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Theta functions on the bounded symmetric domain of type $I_{2,2}$ and the period map of a 4-parameter family of K3 surfaces

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Introduction

This paper is a continuation of [MSY2], in which the period map of the 4-dimensional family of K3-surfaces, the double covers of $\mathbb{P}^2$ branching along six lines, is studied. In this paper the inverse of the period map is given by theta functions on the symmetric domain of type $I_{2,2}$.

Classically the period map of the family of elliptic curves, which are the double covers of $\mathbb{P}^1$ branching at four points, sends the configulation space $\mathbf{X}(2, 4)$ of four points on $\mathbb{P}^1$ to the quotient space of the upper half space $\mathbb{H}$ by the principal congruence subgroup $\Gamma(2)$ of level 2. Any element of $\mathbf{X}(2, 4)$ is represented by a 2×4 $\mathbb{C}$-matrix $x = (x_{jk})$, whose columns are homogeneous coordinates of the points; the space $\mathbf{X}(2, 4)$ can be realized in $\mathbb{P}^2$ by the following map:

$$\iota: x \mapsto [D_{12}(x)D_{34}(x),\ D_{13}(x)D_{24}(x),\ D_{14}(x)D_{23}(x)],$$

where $D_{jk}(x) = \det\begin{pmatrix} x_{1j} & x_{1k} \\ x_{2j} & x_{2k} \end{pmatrix}$.

Since we have the Plücker relation

$$D_{12}(x)D_{34}(x) - D_{13}(x)D_{24}(x) + D_{14}(x)D_{23}(x) = 0$$

of the Grassmannian manifold $G(4, 2)$, consisting of $(2-1)$-dimensional linear subspaces in $\mathbb{P}^{4-1}$, the image of $\mathbf{X}(2, 4)$ is a linear subspace Z of $\mathbb{P}^2$.

On the other hand, for Jacobi's theta constants

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z) = \sum_{n\in\mathbb{Z}} \exp\{\pi i(n+a)^2\tau + 2\pi inb\},$$

where $\tau \in \mathbb{H}$ and $a, b \in \left\{0, \frac{1}{2}\right\}$, $ab = 0$, the functions $\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\tau)^4$ are nowhere-vanishing modular forms of weight 2 relative to $\Gamma(2)$ and they satisfy Jacobi's

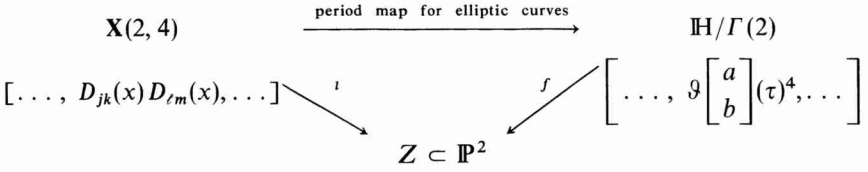
identity:

$$\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}(\tau)^4 - \vartheta\begin{bmatrix}0\\0\end{bmatrix}(\tau)^4 + \vartheta\begin{bmatrix}0\\1/2\end{bmatrix}(\tau)^4 = 0\,.$$

Thus the holomorphic map f

$$f:\tau \mapsto \left[\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}(\tau)^4,\ \vartheta\begin{bmatrix}0\\0\end{bmatrix}(\tau)^4,\ \vartheta\begin{bmatrix}0\\1/2\end{bmatrix}(\tau)^4\right]$$

sends the compactification $\overline{\mathbb{H}/\Gamma(2)}$ of $\mathbb{H}/\Gamma(2)$ to $Z \subset \mathbb{P}^2$. We have the triangle commutative diagram:



namely the map $\iota^{-1} \circ f$ gives the inverse of the period map for elliptic curves. The explicit expression of f determines the structure of the graded ring of modular forms relative to $\Gamma(2)$.

In our case, we can regard the period map *Prd* of the 4-dimensional family of K3-surfaces as a map from the configulation space $\mathbf{X}(3, 6)$ of six points in $\mathbb{P}^2$ to the bounded symmetric domain of type $I_{2,2}$

$$\mathbf{H}_2 := \{W \in M(2, 2; \mathbb{C}) \mid (W - W^*)/2i > 0\}\,,$$

on which the transposition T and the group

$$U(2,2) := \left\{g \in \mathrm{GL}(4, \mathbb{C}) \mid g^* J g = J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}\right\},$$

act as follows:

$$T \cdot W = {}^t W\,,$$

$$g \cdot W = (AW + B)(CW + D)^{-1}, \quad W \in \mathbf{H}_2, \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(2, 2)\,.$$

The monodromy group of the period map is a subgroup $\Gamma_M(1 + i)$ of index 2 of the semi-direct product

$$\Gamma_T(1 + i) := \Gamma(1 + i) \rtimes \langle T \rangle\,,$$

where $\Gamma(1 + i)$ is the principal congruence modular group of level $(1 + i)$

$$\Gamma(1 + i) := \{g \in \mathrm{GL}(4, \mathbb{Z}[i]) \mid g^* J g = J,\ g \equiv I_4 \bmod (1 + i)\}\,,$$

and $\langle T \rangle$ is the group generated by T.

The space $\mathbf{X}(3, 6)$ is not so simple, however it is shown in [MSY2] that there is an involution ♥ on $\mathbf{X}(3, 6)$, which is equivariant under the period map to the action

T on H_2, i.e., $Prd \circ \heartsuit = T \circ Prd$, and that the space $\mathbf{X}(3, 6)/\heartsuit$ is realized as a 4-dimensional linear subspace as follows. Any element of $\mathbf{X}(3, 6)/\heartsuit$ is represented by a 3×6 $\mathbb{C}$-matrix $x = (x_{jk})$, whose columns are homogeneous coordinates of the points; the space $\mathbf{X}(3, 6)/\heartsuit$ can be embedded in $\mathbb{P}^{10-1}$ by the following map:

$$\iota : x \mapsto [\ldots, D_{hjk}(x) D_{\ell mn}(x), \ldots], \{h, \ldots, n\} = \{1, \ldots, 6\},$$

where

$$D_{hjk}(x) = \det \begin{pmatrix} x_{1h} & x_{1j} & x_{1k} \\ x_{2h} & x_{2j} & x_{2k} \\ x_{3h} & x_{3j} & x_{3k} \end{pmatrix}.$$

The Plücker relations of the Grassmannian manifold $G(6, 3)$ tell that the image of $\mathbf{X}(3, 6)/\heartsuit$ is a 4-dimensional linear subspace Y of $\mathbb{P}^{10-1}$. The period map induces a map from $\mathbf{X}(3, 6)/\heartsuit$ to $\mathbf{H}_2/\Gamma_T(1 + i)$. We define theta functions on $\mathbf{H}_2$ with characteristics $a, b \in \frac{1}{2}\mathbb{Z}[i]^2$ as follows:

$$\Theta\begin{bmatrix} a \\ b \end{bmatrix}(W) := \sum_{n \in \mathbb{Z}[i]^2} \mathbf{e}[(n + a)^* W(n + a) + 2\mathrm{Re}\{b^* n\}] .$$

There are ten linearly independent theta functions with characteristics $a, b \in \left\{0, \dfrac{1 + i}{2}\right\}^2$, $a^* b \in \mathbb{Z}$; their squares are modular forms of weight 2 relative to $\Gamma_T(1 + i)$ and they satisfy linear relations. By considering the set of zeros of each thera function, we can code the ten theta functions by using numbers $1, \ldots, 6$ as

$$\Theta\begin{bmatrix} a \\ b \end{bmatrix}(W) = \Theta\begin{pmatrix} hjk \\ \ell mn \end{pmatrix}(W), \{h, \ldots, n\} = \{1, \ldots, 6\}, \left(\binom{6}{3}\Big/2 = 10\right).$$

Then the linear relations among the squares of the ten theta functions, after replacing $\Theta\begin{pmatrix} hjk \\ \ell mn \end{pmatrix}(W)^2$ by $D_{hjk}(x) D_{\ell mn}(x)$, coincide with the Plücker relations of $G(6, 3)$. Define the map $F : \mathbf{H}_2/\Gamma_T(1 + i) \to \mathbb{P}^{10-1}$ by

$$W \mapsto \left[\ldots, \Theta\begin{pmatrix} hjk \\ \ell mn \end{pmatrix}(W)^2, \ldots\right], \{h, \ldots, n\} = \{1, \ldots, 6\} .$$

We have the following triangle diagram:

$$\begin{array}{ccc} \mathbf{X}(3, 6)/\heartsuit & \xrightarrow{\text{period map for K3 surfaces}} & \mathbf{H}_2/\Gamma_T(1 + i) , \\ [\ldots, D_{hjk}(x) D_{\ell mn}(x), \ldots] \searrow^{\iota} & & \swarrow_{F} \left[\ldots, \Theta\begin{bmatrix} a \\ b \end{bmatrix}(W)^2, \ldots\right] \\ & Y \subset \mathbb{P}^{10-1} & \end{array}$$

namely $\iota^{-1} \circ F$ gives an explicit expression of the inverse of the period map $\mathbf{X}(3, 6)/\heartsuit \to \mathbf{H}_2/\Gamma_T(1 + i)$. This expression gives the structure of the graded ring of modular forms relative to $\Gamma_T(1 + i)$.

To give the inverse of the period map $\mathbf{X}(3, 6) \to \mathbf{H}_2/\Gamma_M(1+i)$, we make a modular form Θ of weight 4 relative to

$$\Gamma_T := \{g \in \mathrm{PGL}(4, \mathbb{Z}[i]) \mid g^* J g = J\} \rtimes \langle T \rangle ,$$

which has simple zeros on the branching locus of the covering $\mathbf{H}_2/\Gamma_M(1+i) \to \mathbf{H}_2/\Gamma_T(1+i)$; such a modular form is unique up to constant.

1 Discrete subgroups acting on some symmetric domains

1.1 The symmetric domain $\mathbf{H}_2$ *of type* $I_{2,2}$

For a ring R let $M(m, n; R)$ be the set of $m \times n$ R-matrices; put

$$H(4, 2) := \{X \in M(4, 2; \mathbb{C}) \mid X^*(iJ)X > 0\} ,$$

$$S(4, 2) := \{X \in H(4, 2) \mid {}^t X J X = 0\} ,$$

where $X^* = {}^t\bar{X}$, $i = \sqrt{-1}$ and $J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$. Note that for any element $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \in H(4, 2)$, the regularity of X_1 and X_2 follows from by the positivity of $X^*(iJ)\,X = i(X_1^* X_2 - X_2^* X_1)$. The groups

$$U(2, 2) = \{g \in \mathrm{GL}(4, \mathbb{C}) \mid g^* J g = J\}$$

and

$$\mathrm{Sp}(4, \mathbb{R}) = \{g \in \mathrm{GL}(4, \mathbb{R}) \mid {}^t g J g = J\}$$

act naturally on $H(4, 2)$ and $S(4, 2)$, respectively, from the left, since we have

$$(gX)^*(iJ)(gX) = X^*\{g^*(iJ)g\}\,X = X^*(iJ)\,X > 0,\ X \in H(4, 2) ,$$

$${}^t(gX)J(gX) = {}^t X({}^t g J g)X = {}^t X J X = 0,\ X \in S(4, 2) .$$

The involution

$$T : \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \mapsto \begin{pmatrix} {}^t X_2^{-1} \\ {}^t X_1^{-1} \end{pmatrix}$$

also acts on $H(4, 2)$, since we have

$$T(X)^*\,(iJ)\,T(X) = (\bar{X}_2^{-1}\,{}^t X_1^{-1})^*\,{}^t\{X^*(iJ)X\}\,(\bar{X}_2^{-1}\,{}^t X_1^{-1}) > 0 .$$

The group $\mathrm{GL}(2, \mathbb{C})$ acts on $H(4, 2)$ and $S(4, 2)$ from the right; put

$$\mathscr{D} := H(4, 2)/\mathrm{GL}(2, \mathbb{C}) ,$$

$$\mathscr{F} := S(4, 2)/\mathrm{GL}(2, \mathbb{C}) .$$

Since we have

$$g(Xh) = (gX)h,\ T(Xh) = T(X)\,{}^t h^{-1} ,$$

for $g \in U(2, 2)$, $X \in H(4, 2)$, $h \in \mathrm{GL}(2, \mathbb{C})$, the group $U(2, 2)$ and the involution T naturally act on $\mathscr{D}$. Notice that $\mathscr{F}$ is the set of fixed points of T and that $\mathrm{Sp}(4, \mathbb{R})$

acts on $\mathscr{F}$. By using an action of $\mathrm{GL}(2, \mathbb{C})$, we can represent any element of $\mathscr{D}$ by an unique element of the form $\begin{pmatrix} W \\ I_2 \end{pmatrix}$, where W belongs to the set

$$\mathbf{H}_2 := \left\{ W \in M(2, 2; \mathbb{C}) \,\middle|\, \frac{W - W^*}{2i} > 0 \right\}.$$

Thus we get an isomorphism

$$\iota_1 : \mathscr{D} \to \mathbf{H}_2;$$

the domain $\mathbf{H}_2$ is called the symmetric domain of type $I_{2,2}$. By restricting ι_1 on $\mathscr{F}$, we also get an isomorphism between $\mathscr{F}$ and the Siegel upper half space

$$\mathbf{S}_2 = \{ W \in \mathbf{H}_2 \mid {}^t W = W \}$$

of degree 2. The isomorphism ι_1 induces actions of $U(2, 2)$ and T on $\mathbf{H}_2$ and those of $\mathrm{Sp}(4, \mathbb{R})$ on $\mathbf{S}_2$:

$$g \cdot W = (AW + B)(CW + D)^{-1},\ T \cdot W = {}^t W,\ W \in \mathbf{H}_2,$$

where $W \in \mathbf{H}_2$ or $\mathbf{S}_2$, $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(2, 2)$ or $\mathrm{Sp}(4, \mathbb{R})$, respectively.

Lemma 1.1.1. $(TgT) \cdot W = \bar{g} \cdot W$.

Proof. The condition $g^* J g = J$ says

$$\begin{aligned} -{}^t C \bar{A} + {}^t A \bar{C} = 0, \quad & -{}^t C \bar{B} + {}^t A \bar{D} = I_2, \\ -{}^t D \bar{A} + {}^t B \bar{C} = -I_2, \quad & -{}^t D \bar{B} + {}^t B \bar{D} = 0. \end{aligned} \tag{1.1.1}$$

We have

$$\begin{aligned} (TgT) \cdot W &= {}^t\{(A\,{}^t W + B)(C\,{}^t W + D)^{-1}\} \\ &= (W\,{}^t C + {}^t D)^{-1} (W\,{}^t A + {}^t B) \end{aligned}$$

and

$$\bar{g} \cdot W = (\bar{A} W + \bar{B})(\bar{C} W + \bar{D})^{-1},$$

both of which lead to

$$\begin{aligned} &\{(TgT) \cdot W\}^{-1} \bar{g} \cdot W \\ &= (W\,{}^t A + {}^t B)^{-1} (W\,{}^t C + {}^t D)(\bar{A} W + \bar{B})(\bar{C} W + \bar{D})^{-1}. \end{aligned}$$

On the other hand by (1.1.1) we have

$$\begin{aligned} &(W\,{}^t C + {}^t D)(\bar{A} W + \bar{B}) \\ &= (W\,{}^t A + {}^t B)(\bar{C} W + \bar{D}); \end{aligned}$$

thus we get $\{(T \cdot g \cdot T) \cdot W\}^{-1} \bar{g} \cdot W = I_2$. □

It is known that the group $\mathrm{Aut}(\mathbf{H}_2)$ of automorphisms of $\mathbf{H}_2$ is generated by actions of $U(2,2)$ and T; by Lemma 1.1 we have

$$\mathrm{Aut}(\mathscr{D}) \simeq \mathrm{Aut}(\mathbf{H}_2) \simeq \{U(2,2)/U(1)\} \rtimes \langle T \rangle ,$$

where $\langle T \rangle = \{\mathrm{id}, T\}$ and $U(1) = \{\rho \in \mathbb{C} \,|\, |\rho| = 1\}$ is the center of $U(2,2)$.

1.2 Some discrete subgroups of $\mathrm{Aut}(\mathbf{H}_2)$

Let us define the modular group Γ, a subgroup $S\Gamma$ of Γ, the congruence subgroup $\Gamma(1+i)$ of level $(1+i)$ and a subgroup $S\Gamma(1+i)$ of $\Gamma(1+i)$ as follows:

$$\Gamma := \{g \in \mathrm{PGL}(4, \mathbb{Z}[i]) \,|\, g^* J g = J\} ,$$

$$S\Gamma := \{g \in \Gamma \,|\, \det(g) = 1\} ,$$

$$\Gamma(1+i) := \{g \in \Gamma \,|\, g \equiv I_4 \bmod (1+i)\} ,$$

$$S\Gamma(1+i) := S\Gamma \cap \Gamma(1+i) .$$

We set

$$\Gamma_T := \Gamma \rtimes \langle T \rangle ,$$

$$S\Gamma_T := S\Gamma \rtimes \langle T \rangle ,$$

$$\Gamma_M := \{\mathbf{g} = g T^{\varepsilon} \in \Gamma_T \,|\, \det(g)\mathbf{e}[\varepsilon] = 1\} ,$$

$$\Gamma_T(1+i) := \Gamma(1+i) \rtimes \langle T \rangle ,$$

$$S\Gamma_T(1+i) := S\Gamma(1+i) \rtimes \langle T \rangle ,$$

$$\Gamma_M(1+i) := \Gamma_M \cap \Gamma_T(1+i) ,$$

where $\varepsilon \in \{0, 1\}$ and $\mathbf{e}[x] = \exp(\pi i x)$. Inclusion relations among the above groups are illustrated by the following body consisting of a triangular prism and two tetrahedra attached on the two triangular faces of the prism; two vertices are situated inside the prism. Each vertical segment means that the group below is a normal subgroup of the one above with S_6 as the quotient group, and each diagonal segment means that the group below is a subgroup of index 2 of the one above, while the horizontal segments, the edges of the two triangular faces of the prism, indicate only the levels.

Indeed, we have the following exact sequence:

$$1 \to \Gamma(1+i) \to \Gamma \xrightarrow{p} \mathrm{Sp}(4, \mathbb{Z}/2\mathbb{Z}) \to 1 ,$$

where $\mathrm{Sp}(4, \mathbb{Z}/2\mathbb{Z}) = \{g \in \mathrm{GL}(4, \mathbb{Z}/2\mathbb{Z}) \,|\, {}^t g J g = J\} \simeq S_6$ and p is the natural projection induced from the homomorphism

$$\mathbb{Z}[i] \to \mathbb{Z}[i]/(1+i)\mathbb{Z}[i] \simeq \mathbb{Z}/2\mathbb{Z} .$$

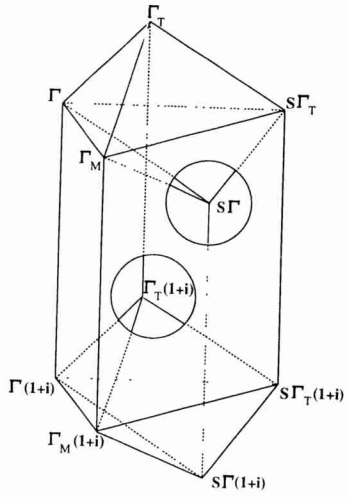


Fig. 1

Notice that

$$\Gamma_T/\Gamma' \simeq S_6 \times (\mathbb{Z}/2\mathbb{Z}), \quad \text{for } \Gamma' = \Gamma(1+i),\ \Gamma_M(1+i),\ S\Gamma_T(1+i)\,;$$

$$\Gamma'/S\Gamma(1+i) \simeq S_6 \times (\mathbb{Z}/2\mathbb{Z}), \quad \text{for } \Gamma' = \Gamma,\ \Gamma_M,\ S\Gamma_T\,.$$

It is known that Γ is generated by matrices of the following three forms:

$$\begin{pmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{pmatrix}, \begin{pmatrix} I_2 & B \\ 0 & I_2 \end{pmatrix} \text{ and } J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix},$$

where $A \in \mathrm{GL}(2, \mathbb{Z}[i])$, $B = B^* \in M(2, 2; \mathbb{Z}[i])$, and that $\Gamma(1+i)$ is generated by matrices of the following three forms:

$$\begin{pmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{pmatrix}, \begin{pmatrix} I_2 & B \\ 0 & I_2 \end{pmatrix} \text{ and } J^{-1}\begin{pmatrix} I_2 & B \\ 0 & I_2 \end{pmatrix} J\,,$$

where $A \in \mathrm{GL}(2, \mathbb{Z}[i])$, $A \equiv I_2 \bmod 1+i$ and $B = B^* \in M(2, 2; (1+i)\mathbb{Z}[i])$.

1.3 The symmetric domain D_{IV} of type IV

We can regard $\mathscr{D}$ and $\mathscr{F}$ as subsets of the Grassmannian manifold

$$G(4,2) := \{X \in M(4,2;\mathbb{C}) \mid {}^3d_{jk}(X) \neq 0\}/\mathrm{GL}(2,\mathbb{C})\,,$$

where $d_{jk}(X) = \det\begin{pmatrix} x_{j1} & x_{j2} \\ x_{k1} & x_{k2} \end{pmatrix}$. By using the Plücker embedding of $G(4,2)$, we will give another realization of $\mathscr{D}$ and $\mathscr{F}$ in a projective space. We define the map $d: M(4,2) \to \mathbb{C}^6$ by

$$X \mapsto {}^t(z_1, \ldots, z_6) = {}^t(d_{34}(X), d_{12}(X), d_{14}(x), d_{23}(X), d_{13}(X), d_{24}(X))\,.$$

There is a quadratic relation among the $d_{jk}(X)$:

$$d_{12}(X)\,d_{34}(X) - d_{13}(X)\,d_{24}(X) + d_{14}(X)d_{23}(X) = 0\,, \tag{1.3.1}$$

which is called the Plücker relation. The map d transforms the condition $X^*(iJ)X > 0$ for $X \in H(2,4)$ into

$$\begin{aligned} &d_{12}(X)\overline{d_{34}(X)} + \overline{d_{12}(X)}d_{34}(X) + d_{13}(X)\overline{d_{24}(X)} + \overline{d_{13}(X)}d_{24}(X) \\ &\qquad + d_{14}(X)\overline{d_{23}(X)} + \overline{d_{14}(X)}d_{23}(X) > 0\,, \\ &\{d_{14}(X)\overline{d_{34}(X)} - \overline{d_{14}(X)}d_{34}(X)\}/2i > 0\,. \end{aligned} \tag{1.3.2}$$

By the property $d_{jk}(Xh) = d_{jk}(X)\det(h)$ for $h \in \mathrm{GL}(2,\mathbb{C})$ and $X \in M(4,2)$, the map d induces $\hat{d}: G(4,2) \to \mathbb{P}^5$, which is called the Plücker embedding of $G(4,2)$; by (1.3.1) the image $\hat{d}(G(4,2))$ is a quadratic hypersurface in $\mathbb{P}^5$. Since $d_{jk}(Xh)\,\overline{d_{\ell m}(Xh)} = d_{jk}(X)\overline{d_{\ell m}(X)}\,|\det(h)|^2$, the image $\hat{d}(\mathscr{D})$ is given by (1.3.2) as a subset of $\hat{d}(G(4,2))$. The group $U(2,2)$ and the involution T naturally act on $\hat{d}(\mathscr{D})$ as follows:

$$\begin{aligned} g\cdot{}^t[\ldots, d_{jk}(X), \ldots] &:= {}^t[\ldots, d_{jk}(gX), \ldots]\,, \\ T\cdot{}^t[\ldots, d_{jk}(X), \ldots] &:= {}^t[\ldots, d_{jk}(T(X)), \ldots]\,, \end{aligned}$$

where $g = (g_{jk}) \in U(2,2)$, $X \in H(4,2)$; an elementary calculation shows

$$\begin{aligned} d_{jk}(gX) &= \sum_{1 \leqq \ell < m \leqq 4} \det\begin{pmatrix} g_{j\ell} & g_{jm} \\ g_{k\ell} & g_{km} \end{pmatrix} d_{\ell m}(X)\,, \\ d_{jk}(T(X)) &= c\,d_{jk}(X) \text{ for } (jk) \in \{(12), (14), (23), (34)\}\,, \\ d_{13}(T(X)) &= -\,c\,d_{24}(X),\ d_{24}(T(X)) = -\,c\,d_{13}(X)\,, \end{aligned}$$

where $c = (d_{12}(X)d_{34}(X))^{-1}$. Since the above actions belong to $\mathrm{PGL}(6,\mathbb{C})$, we have a homomorphism from $\mathrm{Aut}(\mathscr{D})$ to $\mathrm{PGL}(6,\mathbb{C})$. Since $\mathscr{F}$ is the set of fixed points of T in $\mathscr{D}$, the image $\hat{d}(\mathscr{F})$ is the subset of $\hat{d}(\mathscr{D})$ given by the equation $z_5 + z_6 = 0$. The coordinate change ${}^t(z_1', \ldots, z_6') = P\,{}^t(z_1, \ldots, z_6)$ by the matrix

$$P = \begin{pmatrix} 1 & & & & & \\ & -1 & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & \dfrac{-1}{1-i} & \dfrac{-i}{1-i} \\ & & & & \dfrac{-1}{1+i} & \dfrac{i}{1+i} \end{pmatrix}$$

transforms $\hat{d}(\mathscr{D})$ into

$$D_{\mathrm{IV}} := \{z = [z_1', \ldots, z_6'] \in \mathbb{P}^5 \mid {}^t z\,\mathscr{A}\,z = A,\ z^*\mathscr{A}\,z > 0,\ \mathrm{Im}(z_3'/z_1') > 0\}\,,$$

where $\mathscr{A} = \begin{pmatrix} U & & \\ & U & \\ & & -I_2 \end{pmatrix}$, $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The domain D_{IV} is called the symmetric domain of type **IV**. By this coordinate change and $\hat{d}$, we get an isomorphism

$$\iota_2 : \mathscr{D} \to D_{\mathrm{IV}};$$

the set $\mathscr{F}$ is transformed by ι_2 into the subset of D_{IV} defined by $z_5' - z_6' = 0$. Now we have the diagram

$$\begin{array}{ccc} & \mathscr{D} & \\ {\scriptstyle \iota_1}\swarrow & & \searrow{\scriptstyle \iota_2} \\ \mathbf{H}_2 & & D_{\mathrm{IV}}. \end{array}$$

It is known that the group $\mathrm{Aut}(D_{\mathrm{IV}})$ of automorphisms of D_{IV} is given by

$$\mathrm{Aut}(D_{\mathrm{IV}}) = \{g = (g_{jk}) \in \mathrm{GL}(6, \mathbb{R}) \mid {}^t g\, \mathscr{A}\, g = \mathscr{A},\ G(g) > 0\}/\{\pm I_6\},$$

where $G(g) = (g_{11} + g_{12})(g_{33} + g_{34}) - (g_{13} + g_{14})(g_{31} + g_{32})$. The coordinate change P sends $g \in \mathrm{Aut}(\mathscr{D}) \subset \mathrm{PGL}(6, \mathbb{C})$ into $PgP^{-1} \in \mathrm{Aut}(D_{\mathrm{IV}}) \subset \mathrm{PGL}(6, \mathbb{R})$.

1.4 Some discrete subgroups of $\mathrm{Aut}(D_{\mathrm{IV}})$

Let us define some discrete subgroups of $\mathrm{Aut}(D_{\mathrm{IV}})$:

$$\Gamma\mathscr{A} := \{g \in \mathrm{PGL}(6, \mathbb{Z}) \mid {}^t g\, \mathscr{A}\, g = \mathscr{A},\ G(g) > 0\},$$

$$S\Gamma\mathscr{A} := \{g \in \Gamma\mathscr{A} \mid \det(g) = 1\},$$

$$\Gamma\mathscr{A}(2) := \{g \in \Gamma\mathscr{A} \mid g \equiv I_6 \bmod 2\},$$

$$S\Gamma\mathscr{A}(2) := S\Gamma\mathscr{A} \cap \Gamma\mathscr{A}(2).$$

We give a neat description of $\Gamma\mathscr{A}$ and $\Gamma\mathscr{A}(2)$ in terms of reflections, which makes our argument clear. A reflection R_α belonging to a symmetric matrix $\mathscr{A}$, with a root $\alpha = (\alpha_1, \ldots, \alpha_6)$, is a matrix defined by

$$R_\alpha = I_6 - 2\frac{{}^t\alpha\alpha\mathscr{A}}{(\alpha, \alpha)_{\mathscr{A}}},$$

where $(z, w)_{\mathscr{A}} := z\mathscr{A}\,{}^t w$ and the norm $(\alpha, \alpha)_{\mathscr{A}}$ of α is supposed to be negative. Notice that the reflection R_α acts on D_{IV} as

$$z \mapsto z - 2\frac{(\alpha, z)_{\mathscr{A}}}{(\alpha, \alpha)_{\mathscr{A}}}\alpha \quad \text{for } z \in D_{\mathrm{IV}}.$$

The set $\alpha^\perp := \{z \in D_{\mathrm{IV}} \mid (\alpha, z)_{\mathscr{A}} = 0\}$ is called the mirror of a root α. We define twenty roots $\alpha(j, k, \ell)$ $(1 \leqq j < k < \ell \leqq 6)$ of norm -2 and seven roots βm $(1 \leqq m \leqq 7)$ of norm -4 as follows:

$$\alpha(1,2,3) := (0,0,0,0,1,0), \quad \alpha(1,3,5) := (1,1,1,-1,1,0),$$
$$\alpha(2,3,6) := (1,0,0,0,1,0), \quad \alpha(1,3,6) := (1,0,1,0,1,0),$$
$$\alpha(1,2,4) := (0,1,0,0,1,0), \quad \alpha(2,3,5) := (1,0,0,1,1,0),$$
$$\alpha(1,2,6) := (0,0,1,0,1,0), \quad \alpha(1,2,5) := (0,1,1,0,1,0),$$
$$\alpha(2,3,4) := (0,0,0,1,1,0), \quad \alpha(1,3,4) := (0,1,0,1,1,0),$$
$$\alpha(4,5,6) := (0,0,0,0,0,1), \quad \alpha(2,4,6) := (1,1,1,-1,0,1),$$
$$\alpha(1,4,5) := (1,0,0,0,0,1), \quad \alpha(2,4,5) := (1,0,1,0,0,1),$$
$$\alpha(3,5,6) := (0,1,0,0,0,1), \quad \alpha(1,4,6) := (1,0,0,1,0,1),$$
$$\alpha(3,4,5) := (0,0,1,0,0,1), \quad \alpha(3,4,6) := (0,1,1,0,0,1),$$
$$\alpha(1,5,6) := (0,0,0,1,0,1), \quad \alpha(2,5,6) := (0,1,0,1,0,1);$$

$$\beta_1 = (0,0,0,0,1,-1), \quad \beta_2 = (0,1,0,0,1,1),$$
$$\beta_3 = (0,0,1,0,1,1), \quad \beta_4 = (0,1,1,0,1,1),$$
$$\beta_5 = (1,-1,0,0,0,0), \quad \beta_6 = (0,0,1,-1,0,0),$$
$$\beta_7 = (0,1,1,-1,0,0).$$

All reflections $R_{\alpha(j,k,\ell)}$ and $R_{\beta j}$ belong to $\mathrm{GL}(6,\mathbb{Z})$ and $R_{\alpha(j,k,\ell)}$ satisfy $R_{\alpha(j,k,\ell)} \equiv I_6 \bmod 2$. It is shown in [MSY2] that $\Gamma\mathscr{A}(2)$ is generated by the actions of $R_{\alpha(j,k,\ell)}$ $(1 \leqq j < k < \ell \leqq 6)$ and that $\Gamma\mathscr{A}$ is generated by the actions of $R_{\beta m}(1 \leqq m \leqq 7)$ and $R_{\alpha(1,2,3)}$. Since the action of T corresponds to the action of $R_{\beta 1}$, the image $\iota_2(\mathscr{F})$ is the mirror $\beta 1^{\perp}$ of $\beta 1$. We will denote by $\mathbf{H}(j,k,\ell)$ the subset of $\mathbf{H}_2$ corresponding the mirror $\alpha(j,k,\ell)^{\perp}$ by the isomorphism $\iota_1 \circ \iota_2^{-1}$.

1.5 Isomorphisms between discrete subgroups of $\mathrm{Aut}(\mathbf{H}_2)$ *and those of* $\mathrm{Aut}(D_{\mathrm{IV}})$

The isomorphism $\iota_2 \circ \iota_1^{-1} : \mathbf{H}_2 \to D_{\mathrm{IV}}$ induces the following isomorphisms:

Proposition 1.5.1. (1) $\Gamma_T \simeq \Gamma\mathscr{A}$, (2) $\Gamma_M(1+i) \simeq \Gamma\mathscr{A}(2)$, (3) $S\Gamma(1+i) \simeq S\Gamma\mathscr{A}(2)$.

Proof. (1) As we mentioned in Sect. 1.2, Γ is generated by matrices of following three forms

$$\begin{pmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{pmatrix}, \begin{pmatrix} I_2 & B \\ 0 & I_2 \end{pmatrix} \quad \text{and } J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix},$$

where $A \in \mathrm{GL}(2,\mathbb{Z}[i])$, $B = B^* \in M(2,2;\mathbb{Z}[i])$.

We first show that we need only five matrices

$$g1 := \begin{pmatrix} i & & & \\ & 1 & & \\ & & i & \\ & & & 1 \end{pmatrix}, \quad g2 := \begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{pmatrix}, \quad g3 := \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}, \quad g4 := \begin{pmatrix} & & 1 & 0 \\ I_2 & & 0 & 0 \\ & & 0 & I_2 \end{pmatrix}$$

and $J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$ to generate Γ. Since $\mathrm{GL}(2,\mathbb{Z}[i])$ is generated by $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, every matrix of form $\begin{pmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{pmatrix}$ is expressed by a product of some of $g1$, $g2$ and $g3$. The abelian group

$$\left\{ \begin{pmatrix} I_2 & B \\ 0 & I_2 \end{pmatrix} \in M(4, 4; \mathbb{Z}[i]) \mid B^* = B \right\}$$

is generated by four elements which are given by $B_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $B_3 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $B_4 := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. They are expressed by a product of some of $g1$, $g2$, $g3$, and $g4 = \begin{pmatrix} I_2 & B_1 \\ 0 & I_2 \end{pmatrix}$; in fact we have

$$\begin{pmatrix} I_2 & B_2 \\ 0 & I_2 \end{pmatrix} = g3\, g4\, g3\,,$$

$$\begin{pmatrix} I_2 & B_3 \\ 0 & I_2 \end{pmatrix} = g2 \begin{pmatrix} I_2 & B_2 \\ 0 & I_2 \end{pmatrix} g2^{-1}\, g4^{-1} \begin{pmatrix} I_2 & -B_2 \\ 0 & I_2 \end{pmatrix},$$

$$\begin{pmatrix} I_2 & B_4 \\ 0 & I_2 \end{pmatrix} = g1 \begin{pmatrix} I_2 & B_3 \\ 0 & I_2 \end{pmatrix} g1^{-1}\,,$$

which shows our claim.

We next transform the actions of the above five elements on $\mathbf{H}_2$ into those on D_{IV} by the isomorphism $\iota_2 \circ \iota_1^{-1}$. By a direct calculation, we have

$$g1 \mapsto \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & -1 & \\ & & & & & 1 \end{pmatrix}, \quad g2 \mapsto \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & 1 & -1 & -1 \\ & & & 1 & & \\ & & & -1 & 1 & \\ & & & -1 & & 1 \end{pmatrix},$$

$$g3 \mapsto \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}, \quad g4 \mapsto \begin{pmatrix} 1 & & & & & \\ & 1 & 1 & & & \\ -1 & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}, \quad g5 \mapsto \mathscr{A}\,.$$

As we mentioned in Sect. 1.4, the action of T on $\mathbf{H}_2$ corresponds to that of the reflection $R_{\beta 1}$ on D_{IV}; thus the image of Γ_T by $\iota_2 \circ \iota_1^{-1}$ is a subgroup of $\Gamma\mathscr{A}$.

Finally we transform the action of the reflections $R_{\beta j} (j = 2, \ldots, 7)$ and $R_{\alpha(1,2,3)}$ on D_{IV} into those on $\mathbf{H}_2$ by the isomorphism $\iota_1 \circ \iota_2^{-1}$ to show the converse. By a direct calculation, we have

$$R_{\beta 2} \mapsto \begin{pmatrix} 1 & & & -1 \\ & -1 & -1 & \\ & & 1 & \\ & & & -1 \end{pmatrix} T, \quad R_{\beta 3} \mapsto \begin{pmatrix} 1 & -1 & & \\ & -1 & & \\ & & 1 & \\ & & -1 & -1 \end{pmatrix} T,$$

$$R_{\beta 4} \mapsto \begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix} T, \quad R_{\beta 5} \mapsto \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix} T,$$

$$R_{\beta 6} \mapsto \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix} T, \qquad R_{\beta 7} \mapsto \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} T,$$

$$R_{\alpha(1,2,3)} \mapsto g1\ T;$$

this leads to $\Gamma_T \simeq \Gamma\mathscr{A}$.

(2) It is shown in [MSY2] that the reflections $R_{\alpha(j,k,\ell)}$'s are all conjugate in $\Gamma\mathscr{A}$ and that $\Gamma\mathscr{A}/\Gamma\mathscr{A}(2) \simeq S_6 \times \mathbb{Z}/2\mathbb{Z}$. Since we have $R_{\alpha(1,2,3)} \mapsto g1\ T$, we can easily show that the actions of $R_{\alpha(j,k,\ell)}$'s $(1 \leqq j < k < \ell \leqq 6)$, which generate $\Gamma\mathscr{A}(2)$, are transformed into elements of $\Gamma_M(1+i)$ by $\iota_1 \circ \iota_2^{-1}$. By the fact $\Gamma_T/\Gamma_M(1+i) \simeq S_6 \times \mathbb{Z}/2\mathbb{Z}$, we have $\Gamma_T(1+i) \simeq \Gamma\mathscr{A}(2)$.

(3) We can easily show that $S\Gamma\mathscr{A}(2)$ is transformed into a subgroup of $S\Gamma(1+i)$ by $\iota_1 \circ \iota_2^{-1}$. We need only to consider the indices $[\Gamma_M(1+i) : S\Gamma(1+i)]$ and $[\Gamma\mathscr{A}(2) : S\Gamma\mathscr{A}(2)]$, which are both equal to 2. □

2 Theta functions on H_2

2.1 Definition and basic properties of theta functions on $\mathbf{H}_2$

We define theta functions $\Theta\begin{bmatrix} a \\ b \end{bmatrix}$ on $\mathbf{H}_2$ with characteristic $\begin{bmatrix} a \\ b \end{bmatrix}$ $a, b \in \frac{1}{2}\mathbb{Z}[i]^2$, and Riemann's theta functions $\vartheta\begin{bmatrix} c \\ d \end{bmatrix}$ on $\mathbf{S}_m$ with characteristic $\begin{bmatrix} c \\ d \end{bmatrix}$ $c, d \in \frac{1}{2}\mathbb{Z}^m$ as follows:

$$\Theta\begin{bmatrix} a \\ b \end{bmatrix}(W) := \sum_{n \in \mathbb{Z}[i]^2} \mathbf{e}[(n+a)^*\, W(n+a) + 2Re\{b^*\, n\}]\,,$$

where $W \in \mathbf{H}_2$, $a, b \in \frac{1}{2}\mathbb{Z}[i]^2$,

$$\vartheta\begin{bmatrix} c \\ d \end{bmatrix}(\Omega) := \sum_{n \in \mathbb{Z}^m} \mathbf{e}[{}^t(n+c)\Omega(n+c) + 2\,{}^t d\, n]\,,$$

where $\Omega \in \mathbf{S}_m = \{\Omega \in M(m, m; \mathbb{C}) \mid {}^t\Omega = \Omega, \operatorname{Im}(\Omega) > 0\}$, $c, d \in \frac{1}{2}\mathbb{Z}^m$.

The theta functions on $\mathbf{H}_2$ have the following properties.

Lemma 2.1.1. (i) $\Theta\begin{bmatrix} ia \\ ib \end{bmatrix}(W) = \Theta\begin{bmatrix} a \\ b \end{bmatrix}(W)$,

(ii) $\Theta\begin{bmatrix} a+r \\ b+s \end{bmatrix}(W) = \mathbf{e}[2\operatorname{Re}(b^* \cdot r)]\, \Theta\begin{bmatrix} a \\ b \end{bmatrix}(W)$, *where* $r, s \in \mathbb{Z}[i]^2$,

(iii) *if* $2\operatorname{Re}(a^* \cdot b) \notin \mathbb{Z}$, *then* $\Theta\begin{bmatrix} a \\ b \end{bmatrix}(W)$ *vanishes*,

(iv) *if* $a, b \in \frac{1}{1+i}\mathbb{Z}[i]^2$ *and* $(1+i)a^* \cdot b \notin \mathbb{Z}[i]$, *then* $\Theta\begin{bmatrix} a \\ b \end{bmatrix}(W)$ *vanishes*,

(v) $\Theta\begin{bmatrix} a \\ b \end{bmatrix}(W) = \vartheta\begin{bmatrix} \operatorname{Re} a \\ \operatorname{Im} a \\ \operatorname{Re} b \\ \operatorname{Im} b \end{bmatrix}(\tilde{W}) = \mathbf{e}[4\,{}^t(\operatorname{Re} a)(\operatorname{Re} b)]\,\vartheta\begin{bmatrix} \operatorname{Im} a \\ \operatorname{Re} a \\ \operatorname{Im} b \\ \operatorname{Re} b \end{bmatrix}(\tilde{W})$,

where $\tilde{W} = \frac{1}{2}\begin{pmatrix} W + {}^tW & i(W - {}^tW) \\ -i(W - {}^tW) & W + {}^tW \end{pmatrix} \in \mathbf{S}_4$,

(vi) *if* $W \in \mathbf{S}_2$, *then* $\Theta\begin{bmatrix} a \\ b \end{bmatrix}(W) = \vartheta\begin{bmatrix} \operatorname{Re} a \\ \operatorname{Re} b \end{bmatrix}(W)\,\vartheta\begin{bmatrix} \operatorname{Im} a \\ \operatorname{Im} b \end{bmatrix}(W)$.

Proof. The assertions are clear except (iii) and (iv). It is known that $\vartheta\begin{bmatrix} \operatorname{Re} a \\ \operatorname{Im} a \\ \operatorname{Re} b \\ \operatorname{Im} b \end{bmatrix}$ vanishes if $2\{{}^t(\operatorname{Re} a)(\operatorname{Re} b) + {}^t(\operatorname{Im} a)(\operatorname{Im} b)\} \notin \mathbb{Z}$. This fact together with (v) proves (iii). By (i) and (ii), for $a, b \in \frac{1}{1+i}\mathbb{Z}[i]^2$ we have

$$\Theta\begin{bmatrix} a \\ b \end{bmatrix}(W) = \Theta\begin{bmatrix} ia \\ ib \end{bmatrix}(W) = \Theta\begin{bmatrix} a + (i-1)a \\ b + (i-1)b \end{bmatrix}(W)$$

$$= \mathbf{e}[2\operatorname{Re}(b^* \cdot (i-1)a)]\, \Theta\begin{bmatrix} a \\ b \end{bmatrix}(W).$$

Since $b^* \cdot (i-1)a \in \frac{1}{1+i}\mathbb{Z}[i]$, if $b^* \cdot (i-1)a \notin \mathbb{Z}[i]$ then $\mathbf{e}[2\operatorname{Re}(b^* \cdot (i-1)a)] = -1$. Since $b^* \cdot (i-1)a \equiv (1+i)a^* \cdot b \bmod \mathbb{Z}[i]$, we have (iv). □

We give transformation formulae for theta functions on $\mathbf{H}_2$ by the action of Γ_T.

Lemma 2.1.2. (vii) $\Theta\begin{bmatrix} a \\ b \end{bmatrix}(W) = \Theta\begin{bmatrix} \bar{a} \\ \bar{b} \end{bmatrix}({}^tW) = \mathbf{e}[4(\mathrm{Re}\,{}^ta)\cdot(\mathrm{Re}\,b)]\,\Theta\begin{bmatrix} a \\ b \end{bmatrix}({}^tW)$,

(viii) *if* $g \in \Gamma$ *is of the form* $\begin{pmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{pmatrix}$, $A \in \mathrm{GL}(2, \mathbb{Z}[i])$, *then*

$$\Theta\begin{bmatrix} a \\ b \end{bmatrix}(g\cdot W) = \Theta\begin{bmatrix} A^* & a \\ A^{-1} & b \end{bmatrix}(W),$$

(ix) *if* g *is of the form* $\begin{pmatrix} I_2 & B \\ 0 & I_2 \end{pmatrix}$, $B = B^* = (B_{jk}) \in M(2, 2, \mathbb{Z}[i])$, *then*

$$\Theta\begin{bmatrix} a \\ b \end{bmatrix}(g\cdot W) = \mathbf{e}[a^* Ba]\,\Theta\begin{bmatrix} a \\ \tilde{b} \end{bmatrix}(W),\ \tilde{b} = b + Ba + \frac{1}{1+i}\begin{pmatrix} B_{11} \\ B_{22} \end{pmatrix},$$

(x) $\Theta\begin{bmatrix} a \\ b \end{bmatrix}(J\cdot W) = -\,\mathbf{e}[2\mathrm{Re}(a^* b)]\det(W)\,\Theta\begin{bmatrix} b \\ a \end{bmatrix}(W)$.

By the property (ii), it is enough to consider $2^8 (= \{\#(\frac{1}{2}\mathbb{Z}[i]/\mathbb{Z}[i])\}^4)$ theta functions. By the property (iii), $\Theta\begin{bmatrix} a \\ b \end{bmatrix}$ vanishes identically for $a, b \in \frac{1}{2}\mathbb{Z}[i]^2$, $2\mathrm{Re}({}^ta)\cdot\mathrm{Re}(b) + 2\mathrm{Im}({}^ta)\cdot\mathrm{Im}(b) \notin \mathbb{Z}$; so we assume $2\mathrm{Re}({}^ta)\cdot\mathrm{Re}(b) + 2\mathrm{Im}({}^ta)\cdot\mathrm{Im}(b) \in \mathbb{Z}$. By the property (iv), there are only ten linearly independent theta functions with characteristic $\begin{bmatrix} a \\ b \end{bmatrix}$ $a, b \in \frac{1}{1+i}\mathbb{Z}[i]^2$; they are those with characteristics $\begin{bmatrix} a \\ b \end{bmatrix}$ such that $a, b \in \left\{0, \frac{1}{1+i}\right\}^2$ and $a^* b \in \mathbb{Z}$.

2.2 Quadratic relations among the ten theta functions

The ten theta functions satisfy the following quadratic relations.

Proposition 2.2.1.

$$\sum_{\substack{a,b\in\{0,\,(1/1+i)\}^2 \\ a^* b\in\mathbb{Z}}} \Theta\begin{bmatrix} a \\ b \end{bmatrix}(W)^2\,\mathbf{e}[2(d^* a + c^* b)] = 0\,,$$

for $c, d \in \left\{0, \frac{1}{1+i}\right\}^2$, $c^* d \notin \mathbb{Z}$.

Remark 2.2.2. The proposition gives six linear relations among the squares of ten linearly independent theta functions; only five relations among the six are linearly independent.

To prove Proposition 2.2.1, we need the following lemma, which is a direct consequence of the orthogonality of characters.

Lemma 2.2.3. *Let Λ_1 and Λ_2 be two commensurable lattices in $\mathbb{C}^m$. Suppose that f is a function defined on $\Lambda = \Lambda_1 + \Lambda_2$ for which $\sum_{n\in\Lambda} f(n)$ is absolutely convergent. Then the summations of f over Λ_1 and Λ_2 are related as follows:*

$$\sum_{n_1\in\Lambda_1} f(n_1) = [\Lambda:\Lambda_1]^{-1} \sum_{s,r} \left\{ \sum_{n_2\in\Lambda_2} s(n_2+r) f(n_2+r) \right\},$$

where s runs over characters of Λ/Λ_1, r runs over Λ/Λ_2.

Proof of Proposition 2.2.1. Put $\Lambda_1 := \mathbf{M}(2,2;\mathbb{Z}[i])$, $\Lambda_2 := \Lambda_1 U$, where $U = \frac{1}{1+i}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ and $\Lambda := \Lambda_1 + \Lambda_2$; note that $U\,U^* = I_2$ and $[\Lambda : \Lambda_j] = 4$ $(j = 1, 2)$. Elements of Λ mod Λ_2 can be represented by $R_1 = (r_1, 0)$ where $r_1 \in \{0,1\}^2$. Characters of Λ/Λ_1 are given by $s(X) = \mathbf{e}[2\mathrm{Re}\{\mathrm{tr}((R_2)^* X)\}]$ where $R_2 = (r_2, 0)$, $r_2 \in \{0,1\}^2$. Finally put

$$f(X) := \mathbf{e}\Big[\mathrm{tr}\Big(\Big(X + \frac{1}{1+i}M_1\Big)^* W\Big(X + \frac{1}{1+i}M_1\Big)\Big) + 2\mathrm{Re}\Big\{\mathrm{tr}\Big(\Big(\frac{1}{1+i}M_2\Big)^* X\Big)\Big\}\Big]$$

where $X \in M(2,2,\mathbb{C})$, $M_1 = (m_1, m_1)$, $M_2 = (m_2, m_2)$, $m_j \in \mathbb{Z}^2$.

We apply Lemma 2.2.3 for these data and make a straightforward calculation; we get

$$\Theta\begin{bmatrix} \frac{1}{1+i}m_1 \\ \frac{1}{1+i}m_2 \end{bmatrix}(W)^2 = \frac{1}{4}\sum_{r_1,r_2} \Theta\begin{bmatrix} \frac{1}{1+i}r_1 \\ \frac{1}{1+i}r_2 \end{bmatrix}(W)^2\, \mathbf{e}[{}^t m_2 r_1 + {}^t m_1 r_2]\,.$$

In view of (iv), if ${}^t m_1 \cdot m_2$ is odd then $\Theta\begin{bmatrix} \frac{1}{1+i}m_1 \\ \frac{1}{1+i}m_2 \end{bmatrix}(W)$ vanishes; hence we have the required equalities. □

2.3 Coding theta functions

Following [DO], we code 16 characteristics $\begin{bmatrix} c \\ d \end{bmatrix}, \left(c, d \in \left(\frac{1}{2}\mathbb{Z}/\mathbb{Z}\right)^2\right)$ of the theta functions on $\mathbf{S}_2$ by using letters $\{1, \ldots, 6\}$ as follows:

$$\begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}\leftrightarrow\begin{pmatrix}123\\456\end{pmatrix},\quad \begin{bmatrix}1/2\\1/2\\0\\0\end{bmatrix}\leftrightarrow\begin{pmatrix}124\\356\end{pmatrix},\quad \begin{bmatrix}1/2\\0\\0\\0\end{bmatrix}\leftrightarrow\begin{pmatrix}125\\346\end{pmatrix},$$

$$\begin{bmatrix}1/2\\0\\0\\1/2\end{bmatrix}\leftrightarrow\begin{pmatrix}126\\345\end{pmatrix},\quad \begin{bmatrix}0\\1/2\\0\\0\end{bmatrix}\leftrightarrow\begin{pmatrix}134\\256\end{pmatrix},\quad \begin{bmatrix}0\\0\\0\\0\end{bmatrix}\leftrightarrow\begin{pmatrix}135\\246\end{pmatrix},\quad \begin{bmatrix}0\\0\\0\\1/2\end{bmatrix}\leftrightarrow\begin{pmatrix}136\\245\end{pmatrix},$$

$$\begin{bmatrix}0\\0\\1/2\\1/2\end{bmatrix}\leftrightarrow\begin{pmatrix}145\\236\end{pmatrix},\quad \begin{bmatrix}0\\0\\1/2\\0\end{bmatrix}\leftrightarrow\begin{pmatrix}146\\235\end{pmatrix},\quad \begin{bmatrix}0\\1/2\\1/2\\0\end{bmatrix}\leftrightarrow\begin{pmatrix}156\\234\end{pmatrix},$$

$$\begin{bmatrix}0\\1/2\\1/2\\1/2\end{bmatrix}\leftrightarrow\begin{pmatrix}1\\23456\end{pmatrix},\quad \begin{bmatrix}0\\1/2\\0\\1/2\end{bmatrix}\leftrightarrow\begin{pmatrix}2\\13456\end{pmatrix},\quad \begin{bmatrix}1/2\\1/2\\0\\1/2\end{bmatrix}\leftrightarrow\begin{pmatrix}3\\12456\end{pmatrix},$$

$$\begin{bmatrix}1/2\\1/2\\1/2\\0\end{bmatrix}\leftrightarrow\begin{pmatrix}4\\12356\end{pmatrix},\quad \begin{bmatrix}1/2\\0\\1/2\\0\end{bmatrix}\leftrightarrow\begin{pmatrix}5\\12346\end{pmatrix},\quad \begin{bmatrix}0\\1/2\\1/2\\1/2\end{bmatrix}\leftrightarrow\begin{pmatrix}6\\12345\end{pmatrix}.$$

By using the above convention, theta functions on $\mathbf{H}_2$ will be coded by combinations of 6 letters.

For $a, b \in \frac{1}{2}\mathbb{Z}[i]$, $2\mathrm{Re}({}^t a)\cdot\mathrm{Re}(b) \in \mathbb{Z}$, $2\mathrm{Im}({}^t a)\cdot\mathrm{Im}(b) \in \mathbb{Z}$, we put

$$\Theta\begin{pmatrix}hjk\\ \ell mn\end{pmatrix}\begin{pmatrix}pqr\\stu\end{pmatrix} := \Theta\begin{bmatrix}a\\b\end{bmatrix}, \text{ where } \begin{bmatrix}\mathrm{Re}\ a\\ \mathrm{Re}\ b\end{bmatrix}\leftrightarrow\begin{pmatrix}hjk\\ \ell mn\end{pmatrix}, \begin{bmatrix}\mathrm{Im}\ a\\ \mathrm{Im}\ b\end{bmatrix}\leftrightarrow\begin{pmatrix}pqr\\stu\end{pmatrix}.$$

For $a, b \in \frac{1}{2}\mathbb{Z}[i]$, $2\mathrm{Re}({}^t a)\cdot\mathrm{Re}(b) \notin \mathbb{Z}$, $2\mathrm{Im}({}^t a)\cdot\mathrm{Im}(b) \notin \mathbb{Z}$, we put

$$\Theta\begin{pmatrix}h\\ jk\ell mn\end{pmatrix}\begin{pmatrix}p\\qrstu\end{pmatrix} := \Theta\begin{bmatrix}a\\b\end{bmatrix}, \text{ where } \begin{bmatrix}\mathrm{Re}\ a\\ \mathrm{Re}\ b\end{bmatrix}\leftrightarrow\begin{pmatrix}h\\ jk\ell mn\end{pmatrix}, \begin{bmatrix}\mathrm{Im}\ a\\ \mathrm{Im}\ b\end{bmatrix}\leftrightarrow\begin{pmatrix}p\\qrstu\end{pmatrix}.$$

By the property (v), we have

$$\Theta\begin{pmatrix}hjk\\ \ell mn\end{pmatrix}\begin{pmatrix}pqr\\stu\end{pmatrix} = \Theta\begin{pmatrix}pqr\\stu\end{pmatrix}\begin{pmatrix}hjk\\ \ell mn\end{pmatrix},$$

$$\Theta\begin{pmatrix}h\\ jk\ell mn\end{pmatrix}\begin{pmatrix}p\\qrstu\end{pmatrix} = -\Theta\begin{pmatrix}p\\qrstu\end{pmatrix}\begin{pmatrix}h\\ jk\ell mn\end{pmatrix}.$$

There are $6\left(=\binom{6}{1}\right)$ theta functions of type $\Theta\begin{pmatrix} h \\ jk\ell mn\end{pmatrix}\begin{pmatrix} h \\ jk\ell mn\end{pmatrix}$, which vanish identically. There are 10 $\left(=\binom{6}{3}\Big/2\right)$ theta functions of type $\Theta\binom{hjk}{\ell mn}\binom{hjk}{\ell mn}$, which will be denoted simply by $\Theta\binom{hjk}{\ell mn}$; they correspond to ten $\Theta\begin{bmatrix} a \\ b\end{bmatrix}$, where $a, b \in \left\{0, \frac{1}{1+i}\right\}^2$, $a^* b \in \mathbb{Z}$. There are 45 $\left(=\binom{10}{2}\right)$ theta functions of type $\Theta\binom{hjk}{\ell mn}\binom{pqr}{stu}\left(\binom{hjk}{\ell mn} \neq \binom{pqr}{stu}\right)$ and $15\left(=\binom{6}{2}\right)$ theta functions of type $\Theta\begin{pmatrix} h \\ jk\ell mn\end{pmatrix}\begin{pmatrix} p \\ qrstu\end{pmatrix}\left(\begin{pmatrix} h \\ jk\ell mn\end{pmatrix} \neq \begin{pmatrix} p \\ qrstu\end{pmatrix}\right)$. The property (vii) leads

$$\Theta\begin{pmatrix} h \\ jk\ell mn\end{pmatrix}\begin{pmatrix} p \\ qrstu\end{pmatrix}({}^t W) = -\Theta\begin{pmatrix} h \\ jk\ell mn\end{pmatrix}\begin{pmatrix} p \\ qrstu\end{pmatrix}(W),$$

so $\Theta\begin{pmatrix} h \\ jk\ell mn\end{pmatrix}\begin{pmatrix} p \\ qrstu\end{pmatrix}(W)$ vanishes on $\mathbf{S}_2$. Let us define a holomorphic function Θ on $\mathbf{H}_2$ by

$$\Theta := \sum_{1 \leqq h < j \leqq 6} \Theta\begin{pmatrix} h \\ jk\ell mn\end{pmatrix}\begin{pmatrix} j \\ ik\ell mn\end{pmatrix}\Theta\binom{hk\ell}{jmn}\binom{hmn}{jk\ell}$$
$$\Theta\binom{hkm}{j\ell n}\binom{h\ell n}{jkn}\Theta\binom{hkn}{j\ell m}\binom{h\ell m}{jkn};$$

which will play an important role in Sect. 3.2 and 4.4.

Lemma 2.3.1. *The theta function* $\Theta\binom{hjk}{\ell mn}$ *has simple zeros exactly on the* $\Gamma_T(1+i)$*-orbits of the mirrors* $\mathbf{H}(h, j, k)$ *and* $\mathbf{H}(\ell, m, n)$.

Proof. By using transformation formulae (vii) and (viii) in Lemma 2.1.2 for $\Theta\binom{123}{456}$ and $g = \begin{pmatrix} 1 & & & \\ & \pm i & & \\ & & 1 & \\ & & & \pm i \end{pmatrix} T$ which correspond to the reflections $R_{\alpha(1,2,3)}$ and $R_{\alpha(4,5,6)}$, we have

$$\Theta\binom{123}{456}(g \cdot W) = -\Theta\binom{123}{456}(W).$$

This fact implies that $\Theta\binom{123}{456}$ vanishes on the mirrors $\mathbf{H}(1, 2, 3)$ and $\mathbf{H}(4, 5, 6)$. Similarly we can show that $\Theta\binom{hjk}{\ell mn}$ vanishes on the mirrors $\mathbf{H}(h, j, k)$ and

$\mathbf{H}(\ell, m, m)$. By Lemma 2.1.2 we can easily show that $\Theta\begin{pmatrix} hjk \\ \ell mn \end{pmatrix}$ has zeros along the variety $\Gamma_T(1+i)\cdot\mathbf{H}(h, j, k)$; it is proved in [Fr] that they are the only zeros of $\Theta\begin{pmatrix} hjk \\ \ell mn \end{pmatrix}$ and that they are simple. □

3 Modular forms

3.1 Modular forms on $\mathbf{H}_2$

Definition. Let Γ' be a subgroup of Γ_T of finite index, and $\chi : \Gamma' \to \langle i \rangle := \{\pm 1, \pm i\}$ be a homomorphism. A holomorphic function f on $\mathbf{H}_2$ is called a *modular form of weight* $2k$ *relative to* $\Gamma' \subset \Gamma_T$ *with character* χ, if the following condition is satisfied:

$$f(\mathbf{g}\cdot W) = \chi(\mathbf{g})^k \{\det(CW + D)\}^{2k} f(W),$$

where $W \in \mathbf{H}_2$, $\mathbf{g} = g\, T^\varepsilon \in \Gamma'$, $\varepsilon \in \{0, 1\}$, $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

Let $\mathrm{Mod}_\chi^{2k}(\Gamma')$ denote the vector space spanned by modular forms of weight $2k$ relative to Γ' with character χ, and let

$$\mathrm{Mod}_\chi(\Gamma') := \bigoplus_{k \geqq 0} \mathrm{Mod}_\chi^{2k}(\Gamma')$$

be the graded ring of modular forms relative to Γ' with character χ.

Definition. A holomorphic function f on $\mathbf{S}_2$, which satisfies

$$f(g\cdot W) = \{\det(CW + D)\}^{2k} f(W), \text{ for } W \in \mathbf{S}_2,\ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(4, \mathbb{Z}),$$

is called a *Siegel modular form of weight* $2k$ *relative to* $\mathrm{Sp}(4, \mathbb{Z})$.

Proposition 3.1.1. *The ten functions* $\Theta\begin{pmatrix} hjk \\ \ell mn \end{pmatrix}(W)^2$, $\{h, \ldots, n\} = \{1, \ldots, 6\}$, $h < j < k$, $\ell < m < n$, *are (non-zero) modular forms of weight* 2 *relative to* $\Gamma_T(1+i)$ *with the character* $\det : \mathbf{g} \to \det(g)$, $\mathbf{g} = g\, T^\varepsilon \in \Gamma_T(1+i)$, $g \in \Gamma(1+i)$, $\varepsilon \in \{0, 1\}$.

Proof. It is enough to check the transformation formula for the four types:

$$T, \begin{pmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{pmatrix}, \begin{pmatrix} I_2 & B \\ 0 & I_2 \end{pmatrix} \text{ and } J^{-1}\begin{pmatrix} I_2 & B \\ 0 & I_2 \end{pmatrix} J,$$

where $A \in \mathrm{GL}(2, \mathbb{Z}[i])$, $A \equiv I_2 \bmod 1 + i$ and $B = B^* \in M(2, 2, (1+i)\mathbb{Z}[i])$. This can be done by the use of (ii), (vii), (viii), (ix) and (x). □

Since $\Theta\begin{pmatrix} hjk \\ \ell mn \end{pmatrix}(W)$, $\{h, \ldots, n\} = \{1, \ldots, 6\}$ changes into $\zeta \det(CW + D)$ $\Theta\begin{pmatrix} \sigma(h)\sigma(j)\sigma(k) \\ \sigma(\ell)\sigma(m)\sigma(n) \end{pmatrix}(W)$ by the action of $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$, for some ζ ($\zeta^4 = 1$) and

$\sigma \in S_6$, our coding and Proposition 3.1.1 give a homomorphism from $\Gamma/\Gamma(1+i)$ to S_6. We have the following correspondence:

$$J = \begin{pmatrix} & I_2 \\ -I_2 & \end{pmatrix} \leftrightarrow (13)(46), \quad g2 = \begin{pmatrix} 1 & 1 & & \\ 0 & 1 & & \\ & & 1 & 0 \\ & & -1 & 1 \end{pmatrix} \leftrightarrow (12)\,(36)\,(45)\,,$$

$$g3 = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix} \leftrightarrow (16)\,(25)\,(34), \quad g4 = \begin{pmatrix} I_2 & & 1\ 0 \\ & & 0\ 0 \\ & & & I_2 \end{pmatrix} \leftrightarrow (12)\,,$$

where (jk) denotes the permutation of j and k; the homomorphism is an isomorphism. We define the following convention:

$$\Theta\begin{pmatrix} hjk \\ \ell mn \end{pmatrix}^2 = \Theta\begin{pmatrix} \ell mn \\ hjk \end{pmatrix}^2,$$

$$\Theta\begin{pmatrix} hjk \\ \ell mn \end{pmatrix}^2 = \operatorname{sign}\begin{pmatrix} hjk \\ pqr \end{pmatrix} \operatorname{sign}\begin{pmatrix} \ell mn \\ stu \end{pmatrix} \Theta\begin{pmatrix} pqr \\ stu \end{pmatrix}^2,$$

for $\{h, j, k\} = \{p, q, r\}$, $\{\ell, m, n\} = \{s, t, u\}$, then we have

$$\Theta\begin{pmatrix} hjk \\ \ell mn \end{pmatrix}(g \cdot W)^2 = \{\det(CW + D)\}^2\, \Theta\begin{pmatrix} \sigma(h)\sigma(j)\sigma(k) \\ \sigma(\ell)\sigma(m)\sigma(n) \end{pmatrix}(W)^2$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$ and $\sigma \in S_6$ corresponds to g.

Remark 3.1.2. By the above convention the quadratic relations among theta functions in Proposition 2.2.1 are equivalent to the following:

$$\Theta\begin{pmatrix} hjk \\ \ell mn \end{pmatrix}(W)^2 - \Theta\begin{pmatrix} hj\ell \\ mnk \end{pmatrix}(W)^2 + \Theta\begin{pmatrix} hjm \\ nk\ell \end{pmatrix}(W)^2 - \Theta\begin{pmatrix} hjn \\ k\ell m \end{pmatrix}(W)^2 = 0\,.$$

Lemma 3.1.3. *Let $\chi : \Gamma_T \to \{\pm 1\}$ be the homomorphism defined by*

$$\chi : \mathbf{g} \to \det(g)\ \operatorname{sign}(s(g))\ (-1)^{\varepsilon}\,,$$

where $\mathbf{g} = g\,T^{\varepsilon}$, $g \in \Gamma$, $\varepsilon \in \{0, 1\}$ and $s : \Gamma \to \Gamma/\Gamma\ (1+i)\ (\simeq S_6)$ is the natural projection. The function Θ is a modular form of weight 4 relative to Γ_T with the character χ.

Proof. Check the transformation formula for T and J and gj $(1 \leqq j \leqq 4)$. □

Corollary 3.1.4. *The function Θ vanishes identically on $\mathbf{S}_2$.*

Proof. By the above lemma we have

$$\Theta(T \cdot W) = -\Theta(W)\,;$$

note that $\mathbf{S}_2$ is the set of fixed points of T. □

Proposition 3.1.5. *The function* Θ^2 *is expressed in terms of* $\Theta\begin{pmatrix} ijk \\ \ell mn \end{pmatrix}$*'s* ($\{i, \ldots, n\} = \{1, \ldots, 6\}$) *as follows*:

$$\Theta^2 = 2^{-6} \cdot 3 \cdot 5^2 \left\{ \left(\Sigma\, \Theta\begin{pmatrix} hjk \\ \ell mn \end{pmatrix}^4 \right)^2 - 4\Sigma\, \Theta\begin{pmatrix} hjk \\ \ell mn \end{pmatrix}^8 \right\}. \tag{3.1.1}$$

Proof. It is proved in [Fr] that $\dim \mathrm{Mod}_1^8(\Gamma_T) = 2$, where 1 is the trivial character. The space $\mathrm{Mod}_1^8(\Gamma_T)$ is generated by Θ^2, which vanishes along $\mathbf{S}_2$, and a form which does not vanish along $\mathbf{S}_2$. On the other hand it is also shown in [Fr] that $\left(\Sigma\, \Theta\begin{pmatrix} hjk \\ \ell mn \end{pmatrix}^4 \right)^2 - 4\Sigma\, \Theta\begin{pmatrix} hjk \\ \ell mn \end{pmatrix}^8$ vanishes along $\mathbf{S}_2$; thus it is a constant multiple of Θ^2. To determine the constant we will use the Fourier expansion. Since Θ has period 2 for variables w_{11}, w_{22}, $(w_{12} + w_{21})/2$ and $(w_{12} - w_{21})/2i$, it can be expressed by the following Fourier series:

$$\Theta = \Sigma\, c(h)\, \mathbf{e}[\tfrac{1}{2} \mathrm{tr}\, h\, W],$$

where h runs over the set of 2×2 even hermitian $\mathbb{Z}[i]$-matrices, i.e., $\{h = h^* = (h_{jk}) \in M(2, 2; \mathbb{Z}[i]) \mid h_{11}, h_{22} \in 2\mathbb{Z}\}$. We can easily see by the definition of Θ that if h is not semi-positive definite then $c(h)$ vanishes. By direct calculation, we can see that the coefficients $c(h)$ for $h = \begin{pmatrix} 2 & \pm(1+i) \\ \pm(1-i) & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & \pm(1-i) \\ \pm(1+i) & 2 \end{pmatrix}$ are equal to $3 \cdot 5 \cdot 2^4$. Since the above four matrices are the only matrices with the smallest trace and with non-zero Fourier coefficient, the coefficient $c(4I_2)$ of the Fourier series of Θ^2 is equal to $3^2 \cdot 5^2 \cdot 2^{10}$. On the other hand, the Fourier coefficient for $4I_2$ of the expression in curly bracket of the right hand side of (3.1.1) is equal to $3 \cdot 2^{16}$ [Fr]. □

3.2 Modular varieties

Let $\overline{\Gamma_T(1+i)\backslash \mathbf{H}_2}$ be the Satake compactification of the quotient space $\Gamma_T(1+i)\backslash \mathbf{H}_2$. Consider a holomorphic map $F' : \mathbf{H}_2 \to \mathbb{P}^9$ defined by

$$W \mapsto \left[\ldots, \Theta\begin{pmatrix} hjk \\ \ell mn \end{pmatrix}(W)^2, \ldots \right].$$

By Remark 3.1.2, the image lies in a 4-dimensional linear subspace of $\mathbb{P}^9$, which will be denoted by Y. Let $F : \overline{\Gamma_T(1+i)\backslash \mathbf{H}_2} \to Y$ be the map induced by F'.

Theorem 3.2.1. *The map* F *gives an isomorphism between* $\overline{\Gamma_T(1+i)\backslash \mathbf{H}_2}$ *and* $Y (\simeq \mathbb{P}^4)$.

Proof. (1) We first show that the map F is well defined. We already know the zeros of the theta functions; we study the arrangement of the mirrors of the reflections in $\Gamma_T(1+i)$. It is shown in [MSY2] that for any point in D_N, at most four mirrors

pass through the point; thus F is well defined on $\mathbf{H}_2$ modulo $\Gamma_T(1+i)$. On the boundary of type $W = \begin{pmatrix} w & 0 \\ 0 & i\infty \end{pmatrix}$ (Im $w > 0$), theta function $\Theta\begin{bmatrix} a \\ b \end{bmatrix}(W)$ reduces to

$$\vartheta\begin{bmatrix} a1 \\ b1 \end{bmatrix}(w)^2 \ \vartheta\begin{bmatrix} a2 \\ b2 \end{bmatrix}(i\infty)^2 ,$$

where $a = \begin{pmatrix} a1 \\ a2 \end{pmatrix}$, $b = \begin{pmatrix} b1 \\ b2 \end{pmatrix}$ and ϑ is Jacobi's theta constant. Since we have $\vartheta\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}(w) = 0$, $\vartheta\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}(i\infty) = 0$ and $\vartheta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(i\infty) = \vartheta\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}(i\infty) = 1$, $\Theta\begin{pmatrix} hjk \\ \ell mn \end{pmatrix}(W)^2$ degenerates on this boundary as follows:

$$\Theta\begin{pmatrix} hjk \\ \ell 56 \end{pmatrix}(W)^2 = 0, \quad \Theta\begin{pmatrix} h56 \\ jk\ell \end{pmatrix}(W)^2 = 0 ,$$

$$\Theta\begin{pmatrix} 125 \\ 346 \end{pmatrix}(W)^2 = \Theta\begin{pmatrix} 126 \\ 345 \end{pmatrix}(W)^2 = \vartheta\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}(w)^4 ,$$

$$\Theta\begin{pmatrix} 135 \\ 246 \end{pmatrix}(W)^2 = \Theta\begin{pmatrix} 136 \\ 245 \end{pmatrix}(W)^2 = \vartheta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(w)^4 ,$$

$$\Theta\begin{pmatrix} 145 \\ 236 \end{pmatrix}(W)^2 = \Theta\begin{pmatrix} 146 \\ 235 \end{pmatrix}(W)^2 = \vartheta\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}(w)^4 .$$

Thus we can see the lambda-function $\lambda(w) = \vartheta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(w)^4 \Big/ \vartheta\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}(w)^4$ when we restrict F on this boundary; this shows that the map F is well defined on this boundary. Since every $\Gamma_T(1+i)$-rational boundary component can be transformed into the above by an action of Γ_T, we conclude that F is well defined on the Satake compactification.

(2) We next show that F is locally biholomorphic at each point in $\Gamma_T(1+i)\backslash\mathbf{H}_2$. Let $P \in \mathbf{H}_2$ be the intersection of four mirrors which are zero-loci of four theta functions, say, $\Theta\begin{pmatrix} 123 \\ 456 \end{pmatrix}$, $\Theta\begin{pmatrix} 136 \\ 245 \end{pmatrix}$, $\Theta\begin{pmatrix} 156 \\ 234 \end{pmatrix}$ and $\Theta\begin{pmatrix} 125 \\ 346 \end{pmatrix}$. Since each of the four theta functions $\Theta\begin{pmatrix} hjk \\ \ell mn \end{pmatrix}$'s has simple zeros along the corresponding mirrors, which intersect orthogonally, four functions $\Theta\begin{pmatrix} hjk \\ \ell mn \end{pmatrix}^2$ can be regarded as a system of local coordinates of $\Gamma_T(1+i)\backslash\mathbf{H}_2$ at $\bar{P} := P \bmod \Gamma_T(1+i)\backslash\mathbf{H}_2$; indeed, a neighborhood of $\bar{P} \in \Gamma_T(1+i)\backslash\mathbf{H}_2$ is obtained by the quotient of a neighborhood of $P \in \mathbf{H}_2$ by the group ($\simeq (\mathbb{Z}/2\mathbb{Z})^4$) generated by mutually commutative reflections with mirrors $\alpha(1,2,3)^\perp$, $\alpha(1,5,6)^\perp$, $\alpha(2,4,5)^\perp$ and $\alpha(3,4,6)^\perp$. Thus F is locally biholomorphic at $\bar{P}$. By the detailed study of the behavior of F at the boundary component we made in (1), we can easily show that F is biholomorphic around the boundary.

(3) Finally we prove that F is biholomorphic. Since F is an open map, F is a covering of $\mathbb{P}^4$. In the situation of (2), one can see that the point P is the unique intersection point of the four mirrors. Thus we have $F^{-1}(F(\bar{P})) = \{\bar{P}\}$,

which implies that the sheet number of the covering is 1 and that F is biholomorphic. □

Corollary 3.2.2. *Any five linearly independent theta functions* $\Theta\begin{pmatrix} hjk \\ \ell mn \end{pmatrix}(W)^2$ *are free generators of the graded ring* $\mathrm{Mod}_{\det}(\Gamma_T(1+i))$.

Proof. For any $f \in \mathrm{Mod}_{\det}^{2k}(1+i)$ the function $f/\Theta\begin{pmatrix} 123 \\ 456 \end{pmatrix}^{2k}$ is meromorphic on $\overline{\Gamma_T(1+i)\backslash \mathbf{H}_2}$. Pull back this function on $\mathbb{P}^9$ by the isomorphism F; it can be written by $F(\mathbf{x})/x^k$, where $F(\mathbf{x})$ is a homogeneous polynomial of degree k. Thus we have

$$f/\Theta\begin{pmatrix} 123 \\ 456 \end{pmatrix}^{2k} = F\left(\ldots, \Theta\begin{pmatrix} hjk \\ \ell mn \end{pmatrix}^{2k}, \ldots \right) \Big/ \Theta\begin{pmatrix} 123 \\ 456 \end{pmatrix}^{2k}. \qquad \square$$

Remark 3.2.3. Since the function Θ vanishes on $\mathbf{S}_2$, the image $F'(\mathbf{S}_2)$ is included in the variety defined by the right hand side of (3.1.1). It is shown in [Ig1] that they coincide.

We will give a realization of $\Gamma_M(1+i)\backslash \mathbf{H}_2$ in $\mathbb{P}^{45}$ by using Θ and $\Theta\begin{pmatrix} hjk \\ \ell mn \end{pmatrix}$. We note that the space $\Gamma_M(1+i)\backslash \mathbf{H}_2$ is the double cover of $\Gamma_T(1+i)\backslash \mathbf{H}_2$ branching along $\mathbf{S}_2 \bmod \Gamma_T(1+i)$. We define the map $\Psi : \Gamma_M(1+i)\backslash \mathbf{H}_2 \to \mathbb{P}^{45}$ by

$$\Psi : W \mapsto [z_0, \ldots, z_{45}] = \left[\Theta(W), \ldots, \Theta\begin{pmatrix} hjk \\ \ell mn \end{pmatrix}(W)^2\, \Theta\begin{pmatrix} pqr \\ stu \end{pmatrix}(W)^2, \ldots \right],$$

which is well-defined because of Lemmas 3.1.1 and 3.1.2. Since the map $W \mapsto [z_1, \ldots, z_{45}] = \left[\ldots, \Theta\begin{pmatrix} hjk \\ \ell mn \end{pmatrix}(W)^2\ \Theta\begin{pmatrix} pqr \\ stu \end{pmatrix}(W)^2, \ldots \right]$ is the compound map of F and the Segre embedding of $\mathbb{P}^9$ into $\mathbb{P}^{44}$, and we have $\Theta({}^tW) = -\Theta(W)$, the image $\Psi(\Gamma_M(1+i)\backslash \mathbf{H}_2) \subset \mathbb{P}^{45}$ is a double cover of $\Psi(\Gamma_T(1+i)\backslash \mathbf{H}_2)$. Note that $\Theta\begin{pmatrix} hjk \\ \ell mn \end{pmatrix}^4$ is expressed by a linear combination of $\Theta\begin{pmatrix} hjk \\ \ell mn \end{pmatrix}^2 \Theta\begin{pmatrix} pqr \\ stu \end{pmatrix}^2$ by Remark 3.1.2 and that the branching locus is defined by the right hand side of (3.1.1) in view of Remark 3.2.3. Thus we have the following theorem:

Theorem 3.2.4. *The map* Ψ *gives an embedding of the Satake compactification* $\overline{\Gamma_M(1+i)\backslash \mathbf{H}_2}$ *of* $\Gamma_M(1+i)\backslash \mathbf{H}_2$.

4 Relation with the period map of a family of K3 surfaces

In this section we discuss the relation between the modular forms obtained in Sect. 3 and the period map of a family of K3 surfaces. Subsections from 4.1 to 4.3 recall some results in [MSY2], which are needed to state the theorem in Subsect. 4.4.

4.1 A family of K3 *surfaces*

Let

$$L_j = \{(t_1, t_2, t_3) \in \mathbb{P}^2 | x_{1j}t_1 + x_{2j}t_2 + x_{3j}t_3 = 0\} \ (1 \leqq j \leqq 6)$$

be six lines in general position on the projective plane $\mathbb{P}^2$ with homogeneous coordinates (t_1, t_2, t_3); denote by L the system of the ordered six lines. Here six lines are said to be in general position if no three of them intersect at a point. Consider the double cover of $S'(L)$ of $\mathbb{P}^2$ branching along the union of six lines, and $S(L)$ the smooth surface obtained from $S'(L)$ by replacing $\binom{6}{2} = 15$ singular points by 15 non-singular rational curves; the surface $S(L)$ is a K3 surface with the nowhere-vanishing holomorphic 2-form

$$\eta(L) = \prod_{j=1}^{6} (x_{1j}t_1 + x_{2j}t_2 + x_{3j}t_3)^{-1/2} dt \ ,$$

where

$$dt = t^1 dt^2 \wedge dt^3 + t^3 dt^1 \wedge dt^2 + t^2 dt^3 \wedge dt^1 \ .$$

There are 16 algebraic curves on $S(L)$; 15 curves come from the desingularization and the pull back of a generic line in $\mathbb{P}^2$; let us call $AC(L)$ the sublattice, free of rank 16, of the homology group $H_2(S(L), \mathbb{Z})$ spanned by the 16 algebraic curves. There are two systems of cycles

$$c_1(L), \ldots, c_6(L) \text{ and } c'_1(L), \ldots, c'_6(L) \ ,$$

depending continuously on L, such that the cycles $c'_j(L)$ form a $\mathbb{Z}$-basis of the orthogonal complement of $AC(L)$ in $H_2(S(L), \mathbb{Z})$, the cycles $c_j(L)$ are dual to the $c'_j(L)$, i.e., $c_j(L) \cdot c'_k(L) = \delta_{jk}$, and that the intersection matrix $(c'_j(L) \cdot c'_k(L))$ is given by $2\mathscr{A}$. We call the pair $(S(L), c_j(L))$ a marked K3 surface. As the line L_j approaches the point $L_k \cap L_\ell$, turns around the point and travels back, the marking $(c_1(L), \ldots, c_6(L))$ changes $\mathbb{Z}$-linearly. As two lines, among the six, move and interchange each other, the marking changes $\mathbb{Z}$-linearly.

4.2 The period map

The periods of a marked K3 surface $(S(L), c_j(L))$ are defined by

$$z_j(L) = \int_{c_j(L)} \eta(L) \quad (1 \leqq j \leqq 6) \ ;$$

they satisfy the Riemann relation and the Riemann inequality:

$$\sum_{j,k=1}^{6} \mathscr{A}_{jk} z_j(L) z_k(L) = 0, \ \sum_{j,k=1}^{6} \mathscr{A}_{jk} z_j(L) \bar{z}_k(L) > 0 \ .$$

Note that the domain D_{IV} is a connected component of the domain defied by the above. Let $\mathbf{X}(3, 6)$ be the space of systems of ordered six lines in general position in $\mathbb{P}^2$, which can be considered to be a Zariski open set of the affine 4-space. Then the image of the multi-valued period map

$$\mathrm{Prd} : \mathbf{X}(3, 6) \ni L \mapsto (z_1(L), \ldots, z_6(L)) \in \mathbb{P}^5$$

is in D_{IV}. The monodromy group of the multi-valued vector function *Prd* is generated by the linear transformations of the marking caused by the move of the line L_j around the point $L_k \cap L_\ell$ $(1 \leqq j < k < \ell \leqq 6)$ described in the previous subsection; each transformation is conjugate to the reflection with root $\alpha(j, k, \ell)$. As is mentioned in Sect. 1, the monodromy group happens to be equal to the principal congruence subgroup $\Gamma\mathscr{A}(2)$. We can also define a period map from the quotient space $\mathbf{X}(3, 6)/S_6$ to D_{IV}, of which monodromy group is generated by the reflections with roots $\alpha(j, k, \ell)$ and the linear transformations of the marking caused by the interchange of L_j and L_k $(1 \leqq j < k \leqq 6)$, described in the previous subsection.

4.3 Realization of $\mathbf{X}(3, 6)$

Let us define an involution on $\mathbf{X}(3, 6)$. The duality for the projective plane with respect to a non-singular conic C is described as follows:
the dual of a point $p \in \mathbb{P}^2 - C$ is the line joining the two tangent points of the two tangents of C passing through p; if p is on C then its dual is the tangent line of C at p. The involution $\heartsuit$ on $\mathbf{X}(3, 6)$ is defined as follows: for a given system L of six lines $L_1, \ldots, L_6$ in general position on $\mathbb{P}^2$, let us name six intersection points (out of fifteen) as follows:

$$p_1 = L_2 \cap L_3,\ p_2 = L_1 \cap L_3,\ p_3 = L_1 \cap L_2\ ,$$

$$p_4 = L_5 \cap L_6,\ P_5 = L_4 \cap L_6,\ p_6 = L_4 \cap L_5\ ;$$

the system $\heartsuit L$ consists of six lines which are dual, with respect to a conic, to $p_1, \ldots, p_6$. Notice that if one defines the involution by using another conic, the result is the same up to projective motions. Notice also that the set of fixed point of $\heartsuit$ on $\mathbf{X}(3, 6)$ consists of systems of six lines which are tangent to a conic.

Let us give a natural projective structure on the parameter space $\mathbf{X}(3, 6)$. Since $\mathbf{X}(3, 6)$ is the space of ordered six lines in $\mathbb{P}^2$ in general position, we can regard $\mathbf{X}(3, 6)$ as the quotient space

$$\mathrm{GL}(3, \mathbb{C})\backslash M^\circ(3, 6; \mathbb{C})/\{\mathbb{C}^*\}^6,$$

where $M^\circ(3, 6; \mathbb{C})$ is

$$\{x \in M(3, 6; \mathbb{C}) | d_{\ell mn}(x) \neq 0 \quad \text{for all } 1 \leqq \ell < m < n \leqq 6\}\ ,$$

$$d_{\ell mn}(x) := \begin{vmatrix} x_{1\ell} & x_{1\ell} & x_{1\ell} \\ x_{2m} & x_{2m} & x_{2m} \\ x_{3n} & x_{3n} & x_{3n} \end{vmatrix}\ .$$

Here a system L of six lines is represented by a matrix

$$x = \begin{pmatrix} x_{11} & \ldots & x_{16} \\ x_{21} & \ldots & x_{26} \\ x_{31} & \ldots & x_{36} \end{pmatrix} \in M^\circ(3, 6; \mathbb{C})\ ,$$

where columns are homogeneous coordinates of the lines.

On the other hand let us consider a regular tetrahedron whose six edges are labeled by numbers 1, . . . , 6; such a tetrahedron will be called a labeled tetrahedron. We identify two labeled tetrahedra if one can be transformed into the other by an element of the extended tetrahedral group, which is isomorphic to the symmetric group of S_4 of order four. In this way we get $|S_6/S_4| = 30$ labeled tetrahedra. Notice that the Poincaré duality causes an involution on the set of 30 labeled tetrahedra. Let τ be an oriented labeled tetrahedron; there are four oriented triangular faces each of which is bounded by three oriented labeled edges. For a matrix $x \in M^\circ(3, 6; \mathbb{C})$, we let $d_\tau(x)$ denote the product of four 3×3 minors $d_{jk\ell}(x)$, where (j, k, ℓ) is the oriented triple corresponding to one of the four oriented faces of τ. Notice that the quantity $d_\tau(x)$ is independent of the choice of an orientation of τ. The quotient space $\mathbf{X}(3, 6)$ can be realized in the 29-dimensional projective space by the following left $\mathrm{GL}(3, \mathbb{C})$ and right $(\mathbb{C}^*)^6$ invariant map:

$$Trh : M^\circ(3, 6) \ni x \mapsto [\ \ldots, D_\tau(x), \ldots\ ] \in \mathbb{P}^{29},$$

where τ runs through the thirty labeled tetrahedra. The map Trh defines an embedding of $\mathbf{X}(3, 6)$; let us call by Z the closure of the image. We can make a compactification $\overline{\mathbf{X}(3, 6)}$ of $\mathbf{X}(3, 6)$ such that the embedding $Trh : x \to Z$ can extend to a bijection. The Poincaré duality mentioned above defines an involution ♠ on Z. The involutions ♥ on $\mathbf{X}(3, 6)$ and ♠ on Z are equivariant under the isomorphism Trh, i.e.,

$$Trh \heartsuit = \spadesuit Trh\ ;$$

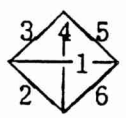
thus the involution ♥ on $\mathbf{X}(3, 6)$ naturally extends on $\overline{\mathbf{X}(3, 6)}$. The quotient space $\overline{\mathbf{X}(3, 6)}/\heartsuit$ happens to be isomorphic to the 4-dimensional projective space, it can be realized by the following map:

$$Pl : M^\circ(3, 6) \ni x \mapsto [\ \ldots, d_{hjk}(x)d_{\ell mn}(x), \ldots\ ] \in \mathbb{P}^9,$$

where (h, j, k, ℓ, m, n) is a permutation of $(1, 2, 3, 4, 5, 6)$ $\left(\text{notice that } \binom{6}{3}\Big/2 = 10\right)$. There are linear relations among the $d_{hjk}(x)d_{\ell mn}(x)$'s which are called the Plücker relations:

$$d_{hjk}(x)d_{\ell mn}(x) - d_{hj\ell}(x)d_{mnk}(x) + d_{hjm}(x)d_{nk\ell}(x) - d_{hjn}(x)d_{k\ell m}(x) = 0; \qquad (4.3.1)$$

thus Pl gives a linear embedding of $\overline{\mathbf{X}(3, 6)}/\heartsuit$ into $\mathbb{P}^9$. The space $\overline{\mathbf{X}(3, 6)}$ can be realized in $\mathbb{P}^{45}$ as the double cover of $Z \simeq \overline{\mathbf{X}(3, 6)}/\heartsuit$ by using the Segre embedding $\mathbb{P}^9 \subsetneq \mathbb{P}^{44}$ and $D_\tau(x) - D_{*\tau}(x)$ where τ is a labeled tetrahedron and $*\tau$ is the Poincaré duality of τ, since we can see that the absolute value of $D_\tau(x) - D_{*\tau}(x)$ does not depend on τ and its signature changes by the action of the involution ♥. For the labeled tetrahedron

$\tau =$ 

,

we have

$$
\begin{aligned}
&D_\tau(x) + D_{*\tau}(x) \\
&= d_{123}(x)d_{456}(x)d_{156}(x)d_{234}(x) + d_{123}(x)d_{456}(x)d_{125}(x)d_{346}(x) \\
&\quad + d_{125}(x)d_{346}(x)d_{134}(x)d_{256}(x) + d_{135}(x)d_{246}(x)d_{156}(x)d_{234}(x) \\
&\quad - d_{134}(x)d_{256}(x)d_{135}(x)d_{246}(x)\,, \\
&D_\tau(x)\,D_{*\tau}(x) \\
&= d_{123}(x)d_{456}(x)d_{126}(x)d_{345}(x)d_{135}(x)d_{246}(x)d_{156}(x)d_{234}(x)\,;
\end{aligned}
$$

these equalities together with (4.3.1) lead

$$
(D_\tau(x) - D_{*\tau}(x))^2 = \tfrac{1}{12}[\{\Sigma\, d_{hjk}(x)^2 d_{\ell mn}(x)^2\}^2 - 4\Sigma\, d_{hjk}(x)^4 d_{\ell mn}(x)^4]\,. \tag{4.3.2}
$$

4.4 Relations between the modular forms and the period map

Theorem 4.4.1. *We have the following commutative diagram:*

$$
\begin{array}{ccc}
\overline{\mathbf{X}(3,6)} & \xrightarrow{Pl} & \mathbb{P}^9 \\
\big\downarrow{\scriptstyle Prd} & & \big\downarrow{\scriptstyle F} \\
\mathrm{D_{IV}} & \xrightarrow{\iota_1 \circ \iota_2^{-1}} & \mathbf{H}_2\,.
\end{array}
$$

Proof. We can see that the $d_{hjk}(x)\mathrm{d}_{\ell mn}(x)$'s and $\Theta\begin{pmatrix} hjk \\ \ell mn \end{pmatrix}^2$'s satisfy the same relations by Remark 3.1.2 and (4.3.1). By studying degeneration of cycles $c_j(L)$ where configuration L is not in general position, we can easily check that $d_{hjk}(x)d_{\ell mn}(x)$ and $\Theta\begin{pmatrix} hjk \\ \ell mn \end{pmatrix}^2 (\iota_1 \circ \iota_2^{-1} \circ Prd(x))$ vanish on the same divisor which consists of two kinds of configurations: one corresponds to lines L_h, L_j and L_k intersecting at a point and another corresponds to lines L_ℓ, L_m and L_n intersecting at another point. These prove the theorem. □

Corollary 4.4.2. *We have the following commutative diagram:*

$$
\begin{array}{ccc}
\overline{\mathbf{X}(3,6)} & \xrightarrow{\Phi} & \mathbb{P}^{45} \\
\big\downarrow{\scriptstyle Prd} & & \big\downarrow{\scriptstyle \Psi} \\
D_{\mathrm{IV}} & \xrightarrow{\iota_1 \circ \iota_2^{-1}} & \mathbf{H}_2
\end{array}
$$

where Φ is given by

$$
\Phi : L \mapsto [D_\tau(x), \ldots, d_{hjk}(x)d_{\ell mn}(x)d_{pqr}(x)d_{stu}(x), \ldots]\,,
$$

$$
D_\tau(x) = \frac{15}{4}\{D_\tau(x) - D_{*\tau}(x)\}\,,\quad \tau = \begin{array}{c} 3\ \ 4\ \ 5 \\ \diamond\!\!-\!\!-\!\!\diamond \\ 2\ \ 1\ \ 6 \end{array}\,.
$$

Proof. For the involution ♥ on $\mathbf{X}(3,6)$, we have

$$
\begin{aligned}
&D_\tau(\heartsuit x) = -\,D_\tau(x)\,, \\
&\Theta[\iota_1 \circ \iota_2^{-1}\{Prd(\heartsuit x)\}] = \Theta[{}^t\{\iota_1 \circ \iota_2^{-1}\{Prd(x)\}\}] - \Theta[\iota_1 \circ \iota_2^{-1}\{Prd(x)\}]\,.
\end{aligned}
$$

On the other hand, we can check that the images of Φ and Ψ in $\mathbb{P}^{45}$ coincide by Theorem 4.4.1, Proposition 3.1.4, identity (4.3.2) and a straightforward calculation. These facts prove the corollary. □

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