter proof of Cartan's first classification.

(2.6) Corollary. *Let $M = G/K$ be a compact irreducible symmetric space which is not a sphere $\mathrm{SO}(n+1)/\mathrm{SO}(n)$ or a quaternionic projective space $\mathrm{Sp}(n+1)/\mathrm{Sp}(n)\mathrm{Sp}(1)$*

and with isotropy representation $\pi \otimes \mathbb{C} = \pi_\lambda$ *or* $\pi_\lambda \oplus \pi_\lambda^*$. *If* B *is the negative of the Killing form of* G *and* α *a simple root of* $\mathfrak{k}$ *with* $B(\lambda, \alpha) \neq 0$, *then* $B(\lambda, \lambda) = 2B(\lambda, \alpha)$.

Proof. If $\pi \otimes \mathbb{C} = \pi_\lambda$, then this follows from (2.3), (2.4), and (2.5). If $\pi \otimes \mathbb{C} = \pi_\lambda \oplus \pi_\lambda^*$, then G/K is hermitian symmetric and hence $\mathfrak{k}$ has a one dimensional center. By effectiveness λ is nonzero on this center and hence $2\lambda - \alpha$ is not a root of $\mathfrak{k}$ and (2.5) applies. □

For hermitian and quaternionic symmetric spaces we obtain a further refinement:

(2.7) Theorem. (a) *Let* G/K *be a compact irreducible hermitian symmetric space other than the complex projective spaces, with* $K = H \cdot \mathrm{U}(1)$ *and isotropy representation* $\pi = [\pi_\lambda]_{\mathbb{R}} = [\pi_{\lambda'} \hat{\otimes} \phi]_{\mathbb{R}}$. *Then* $B(\lambda, \lambda^*) = 0$ *and if* α *is any simple root of* H *such that* $B(\lambda', \alpha) \neq 0$, *we also have* $B(\lambda', \lambda') + B(\lambda', \lambda'^*) = 2B(\lambda', \alpha)$.

(b) *Let* G/K *be a compact quaternionic symmetric space other than the quaternionic projective spaces, with* $K = H \cdot \mathrm{Sp}(1)$ *and isotropy representation* $\pi \otimes \mathbb{C} = \pi_\lambda = \pi_{\lambda'} \hat{\otimes} \nu_2$ *or* $\pi \otimes \mathbb{C} = \pi_\lambda \oplus \pi_\lambda^* = [\pi_{\lambda'} \oplus \pi_{\lambda'}^*] \hat{\otimes} \nu_2$. *If* α *is any simple root of* H *such that* $B(\lambda', \alpha) \neq 0$, *then* $B(\lambda', \lambda') = \frac{3}{2} B(\lambda', \alpha)$.

Proof. (a) In the hermitian case, $\lambda = \lambda' + x$, where x is the weight of ϕ. There are weight vectors v_λ of π_λ, v_{λ^*} and $v_{-\lambda}$ of π_{λ^*} such that $[v_\lambda, v_{-\lambda}] = H_\lambda \in \mathfrak{k}$. Furthermore, $[v_{\lambda^*}, v_{-\lambda}] = 0$ since $\lambda^* - \lambda$ cannot be a root of $\mathfrak{k}$ (it is nonzero on $\mathfrak{u}(1)$). By the Jacobi identity, we have

$$[[v_\lambda, v_{-\lambda}], v_{\lambda^*}] + [[v_{-\lambda}, v_{\lambda^*}], v_\lambda] + [[v_{\lambda^*}, v_\lambda], v_{-\lambda}] = 0$$

or

$$B(\lambda, \lambda^*) v_{\lambda^*} + [[v_{\lambda^*}, v_\lambda], v_{-\lambda}] = 0 .$$

If $\lambda + \lambda^*$ is not a root of $\mathfrak{k}$, then $[v_{\lambda^*}, v_\lambda] = 0$ and so $B(\lambda, \lambda^*) = 0$, as desired. If $\lambda + \lambda^*$ is a root of a simple ideal of $\mathfrak{k}$, since it is dominant and cannot be a basic weight on that simple ideal, it must be the maximal root. Then (2.1.a) implies that $\mathfrak{h} = \mathfrak{su}(n) \oplus \mathfrak{u}(1)$ or $\mathfrak{sp}(n) \oplus \mathfrak{u}(1)$, with $\pi_\lambda = \mu_n \hat{\otimes} \phi$ and $\nu_{2n} \hat{\otimes} \phi$ respectively. In the first case we have the isotropy representation of complex projective space, which is excluded by hypothesis. The second case is ruled out by (2.4.a).

Since $\lambda^* = \lambda'^* - x$, $B(\lambda, \lambda^*) = 0$ implies $B(\lambda', \lambda'^*) = B(x, x)$. Furthermore, by Corollary 2.6 $2B(\lambda', \alpha) = 2B(\lambda, \alpha) = B(\lambda, \lambda) = B(\lambda', \lambda') + B(x, x) = B(\lambda', \lambda') + B(\lambda', \lambda'^*)$.

(b) In the quaternionic case $\lambda = \lambda' + \frac{1}{2}\beta$, where β is the simple root of $\mathrm{Sp}(1)$. (2.6) implies that

$$B(\lambda, \lambda) = 2B(\lambda, \alpha) = 2B(\lambda', \alpha) = B(\lambda', \lambda') + \tfrac{1}{4} B(\beta, \beta)$$

and

$$B(\lambda, \lambda) = 2B(\lambda, \beta) = B(\beta, \beta) .$$

Hence $2B(\lambda', \alpha) = B(\lambda', \lambda') + \frac{1}{2} B(\lambda, \beta) = B(\lambda', \lambda') + \frac{1}{2} B(\lambda', \alpha)$, which implies our claim. □

We now come to our first application of (2.5), which is actually the only one we need in the forward correspondence in Sect. 2.

(2.8) Theorem. *Let G/K be a compact irreducible symmetric space with isotropy representation π and dominant weight λ. If $G/K \neq [\mathrm{SU}(n) \times \mathrm{SU}(n)]/\Delta\mathrm{SU}(n)$ then for every simple ideal $\mathfrak{h} \subset \mathfrak{k}$ there exists a unique simple root α of $\mathfrak{h}$ such that $B(\lambda, \alpha) \neq 0$.*

Proof. Assume that α and β are simple roots of a simple ideal $\mathfrak{h}$ of $\mathfrak{k}$ such that $B(\lambda, \alpha) \neq 0$ and $B(\lambda, \beta) \neq 0$. Then there is a minimal chain of simple roots $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_r = \beta$ connecting α and β. Let $\gamma = \alpha_1 + \ldots + \alpha_r$. Now γ is a root of $\mathfrak{h}$ and $B(\gamma, \lambda) \neq 0$. Since our claim is certainly true for the spheres and quaternionic projective spaces, (2.6) implies that $B(\lambda, \lambda) = 2B(\lambda, \alpha) = 2B(\lambda, \beta)$. We claim that $2\lambda - \gamma$ is a root of $\mathfrak{k}$. If not, we have in addition $B(\lambda, \lambda) = 2B(\lambda, \gamma) = 2B(\lambda, \alpha_1) + \ldots + 2B(\lambda, \alpha_r) \leqq 2B(\lambda, \alpha) + 2B(\lambda, \beta)$, which is a contradiction.

Since $2\lambda - \gamma$ is a root of $\mathfrak{k}$, $\mathfrak{k}$ must be semisimple. We claim next that $\mathfrak{k}$ is simple. Suppose not, then let $\mathfrak{h}'$ be a second simple ideal in $\mathfrak{k}$. Since π is faithful, there exists a simple root α' of $\mathfrak{h}'$ such that $B(\lambda, \alpha') \neq 0$. Now $B(2\lambda - \gamma, \alpha') = 2B(\lambda, \alpha') \neq 0$, so $2\lambda - \gamma|_{\mathfrak{h}} \equiv 0$. On the other hand, $B(2\lambda - \gamma, \alpha) = 2B(\lambda, \alpha) - B(\alpha, \alpha) - B(\alpha_2, \alpha) \neq 0$, a contradiction to the assumption that $2\lambda - \gamma$ is a root.

Therefore, $\mathfrak{k} = \mathfrak{h}$ is simple. As one can verify directly, $2\lambda - \gamma$ is a dominant weight of $\mathfrak{k}$. Moreover, $B(2\lambda - \gamma, \alpha) \neq 0$, $B(2\lambda - \gamma, \beta) \neq 0$ so $2\lambda - \gamma$ is not a basic weight. It is also a root of $\mathfrak{k}$, so by (2.1), $\mathfrak{k} = \mathfrak{su}(n)$, $n \geqq 3$ and $2\lambda - \gamma$ is the maximal root. It follows that α and β must be at the end of the Dynkin diagram and hence $\lambda = \gamma =$ maximal root and $G/K = [\mathrm{SU}(n) \times \mathrm{SU}(n)]/\Delta\mathrm{SU}(n)$. □

We will now show that the conditions on the isotropy representation in (2.6) and (2.7) are so strong that one can easily derive a classification of the symmetric spaces via their isotropy representation. This will also enable us to prove a converse to (2.6) and (2.7). The results in the remainder of this section will mostly not be needed in the proof of the Wall correspondence, except in a minor way in the proof of (3.4) and (4.4). We first prove the following lemma.

(2.9) Lemma. *Let $\mathfrak{h}$ be a compact simple Lie algebra with $\mathrm{ad}(\mathfrak{h})$-invariant inner product $\langle , \rangle$ and $\pi_\lambda : \mathfrak{h} \to \mathfrak{gl}(V)$ an irreducible faithful complex representation of $\mathfrak{h}$.*

(a) *Suppose that α is a simple root of $\mathfrak{h}$ such that $\langle \lambda, \alpha \rangle \neq 0$ and $\langle \lambda, \lambda \rangle \leqq \langle \lambda, \alpha \rangle$. Then up to an automorphism of $\mathfrak{h}$ one of the following holds:*

(i) $\lambda = \lambda^*$, $(\mathfrak{h}, \pi) = (\mathfrak{so}(n), \varrho_n)$, $n \geqq 3$, *or* $(\mathfrak{sp}(n), \nu_{2n})$, $n \geqq 2$, *or* $(\mathfrak{sp}(1), \nu_2)$. *In the first two cases* $\dfrac{\langle \lambda, \lambda \rangle}{\langle \lambda, \alpha \rangle} = 1$ *and in the third case* $\dfrac{\langle \lambda, \lambda \rangle}{\langle \lambda, \alpha \rangle} = \dfrac{1}{2}$.

(ii) $\lambda \neq \lambda^*$, $(\mathfrak{h}, \pi) = (\mathfrak{su}(n), \mu_n)$, $n \geqq 3$ *and* $\dfrac{\langle \lambda, \lambda \rangle}{\langle \lambda, \alpha \rangle} = \dfrac{n-1}{n}$.

(b) *If α is a simple root of $\mathfrak{h}$ such that $\langle \lambda, \alpha \rangle \neq 0$ and $\langle \lambda, \lambda \rangle + \langle \lambda, \lambda^* \rangle \leqq \langle \lambda, \alpha \rangle$, then $(\mathfrak{h}, \pi) = (\mathfrak{su}(n), \mu_n)$, $n \geqq 2$, with $\langle \lambda, \lambda \rangle + \langle \lambda, \lambda^* \rangle = \langle \lambda, \alpha \rangle$.*

Proof. (a) Without loss of generality, we may choose $\langle , \rangle$ to be the multiple of the Killing form of $\mathfrak{h}$ such that the longest root has length $\sqrt{2}$. If $\mathrm{rank}(\mathfrak{h}) = 1$, then $\langle \lambda, \lambda \rangle \leqq \langle \lambda, \alpha \rangle$ implies that $\pi = \nu_2$ or ϱ_3 with $\dfrac{\langle \lambda, \lambda \rangle}{\langle \lambda, \alpha \rangle} = \dfrac{1}{2}$ or 1 respectively. Hence we will assume that $\mathrm{rank}(\mathfrak{h}) \geqq 2$ in the following.

Let $\lambda = \sum_\beta n_\beta \lambda_\beta$, where the sum is over all simple roots and n_β are the non-negative integers $\dfrac{2\langle \lambda, \beta \rangle}{\langle \beta, \beta \rangle}$. We can also write λ as $\sum_\beta r_\beta \beta$ where β runs over all

simple roots of $\mathfrak{h}$ and r_β are positive rational numbers. We first observe that if we write $\lambda_\alpha = \sum_\beta r_{\alpha\beta}\beta$, then

$$r_{\alpha\beta} = \frac{2\langle\lambda_\alpha, \lambda_\beta\rangle}{\langle\beta,\beta\rangle} = \frac{\langle\lambda_\alpha,\lambda_\beta\rangle}{\langle\lambda_\beta,\beta\rangle}.$$

Since $\mathbf{1} \in \pi_\lambda \otimes \pi_{\lambda^*}$, this has the particular consequence that $\dfrac{\langle\lambda_\alpha,\lambda_\alpha\rangle}{\langle\lambda_\alpha,\alpha\rangle} = r_{\alpha\alpha}$ must be an integer or half integer if λ_α is orthogonal or symplectic and that $\dfrac{\langle\lambda_\alpha,\lambda_\alpha\rangle + \langle\lambda_\alpha,\lambda_\alpha^*\rangle}{\langle\lambda_\alpha,\alpha\rangle} = r_{\alpha\alpha} + r_{\alpha\alpha^*}$ must be an integer in general.

We will first show that if (2.9) holds for basic representations of $\mathfrak{h}$ then it holds for all irreducible representations of $\mathfrak{h}$. Suppose that there is a simple root β other than α for which $\langle\lambda,\beta\rangle \neq 0$. Then $\langle\lambda,\lambda\rangle > n_\alpha^2\langle\lambda_\alpha,\lambda_\alpha\rangle + n_\beta^2\langle\lambda_\alpha,\lambda_\beta\rangle$ since $\langle\lambda_\alpha,\lambda_\beta\rangle > 0$. Dividing throughout by $\langle\lambda,\alpha\rangle = n_\alpha\langle\lambda_\alpha,\alpha\rangle$, we obtain $\dfrac{n_\alpha\langle\lambda_\alpha,\lambda_\alpha\rangle}{\langle\lambda_\alpha,\alpha\rangle} < \dfrac{\langle\lambda,\lambda\rangle}{\langle\lambda,\alpha\rangle} \leqq 1$. By assumption, (2.9) holds for λ_α, so $n_\alpha = 1$ and $\mathfrak{h} = \mathfrak{su}(n)$, $n \geqq 3$. Since all roots of $\mathfrak{su}(n)$ have the same length, $\langle\lambda_\alpha,\alpha\rangle = \langle\lambda_\beta,\beta\rangle$. Hence

$$\frac{\langle\lambda_\alpha,\lambda_\alpha\rangle}{\langle\lambda_\alpha,\alpha\rangle} + \frac{n_\beta^2\langle\lambda_\beta,\lambda_\beta\rangle}{\langle\lambda_\beta,\beta\rangle} < \frac{\langle\lambda,\lambda\rangle}{\langle\lambda,\alpha\rangle} \leqq 1.$$

Since $\dfrac{\langle\lambda_\alpha,\lambda_\alpha\rangle}{\langle\lambda_\alpha,\alpha\rangle} = \dfrac{n-1}{n}$, $\dfrac{\langle\lambda_\beta,\lambda_\beta\rangle}{\langle\lambda_\beta,\beta\rangle} < \dfrac{1}{n_\beta^2 n} < \dfrac{1}{2}$, a contradiction to the asumption that (2.9) holds for basic weights. Therefore, the inequality $\dfrac{\langle\lambda,\lambda\rangle}{\langle\lambda,\alpha\rangle} \leqq 1$ holds only if λ is a multiple of a basic weight. Next assume $\lambda = k\lambda_\alpha$ with $k \geqq 2$. Then $\lambda - \alpha$ must be a dominant weight and hence $\langle\lambda,\lambda-\alpha\rangle > 0$ which contradicts the assumption that $\dfrac{\langle\lambda,\lambda\rangle}{\langle\lambda,\alpha\rangle} \leqq 1$.

Hence $\dfrac{\langle\lambda,\lambda\rangle}{\langle\lambda,\alpha\rangle} \leqq 1$ can only hold for a basic weight λ_α and in this case $\dfrac{\langle\lambda_\alpha,\lambda_\alpha\rangle}{\langle\lambda_\alpha,\alpha\rangle} = r_{\alpha\alpha}$. We therefore need to determine which simple roots satisfy $r_{\alpha\alpha} \leqq 1$. For this purpose we make the observation that $\dfrac{\langle\lambda_\alpha,\lambda_\alpha\rangle}{\langle\lambda_\alpha,\alpha\rangle} = \dfrac{2\langle\lambda_\alpha,\lambda_\alpha\rangle}{\langle\alpha,\alpha\rangle}$ are the diagonal entries in the inverse of the Cartan matrix. Indeed, since λ_α is dual to τ_α, $\langle\lambda_\alpha,\lambda_\beta\rangle$ is the inverse matrix of $\langle\tau_\alpha,\tau_\beta\rangle = \dfrac{4\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle\langle\beta,\beta\rangle}$ and hence $\dfrac{2\langle\lambda_\alpha,\lambda_\beta\rangle}{\langle\beta,\beta\rangle}$ is the inverse matrix of the Cartan matrix $g_{\alpha\beta} = \dfrac{2\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle}$. See e.g. [Dy, p. 117, Table 2] for the inverse of the Cartan matrix (but notice that Dynkin uses a different normalization). One can also easily obtain the diagonal entries of the inverse Cartan matrix by using the following observation. If H is a compact, simply connected, simple Lie group with center $Z(H)$ and Cartan matrix $g_{\alpha\beta}$, then $\det g_{\alpha\beta} = |Z(H)|$ (see e.g. [He, X.3.31]). Hence it follows that if α is a simple root of H and if we delete the vertex corresponding to α from the Dynkin diagram of H, we obtain several connected Dynkin diagrams corresponding to simply connected Lie groups H_i and $r_{\alpha\alpha} = g^{\alpha\alpha} = \Big(\prod_i |Z(H_i)|\Big)/|Z(H)|$. In either case one easily shows that the only cases where $r_{\alpha\alpha} \leqq 1$ are the representations listed in (a).

(b) The condition in (b) implies that $\frac{\langle\lambda,\lambda\rangle}{\langle\lambda,\alpha\rangle} < 1$ and hence we can apply (a). If $\pi_\lambda = \varrho_n$ or ν_{2n}, then we have $\lambda = \lambda^*$ and hence $\mathfrak{h} = \mathfrak{sp}(1) = \mathfrak{su}(2)$ with $\pi_\lambda = \nu_2 = \mu_2$. If $\pi_\lambda = \mu_n$ one easily checks that $\frac{\langle\lambda,\lambda\rangle + \langle\lambda,\lambda^*\rangle}{\langle\lambda,\alpha\rangle} = 1$. □

Remarks. (1) One can prove a lemma analogous to (2.9) if $\mathfrak{h}$ is only compact but not necessarily simple if one requires that $\frac{\langle\lambda,\lambda\rangle}{\langle\lambda,\alpha\rangle} \leqq 1$ for all simple roots α with $\langle\lambda,\alpha\rangle \neq 0$ and if one includes $(\mathrm{u}(n), \mu_n)$ as one of the possibilities. Indeed if $\lambda = \sum_i \lambda_i + x$ with $\pi_\lambda = \pi_{\lambda_1} \hat{\otimes} \ldots \hat{\otimes} \pi_{\lambda_k} \hat{\otimes} \phi$, then there exist simple roots α_i with $\langle\lambda,\alpha_i\rangle \neq 0$ and hence $\langle\lambda,\lambda\rangle = \sum_j \langle\lambda_j,\lambda_j\rangle + \langle x,x\rangle \leqq \langle\lambda,\alpha_i\rangle = \langle\lambda_i,\alpha_i\rangle$. If $k \geqq 2$, this implies that $\frac{\langle\lambda_i,\lambda_i\rangle}{\langle\lambda_i,\alpha_i\rangle} \leqq \frac{1}{2}$ for some i. Applying (2.9) we see that this is only possible if $\mathfrak{h} = \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) = \mathfrak{so}(4)$ with $\pi = \nu_2 \hat{\otimes} \nu_2 = \varrho_4$. If $k = 1$ and $x \neq 0$, (2.9) implies that $\mathfrak{h} = u(n)$ with $\pi = \mu_n$.

(2) (2.9) and Remark 1 generalizes [Ko1, Theorem 2.5] where he showed that if an irreducible representation has a weight μ such that μ is orthogonal to all weights $\sigma \neq \pm\mu$, then it must be ν_{2n}, ϱ_n, or μ_n. This easily reduces to the case where μ is the dominant weight λ. If α is a simple root with $\langle\lambda,\alpha\rangle \neq 0$, then $\lambda - \alpha$ is a weight and hence our (2.9) shows that it is sufficient to assume that λ and $\lambda - \alpha$ are orthogonal.

(2.10) Proposition. (a) *Let G/K be a compact irreducible hermitian symmetric space with $K = H \cdot \mathrm{U}(1)$. Then H is simple unless $(\mathfrak{k},\pi) = (\mathfrak{su}(m) \oplus \mathfrak{su}(n) \oplus \mathfrak{u}(1), [\mu_m \hat{\otimes} \mu_n^* \hat{\otimes} \phi]_{\mathbb{R}})$, $m \geqq n \geqq 2$. In general $\pi = [\pi_{\lambda'} \hat{\otimes} \phi]_{\mathbb{R}}$ with $\lambda' \neq \lambda'^*$ unless $G/K = \mathrm{SO}(n+2)/\mathrm{SO}(n)\mathrm{SO}(2)$.*

(b) *Let G/K be a quaternionic symmetric space with $K = H \cdot \mathrm{Sp}(1)$. Then H is simple unless $(\mathfrak{k},\pi) = (\mathfrak{so}(n) \oplus \mathfrak{so}(4) = [\mathfrak{so}(n) \oplus \mathfrak{sp}(1)] \oplus \mathfrak{sp}(1), \varrho_n \hat{\otimes} \varrho_4 = [\varrho_n \hat{\otimes} \nu_2] \hat{\otimes} \nu_2)$ or $(\mathfrak{k},\pi) = (\mathfrak{su}(2) \oplus \mathfrak{su}(n) \oplus \mathfrak{u}(1) = [\mathfrak{su}(n) \oplus \mathfrak{u}(1)] \oplus \mathfrak{sp}(1), [\nu_2 \hat{\otimes} \mu_n \hat{\otimes} \phi]_{\mathbb{R}} = [\mu_n \hat{\otimes} \phi]_{\mathbb{R}} \hat{\otimes} \nu_2)$. In particular the only quaternionic symmetric space that is also hermitian symmetric is the Grassmannian $\mathrm{SU}(n+2)/S(\mathrm{U}(n)\mathrm{U}(2))$.*

(c) *Let G/K be a compact irreducible symmetric space with $\mathfrak{k} = \mathfrak{k}_1 \oplus \ldots \oplus \mathfrak{k}_s$, $s \geqq 2$, where the $\mathfrak{k}_i$ are simple non-abelian ideals. Then $s \leqq 4$ and $(\mathfrak{k},\pi)$ belongs to one of the following:*

(i) *$s = 2$ and $(\mathfrak{k},\pi) = (\mathfrak{so}(m) \oplus \mathfrak{so}(n), \varrho_m \hat{\otimes} \varrho_n)$, $m \geqq n \geqq 3$, $m \neq 4$, $n \neq 4$.*
(ii) *$s = 2$ and $(\mathfrak{k},\pi) = (\mathfrak{sp}(m) \oplus \mathfrak{sp}(n), \nu_{2m} \hat{\otimes} \nu_{2n})$, $m \geqq n \geqq 1$.*
(iii) *$s = 2$ and $(\mathfrak{k},\pi) = (\mathfrak{h} \oplus \mathfrak{sp}(1), \pi_{\lambda'} \hat{\otimes} \nu_2)$ with $\mathfrak{h}$ simple and $\pi_{\lambda'}$ symplectic.*
(iv) *$s = 3$ and $(\mathfrak{k},\pi) = (\mathfrak{so}(n) \oplus \mathfrak{so}(4), \varrho_n \hat{\otimes} \varrho_4)$, $n \geqq 3$, $n \neq 4$.*
(v) *$s = 4$ and $(\mathfrak{k},\pi) = (\mathfrak{so}(4) \oplus \mathfrak{so}(4), \varrho_4 \hat{\otimes} \varrho_4)$.*

Proof. We first prove (c) and start by assuming that $s = 2$. Let λ be the dominant weight of π. Then $\lambda = \lambda_1 + \lambda_2$, where λ_i are dominant weights of $\mathfrak{k}_i$. We denote by α_i a simple root of $\mathfrak{k}_i$ such that $B(\lambda,\alpha_i) \neq 0$. Assume also that $(\mathfrak{k},\pi) \neq (\mathfrak{sp}(n) \oplus \mathfrak{sp}(1), \nu_{2n} \hat{\otimes} \nu_2)$. By (2.6) $B(\lambda,\lambda) = 2B(\lambda,\alpha_1) = 2B(\lambda,\alpha_2)$. This implies that

$$\frac{B(\lambda_1,\lambda_1)}{B(\lambda_1,\alpha_1)} + \frac{B(\lambda_2,\lambda_2)}{B(\lambda_2,\alpha_2)} = 2.$$

By relabelling $\mathfrak{k}_1$ and $\mathfrak{k}_2$, we may assume further that $\dfrac{B(\lambda_1,\lambda_1)}{B(\lambda_1,\alpha_1)} \leqq 1$. Since π_λ is orthogonal, π_{λ_i} are self-contragredient, so by (2.9) either $\dfrac{B(\lambda_i,\lambda_i)}{B(\lambda_i,\alpha_i)} = 1$, in which case $(\mathfrak{k},\pi) = (\mathfrak{so}(m)\oplus\mathfrak{so}(n), \varrho_m \hat{\otimes} \varrho_n)$, $m \geqq n \geqq 3$ or $(\mathfrak{sp}(m)\oplus\mathfrak{sp}(n), \nu_{2m}\hat{\otimes}\nu_{2n})$, $m \geqq n \geqq 2$, or else $\dfrac{B(\lambda_1,\lambda_1)}{B(\lambda_1,\alpha_1)} = \dfrac{1}{2}$, in which case $\mathfrak{k}_1 = \mathfrak{sp}(1)$, $\pi_{\lambda_1} = \nu_2$. In this latter situation, $\mathfrak{k}_2 = \mathfrak{h}$ is simple and $\pi_{\lambda_2} = \pi_{\lambda'}$ is symplectic. Notice that $B(\lambda_2,\lambda_2) = \frac{3}{2}B(\lambda_2,\alpha_2)$ holds.

If $s > 2$, we proceed analogously and obtain $\sum_{i=1}^{s} \dfrac{B(\lambda_i,\lambda_i)}{B(\lambda_i,\alpha_i)} = 2$. If one term equals 1, by (2.9) it must correspond to $(\mathfrak{so}(n),\varrho_n)$, $n \neq 4$. Hence $s = 3$, with the other two terms corresponding to $(\mathfrak{sp}(1),\nu_2)$, i.e. $(\mathfrak{k},\pi) = (\mathfrak{so}(n)\oplus\mathfrak{so}(4), \varrho_n\hat{\otimes}\varrho_4)$. If no term equals 1, then all equal $\frac{1}{2}$ and correspond to $(\mathfrak{sp}(1),\nu_2)$, i.e. $(\mathfrak{k},\pi) = (\mathfrak{so}(4)\oplus\mathfrak{so}(4), \varrho_4\hat{\otimes}\varrho_4)$.

Next we prove (a). Let $\mathfrak{h} = \mathfrak{h}_1\oplus\ldots\oplus\mathfrak{h}_s$ where $\mathfrak{h}_i$ are simple non-abelian ideals and $s \geqq 2$. Then $\lambda = \lambda_1 + \ldots + \lambda_s + x$, where x is the weight on $\mathfrak{u}(1)$. Let α_i be a simple root of $\mathfrak{h}_i$ such that $B(\lambda,\alpha_i) \neq 0$. By (2.6), we have $\sum_{i=1}^{s} B(\lambda_i,\lambda_i) + B(x,x) = 2B(\lambda,\alpha_i)$ for each i. Since G/K is not complex projective space, (2.7) implies that $\sum_{i=1}^{s} B(\lambda_i,\lambda_i^*) - B(x,x) = 0$. Therefore, $\sum_{i=1}^{s}\left(\dfrac{B(\lambda_i,\lambda_i)+B(\lambda_i,\lambda_i^*)}{B(\lambda_i,\alpha_i)}\right) = 2$. If $s \geqq 2$, it follows that there is some i for which $\dfrac{B(\lambda_i,\lambda_i)+B(\lambda_i,\lambda_i^*)}{B(\lambda_i,\alpha_i)} \leqq 1$. By (2.9.b) $\mathfrak{k}_i = \mathfrak{su}(n)$ and $\pi_{\lambda_i} = \mu_n$ with $\dfrac{B(\lambda_i,\lambda_i)+B(\lambda_i,\lambda_i^*)}{B(\lambda_i,\alpha_i)} = 1$. Hence $s = 2$ and $(\mathfrak{k},\pi) = (\mathfrak{su}(m)\oplus\mathfrak{su}(n)\oplus\mathfrak{u}(1), [\mu_m\hat{\otimes}\mu_n\hat{\otimes}\phi]_{\mathbb{R}})$.

If we assume that $\lambda = \lambda' + x$ with $\lambda' = \lambda'^*$, then the above shows that $s = 1$ and then (2.7.a) implies that $B(\lambda',\lambda') = B(\lambda',\alpha)$ and by (2.9) and (2.4.a) we are in the case of the complex quadric $\mathrm{SO}(n+2)/\mathrm{SO}(n)\mathrm{SO}(2)$.

(b) follows by combining (a) and (c). □

(2.8) and (2.10) reduces the study of irreducible symmetric spaces to the following three cases:

(A) $M = (G/H\cdot\mathrm{U}(1),\pi)$ with H simple and $\pi = [\pi_\lambda\hat{\otimes}\phi]_{\mathbb{R}}$, $\lambda = k\lambda_\alpha \neq \lambda^*$ and $\langle\lambda,\lambda\rangle + \langle\lambda,\lambda^*\rangle = 2\langle\lambda,\alpha\rangle$.

(B) $M = (G/H\cdot\mathrm{Sp}(1),\pi)$ with H simple and $\pi\otimes\mathbb{C} = \pi_\lambda\hat{\otimes}\nu_2$, $\lambda = k\lambda_\alpha$ and $\langle\lambda,\lambda\rangle = \frac{3}{2}\langle\lambda,\alpha\rangle$.

(C) $M = (G/K,\pi)$ with K simple and $\pi\otimes\mathbb{C} = \pi_\lambda$, $\lambda = k\lambda_\alpha$ and $\langle\lambda,\lambda\rangle = 2\langle\lambda,\alpha\rangle$.

In the above three cases the equations hold with respect to any inner product $\langle\,,\rangle$ since the group in question is simple. We first show that in most cases $k = 1$.

(2.11) Proposition. *If in the cases* (A)–(C) *above we have* $k > 1$, *then one of the following holds:*

(A) $k = 2$ *with* $(\mathfrak{k}\oplus\mathfrak{u}(1),\pi) = (\mathfrak{u}(n), S^2\mu_n)$ *which is the isotropy representation of the symmetric space* $\mathrm{Sp}(n)/\mathrm{U}(n)$.

(B) $k = 3$ *with* $(\mathfrak{k} \oplus \mathfrak{sp}(1), \pi) = (\mathfrak{sp}(1) \oplus \mathfrak{sp}(1), S^3\nu_2 \hat{\otimes} \nu_2)$ *which is the isotropy representation of the symmetric space* $G_2/SO(4)$.

(C) $k = 2$ *with* $(\mathfrak{k}, \pi) = (\mathfrak{so}(n), S^2\varrho_n - \mathbf{1})$, $n \geqq 5$, *which is the isotropy representation of the symmetric space* $SU(n)/SO(n)$; *or* $(\mathfrak{k}, \pi) = (\mathfrak{sp}(n), S^2\nu_{2n})$, $n \geqq 1$, *which is the isotropy representation of the symmetric space* $[Sp(n) \times Sp(n)]/\Delta Sp(n)$. *Otherwise,* $k = 4$ *with* $(\mathfrak{k}, \pi) = (\mathfrak{so}(3), S^4\nu_2)$, *which is the isotropy representation of the symmetric space* $SU(3)/SO(3)$.

Proof. In case (A) we have $2 = \dfrac{\langle \lambda, \lambda \rangle + \langle \lambda, \lambda^* \rangle}{\langle \lambda, \alpha \rangle} = k \dfrac{\langle \lambda_\alpha, \lambda_\alpha \rangle + \langle \lambda_\alpha, \lambda_\alpha^* \rangle}{\langle \lambda_\alpha, \alpha \rangle}$ and hence (2.9.b) implies that $\mathfrak{k} = \mathfrak{su}(n)$ with $\lambda_\alpha = \mu_n$ and $k = 2$.

In case (B) we have $k \dfrac{\langle \lambda_\alpha, \lambda_\alpha \rangle}{\langle \lambda_\alpha, \alpha \rangle} = \dfrac{3}{2}$ and since $k\lambda_\alpha$ must be symplectic, (2.9.a) implies that $\mathfrak{k} = \mathfrak{sp}(1)$ with $\lambda_\alpha = \nu_2$ and $k = 3$.

In case (C) we have $2 = \dfrac{\langle \lambda, \lambda \rangle}{\langle \lambda, \alpha \rangle} = k \dfrac{\langle \lambda_\alpha, \lambda_\alpha \rangle}{\langle \lambda_\alpha, \alpha \rangle}$. (2.9) together with the fact that $k\lambda_\alpha$ must be orthogonal implies that $\mathfrak{k} = \mathfrak{so}(n)$ with $\lambda_\alpha = \varrho_n$, $n \geqq 5$, and $k = 2$, or $\mathfrak{k} = \mathfrak{sp}(n)$ with $\lambda_\alpha = \nu_n$, $n \geqq 2$, and $k = 2$, or $\mathfrak{k} = \mathfrak{sp}(1)$ with $\lambda_\alpha = \nu_2$ and $k = 4$. □

The classification of symmetric spaces is now reduced to the cases (A)–(C) with λ a basic weight λ_α. We can easily obtain a complete list by using the observation, which we made in the proof of (2.9), that $r_{\alpha\alpha} = \dfrac{\langle \lambda_\alpha, \lambda_\alpha \rangle}{\langle \lambda_\alpha, \alpha \rangle}$ are the diagonal entries in the inverse of the Cartan matrix. Now, as in the proof of (2.9), one immediately enumerates the possibilities where $r_{\alpha\alpha} = \dfrac{\langle \lambda_\alpha, \lambda_\alpha \rangle}{\langle \lambda_\alpha, \alpha \rangle} \leqq 2$ and easily checks that they belong to one of the following cases:

(A) If $\lambda_\alpha \neq \lambda_\alpha^*$ then $\dfrac{\langle \lambda_\alpha, \lambda_\alpha \rangle + \langle \lambda_\alpha, \lambda_\alpha^* \rangle}{\langle \lambda_\alpha, \alpha \rangle} = 2$ or $\dfrac{\langle \lambda_\alpha, \lambda_\alpha \rangle + \langle \lambda_\alpha, \lambda_\alpha^* \rangle}{\langle \lambda_\alpha, \alpha \rangle} = 1$. In the first case one obtains the entries (2), (4), and (5) in our table at the end of this paper and in the second case $\lambda_\alpha = \mu_n$, corresponding to entry (7).

(B) If λ_α is symplectic then $\dfrac{\langle \lambda_\alpha, \lambda_\alpha \rangle}{\langle \lambda_\alpha, \alpha \rangle} = \dfrac{3}{2}$ or $\dfrac{\langle \lambda_\alpha, \lambda_\alpha \rangle}{\langle \lambda_\alpha, \alpha \rangle} = 1$ or $\dfrac{\langle \lambda_\alpha, \lambda_\alpha \rangle}{\langle \lambda_\alpha, \alpha \rangle} = \dfrac{1}{2}$. In the first case one obtains the entries (10)–(13) in our table, in the second case $\lambda_\alpha = \nu_{2n}$, $n \geqq 2$, and in the third case $\lambda_\alpha = \nu_2$, both corresponding to entry (15).

(C) If λ_α is orthogonal then $\dfrac{\langle \lambda_\alpha, \lambda_\alpha \rangle}{\langle \lambda_\alpha, \alpha \rangle} = 2$ or $\dfrac{\langle \lambda_\alpha, \lambda_\alpha \rangle}{\langle \lambda_\alpha, \alpha \rangle} = \dfrac{3}{2}$ or $\dfrac{\langle \lambda_\alpha, \lambda_\alpha \rangle}{\langle \lambda_\alpha, \alpha \rangle} = 1$. In the first case one obtains the entries (17) and (19)–(24) in our table as well as $(\mathfrak{k}, \lambda_\alpha) = (\mathfrak{g}_2, \phi_7)$ which by (2.4) cannot be the isotropy representation of a symmetric space. In the second case $(\mathfrak{k}, \lambda_\alpha) = (\mathfrak{so}(7), \Delta_7)$ which cannot be the isotropy representation of a symmetric space either. In the third case $\lambda_\alpha = \varrho_n$ which corresponds to entry (25).

This finishes the classification of symmetric spaces. These arguments can be easily modified to obtain the following converse to (2.6) and (2.7).

(2.12) Theorem. *Let* $\pi_\lambda : \mathfrak{h} \to \mathfrak{gl}(V)$ *be a faithful irreducible representation of a compact semisimple Lie algebra* $\mathfrak{h}$ *on a complex vector space* V, *and let* $\langle , \rangle$ *be a bi-invariant metric on* $\mathfrak{h}$.

(a) *If* $\pi_\lambda \neq \pi_{\lambda^*}$, *and* $\langle \lambda, \lambda\rangle + \langle \lambda, \lambda^*\rangle = 2\langle \lambda, \alpha\rangle$ *for every simple root* α *such that* $\langle \lambda, \alpha\rangle \neq 0$, *then* $(\mathfrak{h} \oplus \mathfrak{u}(1), [\pi_\lambda \hat{\otimes} \phi]_{\mathbb{R}})$ *is the isotropy representation of a hermitian symmetric space.*

(b) *If* π_λ *is symplectic and* $\langle \lambda, \lambda\rangle = \frac{3}{2}\langle \lambda, \alpha\rangle$ *for every simple root* α *such that* $\langle \lambda, \alpha\rangle \neq 0$, *then* $(\mathfrak{h} \oplus \mathfrak{sp}(1), \pi_\lambda \hat{\otimes} \nu_2)$ *is the isotropy representation of a quaternionic symmetric space.*

(c) *If* π_λ *is orthogonal and* $\langle \lambda, \lambda\rangle = 2\langle \lambda, \alpha\rangle$ *for every simple root* α *such that* $\langle \lambda, \alpha\rangle \neq 0$, *then* $(\mathfrak{h}, \pi_\lambda)$ *is the isotropy representation of an irreducible symmetric space, except in the case* $(\mathfrak{g}_2, \phi_7)$.

Proof. Notice that the proof of (2.10) and (2.11) and the classification after (2.11) only uses the condition on the dominant weights in (a)–(c) and hence carries over directly. But we need to find a different proof of (2.8) since we do not assume the stronger condition in (2.5).

But this can be easily done using (2.9) as in the proof of (2.10). For example, in case (c), if $\alpha \neq \beta$ are two roots in the same simple ideal $\mathfrak{h}$ of $\mathfrak{k}$ with $\langle \lambda, \alpha\rangle \neq 0$ and $\langle \lambda, \beta\rangle \neq 0$, then $\dfrac{\langle \lambda_\alpha, \lambda_\alpha\rangle}{\langle \lambda_\alpha, \alpha\rangle} + \dfrac{\langle \lambda_\beta, \lambda_\beta\rangle}{\langle \lambda_\beta, \beta\rangle} < 2$ since $\langle \lambda_\alpha, \lambda_\beta\rangle > 0$. Hence, after relabeling α, and β, we can assume $\dfrac{\langle \lambda_\alpha, \lambda_\alpha\rangle}{\langle \lambda_\alpha, \alpha\rangle} < 1$ which implies $\mathfrak{h} = \mathfrak{su}(n)$, $\lambda_\alpha = \mu_n$. Since $\langle \lambda, \alpha\rangle = \langle \lambda^*, \alpha^*\rangle = \langle \lambda, \alpha^*\rangle \neq 0$ we can assume that $\beta = \alpha^*$ and hence $\lambda_\beta = \mu_n^*$. If $\lambda = k\lambda_\alpha + k\lambda_\alpha^* + \ldots$, then $\langle \lambda, \lambda\rangle \geqq k^2\langle \lambda_\alpha + \lambda_\beta, \lambda_\alpha + \lambda_\beta\rangle = 2k^2(\langle \lambda_\alpha, \lambda_\alpha\rangle + \langle \lambda_\alpha, \lambda_\alpha^*\rangle) = 2k^2\langle \lambda_\alpha, \alpha\rangle \geqq 2\langle \lambda, \alpha\rangle$, which implies that $\lambda = \lambda_\alpha + \lambda_\beta$ or $\pi_\lambda = \mathrm{Ad}_{\mathrm{SU}(n)}$. This proves (2.8) under the asumption (c). Similarly in the case of (a) and (b). □

Remarks. (1) One can also formulate a similar theorem if $\mathfrak{h}$ is compact but not semisimple. If $[\pi_\lambda]_{\mathbb{R}}$ is an irreducible orthogonal representation of such a Lie algebra with $\langle \lambda, \lambda\rangle = 2\langle \lambda, \alpha\rangle$ for every simple root α with $\langle \lambda, \alpha\rangle \neq 0$ and if in addition $\langle \lambda, \lambda^*\rangle = 0$, then $[\pi_\lambda]_{\mathbb{R}}$ is the isotropy representation of a hermitian symmetric space. Indeed, the center of $\mathfrak{h}$ is 1-dimensional and if $\lambda = \lambda' + x$, then the two conditions imply that $\langle \lambda', \lambda'\rangle + \langle \lambda', \lambda'^*\rangle = 2\langle \lambda', \alpha\rangle$ and hence the claim follows from (2.12.a).

(2) Our proof of Theorem 2.12 is not ideal since it involves case by case arguments. It indirectly uses the classification of symmetric spaces since we need to know that the resulting representations are indeed isotropy representations of symmetric spaces. For this purpose we use the list of the isotropy representations of all irreducible symmetric spaces in [Wo, (8.11.2) and (8.11.5)]. It would be much more desirable to have a direct proof of this theorem that does not use the classification of symmetric spaces. We hope to return to this problem in a separate paper.

3 Strongly isotropy irreducible spaces from irreducible symmetric spaces

We first formulate the correspondence precisely.

(3.1) Theorem (Forward correspondence). (a) *Let* $G/K = G/H \cdot \mathrm{U}(1)$ *be a compact, simple connected, effective, irreducible hermitian symmetric space of real dimension* $2n$ *with isotropy representation* $\pi \otimes \mathbb{C} = \pi_\lambda \oplus \pi_\lambda^*$ *and* $\pi_\lambda = \pi_{\lambda'} \hat{\otimes} \phi$. *Then*

$\pi_{\lambda'}: H \to \mathrm{SU}(n)$ is a faithful unitary representation and the homogeneous manifold $V = \mathrm{SU}(n)/\pi_{\lambda'}(H)$ is strongly isotropy irreducible.

(b) *Let $G/K = G/H \cdot \mathrm{Sp}(1)$ be a compact, simple connected, effective, quaternionic symmetric space of real dimension $4n$ with isotropy representation $\pi \otimes \mathbb{C} = \sigma \hat{\otimes} \nu_2$, where σ is a faithful irreducible quaternionic representation $\sigma: H \to \mathrm{Sp}(n)$ of H. Then the homogeneous manifold $V = \mathrm{Sp}(n)/\sigma(H)$ is strongly isotropy irreducible.*

(c) *Let G/K be a compact, simply connected, effective, irreducible symmetric space of dimension n and assume that either K is simple or that G/K is a quaternionic projective space. Then the isotropy representation $\pi \otimes \mathbb{C} = \pi_\lambda: K \to \mathrm{SO}(n)$ is a faithful, absolutely irreducible orthogonal representation and the homogeneous manifold $V = \mathrm{SO}(n)/\pi(K)$ is strongly isotropy irreducible.*

By applying this construction to the Grassmannians, we can see some complications in this correspondence:

(1) The rank 1 symmetric spaces (a) $\mathrm{SU}(n+1)/\mathrm{S}(\mathrm{U}(n)\mathrm{U}(1))$, (b) $\mathrm{Sp}(n+1)/\mathrm{Sp}(n)\,\mathrm{Sp}(1)$, and (c) $\mathrm{SO}(n+1)/\mathrm{SO}(n)$ give rise to the space V consisting of a single point. It follows from (2.2) that these are actually the only cases in which V can be a point.

(2) The Grassmannian $\mathrm{SU}(n+2)/\mathrm{S}(\mathrm{U}(n)\mathrm{U}(2))$ is both hermitian symmetric and quaternionic symmetric, and we obtain respectively the strongly isotropy irreducible spaces $\mathrm{SU}(2n)/\mathrm{SU}(n)\,\mathrm{SU}(2)$ and $\mathrm{Sp}(n)/\mathrm{U}(n)$, the latter also being a symmetric space. By (2.10.b) this complex Grassmannian is actually the only symmetric space which is both hermitian symmetric and quaternionic symmetric.

(3) Besides the above example there are two more in which V is again a symmetric space. Namely if G/K is the hermitian symmetric space $\mathrm{SO}(n+2)/\mathrm{SO}(n)\,\mathrm{SO}(2)$, then V is the symmetric space $\mathrm{SU}(n)/\mathrm{SO}(n)$. If G/K is the real symmetric space $\mathrm{Sp}(3)/\mathrm{Sp}(2)\,\mathrm{Sp}(1)$, then V is the symmetric space $\mathrm{SO}(8)/[\mathrm{Sp}(2)\,\mathrm{Sp}(1)]/\Delta\mathbb{Z}_2 = (\mathrm{SO}(8)/\mathbb{Z}_2)/\mathrm{SO}(5)\,\mathrm{SO}(3)$. We will be able to prove this directly in (3.4).

(4) On the other hand, real or quaternionic Grassmannians which are not hermitian symmetric or quaternionic symmetric are excluded in the correspondence. This fact will be explained after the proof of (3.1), where we will show that in these cases the isotropy representation of $\mathrm{SO}(n)/\pi(K)$ has as many irreducible subrepresentations as there are simple factors in K.

(5) But we cannot totally exclude the Grassmannians from the correspondence either since we need the complex Grassmannian in (a), the real Grassmannian $\mathrm{SO}(n+4)/\mathrm{SO}(n)\,\mathrm{SO}(4)$ in (b), and the quaternionic Grassmannian $\mathrm{Sp}(n+1)/\mathrm{Sp}(n)\,\mathrm{Sp}(1)$ in case (c).

(6) As we shall see, the only strongly isotropy irreducible quotient of a classical group that is excluded from the range of the correspondence is $\mathrm{SO}(7)/\mathrm{G}_2$. Notice though that (G_2, ϕ_7) was also an exception in several of our results in Sect. 2.

(7) There is one strongly isotropy irreducible space which can arise in two distinct ways from a symmetric space. Namely the symmetric space $\mathrm{SU}(3)/\mathrm{SO}(3)$ gives rise to the strongly isotropy irreducible space $\mathrm{SO}(5)/\mathrm{SO}(3)$ and the quaternionic symmetric space $\mathrm{G}_2/\mathrm{SO}(4)$ gives rise to the strongly isotropy irreducible space $\mathrm{Sp}(2)/\mathrm{Sp}(1) = \mathrm{SO}(5)/\mathrm{SO}(3)$. As we will see, this is the only case where this can occur.

It is amusing to observe that it is precisely these familiar spaces that are the exceptional cases in the correspondence.

The first step in proving Theorem 3.1 is the following fact about Casimir operators. Our convention for the Casimir operator of a representation π is that $\mathscr{C}_\pi = -\sum_i \pi(x_i) \circ \pi(x_i)$ where x_i is an orthonormal basis of $\mathfrak{k}$ with respect to a

given bi-invariant symmetric bilinear form. Furthermore, if we use a positive definite inner product on $\mathfrak{k}$, then Casimir constants are non-negative. For further details, see [WZ2, pp. 568–571].

(3.2) Proposition. *Let G/K be a compact symmetric space with no Euclidean factors, $\mathfrak{g} = \mathfrak{k} \perp \mathfrak{p}$ and π be its isotropy representation. Suppose that B is the negative of the Killing form of G and $\mathscr{C}_\pi$, $\mathscr{C}_{\Lambda^2\pi}$ are the Casimir operators with respect to $B|_{\mathfrak{k}}$. Let $\hat{R}$ be the curvature operator of the normal metric (defined by B) acting on $\Lambda^2\mathfrak{p}$. Then $\mathscr{C}_\pi = \frac{1}{2}\,\mathrm{Id}$ and $\mathscr{C}_{\Lambda^2\pi} = \mathrm{Id} - 2\hat{R}$.*

Proof. By assumption G is semisimple and hence B is positive definite. Let $\{x_i\}$ be an orthonormal basis for $\mathfrak{k}$ and $\{e_p\}$ an orthonormal basis for $\mathfrak{p}$.

The fact that $\mathscr{C}_\pi = \frac{1}{2}\,\mathrm{Id}$ follows easily from [WZ2, Corollaries I.1.6 and I.1.7], but for the convenience of the reader we give here a direct proof. Indeed, for $X, Y \in \mathfrak{p}$ we have:

$$
\begin{aligned}
B(X,Y) &= -\mathrm{tr}[\mathrm{ad}\,X\,\mathrm{ad}\,Y] = -\sum_i B([X,[Y,x_i]],x_i) - \sum_p B([X,[Y,e_p]],e_p)\\
&= \sum_i B([X,x_i],[Y,x_i]) + \sum_p B([X,e_p],[Y,e_p])\\
&= \sum_i B([x_i,X],[x_i,Y]) + \sum_p \sum_i B([X,e_p],x_i)B([Y,e_p],x_i)\\
&= -\sum_i B([x_i,[x_i,X]],Y) + \sum_i \sum_p B([[X,x_i],e_p])B([Y,x_i],e_p)\\
&= B(\mathscr{C}_\pi(X),Y) + \sum_i B([X,x_i],[Y,x_i]) = 2B(\mathscr{C}_\pi(X),Y)\,.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathscr{C}_{\Lambda^2\pi}(X\wedge Y) &= -\sum_i (\mathrm{ad}(x_i)^2X)\wedge Y - \sum_i X\wedge(\mathrm{ad}(x_i)^2Y)\\
&\quad - 2\sum_i (\mathrm{ad}(x_i)X)\wedge(\mathrm{ad}(x_i)Y)\\
&= X\wedge Y - 2\sum_i (\mathrm{ad}(x_i)X)\wedge(\mathrm{ad}(x_i)Y)\,.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
&\sum_i (\mathrm{ad}(x_i)X)\wedge(\mathrm{ad}(x_i)Y)\\
&\quad = \sum_i \sum_{p,q} B([x_i,X],e_p)B([x_i,Y],e_q)e_p\wedge e_q\\
&\quad = \sum_{p,q} B([X,e_p],[Y,e_q])e_p\wedge e_q\\
&\quad = -\sum_{p,q} B(e_p,[X,[Y,e_q]])e_p\wedge e_q
\end{aligned}
$$

$$= \sum_{p,q} B(e_p, [Y,[e_q,X]] + [e_q,[X,Y]])e_p \wedge e_q \text{ by the Jacobi identity}$$

$$= \sum_{p,q} B([e_q,X],[e_p,Y])e_p \wedge e_q + \sum_{p,q} B([e_p,e_q],[X,Y])e_p \wedge e_q$$

$$= -\sum_i (\mathrm{ad}(x_i)X) \wedge (\mathrm{ad}(x_i)Y) - \sum_{p,q} B(R_{X,Y}(e_p),e_q)e_p \wedge e_q .$$

Hence

$$\sum_i \mathrm{ad}(x_i)X \wedge \mathrm{ad}(x_i)Y = -\sum_{p<q} B(R_{X,Y}(e_p),e_q)e_p \wedge e_q$$

$$= \hat{R}(X \wedge Y)$$

where our convention is chosen so that sectional curvature is given by $K(e_i \wedge e_j) = \langle R_{e_i,e_j}(e_j), e_i\rangle$ and $\hat{R} = \mathrm{Id}$ for the standard sphere. The formula for $\mathscr{C}_{\Lambda^2\pi}$ follows immediately. □

(3.3) Corollary. (a) *Let G/K be a compact symmetric space with no euclidean factors and with isotropy representation $\pi: K \to \mathrm{SO}(\mathfrak{p})$. Then $\Lambda^2\pi = \mathrm{Ad}_K \oplus \chi$ where χ is the isotropy representation of $\mathrm{SO}(\mathfrak{p})/\pi(K)$ and $\mathscr{C}_\chi = \mathrm{Id}$, where the Casimir operator is taken with respect to the negative of the Killing form of G restricted to K.*

(b) *Let G/K be a compact symmetric space as in* (3.1.a) *or* (3.1.b) *and let χ be the isotropy representation of the corresponding homogeneous space $V = \mathrm{SU}(n)/\pi_{\lambda'}(H)$ or $V = \mathrm{Sp}(n)/\sigma(H)$ respectively. Then $\mathscr{C}_\chi = \mathrm{Id}$ where the Casimir operator is taken with respect to the negative of the Killing form of G restricted to H.*

Proof. It follows from Sect. 1 that $\Lambda^2\pi = \mathrm{Ad}_K \oplus \chi$ and that χ is the isotropy representation of $\mathrm{SO}(\mathfrak{p})/\pi(K)$. The embedding $\mathrm{ad}(\mathfrak{k}) \subset \Lambda^2(\mathfrak{p}) \cong \mathfrak{so}(\mathfrak{p})$ is given by $\mathrm{ad}(X) \to \sum_{p<q} B(\mathrm{ad}(X)e_p, e_q)e_p \wedge e_q$ and hence $\hat{R}(X \wedge Y)$ is equal to the image of $-\mathrm{ad}[X,Y]$. We can choose the representation space $\mathfrak{m}$ for $\chi \subset \Lambda^2\pi$ to be perpendicular to $\mathrm{ad}(\mathfrak{k})$ and since $\hat{R}$ is a symmetric operator, it keeps the decomposition $\Lambda^2(\mathfrak{p}) = \mathrm{ad}(\mathfrak{k}) \oplus \mathfrak{m}$ invariant and must be 0 to $\mathfrak{m}$. This together with (3.2) proves (a).

To prove (b) we observe that in the case of $G/H \cdot \mathrm{U}(1)$ we have $H \cdot \mathrm{U}(1) \subset \mathrm{SU}(n)\mathrm{U}(1) \subset \mathrm{SO}(2n)$ and hence the isotropy representation of $\mathrm{SU}(n)/\pi_{\lambda'}(H)$ is contained in the isotropy representation of $\mathrm{SO}(2n)/\pi(H \cdot \mathrm{U}(1))$. Similarly for the case of $G/H \cdot \mathrm{Sp}(1)$ and hence (b) follows from (a). □

Remarks. (1) Notice that in (3.3.a) the representation π and hence χ may be reducible, but the Casimir constants with respect to $B|_{\mathfrak{k}}$ of the irreducible sub-representations of χ are all equal. But this does not say that $\mathrm{SO}(\mathfrak{p})/\pi(K)$ is normal homogeneous Einstein (see [WZ2, I.1.7]) since the Casimir constants with respect to $B_{\mathrm{SO}(p)}|_{\mathfrak{k}}$ could be different. See also (5.3).

(2) There is a converse to (3.3), which we proved in [WZ2, III.3.4, II.3.6, III.3.8], assuming all the results of this paper. It states that if $\mathrm{SO}(n)/\pi(H)$, $\mathrm{SU}(n)/\pi(H)$, or $\mathrm{Sp}(n)/\pi(H)$ is a homogeneous space with isotropy representation χ such that $\mathscr{C}_\chi = a \cdot \mathrm{Id}$ for some choice of bi-invariant metric on H and some constant a, then, up to a few exceptions, π, $[\pi \hat{\otimes} \phi]_{\mathbb{R}}$, $\pi \hat{\otimes} \nu_2$ resp. are the isotropy representation of a symmetric space.

Proof of Theorem 3.1. We begin with the hermitian symmetric case. If G/K is the complex projective space, then $V =$ point. So we assume that this is not the case.

By (2.2.a) and (2.4.a) we have $\pi_{\lambda'+\lambda'^*} \subset \chi \subset \pi_{\lambda'} \otimes \pi^*_{\lambda'}$ and by (3.3.b) $\mathscr{C}_\chi = \mathrm{Id}$. Any other irreducible summand in $\pi_{\lambda'} \otimes \pi^*_{\lambda'}$ and hence in χ has dominant weight of the form $\lambda' + \lambda'^* - \left(\sum_\alpha n_\alpha \alpha\right)$ with $n_\alpha \geqq 0$ and some $n_\alpha > 0$. Now in general one easily shows, see e.g. [WZ2, II.2.2], that if μ and ϱ are two dominant weights with $\varrho = \mu - \left(\sum_\alpha n_\alpha \alpha\right)$ and $n_\alpha \geqq 0$ with some $n_\alpha > 0$, then the Casimir constant for π_ϱ must be strictly smaller than the one for π_μ. Hence $\chi = \pi_{\lambda'+\lambda'^*}$ and V is strongly isotropy irreducible.

In the case of a quaternionic symmetric space we have $\sigma = \pi_{\lambda'}$ or $\sigma = \pi_{\lambda'} \oplus \pi^*_{\lambda'}$. In the first case and excluding the quaternionic projective space, we have $\pi_{2\lambda'} \subset \chi \subset S^2\pi_{\lambda'}$ and $\mathscr{C}_\chi = \mathrm{Id}$. Any other summand in χ is of the form $2\lambda' - \left(\sum_\alpha n_\alpha \alpha\right)$ and hence has smaller Casimir constant. Thus $\chi = \pi_{2\lambda'}$ and V is strongly isotropy irreducible. In the second case G/K must also be a hermitian symmetric space. We could now use (2.10.b) to conclude that $\pi_{\lambda'} = \mu_n$ and hence $V = \mathrm{Sp}(n)/\mathrm{U}(n)$ which is strongly isotropy irreducible. We can also argue directly as follows. We have $\chi \subset S^2\sigma = S^2(\pi_{\lambda'} \oplus \pi^*_{\lambda'}) = S^2\pi_{\lambda'} \oplus S^2\pi^*_{\lambda'} \oplus \pi_{\lambda'} \otimes \pi^*_{\lambda'}$ and since $\pi_{\lambda'} \otimes \pi^*_{\lambda'} = \mathrm{Ad}_H \oplus \psi$ we get $\chi = S^2\pi_{\lambda'} \oplus S^2\pi^*_{\lambda'} \oplus \psi$. Since $\pi_{2\lambda'} \subset S^2\pi_{\lambda'}$ and since any other summand in $S^2\pi_{\lambda'}$ would have smaller Casimir constant, it follows that $S^2\pi_{\lambda'} = \pi_{2\lambda'}$ and $S^2\pi^*_{\lambda'} = \pi^*_{2\lambda'}$. If $\pi_{\lambda'} \neq \mu_n$ then $\pi_{\lambda'+\lambda'^*} \subset \psi \subset \chi$ and hence must have the same Casimir constant as $\pi_{2\lambda'}$. But these Casimir constants are equal to $\langle \lambda'+\lambda'^*, \lambda'+\lambda'^*+2\delta\rangle$ and $\langle 2\lambda', 2\lambda'+2\delta\rangle$ respectively, where 2δ is the sum of all positive roots of $\mathfrak{h}$. Hence if they are equal it follows that $\langle \lambda', \lambda'\rangle = \langle \lambda', \lambda'^*\rangle = \langle \lambda'^*, \lambda'^*\rangle$ and thus $\lambda' = \lambda'^*$ which would contradict the assumption that G/K is hermitian symmetric. Hence $\pi_{\lambda'} = \mu_n$ and thus $\chi = \pi_{2\lambda'} \oplus \pi^*_{2\lambda'}$ which means that V is strongly isotropy irreducible.

In the case of (3.1.c) assume first that G/K is not a sphere or a quaternionic projective space or the symmetric space $[\mathrm{SU}(n) \times \mathrm{SU}(n)]/\Delta\mathrm{SU}(n)$. Then, if α is a simple root of K with $\langle \lambda, \alpha\rangle \neq 0$, (2.2.c) and (2.4) imply that $\pi_{2\lambda-\alpha} \subset \chi \subset \Lambda^2\pi$ and by (3.3.a) $\mathscr{C}_\chi = \mathrm{Id}$. Since K is simple, (2.8) says that there exists a unique simple root α with $\langle \lambda, \alpha\rangle \neq 0$. Hence any other irreducible summand in $\Lambda^2\pi$ and hence in χ must have dominant weight $2\lambda - \alpha - \left(\sum_\beta n_\beta \beta\right)$ and hence has smaller Casimir constant. Thus $\chi = \pi_{2\lambda-\alpha}$ and V is strongly isotropy irreducible.

If $G/K = [\mathrm{SU}(n) \times \mathrm{SU}(n)]/\Delta\mathrm{SU}(n)$, then there exist two simple roots α and α^* with $\langle \lambda, \alpha\rangle \neq 0$ and $\langle \lambda, \alpha^*\rangle \neq 0$. Hence $\pi_{2\lambda-\alpha}$ and $\pi_{2\lambda-\alpha^*} \subset \chi$ and the same argument as above shows that $\chi = \pi_{2\lambda-\alpha} \oplus \pi_{2\lambda-\alpha^*}$. But since $(2\lambda-\alpha)^* = 2\lambda - \alpha^*$ this says that χ is irreducible as a real representation.

If $G/K = \mathrm{Sp}(n+1)/\mathrm{Sp}(n)\mathrm{Sp}(1)$, then there exist precisely two simple roots α and β with $\langle \lambda, \alpha\rangle \neq 0$ and $\langle \lambda, \beta\rangle \neq 0$ where α is a root in $\mathrm{Sp}(n)$ and β is the root of $\mathrm{Sp}(1)$. Then by (2.2.c), $\pi_{2\lambda-\alpha} \subset \chi$ and $\pi_{2\lambda-\beta} \subset \mathrm{ad}(\mathfrak{k})$. Any other irreducible summand in χ must have dominant weight $2\lambda - \alpha - \left(\sum_\gamma n_\gamma \gamma\right)$ and hence has smaller Casimir constant. Thus $\chi = \pi_{2\lambda-\alpha}$ and V is strongly isotropy irreducible.

This finishes the proof of (3.1). □

Remarks. (1) We can see from the above proof why the construction (3.1.c) fails to yield isotropy irreducible spaces when G/K is a quaternionic Grassmannian $\mathrm{Sp}(p+q)/\mathrm{Sp}(p)\mathrm{Sp}(q)$, $p \geqq q \geqq 2$ or a real Grassmannian $\mathrm{SO}(p+q)/\mathrm{SO}(p)\mathrm{SO}(q)$, $p \geqq q \geqq 3$, $p \neq 4$, $q \neq 4$. This is because K contains at least two simple ideals and

the proof of (3.1) shows that there are as many irreducible summands of the form $\pi_{2\lambda-\alpha}$ in $\chi \otimes \mathbb{C}$ as there are simple ideals in $\mathfrak{k}$ and these all have the same Casimir constant. The proof also shows that $\mathrm{Sp}(n+1)/\mathrm{Sp}(n)\mathrm{Sp}(1)$ is an exception in (3.1.c), since in this case $\pi_{2\lambda-\beta} \subset \mathrm{ad}(\mathfrak{k})$.

(2) Notice also that the proof of (3.1) gives a simple description of the isotropy representation χ of the strongly isotropy irreducible spaces V. Indeed, assuming that V is not a point nor a symmetric space, it follows that $\chi = \pi_{\lambda'+\lambda'^*}$ in (3.1.a), $\chi = \pi_{2\lambda'}$ in (3.1.b), and in (3.1.c) $\chi = \pi_{2\lambda-\alpha}$ or, in the case of $G/K = [\mathrm{SU}(n)\times\mathrm{SU}(n)]/\Delta\mathrm{SU}(n)$, $\chi = [\pi_{2\lambda-\alpha}]_{\mathbb{R}}$.

We finally determine when V is again a symmetric space.

(3.4) Proposition. *Let G/K be a symmetric space as in* (3.1) *and V the corresponding strongly isotropy irreducible homogeneous space. Then V is an irreducible symmetric space iff*

(a) *G/K is the hermitian symmetric space* $\mathrm{SO}(n+2)/\mathrm{SO}(n)\mathrm{SO}(2)$ *with* $V = \mathrm{SU}(n)/\mathrm{SO}(n)$.

(b) *G/K is the quaternionic symmetric space* $\mathrm{SU}(n+2)/\mathrm{S}(\mathrm{U}(n)\mathrm{U}(2))$ *with* $V = \mathrm{Sp}(n)/\mathrm{U}(n)$.

(c) $G/K = \mathrm{Sp}(3)/\mathrm{Sp}(2)\mathrm{Sp}(1)$ *regarded as a real symmetric space with* $V = \mathrm{SO}(8)/[\mathrm{Sp}(2)\mathrm{Sp}(1)]/\Delta\mathbb{Z}_2 = (\mathrm{SO}(8)/\mathbb{Z}_2)/\mathrm{SO}(5)\mathrm{SO}(3)$.

Proof. If G/K is hermitian symmetric, the proof that V is isotropy irreducible shows that the isotropy representation χ of V is equal to $\pi_{\lambda'+\lambda'^*}$. If $\lambda' \neq \lambda'^*$ and if χ is the isotropy representation of a symmetric space, then (2.8) shows (since H must be semisimple) that $\pi_{\lambda'} = \mu_n$. But in this case $V =$ point. If $\lambda' = \lambda'^*$ then by (2.10.a) $\pi_{\lambda'} = \varrho_n$ and hence $V = \mathrm{SU}(n)/\mathrm{SO}(n)$, which is a symmetric space.

Now let G/K be quaternionic symmetric. If $\sigma = \pi_{\lambda'} \oplus \pi^*_{\lambda'}$, then by (2.10.b) we have $\pi_{\lambda'} = \mu_n$ and hence $V = \mathrm{Sp}(n)/\mathrm{U}(n)$, which is a symmetric space. If $\sigma = \pi_{\lambda'}$, then the proof of (3.1) shows that $\chi = \pi_{2\lambda'}$. If H is not simple, then (2.10.b) shows that $\pi_{\lambda'} = \varrho_n \hat{\otimes} \nu_2$ but (2.10) also shows that $\pi_{2\lambda'}$ cannot be the isotropy representation of a symmetric space.

If H is simple, then we can use (2.6) since all bi-invariant inner products on $\mathfrak{h}$ must be a multiple of each other. Since $\langle\lambda',\lambda'\rangle = \frac{3}{2}\langle\lambda',\alpha\rangle$ we have $\langle 2\lambda', 2\lambda'\rangle = 3\langle 2\lambda', \alpha\rangle$ and hence (2.6) implies that $\pi_{2\lambda'}$ cannot be the isotropy representation of a symmetric space.

We now deal with the case where G/K is neither hermitian symmetric nor quaternionic symmetric. If G/K is the quaternionic projective space, then $\pi_\lambda = \nu_n \hat{\otimes} \nu_2$ and the proof of (3.1) implies that $\chi = \pi_{2\lambda-\alpha_1}$ where $\nu_n = \pi_{\lambda_{\alpha_1}}$ and hence $\chi = \pi_{\lambda_{\alpha_2}} \hat{\otimes} \varrho_3$ where α_2 is adjacent to α_1. But this cannot be the isotropy representation of a symmetric space by (2.10) if $n \geqq 3$. If $n = 2$ it is the isotropy representation of the symmetric space which is the unoriented Grassmannian $(\mathrm{SO}(8)/\mathbb{Z}_2)/\mathrm{SO}(5)\mathrm{SO}(3)$.

If $G/K = [\mathrm{SU}(n) \times \mathrm{SU}(n)]/\Delta\mathrm{SU}(n)$, then the proof of (3.1) implies that $\chi\otimes\mathbb{C} = \pi_{2\lambda-\alpha}\oplus\pi_{2\lambda-\alpha^*}$. But this cannot be the isotropy representation of a symmetric since it would have to be hermitian symmetric with K not semisimple.

In all other case K is simple and the proof of (3.1) implies that $\chi \otimes \mathbb{C} = \pi_{2\lambda-\alpha}$ and that α is the unique simple root with $\langle\lambda,\alpha\rangle \neq 0$. Hence $\lambda = k\lambda_\alpha$. If $\mathrm{rank}\,\mathfrak{h} = 1$, it follows from (2.6) that $k = 4$ and hence $2\lambda - \alpha = 3\alpha = 6\lambda_\alpha$ which cannot be the isotropy representation of a symmetric space by (2.6). Hence we can assume that $\mathrm{rank}\,\mathfrak{h} > 1$ and there must therefore be a root β adjacent to α. Since $\langle 2\lambda-\alpha,\beta\rangle \neq 0$,

if χ were the isotropy representation of a symmetric space, (2.6) would imply that $\langle 2\lambda-\alpha, 2\lambda-\alpha\rangle = 2\langle 2\lambda-\alpha, \beta\rangle$ or $\dfrac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} = -(2k+1)$. Hence $k=1$, $\mathfrak{k}=\mathfrak{g}_2$, and α must be short root of $\mathfrak{g}_2$, i.e. $\pi_\lambda = \phi_7$. But ϕ_7 cannot be the isotropy representation of a symmetric space by (2.4) □

Remark. As mentioned earlier, there is one other phenomena which keeps the correspondence from being one to one. Namely the quaternionic symmetric space $G_2/SO(4)$ and the symmetric space $SU(3)/SO(3)$ give rise to the same strongly isotropy irreducible space $SO(5)/SO(3)$. One easily sees that this is the only case where this can happen. Indeed, if two distinct symmetric spaces are of the same type (3.1.a), (31.b), or (3.1.c), then the corresponding spaces V must also be distinct since the symmetric spaces are determined by their isotropy representation. If two symmetric spaces belong to two different categories, then the corresponding spaces V can be the same only due to the low dimensional isomorphisms $\mathfrak{so}(6) \cong \mathfrak{su}(4)$, and $\mathfrak{so}(5) \cong \mathfrak{sp}(2)$. By listing the possible symmetric spaces that can give rise to spaces V in these low dimensions, one sees that the above is the only case where this can happen.

4 Symmetric spaces from strongly isotropy irreducible quotients of the classical groups

Let V be a strongly isotropy irreducible quotient of a classical group. The embedding of the isotropy subgroup in the classical group is described by a faithful representation $\pi': H \to \mathrm{SU}(N)$, $\pi': H \to \mathrm{Sp}(N)$, or $\pi': H \to \mathrm{SO}(N)$. Since G/H is strongly isotropy irreducible iff G/H^0 is, we may as well assume that H is connected. In the case of $\mathrm{SU}(N)$ and $\mathrm{Sp}(N)$ this is equivalent to requiring that V is simply connected. In the case of $\mathrm{SO}(N)$, V may or may not be simply connected.

In view of the construction in (1.2), the isotropy representation π of the corresponding symmetric space G/K should be given by $[\pi'\hat{\otimes}\phi]_{\mathbb{R}}$, $\pi'\hat{\otimes}\nu_2$, or π' respectively, with the corresponding $\mathfrak{k} = \mathfrak{h}\oplus\mathfrak{u}(1)$, $\mathfrak{h}\oplus\mathfrak{sp}(1)$, or $\mathfrak{h}$. Let $\mathfrak{p}$ be the representation space for the above, so that $\mathfrak{p} = \mathbb{R}^{2N}$, $\mathbb{R}^{4N}$, or $\mathbb{R}^N$ respectively. To recover the symmetric space we let $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$. One's first impulse is to define a suitable Lie bracket for $\mathfrak{g}$ such that $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$. Indeed, we already have a Lie bracket in $\mathfrak{k}$, and if $X\in\mathfrak{k}$, $v\in\mathfrak{p}$, we can define $[X,v] = [-v,X]$ as $\pi(X)(v)$. For $u,v\in\mathfrak{p}$ we can define $[u,v] = -[v,u] = C\,\mathrm{pr}_{\mathfrak{k}(u\wedge v)}$, where C is a suitable constant and $\mathrm{pr}_{\mathfrak{k}}$ is the orthogonal projection associated to the embedding $\pi:\mathfrak{k}\to\mathfrak{so}(\mathfrak{p}) \cong \Lambda^2(\mathfrak{p})$ with respect to the Killing form on $\mathfrak{so}(\mathfrak{p})$. The difficulty with this approach is in proving the Jacobi identity. We do not know whether this approach can be successful or not. Instead, we take advantage of the following lemma of Cartan, which says that a symmetric space can be recovered from its isotropy representation and its curvature tensor. To formulate it, let $\mathscr{R}$ be the space of algebraic curvature tensors on $\mathfrak{p}$, i.e. the space of all tensors R of type $(4,0)$ on $\mathfrak{p}$ which are skew symmetric in the first two and last two arguments and which satisfy the Jacobi identity $R(x_1,x_2,x_3,x_4) + R(x_2,x_3,x_1,x_4) + R(x_3,x_1,x_2,x_4) = 0$. For each $R\in\mathscr{R}$, denote by $\hat{R}$ the corresponding curvature operator, i.e. the symmetric endomorphism $\hat{R}:\Lambda^2(\mathfrak{p})\to\Lambda^2(\mathfrak{p})$, which can also be regarded as an element in $S^2\Lambda^2(\mathfrak{p})$.

(4.1) Lemma (Cartan). *Let $\pi:\mathfrak{k}\to\mathfrak{so}(\mathfrak{p})$ be an orthogonal representation of a compact Lie algebra $\mathfrak{k}$. Suppose that $\mathfrak{p}$ admits a non-trivial $\mathfrak{k}$-invariant algebraic*

curvature tensor R with values in $\pi(\mathfrak{k}) \subset \mathfrak{so}(\mathfrak{p}) \cong \Lambda^2(\mathfrak{p})$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ can be made into a Lie algebra by defining $[X, v] = -[v, X] = \pi(X)(v)$ for $X \in \mathfrak{k}$, $v \in \mathfrak{p}$ and $[u, v] = Y \in \mathfrak{k}$ with $\pi(Y) = -R(u, v)$ for $u, v \in \mathfrak{p}$. Furthermore $(\mathfrak{g}, \mathfrak{k})$ is then a symmetric pair.

See [Ca, p. 225] or [Si, pp. 215–216] for a more modern proof.

Our strategy is therefore to show that for the $\pi : \mathfrak{k} \to \mathfrak{so}(\mathfrak{p})$ we constructed from the strongly isotropy irreducible quotients of the classical groups, there exists a non-trivial invariant curvature with values in $\pi(\mathfrak{k})$. To do this, recall that $S^2\Lambda^2(\mathfrak{p}) = \Lambda^4(\mathfrak{p}) \oplus \mathscr{R}$. From the decomposition of $\Lambda^2(\mathfrak{p})$ into $\mathfrak{k}$-irreducible summands we can deduce the dimension of $\mathfrak{k}$-invariant elements in $S^2\Lambda^2(\mathfrak{p})$. An invariant curvature tensor on $\mathfrak{p}$ is a symmetric endomorphism of $\Lambda^2(\mathfrak{p})$ and if sufficiently many invariant curvature tensors exist, a suitable linear combination will yield a non-trivial one taking values in $\pi(\mathfrak{k})$ in $\Lambda^2(\mathfrak{p})$. Our principal task is therefore to analyse the invariant 4-forms on $\mathfrak{p}$ using the isotropy irreducibility condition given. From the result of this analysis, we will be able to determine the dimension of the space of invariant curvatures on $\mathfrak{p}$. From Lemma 4.1 and the general theory of symmetric spaces it then follows that there exists a simply connected symmetric space G'/K' where G' is simply connected with Lie algebra $\mathfrak{g}$ and K' is the connected subgroup with Lie algebra $\mathfrak{k}$. G'/K' is not yet effective. By construction, the infinitesimal isotropy representation of G'/K' is equal to π and since π is an effective representation of K, the effective version of the symmetric space must have isotropy group K, which will then finish the proof.

After finishing this paper, we were kindly informed by one of the referees that the characterization of the isotropy representations of irreducible symmetric spaces in terms of invariant 4-forms on the tangent spaces has been studied by Kostant, see [Ko2].

A. The orthogonal case

(4.2.) Theorem. *Let $\pi : K \to \mathrm{SO}(N)$ be a faithful, orthogonal representation of a compact connected Lie group K such that $\mathrm{SO}(N)/\pi(K)$ is strongly isotropy irreducible, nonsymmetric, and not $\mathrm{SO}(7)/\mathrm{G}_2$. Then (K, π) is the isotropy representation of a compact, simply connected, irreducible symmetric space G/K. Furthermore, K is a simple Lie group unless $K = \mathrm{Sp}(n)\mathrm{Sp}(1)$ with $\pi = \nu_n \hat{\otimes} \nu_2$.*

Proof. It follows from Sect. 1 that $\Lambda^2\pi = \mathrm{Ad}_K \oplus \chi$ and our assumptions imply that χ is irreducible over $\mathbb{R}$. If π is not irreducible, then $\pi = \pi_1 \oplus \pi_2$. But π_1 and π_2 must both be orthogonal (of dimension p and q) and hence $K \subset \mathrm{SO}(p) \times \mathrm{SO}(q) \subset \mathrm{SO}(N)$ with $N = p + q$. But then the isotropy representation of $[\mathrm{SO}(p) \times \mathrm{SO}(q)]/K$ is contained in χ and hence $K = \mathrm{SO}(p) \times \mathrm{SO}(q)$ and $\pi = \varrho_p \hat{\otimes} \mathbf{1} \oplus \mathbf{1} \hat{\otimes} \varrho_q$. In this case V is the Grassmannian and hence already symmetric.

If π is not absolutely irreducible, then $\pi \otimes \mathbb{C} = \sigma \oplus \sigma^*$ where σ is a unitary n-dimensional representation. But then $K \subset \mathrm{U}(n) \subset \mathrm{SO}(N)$ with $N = 2n$ and as before it follows that $K = \mathrm{U}(n)$ and hence V is a symmetric space.

We can therefore assume that π is absolutely irreducible and hence also that K is semisimple. Let $\mathfrak{p}$ be the representation space on which π acts and denote by b the inner product on $\mathfrak{p}$, invariant under π. Then $\pi : \mathfrak{k} \to \mathfrak{so}(\mathfrak{p}, b)$. Furthermore, $\mathfrak{so}(\mathfrak{p}, b)$ carries a natural symmetric bilinear form given by $\langle A, B \rangle = -\frac{1}{2}\,\mathrm{tr}(AB)$ which induces an inner product on $\mathfrak{k}$, again denoted by $\langle\,,\rangle$, via the embedding π. Let λ be the dominant weight of $\pi \otimes \mathbb{C}$ and α be a simple root of $\mathfrak{k}$ such that $\langle \lambda, \alpha \rangle \neq 0$. It follows from

(2.2.c) that $\pi_{2\lambda-\alpha} \subset \chi \otimes \mathbb{C}$ unless (i) $\chi = 0$, (ii) $\mathfrak{h} = G_2$, $\pi_\lambda = \phi_7$, (iii) $\mathfrak{h} = \mathfrak{so}(7)$, $\pi_\lambda = \Delta_7$, or (iv) $\mathfrak{h} = \mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$, $\pi_\lambda = \nu_{2n} \otimes \nu_2$ and α is the simple root of $\mathfrak{sp}(1)$. In case (i), V is only a point. Case (ii) is excluded by assumption and in case (iii) $V = \mathrm{SO}(8)/\mathrm{Spin}(7) = (\mathrm{SO}(8)/\{\pm \mathrm{Id}\})/\mathrm{SO}(7) = \mathbb{R}\mathrm{P}^7$ which is a symmetric space and hence also excluded. In case (iv) π is already the isotropy representation of the quaternionic symmetric space $\mathbb{H}\mathrm{P}^n$. Notice also that it follows from (3.1.c) that $\mathrm{SO}(4n)/\pi(\mathrm{Sp}(n)\mathrm{Sp}(1))$ is indeed strongly isotropy irreducible.

Hence we can assume that $\pi_{2\lambda-\alpha} \subset \chi \otimes \mathbb{C}$. Isotropy irreducibility implies that either $\chi \otimes \mathbb{C} = \pi_{2\lambda-\alpha}$ or $\chi \otimes \mathbb{C} = \pi_{2\lambda-\alpha} \oplus \pi^*_{2\lambda-\alpha}$. Hence we may assume that α and α^* are the only simple roots β of $\mathfrak{k}$ such that $\langle \lambda, \beta \rangle \neq 0$. By faithfulness, $\mathfrak{k}$ is simple.

Consider now $S^2\Lambda^2\pi = S^2(\mathrm{ad}(\mathfrak{k}) \oplus \chi) = S^2(\mathrm{ad}(\mathfrak{k})) \oplus S^2(\chi) \oplus [\mathrm{ad}(\mathfrak{k}) \otimes \chi]$. Since $\sigma \otimes \varrho$ contains a trivial summand iff $\sigma = \varrho^*$, and since $2\lambda - \alpha$ is not a root of $\mathfrak{k}$, it follows that $\mathrm{ad}(\mathfrak{k}) \otimes \chi$ does not contain a trivial summand. But since $\mathrm{ad}(\mathfrak{k})$ and χ are both orthogonal, $S^2(\mathrm{ad}(\mathfrak{k}))$ and $S^2(\chi)$ both contain a trivial summand coming from the metric. Hence $\dim(S^2\Lambda^2\mathfrak{p})^{\mathfrak{k}} = 2$. But $S^2\Lambda^2\mathfrak{p} = \Lambda^4\mathfrak{p} \oplus \mathscr{R}$. Each $R \in \mathscr{R}$ is a symmetric endomorphism of $\mathrm{ad}(\mathfrak{k}) \oplus \chi$. If R is $\mathfrak{k}$-invariant, then it is a diagonal operator with one eigenvalue for each summand of $\mathrm{ad}(\mathfrak{k}) \oplus \chi$. Suppose that $\dim(\Lambda^4\mathfrak{p})^{\mathfrak{k}} = 0$, then there are two linearly independent invariant curvature tensors R_1 and R_2 in $\mathscr{R}$. It follows that up to a constant there is a unique linear combination of R_1 and R_2 that yields a non-trivial invariant curvature tensor with values in $\mathrm{ad}(\mathfrak{k})$. By (4.1) we would be done. So in the following we will prove that $\dim(\Lambda^4\mathfrak{p})^{\mathfrak{k}} = 0$.

Since π is absolutely irreducible we may work over the complex numbers whenever convenient. As before, we denote by $\langle\,,\rangle$ the natural metric on $\mathfrak{so}(\mathfrak{p})$, which induces a metric on $\mathfrak{k}$ via the embedding π. We also denote by b the extension of b to a symmetric bilinear form on $\mathfrak{p} \otimes \mathbb{C}$. b induces in the usual way an inner product on $\Lambda^2\mathfrak{p}$ and the identification $\Lambda^2\mathfrak{p} \cong \mathfrak{so}(\mathfrak{p})$, defined by $\langle v \wedge w, A \rangle = b(A(v), w)$ for all $v, w \in \mathfrak{p}$, $A \in \mathfrak{so}(\mathfrak{p})$, becomes a linear isometry. The invariance of b easily implies that $b(v_{\varrho_1}, v_{\varrho_2}) = 0$ for all weight vector v_{ϱ_1}, v_{ϱ_2}, unless $\varrho_1 + \varrho_2 = 0$.

We also choose the usual Chevalley basis $\{H_\alpha, X_\alpha, X_{-\alpha}\}$ of $\mathfrak{k} \otimes \mathbb{C}$ so that $\langle X_\alpha, X_{-\alpha} \rangle = 1$ and $[X_\alpha, X_{-\alpha}] = H_\alpha$. Then choose weight vectors v_λ, $v_{-\lambda}$ such that $b(v_\lambda, v_{-\lambda}) = 1$ and define $v_{\lambda-\alpha} = \pi(X_{-\alpha})v_\lambda$, $v_{\alpha-\lambda} = \pi(X_\alpha)v_{-\lambda}$. We then have

$$\begin{aligned} b(v_{\lambda-\alpha}, v_{\alpha-\lambda}) &= b(\pi(X_{-\alpha})v_\lambda, \pi(X_\alpha)v_{-\lambda}) = -b(v_\lambda, \pi(X_{-\alpha})\pi(X_\alpha)v_{-\lambda}) \\ &= b(v_\lambda, \pi([X_{-\alpha}, X_\alpha])v_{-\lambda} = b(v_\lambda, \pi(H_\alpha)v_{-\lambda}) = -\langle \lambda, \alpha \rangle . \end{aligned}$$

We denote by $\mathrm{pr}_{\mathfrak{k}}$ the orthogonal projection from $\Lambda^2\mathfrak{p}$ to $\mathfrak{k}$ via the embedding π. If v_{ϱ_1}, v_{ϱ_2} are weight vectors with respect to a maximal torus $\mathfrak{t} \otimes \mathbb{C} \subset \mathfrak{k} \otimes \mathbb{C}$, we claim that $\mathrm{pr}_{\mathfrak{k}}(v_{\varrho_1} \wedge v_{\varrho_2})$ has weight $\varrho_1 + \varrho_2$, i.e. lies in $\mathfrak{t} \otimes \mathbb{C}$ if $\varrho_1 + \varrho_2 = 0$, is 0 if $\varrho_1 + \varrho_2$ is not a root and is a multiple of $X_{\varrho_1+\varrho_2}$ if it is a root. Indeed, $[\pi(X), \mathrm{pr}_{\mathfrak{k}}(v_{\varrho_1} \wedge v_{\varrho_2})] = \mathrm{pr}_{\mathfrak{k}}(\Lambda^2\pi(X)(v_{\varrho_1} \wedge v_{\varrho_2})) = (\varrho_1(X) + \varrho_2(X))\,\mathrm{pr}_{\mathfrak{k}}(v_{\varrho_1} \wedge v_{\varrho_2})$ for all $X \in \mathfrak{t} \otimes \mathbb{C}$. Hence one easily shows that

$$\begin{aligned} \mathrm{pr}_{\mathfrak{k}}(v_\lambda \wedge v_{-\lambda}) &= H_\lambda \\ \mathrm{pr}_{\mathfrak{k}}(v_{\lambda-\alpha} \wedge v_{\alpha-\lambda}) &= -\langle \lambda, \alpha \rangle H_{\lambda-\alpha} \\ \mathrm{pr}_{\mathfrak{k}}(v_\lambda \wedge v_{\alpha-\lambda}) &= -\langle \lambda, \alpha \rangle X_\alpha \\ \mathrm{pr}_{\mathfrak{k}}(v_{\lambda-\alpha} \wedge v_{-\lambda}) &= \langle \lambda, \alpha \rangle X_{-\alpha} \\ \mathrm{pr}_{\mathfrak{k}}(v_\lambda \wedge v_{\lambda-\alpha}) &= 0 . \end{aligned}$$

For example, the first equation follows from $\langle \mathrm{pr}_{\mathfrak{k}}(v_\lambda \wedge v_{-\lambda}), X\rangle = \langle v_\lambda \wedge v_{-\lambda}, \pi(X)\rangle = b(\pi(X)v_\lambda, v_{-\lambda}) = \lambda(X)$.

Now let η be a $\mathfrak{k}$-invariant 4-form on $\mathfrak{p}$ and let $\hat{\eta}$ denote the corresponding symmetric endomorphism of $\Lambda^2(\mathfrak{p})$. Furthermore, assume that $\hat{\eta} = \begin{pmatrix} a_1\mathbf{1} & 0 \\ 0 & a_2\mathbf{1} \end{pmatrix}$ with respect to $\Lambda^2\pi_\lambda = \mathrm{ad}(\mathfrak{k}) \oplus \chi$.

Since η is a 4-form, we have

$$\eta(v_\lambda, v_{\lambda-\alpha}, v_{-\lambda}, v_{\alpha-\lambda}) = \eta(v_\lambda, v_{\alpha-\lambda}, v_{\lambda-\alpha}, v_{-\lambda}) = \eta(v_\lambda, v_{-\lambda}, v_{\alpha-\lambda}, v_{\lambda-\alpha}).$$

Let us compute each of these explicitly. First

$$\begin{aligned}\eta(v_\lambda, v_{\lambda-\alpha}, v_{-\lambda}, v_{\alpha-\lambda}) &= \langle \hat{\eta}(v_\lambda \wedge v_{\lambda-\alpha}), v_{-\lambda} \wedge v_{\alpha-\lambda}\rangle \\ &= a_2\langle v_\lambda \wedge v_{\lambda-\alpha}, v_{-\lambda} \wedge v_{\alpha-\lambda}\rangle \\ &= a_2(b(v_\lambda, v_{-\lambda})b(v_{\lambda-\alpha}, v_{\alpha-\lambda}) - b(v_\lambda, v_{\alpha-\lambda})b(v_{\lambda-\alpha}, v_{-\lambda})) \\ &= a_2(-\langle\lambda, \alpha\rangle).\end{aligned}$$

Furthermore

$$\begin{aligned}\eta(v_\lambda, v_{\alpha-\lambda}, v_{\lambda-\alpha}, v_{-\lambda}) &= \langle \hat{\eta}(v_\lambda \wedge v_{\alpha-\lambda}), v_{\lambda-\alpha} \wedge v_{-\lambda}\rangle \\ &= \langle -a_1\langle\lambda, \alpha\rangle X_\alpha + a_2 y_1, \langle\lambda, \alpha\rangle X_{-\alpha} + z_1\rangle \\ &= -a_1\langle\lambda, \alpha\rangle^2\langle X_\alpha, X_{-\alpha}\rangle + a_2\langle y_1, z_1\rangle \\ &= -a_1\langle\lambda, \alpha\rangle^2 + a_2\langle y_1, z_1\rangle\end{aligned}$$

where y_1, z_1 are in $\chi_{\mathbb{C}}$.

To evaluate $\langle y_1, z_1\rangle$, consider $\langle v_\lambda \wedge v_{\alpha-\lambda}, v_{\lambda-\alpha} \wedge v_{-\lambda}\rangle = -\langle\lambda, \alpha\rangle^2 + \langle y_1, z_1\rangle = b(v_\lambda, v_{\lambda-\alpha})b(v_{\alpha-\lambda}, v_{-\lambda}) - b(v_\lambda, v_{-\lambda})b(v_{\alpha-\lambda}, v_{\lambda-\alpha}) = \langle\lambda, \alpha\rangle$, since $2\lambda - \alpha$ is not a root. So $\langle y_1, z_1\rangle = \langle\lambda, \alpha\rangle(1 + \langle\lambda, \alpha\rangle)$.

Finally

$$\begin{aligned}\eta(v_\lambda, v_{-\lambda}, v_{\alpha-\lambda}, v_{\lambda-\alpha}) &= \langle \hat{\eta}(v_\lambda \wedge v_{-\lambda}), v_{\alpha-\lambda} \wedge v_{\lambda-\alpha}\rangle \\ &= \langle a_1 H_\lambda + a_2 y_2, \langle\lambda, \alpha\rangle H_{\lambda-\alpha} + z_2\rangle, \qquad y_2, z_2 \in \chi \otimes \mathbb{C} \\ &= a_1\langle\lambda, \alpha\rangle\langle\lambda, \lambda - \alpha\rangle + a_2\langle y_2, z_2\rangle.\end{aligned}$$

To evaluate $\langle y_2, z_2\rangle$ consider

$$\begin{aligned}\langle v_\lambda \wedge v_{-\lambda}, v_{\alpha-\lambda} \wedge v_{\lambda-\alpha}\rangle &= \langle\lambda, \alpha\rangle\langle\lambda, \lambda - \alpha\rangle + \langle y_2, z_2\rangle \\ &= b(v_\lambda, v_{\alpha-\lambda})b(v_{-\lambda}, v_{\lambda-\alpha}) - b(v_\lambda, v_{\lambda-\alpha})b(v_{-\lambda}, v_{\alpha-\lambda}) \\ &= 0.\end{aligned}$$

So $\eta(v_\lambda, v_{-\lambda}, v_{\alpha-\lambda}, v_{\lambda-\alpha}) = (a_1 - a_2)\langle\lambda, \alpha\rangle\langle\lambda, \lambda - \alpha\rangle$.

Combining these computations, we obtain the following linear system that has to be satisfied by a_1 and a_2:

$$\begin{aligned}a_1\langle\lambda, \alpha\rangle - a_2(2 + \langle\lambda, \alpha\rangle) &= 0 \\ a_1\langle\lambda, \lambda - \alpha\rangle + a_2(1 - \langle\lambda, \lambda - \alpha\rangle) &= 0.\end{aligned}$$

The determinant is $2\langle\lambda, \lambda\rangle - \langle\lambda, \alpha\rangle = \langle\lambda, 2\lambda - \alpha\rangle$. Since λ and $2\lambda - \alpha$ are both dominant weights, the determinant is non-zero. Hence $a_1 = a_2 = 0$ and $\dim(\Lambda^4\mathfrak{p})^{\mathfrak{k}} = 0$, as claimed. □

Remark. For the isotropy irreducible space $\mathrm{SO}(7)/\mathrm{G}_2 = \mathbb{R}\mathrm{P}^7$ we have $\pi_\lambda = \phi_7$. But $2\lambda - \alpha$ is precisely the maximal root of G_2 and $\Lambda^2\pi_\lambda = \mathrm{ad}(\mathrm{G}_2) \oplus \pi_\lambda$, and so the above argument fails. Furthermore, one knows that there actually exists a unique 4-form on $\mathbb{R}^7$ invariant under ϕ_7 and hence $\dim(\Lambda^4\mathfrak{p})^{\mathfrak{k}} = 1$, contrary to the conclusion in the above proof.

B. The quaternionic case

(4.3) Theorem. *Let $\pi' : H \to \mathrm{Sp}(N)$ be a faithful, quaternionic representation of a compact connected Lie group H such that $\mathrm{Sp}(N)/\pi'(H)$ is strongly isotropy irreducible and non-symmetric. Then $(K = H \cdot \mathrm{Sp}(1), \pi = \pi' \otimes \nu_2)$ is the isotropy representation of a compact, simply connected, irreducible symmetric space G/K. Moreover, either H is simple or $H = \mathrm{SO}(N)\mathrm{Sp}(1)$ with $\pi' = \varrho_N \hat{\otimes} \nu_2$, $N \geqq 3$.*

Proof. In this case we have $S^2\pi' = \mathrm{Ad}_H \oplus \chi$ with χ irreducible over $\mathbb{R}$. If π' is reducible, it follows as in (4.2) that $H = \mathrm{Sp}(p) \times \mathrm{Sp}(q)$ and $\pi' = \nu_p \hat{\otimes} \mathbf{1} \oplus \mathbf{1} \hat{\otimes} \nu_q$, $N = p + q$, which implies that $\mathrm{Sp}(N)/\pi'(H)$ is symmetric. Similarly, if $\pi' \otimes \mathbb{C} = \sigma \oplus \sigma^*$, then $H = \mathrm{U}(n)$, $N = 2n$, and again $\mathrm{Sp}(N)/\pi'(H)$ is symmetric.

Hence we can assume that H is semisimple and that π' is absolutely irreducible with dominant weight λ'. If H is not simple, then $\pi' = \pi_1 \hat{\otimes} \pi_2$ and we can assume that π_1 is orthogonal and π_2 symplectic. Then $S^2(\pi_1 \hat{\otimes} \pi_2) = S^2\pi_1 \hat{\otimes} S^2\pi_2 \oplus \Lambda^2\pi_1 \hat{\otimes} \Lambda^2\pi_2$. Since π_1 is orthogonal, we have $S^2\pi_1 = \mathbf{1} \oplus \alpha$ and $\Lambda^2\pi_1 = \mathrm{ad}(\mathfrak{h}_1) \oplus \beta$, and since π_2 is symplectic, we have $S^2\pi_2 = \mathrm{ad}(\mathfrak{h}_2) \oplus \gamma$ and $\Lambda^2\pi_2 = \mathbf{1} \oplus \delta$. In order for $S^2(\pi_1 \hat{\otimes} \pi_2)$ to be equal to $\mathrm{ad}(\mathfrak{h}) \oplus \chi$, the only possibility is that $\Lambda^2\pi_1 = \mathrm{ad}(\mathfrak{h}_1)$, $S^2\pi_2 = \mathrm{ad}(\mathfrak{h}_2)$, and $\Lambda^2\pi_2 = \mathbf{1}$ and hence $\chi = \alpha \hat{\otimes} \mathrm{ad}(\mathfrak{h}_2)$. Using (2.2) this implies that, up to an automorphism, $\pi_1 = \varrho_N$ and $\pi_2 = \nu_2$ and hence $H = \mathrm{SO}(N)\mathrm{Sp}(1)$ with $\pi' = \varrho_N \hat{\otimes} \nu_2$. But $[\pi' \hat{\otimes} \nu_2]_{\mathbb{R}}$ is indeed the isotropy representation of the symmetric space $G/K = \mathrm{SO}(N+4)/\mathrm{SO}(N)\mathrm{SO}(4)$.

We can therefore assume from now on that H is simple and $\pi' = \pi_{\lambda'}$. The outline of our proof in this case is the same as before, but further technicalities arise since we now have to consider curvature tensors and 4-forms on $\mathfrak{p}$, a real form of the representation space of $\pi \otimes \mathbb{C} = \pi' \hat{\otimes} \nu_2$. Now $\Lambda^2(\pi_{\lambda'} \hat{\otimes} \nu_2) = \Lambda^2\pi_{\lambda'} \hat{\otimes} \mathrm{ad}(\mathfrak{sp}(1))_{\mathbb{C}} \oplus S^2\pi_{\lambda'} \hat{\otimes} \mathbf{1}$, and since $\pi_{\lambda'}$ is symplectic, $\Lambda^2\pi_{\lambda'} = \mathbf{1} \oplus \tilde{\pi}$, while isotropy irreducibility implies that $S^2\pi_{\lambda'} = \mathrm{ad}(\mathfrak{h})_{\mathbb{C}} \oplus \chi$ with χ irreducible. Therefore, it is important to decide about the nature of the representation $\tilde{\pi}$. We do this below. For this purpose we introduce some more notation. Let V' be the representation space of $\pi' \otimes \mathbb{C}$ and ω' the $\mathfrak{h}$-invariant skew symmetric bilinear form on V'. Hence $\pi' \otimes \mathbb{C} : \mathfrak{h} \otimes \mathbb{C} \to \mathfrak{sp}(V', \omega')$. On $\mathfrak{sp}(V', \omega')$ we have the natural inner product $\langle A, B \rangle = -\frac{1}{2}\,\mathrm{tr}(AB)$ which induces an inner product on $\mathfrak{h} \otimes \mathbb{C}$, again denoted by $\langle\,,\rangle$, via the embedding $\pi' \otimes \mathbb{C}$. ω' induces a symmetric bilinear form on S^2V', which we also denote by $\langle\,,\rangle$ and with respect to this form the equation $\langle v \circ w, A \rangle = \omega'(A(v), w)$ induces a natural isomorphism $S^2V' \cong \mathfrak{sp}(V', \omega')$ which also respects the natural inner products on both sides.

(4.4) Proposition. *With the same assumptions as in* (4.3) *and the assumption that H is simple, we have*

(a) *There is a unique simple root α' of $\mathfrak{h}$ such that $\langle \lambda', \alpha' \rangle \neq 0$. Furthermore, $\langle \lambda', \lambda' \rangle = \frac{3}{2}\langle \lambda', \alpha' \rangle = -3 \dim H / \dim \mathrm{Sp}(N)$.*

(b) $\Lambda^2\pi_{\lambda'} = \mathbf{1} \oplus \pi_{2\lambda' - \alpha'}$.

Proof of (a). As in the beginning of the proof of (4.3), it follows that π' is absolutely irreducible with dominant weight λ'. Also observe that by (2.2.b) it follows that $\chi = \pi_{2\lambda'}$. Furthermore, since $\langle \lambda', \alpha' \rangle \neq 0$, $2\lambda' - \alpha'$ is a weight in $\pi_{2\lambda'}$ and hence does not lie in $\mathrm{ad}(\mathfrak{h}) \otimes \mathbb{C}$.

There is a decomposition $S^2 S^2 V' = S^4 V' \oplus \mathscr{R}'$ invariant under $\mathrm{Sp}(V', \omega')$ analogous to the decomposition for the orthogonal group. Here $\mathscr{R}'$ is the space of algebraic curvature tensors on V', the only difference being that V' is a complex vector space. If $R \in \mathscr{R}'$, we let $\overset{\circ}{R}$ denote the corresponding symmetric endomorphism of $S^2 V'$.

Corresponding to $\pi' \otimes \mathbb{C} : \mathfrak{h} \otimes \mathbb{C} \to \mathfrak{sp}(V', \omega')$ and $\langle\ ,\ \rangle$ is the projection map $\mathrm{pr}_{\mathfrak{h}}$ from $S^2 V'$ to $\mathfrak{h} \otimes \mathbb{C}$. As before we define weight vectors $v_{\lambda'}, v_{-\lambda'}, v_{\lambda'-\alpha'} = \pi'(X_{-\alpha'}) v_{\lambda'}$, $v_{\alpha'-\lambda'} = \pi'(X_{\alpha'}) v_{-\lambda'}$ of π' such that $\omega'(v_{\lambda'}, v_{-\lambda'}) = 1$ and it follows again that $\omega'(v_{\lambda'-\alpha'}, v_{\alpha'-\lambda'}) = -\langle \lambda', \alpha' \rangle$. We also have that $\omega'(v_{\varrho_1}, v_{\varrho_2}) = 0$ unless $\varrho_1 + \varrho_2 = 0$ and that $\mathrm{pr}_{\mathfrak{h}}(v_{\varrho_1} \circ v_{\varrho_2})$ has weight $\varrho_1 + \varrho_2$. Hence one easily shows that

$$\begin{aligned}
\mathrm{pr}_{\mathfrak{h}}(v_{\lambda'} \circ v_{-\lambda'}) &= H_{\lambda'} \\
\mathrm{pr}_{\mathfrak{h}}(v_{\lambda'-\alpha'} \circ v_{\alpha'-\lambda'}) &= -\langle \lambda', \alpha' \rangle H_{\lambda'-\alpha'} \\
\mathrm{pr}_{\mathfrak{h}}(v_{\lambda'} \circ v_{\alpha'-\lambda'}) &= -\langle \lambda', \alpha' \rangle X_{\alpha'} \\
\mathrm{pr}_{\mathfrak{h}}(v_{-\lambda'} \circ v_{\lambda'-\alpha'}) &= \langle \lambda', \alpha' \rangle X_{-\alpha'} \\
\mathrm{pr}_{\mathfrak{h}}(v_{\lambda'} \circ v_{\lambda'-\alpha'}) &= 0 .
\end{aligned}$$

Define

$$\eta(v_1, v_2, v_3, v_4) = \omega'(\mathrm{pr}_{\mathfrak{h}}(v_1 \circ v_2) v_3 + \mathrm{pr}_{\mathfrak{h}}(v_2 \circ v_3) v_1 + \mathrm{pr}_{\mathfrak{h}}(v_3 \circ v_1) v_2, v_4) .$$

Using the fact that

$$\begin{aligned}
\omega'(\mathrm{pr}_{\mathfrak{h}}(v_1 \circ v_2) v_3, v_4) &= \langle v_3 \circ v_4, \mathrm{pr}_{\mathfrak{h}}(v_1 \circ v_2) \rangle \\
&= \langle \mathrm{pr}_{\mathfrak{h}}(v_1 \circ v_2), v_3 \circ v_4 \rangle = \omega'(\mathrm{pr}_{\mathfrak{h}}(v_3 \circ v_4) v_1, v_2)
\end{aligned}$$

it easily follows that $\eta \in S^4 V'$. Furthermore η is $\mathfrak{h}$-invariant and non-trivial since $\eta(v_{\lambda'}, v_{-\lambda'}, v_{\lambda'}, v_{-\lambda'}) = 2\langle \lambda', \lambda' \rangle$. One can also check that any element in $S^4 V'$ considered as a symmetric endomorphism of $S^2 V'$ has trace 0. Since η is $\mathfrak{h}$-invariant, $\overset{\circ}{\eta}$ has diagonal form with respect to the decomposition $S^2 V' = [\mathrm{ad}(\mathfrak{h}) \otimes \mathbb{C}] \oplus [\chi \otimes \mathbb{C}]$. Since $\overset{\circ}{\eta}$ is nonzero and $\mathrm{tr}(\overset{\circ}{\eta}) = 0$, its two eigenvalues are nonzero. By renormalizing η, we may assume that $\overset{\circ}{\eta}|_{\mathrm{ad}(\mathfrak{h}) \otimes \mathbb{C}} = \mathbf{1}$ and $\overset{\circ}{\eta}|_{\chi \otimes \mathbb{C}} = a\mathbf{1}$. We have $\eta(v_{\lambda'}, v_{-\lambda'}, v_{\lambda'}, v_{-\lambda'}) = \eta(v_{\lambda'}, v_{\lambda'}, v_{-\lambda'}, v_{-\lambda'})$ and $\eta(v_{\lambda'}, v_{\alpha'-\lambda'}, v_{-\lambda'}, v_{\lambda'-\alpha'}) = \eta(v_{\lambda'}, v_{\lambda'-\alpha'}, v_{-\lambda'}, v_{\alpha'-\lambda'})$.

Let $v_{\lambda'} \circ v_{-\lambda'} = H_{\lambda'} + x$ and $v_{\lambda'-\alpha'} \circ v_{\alpha'-\lambda'} = -\langle \lambda', \alpha' \rangle H_{\lambda'-\alpha'} + y$, where $x, y \in \chi \otimes \mathbb{C}$. We then have

$$\begin{aligned}
\eta(v_{\lambda'}, v_{-\lambda'} v_{\lambda'}, v_{-\lambda'}) &= \langle \overset{\circ}{\eta}(v_{\lambda'} \circ v_{-\lambda'}), v_{\lambda'} \circ v_{-\lambda'} \rangle \\
&= \langle H_{\lambda'} + ax, H_{\lambda'} + x \rangle = \langle \lambda', \lambda' \rangle + a\langle x, x \rangle \\
\eta(v_{\lambda'}, v_{\lambda'} v_{-\lambda'}, v_{-\lambda'}) &= \langle \overset{\circ}{\eta}(v_{\lambda'} \circ v_{\lambda'}), v_{-\lambda'} \circ v_{-\lambda'} \rangle = a\langle v_{\lambda'} \circ v_{\lambda'}, v_{-\lambda'} \circ v_{-\lambda'} \rangle \\
&= a(\omega(v_{\lambda'}, v_{-\lambda'})^2 + \omega(v_{\lambda'}, v_{-\lambda'})^2) = 2a .
\end{aligned}$$

To determine $\langle x, x \rangle$, we use

$$\begin{aligned}
\langle v_{\lambda'} \circ v_{-\lambda'}, v_{\lambda'} \circ v_{-\lambda'} \rangle &= \omega'(v_{\lambda'}, v_{\lambda'}) \omega'(v_{-\lambda'}, v_{-\lambda'}) + \omega'(v_{\lambda'}, v_{-\lambda'}) \omega'(v_{-\lambda'}, v_{\lambda'}) \\
&= -1 = \langle \lambda', \lambda' \rangle + \langle x, x \rangle .
\end{aligned}$$

Combining these equations, we obtian $a = \dfrac{\langle \lambda', \lambda' \rangle}{3 + \langle \lambda', \lambda' \rangle}$. On the other hand,

$$\begin{aligned}
\eta(v_{\lambda'}, v_{\alpha'-\lambda'}, v_{-\lambda'}, v_{\lambda'-\alpha'}) &= \langle \mathring{\eta}(v_{\lambda'} \circ v_{\alpha'-\lambda'}), v_{-\lambda'} \circ v_{\lambda'-\alpha'} \rangle \\
&= \langle -\langle \lambda', \alpha' \rangle X_{\alpha'} + ax', \langle \lambda', \alpha' \rangle X_{-\alpha'} + y' \rangle \\
&= -\langle \lambda', \alpha' \rangle^2 + a\langle x', y' \rangle \\
\eta(v_{\lambda'}, v_{\lambda'-\alpha'}, v_{-\lambda'}, v_{\alpha'-\lambda'}) &= \langle \mathring{\eta}(v_{\lambda'} \circ v_{\lambda'-\alpha'}), v_{-\lambda'} \circ v_{\alpha'-\lambda'} \rangle \\
&= a\langle v_{\lambda'} \circ v_{\lambda'-\alpha'}, v_{-\lambda'} \circ v_{\alpha'-\lambda'} \rangle \\
&= a\omega'(v_{\lambda'}, v_{-\lambda'})\omega'(v_{\lambda'-\alpha'}, v_{\alpha'-\lambda'}) \\
&= -a\langle \lambda', \alpha' \rangle .
\end{aligned}$$

In addition,

$$\begin{aligned}
\langle v_{\lambda'} \circ v_{\alpha'-\lambda'}, v_{-\lambda'} \circ v_{\lambda'-\alpha'} \rangle &= -\langle \lambda', \alpha' \rangle^2 + \langle x', y' \rangle \\
&= \omega'(v_{\lambda'}, v_{-\lambda'})\omega'(v_{\alpha'-\lambda'}, v_{\lambda'-\alpha'}) \\
&= \langle \lambda', \alpha' \rangle .
\end{aligned}$$

Combining these equations, we get $a = \dfrac{\langle \lambda', \alpha' \rangle}{2 + \langle \lambda', \alpha' \rangle}$. Equating the expressions for a, we get $\langle \lambda', \lambda' \rangle = \frac{3}{2}\langle \lambda', \alpha' \rangle$. Finally, since $\operatorname{tr}(\mathring{\eta}) = 0$, $\dim H + a \dim(\mathrm{Sp}(N)/\pi'(H)) = 0$, or $\dfrac{\langle \lambda', \lambda' \rangle}{3 + \langle \lambda', \lambda' \rangle} = \dfrac{-\dim H}{\dim \mathrm{Sp}(N) - \dim H}$. Hence $\langle \lambda', \lambda' \rangle = \dfrac{-3 \dim H}{\dim \mathrm{Sp}(N)}$.

In the above, α' is any simple root such that $\langle \lambda', \alpha' \rangle \neq 0$. If another simple root has this property, we obtain a contradiction as in the proof of (2.12). This finishes the proof of (a). □

Before we continue with the proof of (4.4.b), we first prove the following general lemma:

(4.5) Lemma. *If π is a unitary representation of a compact Lie algebra $\mathfrak{h}$, then $\operatorname{tr} \mathscr{C}_{\Lambda^2\pi} = (\dim \pi - 2) \operatorname{tr} \mathscr{C}_\pi$, where the Casimir operators are taken with respect to any bi-invariant metric $\langle\ ,\ \rangle$ on $\mathfrak{h}$.*

Proof. Remember that $\mathscr{C}_\pi = -\sum \pi(X_i)\pi(X_i)$, where X_i is an orthonormal basis of $\mathfrak{h} \otimes \mathbb{C}$ with respect to $\langle\ ,\ \rangle$. If g is the invariant metric on the representation space for π with orthonormal basis e_i, then it induces a metric on the representation space of $\Lambda^2\pi$ with orthonormal basis $e_i \wedge e_j$, $i < j$. We then have

$$\begin{aligned}
\operatorname{tr} \mathscr{C}_{\Lambda^2\pi} &= \sum_{i<j} g(\mathscr{C}_{\Lambda^2\pi}(e_i \wedge e_j), e_i \wedge e_j) \\
&= -\tfrac{1}{2} \sum_{i,j} \sum_k g(\Lambda^2\pi(X_k)\Lambda^2\pi(X_k)(e_i \wedge e_j), e_i \wedge e_j) \\
&= -\tfrac{1}{2} \sum_{i,j} \sum_k g\big([\pi(X_k)\pi(X_k)e_i] \wedge e_j + e_i \wedge [\pi(X_k)\pi(X_k)e_j] \\
&\quad + 2\pi(X_k)e_i \wedge \pi(X_k)e_j, e_i \wedge e_j\big)
\end{aligned}$$

$$= \tfrac{1}{2} \sum_{i,j} g\Big([\mathscr{C}_\pi e_i] \wedge e_j + e_i \wedge [\mathscr{C}_\pi e_j] - 2 \sum_k \pi(X_k)e_i \wedge \pi(X_k)e_j, e_i \wedge e_j \Big)$$

$$= (\dim \pi - 1) \sum_i g(\mathscr{C}_\pi e_i, e_i)$$

$$- \sum_{i,j} \sum_k \{ g(\pi(X_k)e_i, e_i) g(\pi(X_k)e_j, e_j)$$

$$- g(\pi(X_k)e_i, e_j) g(\pi(X_k)e_j, e_i) \}$$

$$= (\dim \pi - 1) \operatorname{tr} \mathscr{C}_\pi + \sum_{i,j} \sum_k g(\pi(X_k)e_i, e_j) g(\pi(X_k)e_j, e_i)$$

where we have used the fact that $\pi(H) \subset \mathrm{SU}(\dim \pi)$ and hence $\operatorname{tr} \pi(X_k) = 0$. The least term is equal to $\sum_{k,i} g(\pi(X_k)\pi(X_k)e_i, e_i) = -\operatorname{tr} \mathscr{C}_\pi$ which implies our claim. □

Proof of (4.4.b). Since $\pi_{\lambda'}$ is symplectic, it follows that $\mathbf{1} \in \Lambda^2 \pi_{\lambda'}$, and since $\pi_{\lambda'}$ is irreducible, $\mathbf{1}$ occurs only once in $\Lambda^2 \pi_{\lambda'}$. Furthermore, $\pi_{2\lambda' - \alpha'} \in \Lambda^2 \pi_{\lambda'}$ since $\langle \lambda', \alpha' \rangle \neq 0$. It follows from (4.5) that

$$\operatorname{tr} \mathscr{C}_{\Lambda^2 \pi_{\lambda'}} = (2N - 2) \operatorname{tr} \mathscr{C}_{\pi_{\lambda'}} = -(2N - 2)(2N) \langle \lambda', \lambda' + 2\delta' \rangle .$$

Any irreducible summand in $\Lambda^2 \pi_{\lambda'}$ besides $\mathbf{1}$ and $\pi_{2\lambda' - \alpha'}$ has dominant weight of the form $2\lambda' - \alpha' -$ (sum of simple roots) and hence has a smaller Casimir constant. Hence, if there are any other irreducible summands, we have

$$\operatorname{tr} \mathscr{C}_{\Lambda^2 \pi_{\lambda'}} < -\left(\binom{2N}{2} - 1 \right) \langle 2\lambda' - \alpha', 2\lambda' - \alpha' + 2\delta' \rangle$$

$$= -(2N + 1)(N - 1)[2\langle \lambda', \lambda' + 2\delta' \rangle + 2\langle \lambda', \lambda' - \alpha' \rangle$$

$$- 2\langle \lambda', \alpha' \rangle + \langle \alpha', \alpha' \rangle - 2\langle \delta', \alpha' \rangle] .$$

Using the fact that δ' is also the sum of all basic weights, we see that $\langle \alpha', \alpha' \rangle = 2\langle \delta', \alpha' \rangle$. Using this and $2\langle \lambda', \lambda' \rangle = 3\langle \lambda', \alpha' \rangle$ we obtain

$$-2N(2N - 2)\langle \lambda', \lambda' + \delta' \rangle < -(2N + 1)(N - 1)[2\langle \lambda', \lambda' + 2\delta' \rangle - \langle \lambda', \alpha' \rangle] ,$$

or

$$2\langle \lambda', \lambda' + 2\delta' \rangle < (2N + 1)\langle \lambda', \alpha' \rangle .$$

Since $-\langle \lambda', \lambda' + 2\delta' \rangle$ is the Casimir constant of $\mathscr{C}_{\pi_{\lambda'}}$ we obtain immediately that $-2N\langle \lambda', \lambda' + 2\delta' \rangle = \operatorname{tr} \mathscr{C}_{\pi_{\lambda'}} = -\sum_i \operatorname{tr}(\pi_{\lambda'}(X_i)\pi_{\lambda'}(X_i)) = 2 \sum_i \langle X_i, X_i \rangle = 2 \dim H$. Using the expression for $\langle \lambda', \alpha' \rangle$ in (a), we finally obtain

$$\frac{2 \dim H}{-N} < \frac{-2 \dim H}{N} .$$

a contradiction. Therefore, there cannot be any further irreducible summands in $\Lambda^2 \pi_{\lambda'}$. □

Proof of (4.3) continued. Using (4.4), we see that

$$\Lambda^2(\pi \otimes \mathbb{C}) = \Lambda^2(\pi_{\lambda'} \hat{\otimes} \nu_2) = \Lambda^2 \pi_{\lambda'} \hat{\otimes} \operatorname{ad}(sp(1))_{\mathbb{C}} \oplus S^2 \pi_{\lambda'} \hat{\otimes} \mathbf{1}$$

$$= \operatorname{ad}(\mathfrak{h})_{\mathbb{C}} \hat{\otimes} \mathbf{1} \oplus \mathbf{1} \hat{\otimes} \operatorname{ad}(sp(1))_{\mathbb{C}} \oplus [\pi_{2\lambda' - \alpha'} \hat{\otimes} \operatorname{ad}(sp(1))_{\mathbb{C}}] \oplus [\pi_{2\lambda'} \hat{\otimes} \mathbf{1}] .$$

Every $\mathfrak{k}$-invariant element of $S^2\Lambda^2(\mathfrak{p})$ is determined by its four eigenvalues a_1, a_2, b_1, b_2 corresponding to the above decomposition for $\Lambda^2(\pi_{\lambda'} \hat{\otimes} \nu_2)$. Since the four irreducible summands of $\Lambda^2(\pi_{\lambda'} \hat{\otimes} \nu_2)$ are mutually inequivalent and absolutely irreducible, we see that $\dim(S^2\Lambda^2\mathfrak{p})^{\mathfrak{k}} = \dim_{\mathbb{C}}[S^2\Lambda^2(\pi_{\lambda'} \hat{\otimes} \nu_2)]^{\mathfrak{k}} = 4$. Because the symmetric space we are attempting to construct would be quaternionic symmetric, we should expect that $\dim[\Lambda^4(\mathfrak{p})]^{\mathfrak{k}} = 1$. In that event, the space of invariant curvature tensors on $\mathfrak{p}$ would have dimension 3. Hence there would be a linear combination of the three basis elements of $\mathscr{R}$ that gave an invariant curvature tensor with values in $\mathfrak{k}$. By (4.1) we would be done. Accordingly, we will prove that $\dim[\Lambda^4(\mathfrak{p})]^{\mathfrak{k}} = 1$.

Next we describe the various bilinear forms that we will use in the calculations below. First, just as $\pi_{\lambda'}$ has the representation space V' with an $\mathfrak{h}$-invariant skew-symmetric form ω', ν_2 also has a representation space V'' with an $\mathfrak{sp}(1)$-invariant skew form ω''. $V' \otimes V''$ is then the representation space of $\pi_{\lambda'} \hat{\otimes} \nu_2$ and it has a $\mathfrak{k}$-invariant non-degenerate symmetric bilinear form q given by $q(x' \otimes x'', y' \otimes y'') = \omega'(x', y')\omega''(x'', y'')$. The space $\mathfrak{p}$ is a K-invariant real form of $V' \otimes V''$ on which q is positive definite. q in turn induces a $\mathfrak{k}$-invariant symmetric bilinear form on $\Lambda^2(V' \otimes V'')$, to be denoted by $(\,,\,)_{\Lambda^2(V' \otimes V'')}$. On the other hand, there is an isomorphism $\theta: \Lambda^2(V' \otimes V'') \cong S^2V' \otimes \Lambda^2V'' \oplus \Lambda^2V' \otimes S^2V''$ where $(x' \otimes x'') \wedge (y' \otimes y'')$ is mapped to $((x' \circ y') \otimes (x'' \wedge y''), (x' \wedge y') \otimes (x'' \otimes y''))$. S^2V', Λ^2V', S^2V'', Λ^2V'' all have invariant non-degenerate symmetric bilinear forms induced by ω' and ω''. These in turn induce a non-degenerate symmetric bilinear form on $S^2V' \otimes \Lambda^2V'' \oplus \Lambda^2V' \otimes S^2V''$. Under the above isomorphism, one can check that $(\,,\,)_{\Lambda^2(V' \otimes V'')}$ corresponds to $\frac{1}{2}$ times the form $\langle\,,\,\rangle_{S^2V'} \otimes \langle\,,\,\rangle_{\Lambda^2V''} \perp \langle\,,\,\rangle_{\Lambda^2V'} \otimes \langle\,,\,\rangle_{S^2V''}$ on the right.

Let λ'' be the dominant weight of ν_2 and α', α'' be respectively the unique simple root of $\mathfrak{h}$ and $\mathfrak{sp}(1)$ such that $\langle \lambda', \alpha' \rangle \neq 0$, $\langle \lambda'', \alpha'' \rangle \neq 0$. Note that $2\lambda'' = \alpha''$. Let $\lambda = \lambda' + \lambda''$ be the dominant weight of $\pi_{\lambda'} \otimes \nu_2$. Next we choose weight vectors $v_{\lambda'}, v_{-\lambda'}, v_{\lambda'-\alpha'}, v_{\alpha'-\lambda'}$ of $\pi_{\lambda'}$ and $v_{\lambda''}, v_{-\lambda''}$ of ν_2 with the following properties:

$$\omega'(v_{\lambda'}, v_{-\lambda'}) = 1\,, \qquad \omega''(v_{\lambda''}, v_{-\lambda''}) = 1$$

$$v_{\lambda'-\alpha'} = \pi(X_{-\alpha'})v_{\lambda'}\,, \qquad v_{\alpha'-\lambda'} = \pi(X'_{\alpha})v_{\lambda'}$$

$$\omega'(v_{\lambda'-\alpha'}, v_{\alpha'-\lambda'}) = -\langle \lambda', \alpha' \rangle\,.$$

Since $\lambda'' - \alpha'' = -\lambda''$, we define $v_{\lambda''-\alpha''} = v_{-\lambda''}$ and $v_{\alpha''-\lambda''} = v_{\lambda''}$ and hence

$$\omega''(v_{\lambda''-\alpha''}, v_{\alpha''-\lambda''}) = -1\,.$$

As before it follows that

$$\mathrm{pr}_{\mathfrak{h}}(v_{\lambda'} \circ v_{-\lambda'}) = H_{\lambda'}$$

$$\mathrm{pr}_{\mathfrak{h}}(v_{\lambda'-\alpha'} \circ v_{\alpha'-\lambda'}) = -\langle \lambda', \alpha' \rangle H_{\lambda'-\alpha'}$$

$$\mathrm{pr}_{\mathfrak{h}}(v_{\lambda'} \circ v_{\alpha'-\lambda'}) = -\langle \lambda', \alpha' \rangle X_{\alpha'}$$

$$\mathrm{pr}_{\mathfrak{h}}(v_{\lambda'-\alpha'} \circ v_{-\lambda'}) = \langle \lambda', \alpha' \rangle X_{-\alpha'}$$

$$\mathrm{pr}_{\mathfrak{h}}(v_{\lambda''} \circ v_{-\lambda''}) = H_{\lambda''}$$

$$\mathrm{pr}_{\mathfrak{h}}(v_{\lambda''-\alpha''} \circ v_{\alpha''-\lambda''}) = H_{\lambda''}\,.$$

We denote by pr_1 the projection of $\Lambda^2V' = \mathbf{1} \oplus \pi_{2\lambda'-\alpha'}$ onto the trivial summand. Let Ω' and Ω'' respectively be the generators of the trivial representations in Λ^2V', Λ^2V'' subject to the normalization condition

$$\omega'(x', y') = \langle \Omega', x' \wedge y' \rangle_{\Lambda^2V'}\,, \qquad \omega''(x'', y'') = \langle \Omega'', x'' \wedge y'' \rangle_{\Lambda^2V''}\,.$$

It follows from this that $\langle \Omega', \Omega' \rangle = N$ and $\langle \Omega'', \Omega'' \rangle = 1$.

Now one easily shows

$$\begin{aligned}
\mathrm{pr}_1(v_{\lambda'} \wedge v_{-\lambda'}) &= \frac{1}{N}\,\Omega' \\
\mathrm{pr}_1(v_{\alpha'-\lambda'} \wedge v_{\lambda'-\alpha'}) &= \frac{1}{N}\,\langle \lambda', \alpha' \rangle \Omega' \\
\mathrm{pr}_1(v_{\lambda'} \wedge v_{\alpha'-\lambda'}) &= 0 \\
\mathrm{pr}_1(v_{\lambda'-\alpha'} \wedge v_{-\lambda'}) &= 0 \\
v_{\lambda''} \wedge v_{-\lambda''} &= \Omega'' \\
v_{\alpha''-\lambda''} \wedge v_{\lambda''-\alpha''} &= \Omega''\,.
\end{aligned}$$

Let $\eta \in \Lambda^4(\pi_{\lambda'} \hat{\otimes} \pi_{\lambda''})^{\mathfrak{k}}$ and define $v_{\lambda-\alpha'} = v_{\lambda'-\alpha'} \otimes v_{\lambda''}$ and $v_{\lambda-\alpha''} = v_{\lambda'} \otimes v_{\lambda''-\alpha''}$, and so forth. Since η is a 4-form, we have for $\beta = \alpha'$ or α''.

$$\eta(v_\lambda, v_{\lambda-\beta}, v_{-\lambda}, v_{\beta-\lambda}) = \eta(v_\lambda, v_{-\lambda}, v_{\beta-\lambda}, v_{\lambda-\beta}) = \eta(v_\lambda, v_{\beta-\lambda}, v_{\lambda-\beta}, v_{-\lambda})\,.$$

These give rise to 4 homogeneous linear equations in a_1, a_2, b_1, b_2. We will show that this system has rank 3, from which our claim follows. For this purpose we produce a 3×3 minor with non-vanishing determinant.

(a) $\eta(v_\lambda, v_{\lambda-\alpha'}, v_{-\lambda}, v_{\alpha'-\lambda}) = (\hat{\eta}(v_\lambda \wedge v_{\lambda-\alpha'}), v_{-\lambda} \wedge v_{\alpha'-\lambda})_{\Lambda^2(V' \otimes V'')}$.

Now

$$\begin{aligned}
\theta(v_\lambda \wedge v_{\lambda-\alpha'}) &= \theta((v_{\lambda'} \otimes v_{\lambda''}) \wedge (v_{\lambda'-\alpha'} \otimes v_{\lambda''}) \\
&= (v_{\lambda'} \wedge v_{\lambda'-\alpha'}) \otimes (v_{\lambda''} \circ v_{\lambda''}) \in \pi_{2\lambda'-\alpha'} \hat{\otimes}\, \mathrm{ad}(\mathfrak{sp}(1))_{\mathbb{C}}\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\eta(v_\lambda, v_{\lambda-\alpha'}, v_{-\lambda}, v_{\alpha'-\lambda}) \\
&\quad = b_1(v_\lambda \wedge v_{\lambda-\alpha'}, v_{-\lambda} \wedge v_{\alpha'-\lambda})_{\Lambda^2(V' \otimes V'')} \\
&\quad = b_1(q(v_\lambda, v_{-\lambda})q(v_{\lambda-\alpha'}, v_{\alpha'-\lambda}) - q(v_\lambda, v_{\alpha'-\lambda})q(v_{\lambda-\alpha'}, v_{-\lambda})) \\
&\quad = b_1(\omega'(v_{\lambda'-\alpha'}, v_{\alpha'-\lambda'}) - 0) = -b_1\langle \lambda', \alpha' \rangle\,.
\end{aligned}$$

(b) In an analogous fashion, $\eta(v_\lambda, v_{\lambda-\alpha''}, v_{-\lambda}, v_{\alpha''-\lambda}) = -b_2$.

(c) $\eta(v_\lambda, v_{-\lambda}, v_{\alpha'-\lambda}, v_{\lambda-\alpha'}) = (\hat{\eta}(v_\lambda \wedge v_{-\lambda}), v_{\alpha'-\lambda} \wedge v_{\lambda-\alpha'})_{\Lambda^2(V' \otimes V'')}$.

Now

$$\begin{aligned}
\theta(v_\lambda \wedge v_{-\lambda}) &= \theta((v_{\lambda'} \otimes v_{\lambda''}) \wedge (v_{-\lambda'} \otimes v_{-\lambda''})) \\
&= (v_{\lambda'} \circ v_{-\lambda'}) \otimes (v_{\lambda''} \wedge v_{-\lambda''}) + (v_{\lambda'} \wedge v_{-\lambda'}) \otimes (v_{\lambda''} \circ v_{-\lambda''}) \\
&= (H_{\lambda'} + s_1) \otimes \Omega'' + \left(\frac{1}{N}\,\Omega' + x_1\right) \otimes H_{\lambda''}
\end{aligned}$$

with $s_1 \in \pi_{2\lambda'}$ and $x_1 \in \pi_{2\lambda'-\alpha'}$.

So $\theta(\hat{\eta}(v_\lambda \wedge v_{-\lambda})) = a_1 H_{\lambda'} \otimes \Omega'' + \dfrac{a_2}{N}\,\Omega' \otimes H_{\lambda''} + b_1 x_1 \otimes H_{\lambda''} + b_2 s_1 \otimes \Omega''$.

Next

$$\begin{aligned}
&\theta(v_{\alpha'-\lambda} \wedge v_{\lambda-\alpha'}) \\
&\quad = \theta((v_{\alpha'-\lambda'} \otimes v_{-\lambda''}) \wedge (v_{\lambda'-\alpha'} \otimes v_{\lambda''})) \\
&\quad = -(v_{\alpha'-\lambda'} \circ v_{\lambda'-\alpha'}) \otimes (v_{\lambda''} \wedge v_{-\lambda''}) + (v_{\alpha'-\lambda'} \wedge v_{\lambda'-\alpha'}) \otimes (v_{\lambda''} \circ v_{-\lambda''}) \\
&\quad = (\langle \lambda', \alpha' \rangle H_{\lambda'-\alpha'} - s_2) \otimes \Omega'' + \left(\frac{1}{N}\,\langle \lambda', \alpha' \rangle \Omega' + x_2\right) \otimes H_{\lambda''}
\end{aligned}$$

with $s_2 \in \pi_{2\lambda'}$ and $x_2 \in \pi_{2\lambda'-\alpha'}$. Hence

$$\begin{aligned}\eta(v_\lambda, v_{-\lambda}, v_{\alpha'-\lambda}, v_{\lambda-\alpha'}) &= \frac{1}{2}\langle\theta(\hat{\eta}(v_\lambda \wedge v_{-\lambda})), \theta(v_{\alpha'-\lambda} \wedge v_{\lambda-\alpha'})\rangle \\ &= \frac{1}{2}\Big\{a_1\langle\lambda', \alpha'\rangle\langle\lambda' - \alpha', \lambda'\rangle - b_2\langle s_1, s_2\rangle_{\Lambda^2V'} \\ &\quad + \frac{a_2}{N}\langle\lambda', \alpha'\rangle\langle\lambda'', \lambda''\rangle + b_1\langle x_1, x_2\rangle_{\Lambda^2V'}\langle\lambda'', \lambda''\rangle\Big\}.\end{aligned}$$

In order to compute $\langle s_1, s_2\rangle_{S^2V'}$ and $\langle x_1, x_2\rangle_{\Lambda^2V'}$, we examine $\langle v_{\lambda'} \wedge v_{-\lambda'}, v_{\alpha'-\lambda'} \wedge v_{\lambda'-\alpha'}\rangle_{\Lambda^2V'} = 0$ and $\langle v_{\lambda'} \circ v_{-\lambda'}, v_{\alpha'-\lambda'} \circ v_{\lambda'-\alpha'}\rangle_{S^2V'} = 0$ and obtain $\langle x_1, x_2\rangle_{\Lambda^2V'} = -\dfrac{\langle\lambda', \alpha'\rangle}{N}$ and $\langle s_1, s_2\rangle_{S^2V'} = \langle\lambda', \alpha'\rangle\langle\lambda', \lambda' - \alpha'\rangle$. Substituting these in the above, we get

$$\begin{aligned}\eta(v_\lambda, v_{-\lambda}, v_{\alpha'-\lambda}, v_{\lambda-\alpha'}) &= \frac{1}{2}\ \langle\lambda', \alpha'\rangle\Big\{a_1\langle\lambda' - \alpha', \lambda'\rangle - b_2\langle\lambda' - \alpha', \lambda'\rangle \\ &\quad + \frac{a_2}{N}\langle\lambda'', \lambda''\rangle - \frac{b_1}{N}\langle\lambda'', \lambda''\rangle\Big\}.\end{aligned}$$

(d)

$$\begin{aligned}\eta(v_\lambda, v_{-\lambda}, v_{\alpha''-\lambda}, v_{\lambda-\alpha''}) &= (\hat{\eta}(v_\lambda \wedge v_{-\lambda}), v_{\alpha''-\lambda} \wedge v_{\lambda-\alpha''})_{\Lambda^2(V'\otimes V'')} \\ &= \frac{1}{2}\langle\theta(\hat{\eta}(v_\lambda \wedge v_{-\lambda})), \theta(v_{\alpha''-\lambda} \wedge v_{\lambda-\alpha''})\rangle.\end{aligned}$$

The first argument has been computed in (c). Now

$$\begin{aligned}&\theta(v_{\alpha''-\lambda} \wedge v_{\lambda-\alpha''}) \\ &\quad = \theta((v_{-\lambda'} \otimes v_{\alpha''-\lambda''}) \wedge (v_{\lambda'} \otimes v_{\lambda''-\alpha''})) \\ &\quad = -(v_{\lambda'} \circ v_{-\lambda'}) \otimes (v_{\lambda''-\alpha''} \wedge v_{\alpha''-\lambda''}) - (v_{\lambda'} \wedge v_{-\lambda'}) \otimes (v_{\lambda''-\alpha''} \circ v_{\alpha''-\lambda''}) \\ &\quad = (H_{\lambda'} + s_1) \otimes \Omega'' - \left(\frac{1}{N}\Omega' + x_1\right) \otimes H_{\lambda''}.\end{aligned}$$

Also, $\langle v_{\lambda'} \wedge v_{-\lambda'}, v_{\lambda'} \wedge v_{-\lambda'}\rangle_{\Lambda^2V'} = 1$ and $\langle v_{\lambda'} \circ v_{-\lambda'}, v_{\lambda'} \circ v_{-\lambda'}\rangle_{S^2V'} = -1$ from which we get $\langle x_1, x_1\rangle_{\Lambda^2V'} = 1 - \dfrac{1}{N}$ and $\langle s_1, s_1\rangle_{S^2V'} = -1 - \langle\lambda', \lambda'\rangle$. Therefore,

$$\begin{aligned}\eta(v_\lambda, v_{-\lambda}, v_{\alpha''-\lambda}, v_{\lambda-\alpha''}) &= \frac{1}{2}\Big\{a_1\langle\lambda', \lambda'\rangle - b_2(1 + \langle\lambda', \lambda'\rangle) \\ &\quad - \frac{a_2}{N}\langle\lambda'', \lambda''\rangle - b_1\left(1 - \frac{1}{N}\right)\langle\lambda'', \lambda''\rangle\Big\}.\end{aligned}$$

(e) A similar computation shows that

$$\begin{aligned}&\eta(v_\lambda, v_{\alpha'-\lambda}, v_{\lambda-\alpha'}, v_{-\lambda}) \\ &\quad = \frac{\langle\lambda', \alpha'\rangle}{2}\{-a_1\langle\lambda', \alpha'\rangle - b_1\langle\lambda'', \lambda''\rangle + b_2(1 + \langle\lambda', \alpha'\rangle)\}.\end{aligned}$$

Combining these computations and using the fact that $\langle \lambda', \lambda' \rangle = \frac{3}{2} \langle \lambda', \alpha' \rangle$ and setting $c' = \frac{1}{2} \langle \lambda', \alpha' \rangle$, $c'' = \langle \lambda'', \lambda'' \rangle$ we obtain the following linear system for a_1, a_2, b_1, b_2:

$$\begin{pmatrix} c' & \frac{c''}{N} & 2 - \frac{c''}{N} & -c' \\ 2c' & 0 & c'' - 2 & -(1+2c') \\ 3c' & -\frac{c''}{N} & -\left(1 - \frac{1}{N}\right)c'' & 1 - 3c' \\ * & * & * & * \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = 0 .$$

It easily follows that the 3×3 minor consisting of the first three rows and columns 1, 2, and 4 has non-zero determinant and hence $\dim(\Lambda^4(\mathfrak{p})^{\mathfrak{k}}) = \dim_{\mathbb{C}}[\Lambda^4(V' \otimes V'')]^{\mathfrak{k}} \leqq 1$. But equality holds since we may define $\eta(u_1 \otimes v_1, u_2 \otimes v_2, u_3 \otimes v_3, u_4 \otimes v_4)$ by

$$\begin{aligned} &\omega'(u_1, u_2)\omega''(v_3, v_4) + \omega'(u_1, u_4)\omega''(v_2, v_3) - \omega'(u_1, u_3)\omega''(v_2, v_4) \\ &\quad + \omega'(u_3, u_4)\omega''(v_1, v_2) + \omega'(u_2, u_3)\omega''(v_1, v_4) - \omega'(u_2, u_4)\omega''(v_1, v_3) \end{aligned}$$

for $u_i \in V'$, $v_i \in V''$ and one can check that η is $\mathfrak{k}$-invariant since ω', ω'' are invariant, and $\eta(v_{\lambda'} \otimes v_{\lambda''}, v_{-\lambda'} \otimes v_{-\lambda''}, v_{-\lambda'} \otimes v_{\lambda''}, v_{\lambda'-\alpha} \otimes v_{-\lambda''}) = 1$ so that $\eta \neq 0$. This completes the proof of (4.3). □

C. *The hermitian case*

(4.6) Theorem. *Let $\pi' = H \to \mathrm{SU}(N)$ be a faithful, unitary representation of a compact connected Lie group H such that $\mathrm{SU}(N)/\pi'(H)$ is strongly isotropy irreducible and non-symmetric. Then $(K = H \cdot \mathrm{U}(1), \pi = [\pi' \hat{\otimes} \phi]_{\mathbb{R}})$ is the isotropy representation of a compact, simply connected, irreducible Hermitian symmetric space. Moreover, either H is simple or else $H = \mathrm{SU}(p) \cdot \mathrm{SU}(q)$, $p \geqq q \geqq 2$, $\pi' = \mu_p \hat{\otimes} \mu_q^*$.*

Proof. In this case we have $\pi' \otimes \pi'^* = \mathbf{1} \oplus \mathrm{Ad}_H \oplus \chi$ with χ irreducible over $\mathbb{R}$. Furthermore, π' is already a complex representation. If it is not irreducible, then it follows as in (4.2) that $H = S(\mathrm{U}(p) \times \mathrm{U}(q))$ and $\pi' = \mu_p \hat{\otimes} \mathbf{1} \oplus \mathbf{1} \hat{\otimes} \mu_q$, $N = p + q$, which implies that $\mathrm{SU}(N)/\pi'(H)$ is symmetric. Hence π' is an irreducible complex representation with dominant weight λ'. Since π' is faithful, this also implies that H must be semisimple. If $\pi' \cong \pi'^*$, then π' would be orthogonal or symplectic and hence $H \subset \mathrm{SO}(n) \subset \mathrm{SU}(n)$ or $H \subset \mathrm{Sp}(n) \subset \mathrm{SU}(2n)$. Isotropy irreducibility then implies that $H = \mathrm{SO}(n)$ or $H = \mathrm{Sp}(n)$ and hence $\mathrm{SU}(N)/\pi'(H)$ is symmetric. We can therefore also assume that $\lambda' \neq \lambda'^*$. If H is not simple, it follows as in the proof of (4.3) that up to an automorphism $H = \mathrm{SU}(p)\mathrm{SU}(q)$, $p \geqq q \geqq 2$ and $\pi' = \mu_p \hat{\otimes} \mu_q^*$. But then $[\pi' \hat{\otimes} \phi]_{\mathbb{R}}$ is indeed the isotropy representation of the symmetric space $\mathrm{SU}(p+q)/S(\mathrm{U}(p)\mathrm{U}(q))$. We can therefore assume from now on that H is simple and $\pi' = \pi_{\lambda'}$, with $\lambda' \neq \lambda'^*$.

As in the previous proofs, we have to consider curvature tensors on

$$\begin{aligned} &\Lambda^2(\pi_{\lambda'} \hat{\otimes} \phi \oplus \pi_{\lambda'}^* \hat{\otimes} \phi^*) \\ &\quad = \Lambda^2 \pi_{\lambda'} \hat{\otimes} \phi^2 \oplus \Lambda^2 \pi_{\lambda'}^* \hat{\otimes} (\phi^*)^2 \oplus \mathbf{1} \hat{\otimes} \mathbf{1} \oplus \mathrm{ad}(\mathfrak{h})_{\mathbb{C}} \hat{\otimes} \mathbf{1} \oplus \chi_{\mathbb{C}} \hat{\otimes} \mathbf{1} . \end{aligned}$$

We again must first analyse $\Lambda^2 \pi_{\lambda'}$. For this purpose we need the following general lemma.

(4.7) Lemma. *Let $\pi_\lambda : H \to \mathrm{SU}(N)$ be an irreducible unitary representation of a compact connected simple Lie group H. Then $S^2\pi_\lambda$ is irreducible* iff *$H = \mathrm{SU}(N)$ and $\pi_\lambda = \mu_N$ or μ_N^* or $H = \mathrm{Sp}(n)$ and $\pi_\lambda = \nu_{2n}$ with $N = 2n$.*

Proof. Let $V \cong \mathbb{C}^N$ denote the representation space of π_λ and $\langle\,,\rangle$ and $\mathrm{ad}(\mathfrak{h})$-invariant inner product on $\mathfrak{h}$. Furthermore, let $\{H_1, \ldots, H_r\}$ and $\{H_1^*, \ldots, H_r^*\}$ be dual bases of $\mathfrak{t} \otimes \mathbb{C}$ with respect to $\langle\,,\rangle$, where $\mathfrak{t}$ is a maximal abelian subalgebra of $\mathfrak{h}$, and let $\{H_i, X_\alpha, X_{-\alpha}\}$ form a Weyl-Chevalley basis of $\mathfrak{h} \otimes \mathbb{C}$. Then the Casimir operator $\mathscr{C}_{\pi_\lambda}$ with respect to $\langle\,,\rangle$ is $-\sum_{i=1}^{r} \pi_\lambda(H_i)\pi_\lambda(H_i^*) - \sum_{\alpha>0} (\pi_\lambda(X_\alpha)\pi_\lambda(X_{-\alpha}) + \pi_\lambda(X_{-\alpha})\pi_\lambda(X_\alpha))$, where α ranges over all positive roots of $\mathfrak{h} \otimes \mathbb{C}$.

Consider $v_\lambda \circ v_{-\lambda^*} \in S^2V$. If $S^2\pi_\lambda$ is irreducible, then

$$\mathscr{C}_{S^2\pi_\lambda}(v_\lambda \circ v_{-\lambda^*}) = -\langle 2\lambda, 2\lambda + 2\delta\rangle\,(v_\lambda \circ v_{-\lambda^*})\,.$$

On the other hand,

$$\begin{aligned}\mathscr{C}_{S^2\pi_\lambda}(v_\lambda \circ v_{-\lambda^*}) &= \mathscr{C}_{\pi_\lambda}(v_\lambda) \circ v_{-\lambda^*} + v_\lambda \circ \mathscr{C}_{\pi_\lambda}(v_{-\lambda^*}) \\ &\quad + \sum_i [\lambda(H_i)\lambda^*(H_i^*) + \lambda^*(H_i)\lambda(H_i^*)] v_\lambda \circ v_{-\lambda^*} \\ &\quad - 2\sum_{\alpha>0} \pi_\lambda(X_{-\alpha})v_\lambda \circ \pi_\lambda(X_\alpha)v_{-\lambda^*} \\ &= (-2\langle\lambda, \lambda + 2\delta\rangle + 2\langle\lambda, \lambda^*\rangle) v_\lambda \circ v_{-\lambda^*} \\ &\quad - 2\sum_{\alpha>0} \pi_\lambda(X_{-\alpha})v_\lambda \circ \pi_\lambda(X_\alpha)v_{-\lambda^*}\,.\end{aligned}$$

It follows that $\sum_{\alpha>0} \pi_\lambda(X_{-\alpha})v_\lambda \circ \pi_\lambda(X_\alpha)v_{-\lambda^*} = a(v_\lambda \circ v_{-\lambda^*})$ with $a = \langle\lambda, \lambda + \lambda^*\rangle < 0$. Let $\phi_{-\lambda}, \phi_{\lambda^*} \in V^*$ be dual to v_λ and $v_{-\lambda^*}$. Applying $\phi_{-\lambda} \circ \phi_{\lambda^*}$ to $\sum_{\alpha>0} \pi_\lambda(X_{-\alpha})v_\lambda \circ \pi_\lambda(X_\alpha)v_{-\lambda^*}$ we get $0 \neq a = \sum_{\alpha>0} \phi_{\lambda^*}(X_{-\alpha}v_\lambda)\phi_{-\lambda}(X_\alpha v_{-\lambda^*})$ since $\phi_{\varrho_1}(v_{\varrho_2}) = 0$ unless $\varrho_1 + \varrho_2 = 0$. In particular this also means that there is a positive root α such that $\lambda + \lambda^* = \alpha$. By (2.1) we conclude that $H = \mathrm{SU}(N)$ and $\pi_\lambda = \mu_N$ or μ_N^* or $H = \mathrm{Sp}(n)$ and $\pi_\lambda = \nu_{2n}$. □

(4.8) Proposition. *With the same assumptions as in* (4.6) *and the assumption that H is simple we have*

(a) *$\Lambda^2\pi'$ is irreducible and $S^2\pi'$ consists of exactly two irreducible inequivalent sub-representations.*

(b) *There exists a unique simple root α of $\mathfrak{h}$ with $\langle\lambda', \alpha\rangle \neq 0$ and hence $\Lambda^2\pi' = \pi_{2\lambda'-\alpha}$.*

Proof. As in the beginning of the proof of (4.6), it follows that $\pi' = \pi_{\lambda'}$ is irreducible with $\lambda' \neq \lambda'^*$. Since $\mathrm{SU}(N)/\pi'(H)$ is not a point, we can also assume that $\pi' \neq \mu_n, \mu_n^*$. Hence (4.7) implies that $S^2\pi_{\lambda'}$ is reducible. Consider

$$S^2(\pi_{\lambda'} \otimes \pi_{\lambda'}^*) = S^2\pi_{\lambda'} \otimes S^2\pi_{\lambda'}^* \oplus \Lambda^2\pi_{\lambda'} \otimes \Lambda^2\pi_{\lambda'}^*\,.$$

This is also given by

$$S^2(\mathbf{1} \oplus \mathrm{ad}(\mathfrak{h})_{\mathbb{C}} \oplus \chi_{\mathbb{C}}) = \mathbf{1} \oplus S^2(\mathrm{ad}(\mathfrak{h})_{\mathbb{C}}) \oplus S^2\chi_{\mathbb{C}} \oplus \mathrm{ad}(\mathfrak{h})_{\mathbb{C}} \oplus \chi_{\mathbb{C}} \oplus [\mathrm{ad}(\mathfrak{h})_{\mathbb{C}} \otimes \chi_{\mathbb{C}}]\,.$$

By (2.2.a), $\chi_{\mathbb{C}} = \pi_{\lambda'+\lambda'^*} \neq \mathrm{ad}(\mathfrak{h})_{\mathbb{C}}$. Hence there are exactly 3 trivial summands in $S^2(\pi_{\lambda'} \otimes \pi^*_{\lambda'})$. This forces $S^2\pi_{\lambda'}$ to have 2 irreducible inequivalent summands and $\Lambda^2\pi_{\lambda'}$ to be irreducible. (b) follows from (a) since for every simple root α with $\langle \lambda', \alpha\rangle \neq 0$ the representation $\pi_{2\lambda'-\alpha}$ is a sub-representation of $\Lambda^2\pi'$. □

Proof of (4.6) continued. Let $V' \cong \mathbb{C}^N$ be the representation space of π' and $V'' \cong \mathbb{C}$ be that for ϕ and define $V = V' \otimes V''$. Let λ' be the dominant weight of π' and λ'' that for ϕ. Hence $\lambda = \lambda' + \lambda''$ is the dominant weight for $\pi = \pi' \hat{\otimes} \phi$. From the natural pairing between V^* and V one obtains a non-degenerate symmetric bilinear form q on $V \oplus V^*$ given by $q((v_1, \phi_1), (v_2, \phi_2)) = \phi_1(v_2) + \phi_2(v_1)$. Then $\mathfrak{p}$ is a K-invariant real form of $V \oplus V^*$ on which q is positive definite.

It follows from (4.8) that

$$\begin{aligned}\Lambda^2(\mathfrak{p}) &= [\mathrm{ad}(\mathfrak{h}) \hat{\otimes} \mathbf{1}] \oplus \mathbf{1} \hat{\otimes} \mathbf{1} \oplus [\pi_{\lambda'+\lambda'^*} \hat{\otimes} \mathbf{1}]_{\mathbb{R}} \oplus [\pi_{2\lambda'-\alpha} \hat{\otimes} \phi^2]_{\mathbb{R}} \\ &= \mathrm{ad}(\mathfrak{k}) \oplus [\pi_{\lambda+\lambda^*}]_{\mathbb{R}} \oplus [\pi_{2\lambda-\alpha}]_{\mathbb{R}} .\end{aligned}$$

Every element of $S^2\Lambda^2(\mathfrak{p})$ that is $\mathfrak{k}$-invariant has diagonal form with respect to this decomposition of $\Lambda^2(\mathfrak{p})$. Let a_1, a_2, b_1, b_2 be respectively the eigenvalues. Clearly, $\dim[S^2\Lambda^2(\mathfrak{p})]^{\mathfrak{k}} = 4$. Hence, as in the quaternionic case, we will show that $\dim[\Lambda^4(\mathfrak{p})]^{\mathfrak{k}} = 1$, and we will be done.

We first establish some notation for future use. There is a non-degenerate symmetric bilinear form $\langle\,,\rangle$ on $V \otimes V^*$ given by $\langle v_1 \otimes \phi_1, v_2 \otimes \phi_2\rangle = -\phi_1(v_2)\phi_2(v_1)$. Under the isomorphism $V \otimes V^* \approx \mathfrak{gl}(V)$ where $v \otimes \phi \mapsto A_{v\otimes\phi}$ with $A_{v\otimes\phi}(x) = \phi(x)v$, $\langle\,,\rangle$ corresponds to the form on $\mathfrak{gl}(V)$ given by $-\mathrm{tr}(AB)$. Furthermore, q induces a symmetric bilinear form $(\,,)_{\Lambda^2(V\oplus V^*)}$ on $\Lambda^2(V \oplus V^*)$. The isomorphism $\theta: \Lambda^2(V \oplus V^*) \cong \Lambda^2 V \oplus \Lambda^2 V^* \oplus V \otimes V^*$ given by $(v_1, \phi_1) \wedge (v_2, \phi_2) \mapsto (v_1 \wedge v_2, \phi_1 \wedge \phi_2, v_1 \otimes \phi_2 - v_2 \otimes \phi_1)$ becomes an isometry if we choose as our invariant symmetric bilinear form $\hat{q} \perp \langle\,,\rangle$ where $\hat{q}$ comes from the duality of $\Lambda^2 V$ and $\Lambda^2 V^*$ as q does.

Using the embedding $\pi: \mathfrak{k} = \mathfrak{h} \oplus \mathfrak{u}(1) \to \mathfrak{gl}(V) \cong V \otimes V^*$, the inner product on $V \otimes V^*$ induces an inner product on $\mathfrak{k}$, again denoted by $\langle\,,\rangle$ and a projection $\mathrm{pr}_{\mathfrak{k}}$ from $V \otimes V^*$ to $\mathfrak{k}$. We now select weight vectors $v_\lambda, v_{-\lambda^*}, v_{\lambda-\alpha} = \pi(X_{-\alpha})V_\lambda$ in V and $\phi_{\lambda^*}, \phi_{-\lambda}, \phi_{\alpha-\lambda} = \pi(X_\alpha)\phi_{-\lambda}$ in V^* such that

$$\begin{aligned}\phi_{-\lambda}(v_\lambda) &= 1 = \phi_{\lambda^*}(v_{-\lambda^*}) \\ \phi_{\alpha-\lambda}(v_{\lambda-\alpha}) &= -\langle\lambda, \alpha\rangle \\ \mathrm{pr}_{\mathfrak{k}}(v_\lambda \otimes \phi_{\alpha-\lambda}) &= \langle\lambda, \alpha\rangle X_\alpha \\ \mathrm{pr}_{\mathfrak{k}}(v_{\lambda-\alpha} \otimes \phi_{-\lambda}) &= -\langle\lambda, \alpha\rangle X_{-\alpha} \\ \mathrm{pr}_{\mathfrak{k}}(v_\lambda \otimes \phi_{-\lambda}) &= -H_\lambda \\ \mathrm{pr}_{\mathfrak{k}}(v_{\lambda-\alpha} \otimes \phi_{\alpha-\lambda}) &= \langle\lambda, \alpha\rangle H_{\lambda-\alpha} \\ \mathrm{pr}_{\mathfrak{k}}(v_{-\lambda^*} \otimes \phi_{\lambda^*}) &= H_{\lambda^*} .\end{aligned}$$

Let $\eta \in [\Lambda^4(\mathfrak{p})]^{\mathfrak{k}}$ and consider its complexification. Let $\hat{\eta}$ be the corresponding symmetric endomorphism of $\Lambda^2(\mathfrak{p})$. Then we have

$$\begin{aligned}\eta(v_\lambda, v_{\lambda-\alpha}, \phi_{-\lambda}, \phi_{\alpha-\lambda}) &= \eta(v_\lambda, \phi_{-\lambda}, \phi_{\alpha-\lambda}, v_{\lambda-\alpha}) = \eta(v_\lambda, \phi_{\alpha-\lambda}, v_{\lambda-\alpha}, \phi_{-\lambda}) \\ \eta(v_\lambda, \phi_{-\lambda}, \phi_{\lambda^*}, v_{-\lambda^*}) &= \eta(v_\lambda, \phi_{\lambda^*}, v_{-\lambda^*}, \phi_{-\lambda}) = \eta(v_\lambda, v_{-\lambda^*}, \phi_{-\lambda}, \phi_{\lambda^*})\end{aligned}$$

which yields a homogeneous linear system of order 4 in a_1, a_2, b_1, b_2. To obtain the system, we compute as follows:

(a) $$\begin{aligned}\eta(v_\lambda, v_{\lambda-\alpha}, \phi_{-\lambda}, \phi_{\alpha-\lambda}) &= (\hat{\eta}(v_\lambda \wedge v_{\lambda-\alpha}), \phi_{-\lambda} \wedge \phi_{\alpha-\lambda})_{\Lambda^2(V\oplus V^*)} \\ &= \langle \theta(\hat{\eta}(v_\lambda \wedge v_{\lambda-\alpha})), \theta(\phi_{-\lambda} \wedge \phi_{\alpha-\lambda})\rangle .\end{aligned}$$

Now $\theta(v_\lambda \wedge v_{\lambda-\alpha}) = v_\lambda \wedge v_{\lambda-\alpha} \in \pi_{2\lambda-\alpha}$ and $\theta(\phi_{-\lambda} \wedge \phi_{\alpha-\lambda}) = \phi_{-\lambda} \wedge \phi_{\alpha-\lambda} \in \pi^*_{2\lambda-\alpha}$. So

$$\begin{aligned}\eta(v_\lambda, v_{\lambda-\alpha}, \phi_{-\lambda}, \phi_{\alpha-\lambda}) &= b_2 \hat{q}(v_\lambda \wedge v_{\lambda-\alpha}, \phi_{-\lambda} \wedge \phi_{\alpha-\lambda}) \\ &= b_2(\phi_{-\lambda} \wedge \phi_{\alpha-\lambda})(v_\lambda \wedge v_{\lambda-\alpha}) \\ &= -b_2\langle \lambda, \alpha\rangle .\end{aligned}$$

(b) Likewise, $\eta(v_\lambda, v_{-\lambda^*}, \phi_{-\lambda}, \phi_{\lambda^*}) = b_2$ and $\eta(v_\lambda, \phi_{\lambda^*}, v_{-\lambda^*}, \phi_{-\lambda}) = -b_1$, where we have used the fact that $\theta(v_\lambda \wedge \phi_{\lambda^*}) = v_\lambda \otimes \phi_{\lambda^*}$ which lies in $\pi_{\lambda+\lambda^*}$.

(c) $$\begin{aligned}\eta(v_\lambda, \phi_{-\lambda}, \phi_{\alpha-\lambda}, v_{\lambda-\alpha}) &= (\hat{\eta}(v_\lambda \wedge \phi_{-\lambda}), \phi_{\alpha-\lambda} \wedge v_{\lambda-\alpha})_{\Lambda^2(V\oplus V^*)} \\ &= \langle \theta(\hat{\eta}(v_\lambda \wedge \phi_{-\lambda})), \theta(\phi_{\alpha-\lambda} \wedge v_{\lambda-\alpha})\rangle .\end{aligned}$$

We have $\theta(v_\lambda \wedge \phi_{-\lambda}) = v_\lambda \otimes \phi_{-\lambda} = -H_\lambda + x_1$ and $\theta(\phi_{\alpha-\lambda} \wedge v_{\lambda-\alpha}) = -v_{\lambda-\alpha} \otimes \phi_{\alpha-\lambda} = -\langle \lambda, \alpha\rangle H_{\lambda-\alpha} + x_2$, with $x_1, x_2 \in \pi_{\lambda+\lambda^*}$. Furthermore, $\langle v_\lambda \otimes \phi_{-\lambda}, v_{\lambda-\alpha} \otimes \phi_{\alpha-\lambda}\rangle = -\phi_{-\lambda}(v_{\lambda-\alpha})\phi_{\alpha-\lambda}(v_\lambda) = 0 = \langle \lambda, \alpha\rangle\langle \lambda, \lambda - \alpha\rangle + \langle x_1, x_2\rangle$. Hence $\langle x_1, x_2\rangle = -\langle \lambda, \alpha\rangle\langle \lambda, \lambda - \alpha\rangle$. It follows that

$$\begin{aligned}&\eta(v_\lambda, \phi_{-\lambda}, \phi_{\alpha-\lambda}, v_{\lambda-\alpha}) \\ &\quad = \langle -a_1 H_{\lambda'} - a_2 H_{\lambda''} + b_1 x_1, -\langle \lambda, \alpha\rangle H_{\lambda'-\alpha} - \langle \lambda, \alpha\rangle H_{\lambda''} + x_2\rangle \\ &\quad = a_1\langle \lambda, \alpha\rangle\langle \lambda', \lambda' - \alpha\rangle + a_2\langle \lambda, \alpha\rangle\langle \lambda'', \lambda''\rangle - b_1\langle \lambda, \alpha\rangle\langle \lambda, \lambda - \alpha\rangle .\end{aligned}$$

(d) A similar computation shows that

$$\eta(v_\lambda, \phi_{\alpha-\lambda}, v_{\lambda-\alpha}, \phi_{-\lambda}) = -a_1\langle \lambda, \alpha\rangle^2 + b_1\langle \lambda, \alpha\rangle(1 + \langle \lambda, \alpha\rangle) .$$

Combining the above calculations we obtian the following linear system:

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ \langle \lambda', \lambda' - \alpha\rangle & \langle \lambda'', \lambda''\rangle & -\langle \lambda, \lambda - \alpha\rangle & 1 \\ -\langle \lambda, \alpha\rangle & 0 & 1 + \langle \lambda, \alpha\rangle & 1 \\ * & * & * & * \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = 0$$

Since the 3×3 minor consisting of the first three rows and columns has non-zero determinant, we have $\dim[\Lambda^4(\mathfrak{p})]^{\mathfrak{k}} \leqq 1$. Finally, observe that $\Lambda^4(V \oplus V^*) \supset \Lambda^2(V) \otimes \Lambda^2(V^*)$, which contains a trivial summand, and so $\dim[\Lambda^4(\mathfrak{p})^{\mathfrak{k}} = 1$. This completes the proof of (4.6). □

4 Some applications

We end our paper with some applications to the isotropy representation of quaternionic and hermitian symmetric spaces that are of independent interest.

(5.1) Theorem. *Let $G/K = G/H \cdot \mathrm{Sp}(1)$ be a compact, simply connected, irreducible quaternionic symmetric space with isotropy representation $\pi = \pi' \hat{\otimes} \nu_2$ and such that H is simple. Then $\Lambda^2\pi' = \mathbf{1} \oplus \sigma$ with σ irreducible. If H is not simple, $\Lambda^2\pi' - \mathbf{1}$ consists of 2 or 3 irreducible summands.*

Proof. If H is not simple, then (2.10.b) implies that $(\mathfrak{h}, \pi') = (\mathfrak{so}(n) \oplus \mathfrak{sp}(1), \varrho_n \hat{\otimes} \nu_2)$ or $(\mathfrak{u}(n), \mu_n)$. In both cases one easily shows that $\Lambda^2\pi' - \mathbf{1}$ consists of 2 irreducible summands, unless $n = 4$ in the first case, where it consists of 3 irreducible summands. If G/K is the quaternionic projective space, we have $(\mathfrak{h}, \pi') = (\mathfrak{sp}(n), \nu_{2n})$ and it follows that $\Lambda^2\pi' - \mathbf{1}$ is irreducible. If G/K is not the quaternionic projective space, it follows from (3.1.b) that $\mathrm{Sp}(N)/\pi'(H)$, $4N = \dim G/K$, is isotropy irreducible and positive dimensional. Assuming that and that H is simple, a proof identical to the one in (4.4) shows that $\Lambda^2\pi' - \mathbf{1}$ is irreducible. One only needs to observe, which is needed in the proof of (4.4), that π' is absolutely irreducible, since G/K is an irreducible symmetric space. □

(5.2) Theorem. *Let $G/K = G/H \cdot \mathrm{U}(1)$ be a compact, simply connected, irreducible hermitian symmetric space with isotropy representation $\pi = [\pi' \hat{\otimes} \phi]_{\mathbb{R}}$. Then we have*

(a) *If G/K is not a projective space, then $S^2\pi'$ contains exactly two irreducible inequivalent sub-representations: otherwise $S^2\pi' = S^2\mu_n$ is irreducible.*

(b) *If H is simple,* i.e. *G/K is not a complex Grassmannian $\mathrm{SU}(p+q)/S(\mathrm{U}(p) \times \mathrm{U}(q))$, $p \geqq q \geqq 2$, then $\Lambda^2\pi'$ is irreducible; otherwise it contains exactly two irreducible summands.*

Proof. If G/K is $\mathrm{SU}(n+1)/S(\mathrm{U}(n)\mathrm{U}(1))$, then $\pi' = \mu_n$ and so $S^2\pi'$ and $\Lambda^2\pi'$ are irreducible. If not, (3.1.a) implies that $\mathrm{SU}(N)/\pi'(H)$ is positive dimensional and isotropy irreducible, where $N = \frac{1}{2} \dim G/K$. If H is not simple, then (2.10.a) implies that $(\mathfrak{h}, \pi') = (\mathfrak{su}(p) \oplus \mathfrak{su}(q), \mu_p \hat{\otimes} \mu_q)$ with $p \geqq q \geqq 2$, up to an automorphism of $\mathfrak{h}$. In this case both $S^2\pi'$ and $\Lambda^2\pi'$ contain exactly two irreducible sub-representations. In all other cases a proof identical to the one in (4.8) shows that $S^2\pi'$ contains precisely 2 irreducible summands and $\Lambda^2\pi'$ is irreducible. One only needs to observe the following two facts, which are needed in the proof of (4.8). First, π' must be absolutely irreducible, since G/K is an irreducible symmetric space. Second, (4.7) implies that $S^2\pi'$ must be reducible, since $\pi' = \mu_n$ was excluded above and $\pi' = \nu_{2n}$ is excluded since $[\nu_{2n} \hat{\otimes} \phi]_{\mathbb{R}}$ cannot be the isotropy representation of a symmetric space by (2.4). □

Remarks. (1) The fact in (5.2.a), that $S^2\pi'$ consists of exactly two irreducible representations was first observed in [Bo, Lemma 3.3], where a proof was given by verifying it in each case separately. To our knowledge, the above is the first direct proof of this fact. It is used in various vanishing theorems, see e.g. [CV, Bo, Mo]. This fact also has the consequence that the curvature operator on $S^2\pi'$ has at most 2 different eigenvalues. A direct proof of this consequence can also be found in [Gu].

(2) The statements in (5.1) and (5.2) should be seen as complementary to the statements that $S^2\pi'$ in (5.1) and $\pi' \otimes \pi'^* - \mathbf{1}$ in (5.2) consist of 2 irreducible

summands, as follows from (3.1). There is no corresponding fact for irreducible symmetric spaces which are not hermitian symmetric or quaternionic symmetric. In this case it follows from (3.1.c) that $\Lambda^2\pi$ consists of 2 irreducible summands. But $S^2\pi$ can consist of several irreducible summands.

(3) In our forthcoming paper on maximal subgroups we will see that (5.1) and (5.2.b) characterize hermitian and quaternionic symmetric spaces in the following sense. If π is a unitary representation such that $\Lambda^2\pi - \mathbf{1}$ is irreducible, then $\pi \hat{\otimes} \nu_2$ is the isotropy representation of a quaternionic symmetric space and if π is a unitary representation such that $\Lambda^2\pi$ is irreducible, then $[\pi \hat{\otimes} \phi]_{\mathbb{R}}$ is the isotropy representation of a hermitian symmetric space.

The following result was used extensively in [WZ2, p. 585–589]. The fact about the Killing forms in (a) was also proved in [DZ, Theorem 11].

(5.3) Theorem. *Let G/K be a compact irreducible symmetric space and B_G, B_K the negative of the Killing forms of G and K respectively. As in* (3.1) *we let χ be the isotropy representation of the corresponding strongly isotropy irreducible space.*

(a) *If K is simple and $n = \dim G/K$, then $B_K = cB_G|_{\mathfrak{k}}$ with $c = 1 - \dfrac{n}{2\dim K}$. Furthermore, $\mathscr{C}_\chi = \dfrac{2\dim K}{n(n-2)}\,\mathrm{Id}$, where the Casimir operator is computed with respect to the inner product $B_{\mathrm{SO}(n)}|_{\mathfrak{k}}$.*

(b) *If $G/K = G/H \cdot \mathrm{U}(1)$ is hermitian symmetric of dimension $2n$ with H simple, then $B_H = cB_G|_{\mathfrak{h}}$ with $c = 1 - \dfrac{n-1}{\dim H}$. Furthermore, $\mathscr{C}_\chi = \dfrac{\dim H}{n(n-1)}\,\mathrm{Id}$, where the Casimir operator is computed with respect to $B_{\mathrm{SU}(n)}|_{\mathfrak{h}}$.*

(c) *If $G/K = G/H \cdot \mathrm{Sp}(1)$ is quaternionic symmetric of dimension $4n$ with H simple, then $B_H = cB_G|_{\mathfrak{h}}$ with $c = 1 - \dfrac{n(2n+1)}{(n+2)\dim H}$. Furthermore, $\mathscr{C}_\chi = \dfrac{(n+2)\dim H}{n(n+1)(2n+1)}\,\mathrm{Id}$, where the Casimir operator is computed with respect to $B_{\mathrm{Sp}(n)}|_{\mathfrak{h}}$.*

Proof. (a) If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $X \in \mathfrak{k}$ we have

$$B_G(X,X) = -\operatorname{tr}_{\mathfrak{g}} \operatorname{ad} X \operatorname{ad} X = B_K(X,X) - \operatorname{tr}_{\mathfrak{p}} \operatorname{ad} X \operatorname{ad} X$$

or $(1-c)B_G(X,X) = -\operatorname{tr}_{\mathfrak{p}} \operatorname{ad} X \operatorname{ad} X = -\operatorname{tr} \pi(X)\pi(X)$. Hence, if H_i is an orthonormal basis of $\mathfrak{k}$ with respect to $B_G|_{\mathfrak{k}}$, then

$$(1-c)\dim K = (1-c)\sum_i B_G(H_i,H_i) = -\sum_i \operatorname{tr} \pi(H_i)\pi(H_i) = \operatorname{tr} \mathscr{C}_\pi$$

and the first claim follows from (3.2).

Since $B_{\mathrm{SO}(n)}(A,B) = -(n-2)\operatorname{tr} AB$ we have for $X \in \mathfrak{k}$

$$\begin{aligned} B_{\mathrm{SO}(n)}(X,X) &= -(n-2)\operatorname{tr} \pi(X)\pi(X) = (n-2)(B_G - B_K)(X,X) \\ &= \frac{n(n-2)}{2\dim H} B_G|_{\mathfrak{k}}(X,X) \end{aligned}$$

and hence our second claim follows from (3.3).

(b) Letting $X = i \in \mathfrak{u}(1)$, we get $B_G(X,X) = -\operatorname{tr}_{\mathfrak{p}} \operatorname{ad} X \operatorname{ad} X = -\operatorname{tr} \pi(X)\pi(X) = 2n$. Hence we can choose an orthonormal basis H_i of $\mathfrak{k}$ with $H_1 = X/\sqrt{2n}$ and

obtain together with (3.2)

$$(1-c)\dim H = (1-c)\sum_{i\geqq 2} B_G(H_i,H_i) = -\sum_{i\geqq 2}\operatorname{tr}\pi(H_i)\pi(H_i)$$
$$= \operatorname{tr}\mathscr{C}_\pi + \operatorname{tr}\pi(H_1)\pi(H_1) = n-1$$

which implies our first claim.

Since $B_{\mathrm{SU}(n)}(A,B) = -2n\operatorname{tr}AB$ we have for $X\in\mathfrak{h}$

$$B_{\mathrm{SU}(n)}(X,X) = -2n\operatorname{tr}\pi_{\lambda'}(X)\pi_{\lambda'}(X) = -n\operatorname{tr}\pi(X)\pi(X)$$

where we have used the fact that $\pi|_{\mathfrak{h}} = \pi_{\lambda'}\oplus\pi_{\lambda'}^*$. Hence $B_{\mathrm{SU}(n)}|_{\mathfrak{h}} = n(B_G - B_H) = \dfrac{n(n-1)}{\dim H}B_G|_{\mathfrak{h}}$ which implies our second claim.

(c) If X is equal to i,j, or $k\in\mathfrak{sp}(1)$, we get $B_G(X,X) = B_{\mathrm{Sp}(1)}(X,X) - \operatorname{tr}\pi(X)\pi(X) = 8+4n = 4(n+2)$. Hence we can choose an orthonormal basis H_i of $\mathfrak{k}$ with $2\sqrt{n+2}H_1 = i$, $2\sqrt{n+2}H_2 = j$, $2\sqrt{n+2}H_3 = k$. Then

$$(1-c)\dim H = (1-c)\sum_{i\geqq 4} B_G(H_i,H_i)$$
$$= \operatorname{tr}\mathscr{C}_\pi + \sum_{i=1}^{3}\operatorname{tr}\pi(H_i)\pi(H_i) = 2n - \frac{3n}{n+2}$$

from which our first claim follows.

$B_{\mathrm{Sp}(n)}(A,B) = -2(n+1)\operatorname{tr}AB$ where we regard A,B as complex matrices via the embedding $\mathfrak{sp}(n)\subset\mathfrak{su}(2n)$. Hence, for $X\in\mathfrak{h}$ we have

$$B_{\mathrm{Sp}(n)}(X,X) = -2(n+1)\operatorname{tr}\pi_{\lambda'}(X)\pi_{\lambda'}(X) = -(n+1)\operatorname{tr}\pi(X)\pi(X)$$

where we have used the fact that $\pi|_{\mathfrak{h}} = \pi_{\lambda'}\oplus\pi_{\lambda'}$. Hence $B_{\mathrm{Sp}(n)}|_{\mathfrak{h}} = (n+1)(B_G - B_H) = \dfrac{n(n+1)(2n+1)}{(n+2)\dim H}B_G|_{\mathfrak{h}}$ which implies our second claim. □

We finally mention the following consequences of (2.8), (2.10), and (2.11), which were proved without using the classification of symmetric spaces.

(5.4) Theorem. *Let G/K be a compact irreducible symmetric space with isotropy representation π.*

(a) *K is a simple Lie group, unless G/K is hermitian symmetric, quaternionic symmetric, or a real or quaternionic Grassmannian.*

(b) *If G/K is a hermitian symmetric space $G/H\cdot\mathrm{U}(1)$ or a quaternionic symmetric space $G/H\cdot\mathrm{Sp}(1)$, then H is simple unless G/K is a complex or quaternionic Grassmannian.*

(c) *Let $\mathfrak{h}\subset\mathfrak{k}$ be a simple non-abelian ideal such that $\pi|_{\mathfrak{h}}$ is not a basic representation. Then $(\mathfrak{k},\pi)$ must be one of the following.*

(i) $(\mathfrak{so}(n)\oplus\mathfrak{so}(3), \varrho_n\hat{\otimes}\varrho_3)$ *corresponding to the symmetric space* $\mathrm{SO}(n+3)/\mathrm{SO}(n)\mathrm{SO}(3)$.

(ii) $\mathfrak{su}(n)$, ad) *corresponding to the symmetric space* $[\mathrm{SU}(n)\times\mathrm{SU}(n)]/\Delta\mathrm{SU}(n)$.

(iii) $(\mathfrak{sp}(n),\mathrm{ad})$ *corresponding to the symmetric space* $[\mathrm{Sp}(n)\times\mathrm{Sp}(n)]/\Delta\mathrm{Sp}(n)$.

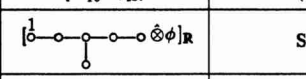
(iv) $(\mathfrak{so}(n), S^2\varrho_n - \mathbf{1})$ *corresponding to the symmetric space* $\mathrm{SU}(n)/\mathrm{SO}(n)$.

(v) $(\mathrm{u}(n), S^2\mu_n)$ *corresponding to the symmetric space* $\mathrm{Sp}(n)/\mathrm{U}(n)$.

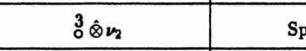
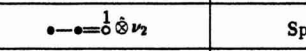
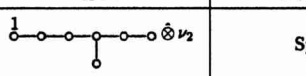
(vi) $(\mathfrak{sp}(1)\oplus\mathfrak{sp}(1), S^3\nu_2\hat{\otimes}\nu_2)$ *corresponding to the symmetric space* $\mathrm{G}_2/\mathrm{SO}(4)$.

Correspondence between irreducible symmetric spaces and strongly isotropy irreducible spaces

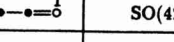
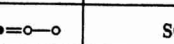
1. Hermitian symmetric spaces

No.	Irreducible symmetric space G/K			Isotropy representation	Strongly isotropy irreducible space
	G	K	Conditions		
1	$SU(p+q)/\mathbf{Z}_{p+q}$	$S(U(p)U(q))/\Delta\mathbf{Z}_{p+q}$	$p \geq q \geq 2$ $pq \neq 4$	$[\mu_p \hat{\otimes} \mu_q^*]_{\mathbf{R}}$	$SU(pq)/([SU(p)SU(q)]/\Delta\mathbf{Z}_{(p,q)})$
2	$SO(2n)/\mathbf{Z}_2$	$U(n)/\mathbf{Z}_2$	$n \geq 5$	$[\Lambda^2 \mu_n]_{\mathbf{R}}$	$SU(\frac{n(n-1)}{2})/[SU(n)/\mathbf{Z}_{(2,n)}]$
3	$Sp(n)/\mathbf{Z}_2$	$U(n)/\mathbf{Z}_2$	$n \geq 3$	$[S^2 \mu_n]_{\mathbf{R}}$	$SU(\frac{n(n+1)}{2})/[SU(n)/\mathbf{Z}_{(2,n)}]$
4	$E_6/\mathbf{Z}_3$	$[Spin(10)U(1)]/\Delta\mathbf{Z}_4$		$[\Delta_{10}^+ \hat{\otimes} \phi]_{\mathbf{R}}$	$SU(16)/Spin(10)$
5	$E_7/\mathbf{Z}_2$	$[E_6 U(1)]/\Delta\mathbf{Z}_3$			$SU(27)/E_6$
6	$SO(n+2)/\mathbf{Z}_{(2,n)}$	$[SO(n)SO(2)]/\Delta\mathbf{Z}_{(2,n)}$	$n \geq 3$	$[\rho_n \hat{\otimes} \phi]_{\mathbf{R}}$	$SU(n)/SO(n)$ (symmetric space)
7	$SU(n+1)/\mathbf{Z}_{n+1}$	$S(U(n)U(1))/\Delta\mathbf{Z}_{n+1}$	$n \geq 1$	$[\mu_n \hat{\otimes} \phi^{-1}]_{\mathbf{R}}$	*

2. Quaternionic symmetric spaces

No.	Irreducible symmetric space G/K			Isotropy representation	Strongly isotropy irreducible space
	G	K	Cond.		
8	$SO(n+4)/\mathbf{Z}_{(2,n)}$	$[SO(n)SO(4)]/\Delta\mathbf{Z}_{(2,n)}$	$n \geq 3$	$\rho_n \hat{\otimes} \rho_4 = [\rho_n \hat{\otimes} \nu_2] \hat{\otimes} \nu_2$	$Sp(n)/([SO(n)Sp(1)]/\Delta\mathbf{Z}_{(2,n)})$
9	G_2	$SO(4)$			$Sp(2)/Sp(1)$
10	F_4	$[Sp(3)Sp(1)]/\Delta\mathbf{Z}_2$			$Sp(7)/Sp(3)$
11	$E_6/\mathbf{Z}_3$	$[(SU(6)/\mathbf{Z}_3)Sp(1)]/\Delta\mathbf{Z}_2$		$\Lambda^3 \mu_6 \hat{\otimes} \nu_2$	$Sp(10)/[SU(6)/\mathbf{Z}_3]$
12	$E_7/\mathbf{Z}_2$	$[SO'(12)Sp(1)]/\Delta\mathbf{Z}_2$		$\Delta_{12}^+ \hat{\otimes} \nu_2$	$Sp(16)/SO'(12)$
13	E_8	$[E_7 Sp(1)]/\Delta\mathbf{Z}_2$			$Sp(28)/E_7$
14	$SU(n+2)/\mathbf{Z}_{n+2}$	$S(U(n)U(2))/\Delta\mathbf{Z}_{n+2}$	$n \geq 1$	$[\mu_n \hat{\otimes} \mu_2^*]_{\mathbf{R}} = [\mu_n]_{\mathbf{R}} \hat{\otimes} \nu_2$	$Sp(n)/U(n)$ (symmetric space)
15	$Sp(n+1)/\mathbf{Z}_2$	$[Sp(n)Sp(1)]/\Delta\mathbf{Z}_2$	$n \geq 1$	$\nu_{2n} \hat{\otimes} \nu_2$	*

3. Other symmetric spaces

No.	Irreducible symmetric space G/K			Isotropy representation	Strongly isotropy irreducible space
	G	K	Conditions		
16	$\mathrm{Sp}(n+1)/\mathbf{Z}_2$	$[\mathrm{Sp}(n)\mathrm{Sp}(1)]/\Delta\mathbf{Z}_2$	$n \geq 2$	$\nu_{2n}\hat{\otimes}\nu_2$	$\mathrm{SO}(4n)/([\mathrm{Sp}(n)\mathrm{Sp}(1)]/\Delta\mathbf{Z}_2)$ (symmetric space if $n = 2$)
17	$\mathrm{SU}(2n)/\mathbf{Z}_2$	$\mathrm{Sp}(n)/\mathbf{Z}_2$	$n \geq 3$	$\Lambda^2\nu_{2n} - 1$	$\mathrm{SO}(\frac{2n(2n-1)-2}{2})/[\mathrm{Sp}(n)/\mathbf{Z}_2]$
18	$\mathrm{SU}(n)/\mathbf{Z}_{(2,n)}$	$\mathrm{SO}(n)/\mathbf{Z}_{(2,n)}$	$n \geq 3, n \neq 4$	$S^2\rho_n - 1$	$\mathrm{SO}(\frac{n(n+1)-2}{2})/[\mathrm{SO}(n)/\mathbf{Z}_{(2,n)}]$
19	$K \times K$	ΔK	K simple, centerless $K \neq \mathrm{SO}(3)$	Ad_K	$\mathrm{SO}(\dim K)/K$
20	$\mathbf{F}_4$	$\mathrm{Spin}(9)$		Δ_9	$\mathrm{SO}(16)/\mathrm{Spin}(9)$
21	$\mathbf{E}_6$	$\mathrm{Sp}(4)/\mathbf{Z}_2$			$\mathrm{SO}(42)/[\mathrm{Sp}(4)/\mathbf{Z}_2]$
22	$\mathbf{E}_6$	$\mathbf{F}_4$			$\mathrm{SO}(26)/\mathbf{F}_4$
23	$\mathbf{E}_7/\mathbf{Z}_2$	$\mathrm{SU}(8)/\mathbf{Z}_4$		$\Lambda^4\mu_8$	$\mathrm{SO}(70)/[\mathrm{SU}(8)/\mathbf{Z}_4]$
24	$\mathbf{E}_8$	$\mathrm{SO}'(16)$		Δ_{16}^+	$\mathrm{SO}(128)/\mathrm{SO}'(16)$
25	$\mathrm{SO}(n+1)$	$\mathrm{SO}(n)$	$n \geq 2$	ρ_n	*
26	$\mathrm{SO}(p+q)/\mathbf{Z}_k$ $k = 2$ if p and q even $k = 1$ otherwise	$[\mathrm{SO}(p)\mathrm{SO}(q)]/\Delta\mathbf{Z}_k$	$p \geq q \geq 3$ $p \neq 4, q \neq 4$	$\rho_p\hat{\otimes}\rho_q$	not strongly isotropy irreducible
27	$\mathrm{Sp}(p+q)/\mathbf{Z}_2$	$[\mathrm{Sp}(p)\mathrm{Sp}(q)]/\Delta\mathbf{Z}_2$	$p \geq q \geq 2$	$\nu_{2p}\hat{\otimes}\nu_{2q}$	not strongly isotropy irreducible
28	none	none			$\mathrm{SO}(7)/\mathrm{G}_2 = \mathbf{RP}^7$

In the above tables the symmetric space G/K is compact, simply connected, irreducible, and effective with $G = I_0/(G/K)$. The values of n are chosen so that no repetitions occur for the symmetric spaces within each individual table, but there is some unavoidable overlap between the three tables. The fourth column describes the isotropy representations of the symmetric space.

The strongly isotropy irreducible spaces are not necessarily effective (although they are always almost effective), nor are they necessarily simply connected. The inclusion of the subgroup as well as the isotropy representation of the strongly isotropy irreducible space can be read off from the isotropy representation of the corresponding symmetric space by using (3.1) and Remark 2 after the proof of (3.1.).

Finally, in the tables $\mathrm{SO}'(4n)$ denotes the quotient $\mathrm{Spin}(4n)/\mathbb{Z}_2$ which is not isomorphic to $\mathrm{SO}(4n)$.

References

[Be] Besse, A.: Einstein manifolds. (Ergeb. Math. Grenzgeb., 3. Folge, Bd. 10) Berlin Heidelberg New York: Springer 1987

[Bo] Borel, A.: On the curvature tensor of the Hermitian symmetric manifolds. Ann. Math. **71**, 508–521 (1960)

[BK] Bourguignon, J.P., Karcher, H.: Curvature operators: Pinching estimates and geometric examples. Ann. Sci. Ec. Norm. Supér. **11**, 71–92 (1978)

[CV] Calabi, E., Vesentini, E.: On compact locally symmetric Kähler manifolds. Ann. Math. **71**, 472–507 (1960)

[Ca] Cartan, E.: Sur une classe remarquable d'espaces de Riemann. Bull. Soc. Math. Fr. **54**, 214–264 (1926); **55**, 114–134 (1927)

[Dy] Dynkin, E.B.: Maximal subgroups of the classical groups. Transl., II., Ser., Am. Math. Soc. **6**, 111–244 (1957)

[DZ] D'Atri, J.E., Ziller, W.: Naturally reductive metrics and Einstein metrics on compact Lie groups. Mem. Am. Math. Soc. **18**, no. 215 (1979)

[Gu] Guan, Z.: Curvature on the Hermitian symmetric spaces. Acta Math. Sin. **4**, 270–283 (1988)

[He] Helgason, S.: Differential geometry, Lie groups, and symmetric spaces. (Pure Appl. Math., vol. 80) New York, London: Academic Press 1978

[Ko1] Kostant, B.: A characterization of the classical groups. Duke Math. J. **25**, 107–123 (1958)

[Ko2] Kostant, B.: On invariant skew tensors. Proc. Natl. Acad. Sci. **42**, 148–151 (1956)

[Kr] Krämer, M.: Eine Klassifikation bestimmter Untergruppen kompakter zusammenhängender Liegruppen. Commun. Algebra **3**, 691–737 (1975)

[M1] Manturov, O.V.: Homogeneous asymmetric Riemannian spaces with an irreducible group of motions. Dokl. Akad. Nauk SSSR **141**, 792–795 (1961)

[M2] Manturov, O.V.: Riemannian spaces with orthogonal and symplectic groups of motions and an irreducible group of rotations. Dokl. Akad. Nauk. SSSR **141**, 1034–1037 (1961)

[M3] Manturov, O.V.: Homogeneous Riemannian manifolds with irreducible isotropy group. Tr. Semin. Vector Tensor Anal. **13**, 68–145 (1966)

[Mo] Mok, N.: Uniqueness theorems of Kähler metrics of semipositive bisectional curvature on compact Hermitian symmetric spaces. Math. Ann. **276**, 177–204 (1987)

[Si] Simons, J.: On the transitivity of holonomy systems. Ann. Math. **76**, 213–234 (1962)

[WZ1] Wang, M., Ziller, W.: On the isotropy representation of a symmetric space. Rend. Semin. Mat., Torino, Fasc. Speciale 253–261 (1985)

[WZ2] Wang, M., Ziller, W.: On normal homogeneous Einstein manifolds. Ann. Sci. Ec. Norm. Supér., IV. Ser. **18**, 563–633 (1985)

[WZ3] Wang, M., Ziller, W.: On isotropy irreducible Riemannian manifolds. Acta Math. **166**, 223–261 (1991)

[Wo1] Wolf, J.A.: The geometry and structure of isotropy irreducible homogeneous spaces. Acta Math. **120**, 59–148 (1968); correction, Acta Math. **152**, 141–142 (1984)

[Wo2] Wolf, J.A.: Spaces of constant curvature, 4th ed. Wilmington: Publish or Perish 1977