Math. Ann. 277, 213–232 (1987)

Mathematische Annalen

Nonstrictly Hyperbolic Nonlinear Systems*

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1. Introduction

This paper addresses the well posedness of the initial value problem, locally in time, for a class of nonlinear hyperbolic systems. These systems are neither required to be strictly hyperbolic nor to be symmetrizable hyperbolic, and the admissible class includes systems with possible multiple characteristics.

With $u(x,t)$ an N-component vector for $x \in R^n$, $t \in R^+$ we study the problem

$$\begin{aligned} &\frac{\partial^2}{\partial t^2} u + F\left(bu, au, c\frac{\partial}{\partial t}u; x, t\right) = 0 \\ &u(x,0) = f(x), \quad \frac{\partial}{\partial t} u(x,0) = g(x). \end{aligned} \tag{1.1}$$

Here $b\left(x, t, D_x, \frac{\partial}{\partial t}\right)$ is a second order differential operator, and $a(x,t,D_x)$, $c(x,t,D_x)$ are classical pseudodifferential operators of orders $0 \leqq d < 2$ and 0 respectively. Several conditions will be imposed upon the pseudodifferential operators and the function F, essentially insuring hyperbolicity and a sign condition on the subprincipal term of the linearized equation. The initial data is assumed to belong to some Sobolev class; more exact statements of the hypotheses on the system and the initial data will appear in Sects. 2–4 below.

Recently nonlinear problems which are hyperbolic but not necessarily strictly hyperbolic have arisen in several problems of continuum mechanics. One of these is known as the water waves problem, concerning the motion of the free surface of a body of an incompressible irrotational fluid under the influence of gravity. In two dimensions the linear dispersion relation is $\omega^2 = gk\tanh(hk)$, from which it is easy to see that the linearized problem about equilibrium has a double characteristic, with characteristic velocity zero. The inherent problems of loss of differentiability

* Research supported by the Office of Naval Research, the Air Force Office of Scientific Research, the Army Research Office, and the National Science Foundation

of solutions of the initial value problem persist for the nonlinear problem. However it turns out that in Lagrangian coordinates the initial value problem is well posed for data in a sufficiently high Sobolev class; these are results of Nalimov [11], Yosihara [14], and Craig [4]. The energy methods used in the present paper were motivated by this work, and a model problem exhibiting the basic idea used for the problem of water waves is presented in Sect. 4.

The approach to the initial value problem for Sobolev initial data is based on the existence of an energy estimate which reflects the loss of differentiability of solutions in the case that characteristics of the linearized system do degenerate. An outline of the procedure goes as follows. By differentiation one constructs an equivalent quasilinear problem, which is a candidate for an iterative scheme similar to the classical one of Leray [10]. An energy norm is constructed out of the principal and subprincipal operators of an appropriate linearized problem; it is here that hypotheses on the principal operator and the sign condition on the subprincipal operator are strongly used.

The basic requirements on the linearized operator can be summarized as follows; (i) Setting $B_1 = \sum_{i,j=1}^{n} \operatorname{diag}(dF_{bu}b)_{ij}$, for sufficiently smooth functions w

$$\int \langle B_1 w, w\rangle dx \geqq 0 . \tag{1.2}$$

This quadratic form however need not be positive definite. (ii) The coefficient matrices of $(dF_{bu}b)_{0j}$ should be symmetric. These hypotheses on our class of systems imply the possession of real bicharacteristics, this is the assumption of hyperbolicity. (iii) Denoting $A = \operatorname{diag}(dF_{au}a(x,t,D)) + \operatorname{diag}\left(dF_{bu}\left[\frac{\partial}{\partial x_j}, b\right]\right)$ the relevant subprincipal operator, it is required that $A_1 = \frac{1}{2}(A + A^*)$ satisfy

$$\int \langle A_1 w, w\rangle dx \geqq 0 . \tag{1.3}$$

This sign condition essentially insures that dispersion is sufficiently strong in the presence of the degeneracies, that is, higher multiplicities of characteristics of the principal operator.

A comparison of the strength of this dispersion is in hypothesis (iv); denoting $A_2 = A - A_1$ there must be a constant such that

$$|\int \langle A_2 w, w\rangle dx| + \left|\int \left\langle \left(\frac{\partial}{\partial t} A_1\right) w, w\right\rangle dx\right| \leqq \text{const} \int \langle B_1 w, w\rangle + \langle A_1 w, w\rangle dx . \tag{1.4}$$

As well several technical hypotheses are needed for higher Sobolev estimates; a description of these appears in Sect. 3, where the main estimates are carried out. When initial data is in a sufficiently regular Sobolev class, these energy estimates are used to demonstrate the convergence of the iterative scheme of Leray, proving the well posedness statement Theorem 3.1.

The theory of linear equations with multiple characteristics is well developed, and sign conditions on the subprincipal symbol play an important rôle. Furthermore one may microlocalize the problem near multiple bicharacteristics and obtain normal forms and parametrices, tools that are at present unavailable for general nonlinear problems. The technique presented in this article depends upon the analogous sign condition on the subprincipal term in the linearized equation.

There is a fair amount of previous work on the initial value problem for nonlinear non strictly hyperbolic systems. Mostly this has been within the class of analytic data and analytic equations, involving variants of the Cauchy-Kowalewski theorem. For pseudodifferential equations there is an abstract form of the theorem due to Nirenberg [12] and Nishida [13], which appears as well in the work of Baouendi and Goulaouic [1]. In another paper [2], Baouendi and Goulaouic also address the case of non-involutive multiple characteristics in nonlinear equations, where the dominant terms are of Fuchsian type; the results of the present paper are not applicable in this situation. Finally, by using Cauchy majorant methods Leray and Ohya [9] addressed the problem for data within a Gevrey class, weakening the assumption of analyticity. The present work considers a restricted class of nonstrictly hyperbolic systems, however with only Sobolev assumptions on the initial data. All of the above results are of course for an existence theory local in time.

Consider hyperbolic systems in first order form. Recent results on linear systems of Lax [6] and Friedland et al. [5] imply that strict hyperbolicity is topologically forbidden in a large class of systems. In the symmetrizable case the theory of symmetric hyperbolic systems is well developed, and gives the well-posedness results locally in time for initial data in an appropriate Sobolev class. On the other hand there are the cases in which the linearized system may possess nontrivial Jordan blocks or other nonstrictly hyperbolic features; little is known for such nonlinear problems. Solutions to the linearized system in general may lose differentiability, and since these linear problems typically arise from an iterative scheme for nonlinear problems, it may not be feasable to know delicate properties of the coefficients or of the degeneracies of the coefficients. A linear system with higher multiplicities of characteristics is particularly sensitive to perturbation; the principal operator must avoid the development of complex characteristics for example. This is essentially the hypothesis of hyperbolicity. Furthermore the subprincipal operator plays an important role in a well posedness result. In the results of this paper the sign condition (1.3) is imposed on the subprincipal operator in order to provide sufficient dispersion near degeneracies of characteristics, and to avoid the possibility of lines of constant group velocity becoming complex as well. The hypotheses (1.2), (1.3), and (1.4), with technical estimate (3.4v) quantify these requirements in the general case, while in Sect. 2 an example of the method is presented for a scalar second order equation in one space variable for which these hypotheses and the method of proof are more clearly presented. Notation in this paper is restricted to more or less standard Sobolev norms, continuous norms and multi-indices, none of which require special note.

I would like to thank Jürgen Moser for his kind invitation to the Forschungsinstitut für Mathematik at the ETH-Zürich, where part of this work was performed.

2. An Illustrative Example

Consider the initial value problem in one spatial dimension for a nonlinear equation in the following form

$$\begin{aligned} &\frac{\partial^2}{\partial t^2} u + F\left(\frac{\partial}{\partial x} Hu, \frac{\partial}{\partial t} u, u; x, t\right) = 0 \\ &u(x,0) = f(x), \quad \frac{\partial}{\partial t} u(x,0) = g(x). \end{aligned} \tag{2.1}$$

The operator H is the Hilbert transform for the upper half plane

$$Hu(x) = \frac{1}{\pi} \text{p.v.} \int \frac{u(y)}{x-y} dy = i \operatorname{sgn}(D) u,$$

where $D = \frac{1}{i}\frac{\partial}{\partial x}$ and $i\operatorname{sgn}(D)$ is a Fourier multiplier operator. Nonlinear problems of this sort have been studied recently in several contexts, in particular in the description of time dependent solutions to free boundary problems that arise in fluid mechanics [4]. We note that form (2.1) may be achieved for a nonlinear functional $G(u_{tt}, Hu_x, u_t, u) = 0$ as long as $G_{u_{tt}} \neq 0$, by an application of the implicit function theorem. The equation is not strictly hyperbolic; the linearized equation possesses double characteristics that are parallel to the t-axis. Since F depends upon x-derivatives of u, without further hypotheses in general the problem will be ill posed. With an additional condition upon F however, we will show that the initial value problem is well posed locally in time for Sobolev initial data, with solutions of the linearized initial value problem exhibiting only a mild loss of differentiability. The condition:

$$F_{Hu_x}(s) \geqq \gamma > 0, \tag{2.2}$$

where s varies over all values of (Hu_x, u_t, u) under consideration.

The implications of (2.2) for the linearized equation

$$v_{tt} + F_{Hu_x} H v_x + F_{u_t} v_t + F_u v = g$$

can be put roughly that characteristics are always real, and furthermore the lines of constant local group velocity remain real for relevant values of s. With these conditions we have the existence result.

Theorem 2.1 *If the initial data* $(f(x), g(x)) \in H^{r+3/2} \times H^{r+1}$ *for* $r \geqq 2$, *then solutions to the initial value problem exist in a time interval* $[0, T]$. *Furthermore if* $F(0; x, t) = 0$ *and* $F_u(0; x, t) = 0$ *then*

$$T \geqq O((\|f\|_{H, r+3/2} + \|g\|_{H, r+1})^{-1}).$$

The solution obeys the energy estimate

$$\begin{aligned} \|(u_t, |D|^{1/2} u)(t)\|_{H,r} &\leqq \exp\left(\text{const}(r) \int_0^t |u(t)|_{C,2} d\tau\right) \\ &\times \left(\|(g, |D|^{1/2} f)\|_{H,r} + \text{const}(r) \int_0^t \|F(\cdot; x, \tau)\|_{H,r} d\tau \right). \end{aligned} \tag{2.3}$$

In particular solutions of the initial value problem are unique.

Proof of this result is via energy estimates, made feasable by the sign condition (2.2) and the fact that F is independent of both second spatial derivatives or pure first spatial derivatives. In Sect. 3 the general existence result will be indicated, which indeed will cover this case. However the energy methods of this paper are in principle quite simple, and more clearly presented by hand in this scalar example in one spatial variable.

Form a quasilinear problem by differentiating the equation with respect to (x,t); one obtains the equation for $v=(u_x, u_t)$

$$v_{tt}+a(u)\frac{\partial}{\partial x}Hv+c(u)v_t+d(u)v+h(u)=0 \tag{2.4}$$

with initial data

$$v(x,0)=\frac{\partial}{\partial x}f(x), \qquad v_t(x,0)=\frac{\partial}{\partial x}g(x),$$

respectively,

$$v(x,0)=g(x)=f_1(x),$$

$$v_t(x,0)=u_{tt}(x,0)=-F(Hf_x, g, f; x, 0)=g_1(x).$$

The hypotheses on F_{Hu_x} imply that $a(u)\geqq\gamma$.

As usual with hyperbolic equations, one considers the solution map for linear equations of the form

$$\begin{aligned} &w_{tt}+a(u)\frac{\partial}{\partial x}Hw+cw_t+dw=h(x,t),\\ &w(x,0)=f_1, \qquad w_t(x,0)=g_1 \end{aligned} \tag{2.5}$$

with coefficients derived from (2.4). If the solution map S; $u\to u^+$, where $(u_x^+, u_t^+)=w$, exhibits a fixed point in an appropriate function space, this gives a solution to the nonlinear equation (2.1). To this end we proceed with energy estimates for solutions of the linear equation (2.5).

Consider $w(x,t)$ a solution of (2.5) which decays with several derivatives as $|x|\to\infty$.

$$\int_{-\infty}^{\infty} w_t\left(w_{tt}+a\frac{\partial}{\partial x}Hw+cw_t+dw\right)dx=\int_{-\infty}^{\infty} w_t h\,dx. \tag{2.6}$$

The analog of an integration by parts is required for the term $\int w_t a\frac{\partial}{\partial x}Hw\,dx$.

Lemma 2.2. (i)

$$\begin{aligned}\int w_t\left(a(x,t)\frac{\partial}{\partial x}Hw\right)dx=&\frac{1}{2}\frac{\partial}{\partial t}\int a(x,t)(|D|^{1/2}w)^2dx\\ &-\frac{1}{2}\int\frac{\partial a}{\partial t}(|D|^{1/2}w)^2dx+\int w_t[a,|D|^{1/2}]\,|D|^{1/2}w\,dx.\end{aligned}$$

(ii)

$$\|[a,|D|^{1/2}]u\|_{L^2}\leqq\text{const}\left|\frac{\partial a}{\partial x}\right|_{L^\infty}\|u\|_{L^2}.$$

Proof. Statement (i) follows whenever w possesses sufficient Sobolev regularity, and is precisely the integration by parts formula. (ii) represents a simple (and not sharp) commutator estimate, similar to the classical result of Calderon [3].

Using the lemma in place of more standard energy methods,

$$\frac{1}{2}\frac{\partial}{\partial t}\int w_t^2 + a(x,t)(|D|^{1/2}w)^2 dx$$
$$= \frac{1}{2}\int \frac{\partial a}{\partial t}(|D|^{1/2}w)^2 dx - \int w_t[a, |D|^{1/2}]\,|D|^{1/2}w dx$$
$$-\int c w_t^2 + d w_t w - h w_t dx \leqq \frac{1}{2}\left|\frac{\partial a}{\partial t}\right|_{L^\infty} \| |D|^{1/2} w\|_{L^2}^2$$
$$+ \left(\left|\frac{\partial a}{\partial x}\right|_{L^\infty} \| |D|^{1/2} w\|_{L^2} + |c|_{L^\infty}\|w_t\|_{L_2} + |d|_{L^\infty}\|w\|_{L^2} + \|h\|_{L^2}\right)\|w_t\|_{L^2}.$$

Since $a(x,t) \geqq \gamma > 0$ this gives a differential inequality for the energy integral

$$\frac{\partial}{\partial t}\left(\int w_t^2 + a(|D|^{1/2}w)^2 dx\right)^{1/2}$$
$$\leqq \text{const}\left|\left(\frac{\partial a}{\partial t}, \frac{\partial a}{\partial x}, c\right)\right|_{L^\infty}\left(\int w_t^2 + a(|D|^{1/2}w)^2 dx\right)^{1/2} + 2(|d|_{L^\infty}\|w\|_{L_2} + \|h\|_{L_2}). \tag{2.7}$$

Using the Gronwall argument to integrate this up, and denoting $\|w\|_E^2 = \int w_t^2 + a(|D|^{1/2}w)^2 dx$,

$$\|w(t)\|_E \leqq \exp\left(\text{const}\int_0^t \left|\left(\frac{\partial a}{\partial t}, \frac{\partial a}{\partial x}, c\right)\right|_{L^\infty} d\tau\right)$$
$$\times\left(\|w(0)\|_E + \int_0^t |d|_{L^\infty}\|w\|_{L^2} + \|h\|_{L^2} d\tau\right).$$

The L^2 norm of $w(x,t)$ may be obtained from integration in time,

$$\|w(t)\|_{L^2} \leqq \|w(0)\|_{L^2} + \int_0^t \|w_t(\tau)\|_{L^2} d\tau.$$

Notice that the energy norm controls one t derivative and one half an x derivative. For strict hyperbolic problems a full x derivative appears in the norm, the difference of one half is the manifestation of loss of differentiability due to the multiple characteristics.

Higher Sobolev estimates come from estimating derivatives of (2.5). Let $w^{(\alpha)}$ denote $\left(\frac{\partial}{\partial x}\right)^\alpha w$, we obtain

$$w_{tt}^{(\alpha)} + a\frac{\partial}{\partial x}Hw^{(\alpha)} + c w_t^{(\alpha)} = \Gamma_\alpha. \tag{2.8}$$

The right hand side consists of terms of at most order α;

$$\Gamma_\alpha = \left[a, \left(\frac{\partial}{\partial x}\right)^\alpha\right]\frac{\partial}{\partial x}Hw + \left[c, \left(\frac{\partial}{\partial x}\right)^\alpha\right]w_t - \left(\frac{\partial}{\partial x}\right)^\alpha d + \left(\frac{\partial}{\partial x}\right)^\alpha h.$$

As above this gives an energy inequality

$$\frac{1}{2}\frac{\partial}{\partial t}\int (w_t^{(\alpha)})^2 + a(|D|^{1/2}w^{(\alpha)})^2 dx \leq \frac{1}{2}\,\mathrm{const}(\alpha)\left|\left(\frac{\partial a}{\partial t}\,\frac{\partial a}{\partial x}, c\right)\right|_{L^\infty} \|w^{(\alpha)}\|_E^2 + \|\Gamma_\alpha\|_{L^2}\|w^{(\alpha)}\|_E\,.$$

Lemma 2.3.

(i) $$\left\|\left[a, \left(\frac{\partial}{\partial x}\right)^\alpha\right] Hu\right\|_{L^2} \leqq \mathrm{const}(\alpha)\left(\left|\frac{\partial a}{\partial x}\right|_{L^\infty}\left\|\left(\frac{\partial}{\partial x}\right)^{\alpha-1} u\right\|_{L^2} + |Hu|_{L^\infty}\left\|\left(\frac{\partial}{\partial x}\right)^\alpha a\right\|_{L^2}\right)$$

(ii) $$\left\|\left[c, \left(\frac{\partial}{\partial x}\right)^\alpha\right] u\right\|_{L^2} \leqq \mathrm{const}(\alpha)\left(\left|\frac{\partial c}{\partial x}\right|_{L^\infty}\left\|\left(\frac{\partial}{\partial x}\right)^{\alpha-1} u\right\|_{L^2} + |u|_{L^\infty}\left\|\left(\frac{\partial}{\partial x}\right)^\alpha c\right\|_{L^2}\right)$$

thus

(iii) $$\|\Gamma_\alpha\|_{L^2} \leqq \mathrm{const}(\alpha)\left(\left|\left(\frac{\partial a}{\partial x}, \frac{\partial c}{\partial x}, d\right)\right|_{L^\infty}\left\|\left(\frac{\partial}{\partial x}\right)^\alpha w\right\|_{L^2} + |(w, Hw)|_{L^\infty}\left\|\left(\frac{\partial}{\partial x}\right)^\alpha (a, c, d)\right\|_{L^2} + \left\|\left(\frac{\partial}{\partial x}\right)^\alpha h\right\|_{L^2}\right).$$

Proof. Use of the Gagliardo-Nirenberg inequalities when estimating terms involving commutators allows one to keep only low derivatives in L^∞ norms. Furthermore H is bounded on L^2, although not on L^∞. Inspection of Γ_α completes the proof.

Using Lemma 2.3 and the Gronwall argument on inequality (2.9)

$$\|w^{(\alpha)}(t)\|_E \leqq \exp\left(\mathrm{const}(\alpha)\int_0^t |(a, c, d)|_{C,1} d\tau\right) \times \left(\|w^{(\alpha)}(0)\|_E + \int_0^t \mathrm{const}(\alpha)\,|(w, Hw)|_{L^\infty}\left\|\left(\frac{\partial}{\partial x}\right)^\alpha (a, c, d)\right\|_{L^2} + \left\|\left(\frac{\partial}{\partial x}\right)^\alpha h\right\|_{L^2} d\tau\right). \quad (2.10)$$

With more care the constant in the exponent will depend only linearly upon α. The Sobolev inequality bounds $|(w, Hw)|_{L^\infty} \leqq \mathrm{const}\,\|w\|_{H,1}$, which is in turn bounded by $(\|w(t)\|_{L^2}^2 + \|w(t)\|_E^2 + \|w^{(1)}(t)\|_E^2)^{1/2}$. Let $\|w\|_{E,r}^2$ denote the sum of squares of $\|w^{(\alpha)}\|_E$ for $\alpha \leqq r$. Putting estimates (2.10) together the energy estimate is obtained.

$$\|w(t)\|_{E,1} + \|w(t)\|_{L^2} \leqq \exp\left(\mathrm{const}(1)\int_0^t |(a, c, d)|_{C,1} d\tau\right) \times \left(\|w(0)\|_{E,1} + \|w(0)\|_{L^2} + \int_0^t \|h(\tau)\|_{H,1} d\tau\right). \quad (2.11)$$

and for $r \geqq 2$

$$\|w(t)\|_{E,r} \leqq \exp\left(\mathrm{const}(r)\int_0^t |(a, c, d)|_{C,1} d\tau\right) \times \left(\|w(0)\|_{E,r} + \int_0^t \mathrm{cost}(r)\,\|(a, c, d)\|_{H,r}(\|w(0)\|_{E,1} + \|w(0)\|_{L^2}) + \|h\|_{H,r} d\tau\right).$$

Given the above, we may proceed with the nonlinear existence result. The method follows the line of a standard argument, with the only difference being the small loss of differentiability of solutions of the linear equation. The effect of this loss is the need for $\frac{1}{2}$ higher Sobolev derivative than is needed for the strictly hyperbolic case.

Consider the solution map S, with coefficients (a, c, d, h) depending upon $u \in H^r$ derived from (2.4).

$$\begin{aligned} a(x,t) &= F_{Hu_x}(Hu_x, u_t, u; x, t) \\ c(x,t) &= F_{u_t} \\ d(x,t) &= F_u \\ h(x,t) &= F_x, \quad \text{respectively} \quad F_t. \end{aligned}$$

Estimates of norms of these coefficients will give (2.11) as a bound on S.

Lemma 2.4. *From the composition of functions inequality* [10] *it follows*

(i) $\|(a,c,d)\|_{H,r} \leqq \text{const}(r)(1 + |(u, u_t, Hu_x)|_{L^\infty}) \|(u, u_t, u_x)\|_{H,r}$

(ii) $\|h(x,t)\|_{H,r} \leqq \text{const}(r)(\|F(s; x,t)\|_{H,r+1} + (1 + |(u, u_t, Hu_x)|_{L^\infty}) \|(u, u_t, u_x)\|_{H,r})$.

Consider initial data such that $\|f\|_{H,r+3/2} + \|g\|_{H,r+1} < \delta(r)$ and $\|F_x(x; x,t), F_t(s; x,t)\|_{H,r} < \delta(r)$. Suppose that $\|(u, u_x, u_t)(x,t)\|_{H,r} < R(r)$ for $r \geqq 2$. It follows from (2.11) that a solution of the linearized equation will satisfy

$$\|w(x,t)\|_{E,r} \leqq \exp(\text{const}(r)tR(2))(\delta(r) + \text{const}(r)t(\delta(z)R(r) + R(r) + \delta(r)). \tag{2.12}$$

Recovering $u^+(x,t) = f(x) + \int_0^t w(x,t)d\tau$ from (2.5) we may compare the Sobolev r-norm with that of $u(x,t)$:

$$\|u^+(t)\|_{E,r} \leqq \delta(r) + \int_0^t \|w(t)\|_{E,r} d\tau < R(r) \tag{2.13}$$

for $t < t(r)$ chosen sufficiently small, for $\delta(r)\exp(\text{const}(r)tR(2)) < R(r)$. Since the energy norm does not control a full spatial derivative of u_t, estimate (2.12) is used again, for an x derivative of (2.1);

$$\left\| \left(\frac{\partial}{\partial x}\right)^{r+1} u^+(t) \right\|_{L^2} < R(r)$$

for perhaps somewhat smaller $t(r)$. Thus on a Banach space whose norm is

$$| * |_{B,r} = \sup_{[0,t(r)]} \left(\| * \|_{E,r} + \left\| \left(\frac{\partial}{\partial x}\right)^{r+1} * \right\|_{L^2} \right)$$

the solution map takes the ball of radius $R(r)$ to itself.

A contraction argument in the lowest E-norm, along with an interpolation between E and E^r norms will complete the existence proof. Let u_1 and u_2 be two functions such that $|u_j|_{B,r} < R(r)$, satisfying the initial data, and denote $\delta u = u_2 - u_1$.

The equation for the difference $\delta w = \delta u_t^+$ is from (2.5)

$$(\delta w)_{tt} + a(u_1)\frac{\partial}{\partial x} H(\delta w) + c(u_1)(\delta w)_t + d(u_1)(\delta w)$$
$$= (a(u_1) - a(u_2))\frac{\partial}{\partial x} H w_2 + (c(u_1) - c(u_2)) w_{2t}$$
$$+ (d(u_1) - d(u_2)) w_2 + (h(u_1) - h(u_2))$$
$$= G.$$

Applying the energy estimate (2.7) yields

$$\|\delta w(t)\|_E + \left\|\frac{\partial}{\partial x}\delta w(t)\right\|_{L^2} \leqq \exp(\operatorname{const} t R(2)) \int_0^t \|G\|_{L^2} d\tau$$

with furthermore

$$\|G\|_{L^2} \leqq \operatorname{const} R(2)\,\|(\delta u, \delta u_t, \delta u_x)\|_{L^2}$$
$$\leqq \operatorname{const} R(2)\left(\|\delta u\|_E + \left\|\frac{\partial}{\partial x}\delta u\right\|_{L^2}\right).$$

For sufficiently small time this provides a contraction estimate in B^0; for some $\theta < 1$

$$|\delta u^+|_{B^0} \leqq \operatorname{const} R(2) t\, |\delta u|_{B^0} \leqq \theta |\delta u|_{B^0}.$$

To finish the argument, iteration of the solution map S provides a sequence of functions which converge in B^0, which are bounded in B^r. Interpolation them shows that the sequence converges in B^s for and $s < r$; the limit is a fixed point of S and a classical solution to the nonlinear equation (2.1).

If $F(s; x, t)$ and $f(x)$, $g(x)$ are C^∞ in their dependence upon (x, t) and the parameter s, the solution provided by these arguments is C^∞ as well. For given $\varrho > r$ the time $t(\varrho)$ for which one may show boundedness of S is perhaps shorter than $t(r)$, but by a familiar extension argument the solution may be continued up to the full interval of time for which the contraction in B^0 is available. The lowest r for which this gives classical solutions is $r = 2$.

3. General Results

The existence and uniqueness results for the initial value problem hold more generally for systems of hyperbolic equations which are not necessarily strictly hyperbolic. For $x \in \mathbf{R}^n$ we will consider functions $u(x, t) = (u_1(x, t), \ldots, u_N(x, t))^t$. Let $b\left(x, t, D, \frac{\partial}{\partial t}\right)$ be a second order differential operator, and $a(x, t, D)$, $c(x, t, D)$ be classical pseudo-differential operators of orders $0 \leqq d < 2$ and 0 respectively. The systems that we address are in the form

$$\frac{\partial^2}{\partial t^2} u + F\left(bu, au, c\frac{\partial}{\partial t}u; x, t\right) = 0,$$
$$u(x, 0) = f(x), \qquad \frac{\partial}{\partial t} u(x, 0) = g(x). \tag{3.1}$$

The nonlinear functions $F_j, j=1, \ldots, N$ are assumed to be smooth in all arguments, satisfying some nonnegativity conditions analogous to (2.2) to be described below.

Form a larger quasilinear system by differentiating (3.1) with respect to all independent variables; for $v=((u_t)^t, (u_{x_1})^t \ldots (u_{x_n})^t)^t$

$$\frac{\partial^2}{\partial t^2} v + Bv + Av + C\frac{\partial}{\partial t} v + Dv + G = 0. \tag{3.2}$$

As in Sect. 2 the coefficients (A, B, C, D, G) of this quasilinear system will be considered as functions of v, and the solution map $S; v \to w$ of the resulting linear system will be shown to exhibit a fixed point in an appropriate function space. For functions $v(x,t)$ under consideration we will ask that the coefficients satisfy some nonnegativity and symmetry conditions. In (3.2) $B = \operatorname{diag}(dF_{bu}b) = B_0 + B_1$ is a block diagonal second order differential operator, involving at most one time derivative. For $x_0 = t$

$$B_0 = \sum_{j=1}^{n} \operatorname{diag}(dF_{bu}b)_{0j},$$

$$B_1 = \sum_{i,j=1}^{n} \operatorname{diag}(dF_{bu}b)_{ij},$$

which we assume satisfies

$$\int \langle B_1 w, w \rangle dx \geqq 0 \tag{3.3}$$

but which need not necessarily be positive definite. The operator

$$A = \operatorname{diag}(dF_{au}a(x,t,D)) + \begin{pmatrix} dF_{bu}\left[\frac{\partial}{\partial t}, b\right] & & \\ & \ldots & \\ & & dF_{bu}\left[\frac{\partial}{\partial x_n}, b\right] \end{pmatrix}$$

is of order $\max(d, 1)$, and would be a classical pseudodifferential operator if $v(x,t)$ were smooth. Following terms of (3.2) are

$$C = \operatorname{diag}(dF_{cu}c(x,t,D)),$$

$$\mathrm{D} = \begin{pmatrix} dF_{au}\left[\frac{\partial}{\partial t}, a\right] & & \\ & \ldots & \\ & & dF_{au}\left[\frac{\partial}{\partial x_n}, a\right] \end{pmatrix} + \begin{pmatrix} dF_{cu_t}\left[\frac{\partial}{\partial t}, c\right]\frac{\partial}{\partial t} & & \\ & \ldots & \\ & & dF_{cu_t}\left[\frac{\partial}{\partial x_n}, c\right]\frac{\partial}{\partial t} \end{pmatrix}$$

and G is the term resulting from derivatives falling on the explicit dependence of F on (x,t).

An energy estimate analogous to (2.7) will motivate the following conditions on (A, B);

$$\sum_{i,j=1}^{n} \int \langle (dF_{bu}b)_{ij} w, w \rangle dx = \sum_{i,j=1}^{n} \int \langle B_{ij} w_{x_i}, w_{x_j} \rangle dx > 0. \tag{3.4i}$$

This is a symmetry condition on the second spatial derivatives in (3.2), which along with nonnegativity implies (3.3). This condition is stronger than hyperbolicity of the linearized operator, however it plays a role in the nonlinear results presented in this paper.

$$\sum_{j=1}^{n} \int \left\langle (dF_{bu})_{0j} w, \frac{\partial w}{\partial t} \right\rangle dx = -\frac{1}{2} \sum_{j=1}^{n} \int \left\langle \frac{\partial}{\partial x_j} (dF_{bu} b_{0j}) \frac{\partial w}{\partial t}, \frac{\partial w}{\partial t} \right\rangle dx \qquad (3.4\text{ii})$$

which is a symmetry condition on the coefficient matrices of $(dF_{bu}b)_{0j}$. The operator A may be decomposed into its symmetric and skew symmetric components, $A = A_1 + A_2$ respectively; then

$$\int \langle A_1(x,t,D)w, w\rangle dx \geqq 0 . \qquad (3.4\text{iii})$$

This condition could in fact be somewhat weakened by invoking the sharp Garding inequality. I thank one of the referees for pointing this out.

The operators A_1, B_1 are used in the construction of an energy norm. The final conditions upon the operators (A, B) are to insure that this energy norm, although not necessarily controlling $\|Dw\|_{L^2}$ or $\| |D|^{1/2} \|_{L^2}$, still is large enough to produce an energy inequality. Define

$$\|w\|_E^2 = \int \langle w_t, w_t\rangle + \langle A_1 w, w\rangle + \langle B_1 w, w\rangle dx .$$

It is asked that for some constant depending only upon L^∞ bounds of $v(x,t)$ and its derivatives

$$\int \langle A_2 w, w_t\rangle dx \leqq \text{const}(v) \|w\|_E^2 ,$$
$$\int \left\langle \frac{\partial}{\partial t} A_1 w, w \right\rangle dx \leqq \text{const}(v) \|w\|_E^2 , \qquad (3.4\text{iv})$$

and as well

$$\int \left\langle \frac{\partial}{\partial t} B_1 w, w \right\rangle dx \leqq \text{const}(v) \|w\|_E^2 .$$

This is the most restrictive requirement on the nonlinear system (1.1). For a nonlinear system which is not symmetrizable hyperbolic, this estimate guarantees that the dispersive effects of lower order terms are strong enough to control error terms in a nonlinear result. The energy norm may not control the H^1 norm of w; (3.4) requires that it be strong enough to control skew symmetric contributions and other lower order terms. From the functional dependence of F on up to second derivatives of u, we will have $\text{const}(v) \leqq \text{const}(1 + |v|_{C,3})$. Finally, in order to obtain higher Sobolev estimates from system (3.2) we require that

$$\left\| \left(\frac{\partial}{\partial x_j} B_1 \right)_{ki} \frac{\partial}{\partial x_i} w \right\|_{L^2}^2 \leqq \text{const}(v) \int \langle B_1 w, w\rangle dx ,$$
$$\left\| \left[\frac{\partial}{\partial x_j}, A \right] w \right\|_{L^2}^2 \leqq \text{const}(v) \int \langle A_1 w, w\rangle + \langle B_1 w, w\rangle dx . \qquad (3.4\text{v})$$

It is known in the theory of linear hyperbolic differential equations that the reality of the principal operator, and in the presence of degenerate characteristics a sign condition on the subprincipal operator play a role in the well posedness of the initial value problem. In this class of nonlinear problems conditions (3.4i) through (v) play a similar role; they imply in particular that the linearization of (3.1) about differing $v(x,t)$ will not lead to an ill posed linear initial value problem. Conditions (3.4i, ii) are akin to the requirement of hyperbolicity for linear systems, and imply that the characteristics of the system are real. The conditions do not exclude multiply characteristics, or Jordon blocks in the equivalent first order system; they are of course weaker than symmetric hyperbolicity. However they do exclude the possibility that characteristics become complex. Condition (3.4iii) addresses the lines of constant group velocity; the nonnegativity of the subprincipal operator insures that they remain real as well. Essentially it requires dispersion to be sufficiently strong, useful in the energy norm near degeneracies of B_1, it is the analog of the nonnegativity condition (2.2). We remark that in the equations of linear waves in the surface of a fluid, nonnegativity of the subprincipal operator is precisely the condition of nonnegativity of the constant of gravitation.

The requirement (3.4iv) comes from the necessity of controlling error terms arising either from B or from derivatives of A_1. Furthermore it limits the support of A_2 to avoid degeneracies of B_1 or the compliment of the support of A_1. It is the most stringent condition of (3.4). The final requirements are somewhat technical in nature. First, control of first derivatives of the coefficients of B_1 are required to be controlled by B_1 itself. Within regions of positivity of B_1 this is usual. Near points of degeneracy of B_1 we note that the lhs of (3.4v) is quadratic in B_1, while the rhs is linear, so the condition is satisfied for typical smooth nonnegative coefficients. The second condition is that a commutator is supported only in regions controlled by the energy norm. Under the above assumptions we have the following result.

Theorem 3.1. *Given a system satisfying* (3.4i) *through* (v) *and initial data* $\|f\|_{H,r+d/2+1}+\|g\|_{H,r+1}=\delta(r)<\infty$ *for* $r=r_0=\frac{n}{2}+3$ *the minimum Sobolev index, then there exists a solution to* (3.1) *in a time interval* $[0,T]$. $T>0$ *depends upon* r_0, $\delta(r_0)$ *and constants otherwise independent of the data. Furthermore, if* $F(0;x,t)=0$ *then at least*

$$T>O(\delta^{-2}(r_0)).$$

Solutions obey the energy estimate

$$\|v(t)\|_{E,r}\leqq\exp\left(\text{const}(r)\int_0^t(1+|v|_{C,1})dt\right)\left(\|v(0)\|_{E,r}+C_r\int_0^t\|F(s;x,t)\|_{H,r+1}d\tau\right).$$

If $F(s;x,t)$, a, b, c *and the data are* C^∞, *then the solution is* C^∞ *in the entire interval* $[0,T]$.

Energy estimates for (3.2) proceed along the same lines as in Sect. 2. Consider the solution map $S:v\to w$ of the linearized problem, where the coefficients (A,B,C,D,G) are evaluated at v, and solve the initial value problem for $w(x,t)$ satisfying the correct initial data

$$w(x,0)=\left(\frac{\partial}{\partial x_j}f\right),\quad w_t(x,0)=\left(\frac{\partial}{\partial x_j}g\right)\quad j=0,1,\dots,n.$$

Integrating the linearized equation against $w_t(x,t)$;

$$\int\langle(w_{tt}+Bw+Aw+Cw_t+Dw),w_t\rangle dx=-\int\langle G,w_t\rangle dx.$$

Using (3.4i)

$$\frac{1}{2}\frac{\partial}{\partial t}\int\langle w_t, w_t\rangle+\langle A_1 w, w\rangle+\langle B_1 w, w\rangle dx$$
$$=\frac{1}{2}\int\left\langle\left(\frac{\partial}{\partial t}A_1+\frac{\partial}{\partial t}B_1\right)w, w\right\rangle dx$$
$$-\int\langle A_2 w, w_t\rangle+\langle B_0 w, w_t\rangle+\langle C w_t, w_t\rangle+\langle D w, w_t\rangle dx-\int\langle G, w_t\rangle dx. \quad (3.5)$$

From hypotheses (3.4ii) through (iv)

$$\leqq \mathrm{const}(v)(\|w(t)\|_E^2+\|w\|_{L^2}^2+\|w\|_E\,\|G\|_{L^2}).$$

This differential inequality gives directly

$$\|w(t)\|_E \leqq \exp\left(\int_0^t \mathrm{const}(v(t))d\tau\right)\left(\|w(0)\|_E+\int_0^t \|w(\tau)\|_{L^2}+\|G(\tau)\|_{L^2}d\tau\right). \quad (3.6)$$

By a further integration in time

$$\|w(t)\|_{L^2}\leqq\|w(0)\|_{L^2}+\int_0^t \|w(\tau)\|_E d\tau \quad (3.7)$$

which serves as bounds on $\|w(0)\|_{L^2}$ wherever either the quadratic form $\int\langle A_1 w, w\rangle+\langle B_1 w, w\rangle dx$ degenerates, or where control on L^2 norms withouth derivatives are necessary. The notation that is used is

$$\|w(0)\|_E^2=\int\langle g, g\rangle+\langle A_1, f, f\rangle+\langle B_1 f, f\rangle dx.$$

Higher Sobolev estimates are obtained from differentiations of the system (3.2) with respect to spatial variables x_j. Denote $w^{(\alpha)}=\left(\frac{\partial}{\partial x}\right)^\alpha w$ with the usual multiindices,

$$w_{tt}^{(\alpha)}+Bw^{(\alpha)}+Aw^{(\alpha)}+Cw_t^{(\alpha)}+D_\alpha w+\Gamma_\alpha(w)=0 \quad (3.8)$$

where lower order terms D_α, Γ_α are

$$D_\alpha w=Dw^{(\alpha)}+\sum_{\substack{\alpha-\beta\geqq 0\\|\beta|=1}}\left(\left(\frac{\partial}{\partial x}\right)^\beta B\right)w^{(\alpha-\beta)}+\left(\left(\frac{\partial}{\partial x}\right)^\beta A\right)w^{(\alpha-\beta)}$$
$$+\sum_{\substack{\alpha-\beta\geqq 0\\|\beta|=2}}\left(\left(\frac{\partial}{\partial x}\right)^\beta B\right)w^{(\alpha-\beta)}$$

$$\Gamma_\alpha(w)=\left(\frac{\partial}{\partial x}\right)^\alpha G+\left[\left(\frac{\partial}{\partial x}\right)^\alpha, D\right]w+\left[\left(\frac{\partial}{\partial x}\right), C\right]w_t$$
$$+\sum_{\substack{\alpha-\beta\geqq 0\\|\beta|=1}}\left[\left(\frac{\partial}{\partial x}\right)^{\alpha-\beta}, \left(\frac{\partial}{\partial x}\right)^\beta (A+B)\right]w$$
$$+\sum_{\substack{\alpha-\beta\geqq 0\\|\beta|=2}}\left[\left(\frac{\partial}{\partial x}\right)^{\alpha-\beta}, \left(\frac{\partial}{\partial x}\right)^\beta B\right]w.$$

Integration of (3.8) against $\frac{\partial}{\partial t} w^{(\alpha)}$ gives the usual energy estimate

$$\frac{1}{2}\frac{\partial}{\partial t}\|w^{(\alpha)}(t)\|_E=\frac{1}{2}\int\left\langle\left(\frac{\partial}{\partial t}A_1+\frac{\partial}{\partial t}B_1\right)w^{(\alpha)},w^{(\alpha)}\right\rangle dx$$
$$-\int\langle(A_2w^{(\alpha)}+B_0w^{(\alpha)}+Cw_t^{(\alpha)}),w_t^{(\alpha)}\rangle dx$$
$$-\int\langle D_\alpha w+\Gamma_\alpha(w),w_t^{(\alpha)}\rangle dx\,. \tag{3.10}$$

Hypotheses (3.4ii–iv) are used to address the first two terms of the right hand side, it is the third term which needs more work.

Lemma 3.2. *Estimates on $D_\alpha w$ and $\Gamma_\alpha(w)$ for $|\alpha|=r$*

(i) $$\|D_\alpha w\|_{L^2}\leqq \mathrm{const}(r)(1+|v|_{C,3})\sum_{|\gamma|=r}\|w^{(\gamma)}\|_E$$

(ii) $$\|\Gamma_\alpha(w)\|_{L^2}\leqq C_r(1+|v|_{C,3})\left(\|w\|_{E,r-1}+(1+|aw|_{L^\infty}+|w|_{C,2}+|w_t|_{C,1})\|v\|_{E,r}+\left\|\left(\frac{\partial}{\partial x}\right)^\alpha F_{x_j}(s;x,t)\right\|_{L^2}\right).$$

The constant const(*r*) *is at most quadratic in r.*

This is a technical lemma, whose proof is defered to the end of this section. By $\left\|\left(\frac{\partial}{\partial x}\right)^\alpha F(s;x,t)\right\|_{L^2}$ we denote $\|F_{x^\alpha}(u;x,t)\|_{L^2}$, that is the norm when only partial derivatives of F in (x,t) are performed. Using this in (3.10) and summing over $|\alpha|=r$,

$$\frac{\partial}{\partial t}\sum_\alpha\|w^{(\alpha)}(t)\|_E\leqq \mathrm{const}(r)(1+|v|_{C,3})\sum_\alpha\|w^{(\alpha)}(t)\|_E$$
$$+C_r((1+|v|_{C,3})\|w\|_{E,r-1}+(1+|aw|_{L^\infty}+|w|_{C,2}+|w_t|_{C,1})\|v\|_{E,r})$$
$$+C_r\|F(s;x,t)\|_{E,r+1}\,. \tag{3.11}$$

The Gronwall argument and a summation over r gives the higher Sobolev estimates

$$\|w(t)\|_{E,r}\leqq\exp\left(\mathrm{const}(r)\int_0^t(1+|v|_{C,3})d\tau\right)$$
$$+\left(C_r\|w(0)\|_{E,r}+C_r\int_0^t(1+|aw|_{L^\infty}+|w|_{C,2}+|w_t|_{C,1})\|v\|_{E,r}+\|F(s;x,t)\|_{H,r+1}d\tau\right). \tag{3.12}$$

Because of possible degeneracies of $B_1\left(x,t,D,\frac{\partial}{\partial t}\right)$ and $A_1(x,t,D)$ the Sobolev estimate is not directly applicable. Instead

$$|v(t)|_{C,3}\leqq\mathrm{const}\|v\|_{H,r_0}\leqq\mathrm{const}\left(\|v(0)\|_{H,r_0}+\int_0^t\|v(\tau)\|_{E,r_0}d\tau\right)$$

$$|aw(t)|_{L^\infty}+|w(t)|_{C,2}+|w_t(t)|_{C,1}\leqq\mathrm{const}\left(\|w(0)\|_{H,r_0}+\int_0^t\|w(\tau)\|_{E,r_0}d\tau\right)$$

where $r_0 = \frac{n}{2} + 3$. Thus we arrive in a standard situation of a priori estimates of solutions $w(x,t)$ to the initial value problem, with coefficients depending upon $v(x,t)$. The energy norm is adapted to the possible degeneracies of the problem, and gives estimate (3.12) provided (3.4i) through (v) are satisfied. Following the same procedure as outlined in Sect. 2, we consider initial data (f, g) such that $\|f\|_{H,r+d/2+1} + \|g\|_{H,r+1} < \delta(r)$, and consider functions $v(x,t)$ satisfying the initial data for (3.2) such that $\|v\|_{E,r} < R(r)$. The Sobolev index is taken $r \geqq r_0$. Ask further that $\left\|\left(\frac{\partial}{\partial x_j}\right) F(s; x, t)\right\|_{H,r+1} < \delta(r)$. A simple argument will shows that a solution to the linear initial value problem will exist, which by the energy estimates (3.12) obeys

$$\|w(t)\|_{E,r} \leqq \exp\left(\operatorname{const}(r) \int_0^t \left(1 + \|v(0)\|_{H,r_0} + \int_0^\tau \|v(\sigma)\|_{E,r_0} d\sigma\right) d\tau\right)$$

$$\times \left(C_r \|w(0)\|_{E,r} + C_r \int_0^t \left(\|w(0)\|_{H,r_0} + \int_0^\tau \|w(\sigma)\|_{E,r_0} d\sigma\right) \|v(\tau)\|_{E,r}\right.$$

$$\left. + \|F(s; x, \tau)\|_{H,r+1} d\tau\right)$$

$$\leqq \exp(\operatorname{const}(r) t(1+t) R(r_0)) (C_r \delta(r) (1 + tR(r)) + t^2 R(r_0) R(r))$$

$$< R(r) \tag{3.13}$$

for sufficiently short time intervals. The solution map $S; v \to w$ is thus bounded on balls of radius $R(r)$, and can be shown to contract in a lower norm, proving Theorem 3.1. Results on C^∞ solutions for C^∞ data follow from the above energy estimates for all integers r, plus the comment that for any $t < T$ (the maximum time of existence of solutions with finite E^{r_0} norm) the local existence results for E^r, $r > r_0$ may be invoked to extend the existence of a solution slightly past t.

It remains to prove Lemma 3.2. First the bounds upon $D_\alpha w$ are addressed. From (3.9) we consider term by term

$$\|Dw^{(\alpha)}\|_{L^2} = \left\| dF_{au}\left[\frac{\partial}{\partial x_j}, a\right] w^{(\alpha)} + dF_{cu_t}\left[\frac{\partial}{\partial x_j}, c\right] \frac{\partial}{\partial t} w^{(\alpha)} \right\|_{L^2}$$

$$\leqq \operatorname{const}(1 + |v|_{C,1}) \left\| \left(\left[\frac{\partial}{\partial x_j}, a\right] w^{(\alpha)}, \left[\frac{\partial}{\partial x_j}, c\right] \frac{\partial}{\partial t} w^{(\alpha)} \right) \right\|_{L^2}$$

$$\leqq \operatorname{const}(1 + |v|_{C,1}) \|w^{(\alpha)}\|_E, \tag{3.14}$$

where hypothesis (3.4v) is involved to control $\left[\frac{\partial}{\partial x_j}, a\right] w^{(\alpha)}$.

$$\left\| \left(\left(\frac{\partial}{\partial x}\right)^\beta B \right) w^{(\alpha-\beta)} \right\|_{L^2} \leqq \operatorname{const}(1 + |v|_{C,3}) \|w^{(\alpha)}\|_E \tag{3.15}$$

by hypothesis (3.4v).

$$\left\|\left(\left(\frac{\partial}{\partial x}\right)^{\beta} A\right) w^{(\alpha-\beta)}\right\|_{L^2} \leqq \left\|\left(\left(\frac{\partial}{\partial x}\right)^{\beta} dF_{au} a w^{(\alpha-\beta)}\right\|_{L^2}$$

$$+ \left\| dF_{au}\left(\left(\frac{\partial}{\partial x}\right)^{\beta} a\right) w^{(\alpha-\beta)}\right\|_{L^2}$$

$$+ \left\|\left(\left(\frac{\partial}{\partial x}\right)^{\beta} dF_{bu}\right)\left[\frac{\partial}{\partial x_j}, b\right] w^{(\alpha-\beta)}\right\|_{L^2}$$

$$+ \left\| dF_{bu}\left[\frac{\partial}{\partial x_j}, \left(\frac{\partial}{\partial x}\right)^{\beta} b\right] w^{(\alpha-\beta)}\right\|_{L^2}$$

$$\leqq \mathrm{const}(1+|v|_{C,2})\left(\sum_{|\gamma|=r} \|w^{(\gamma)}\|_E\right). \tag{3.16}$$

These are the most difficult terms of (i). Similar care controls (ii), where we remark that

$$\left\|\left(\frac{\partial}{\partial x}\right)^{\alpha} G(bu, au, cu_t; x, t)\right\|_{L^2}$$

$$\leqq \mathrm{const}(1+|v|_{C,1})\left(\left\|\left(v^{(\alpha)}, v_t^{(\alpha)}, \frac{\partial}{\partial x_j} b_{ij} v^{(\alpha)}\right)\right\|_{L^2}\right.$$

$$\left. + \left\|\left(\frac{\partial}{\partial x}\right)^{\alpha} F_{x_j}(s; x, t)\right\|_{L^2}\right)$$

$$\leqq \mathrm{const}(1+|v|_{C,1})(\|v^{(\alpha)}\|_E + \|F(s; x, t)\|_{H, r+1}) \tag{3.17}$$

using hypothesis (3.4ii) through (v). If $F(0; x, t) = 0$ then this estimate enters linearly in $\|v^{(\alpha)}\|_E$, leading to the lower bound on the duration of validity of these existence results.

4. Additional Cases

There are other cases not addressed in Sect. 3 which arise in several circumstances, for which it can be shown that the initial value problem is well posed, locally in time. One major example is the equation for the problem of water waves; waves in a free surface of an irrotational ideal fluid under the influence of gravity. The two dimensional problem in Lagrangian coordinates may be written

$$(1+\kappa_{1x})\kappa_{1tt} + \kappa_{2x}(g+\kappa_{2tt}) = 0, \qquad \kappa_{2t} = K(\kappa)\kappa_{1t}. \tag{4.1}$$

The curve $(x+\kappa_1(x,t), \kappa_2(x,t))$ describes the free surface, the Euler coordinates of Lagrangian fluid particles. The gravitation constant is g, and the operator $K(\kappa)$ is a singular integral operator, the Hilbert transform for the fluid region. This system is addressed in [4, 11, 14], and differs from our considerations in this paper by the fact that the operator $K(\kappa)$ depends nonlinearly upon the dependent variables $\kappa(x,t)$. We present here a model situation, involving a linear operator $K(x,t,D)$, which exhibits the main idea of estimation of solutions of equation (4.1) while avoiding some of the technicalities. Consider

$$\begin{gathered} F(u_x, u_{tt}, Ku_x, Ku_{tt}) = 0, \\ u(x,0) = f(x), \qquad u_t(x,0) = g(x), \end{gathered} \tag{4.2}$$

where $K(x,t,D)$ is an invertible zero order pseudodifferential operator with properties to be described below. An example is

$$Ku(x) = \frac{\text{p.v.}}{\pi} \int_{-\infty}^{\infty} \frac{u(y)}{x-y}\, dy. \tag{4.3}$$

Besides being smooth we ask of the function $F(s)$ that

$$dF(0)v = v_{tt} + Kv_x. \tag{4.4}$$

We will study solutions to the nonlinear initial value problem $u(x,t)$ such that $|(u_x, u_{tt}, Ku_x, Ku_{tt})|_{L^\infty} < R$ small enough so that the character of the equation linearized about $u(x,t)$ does not change drastically from (4.4). The most important property that is asked of F is that

$$(F_{u_x} F_{u_{tt}} + F_{Ku_x} F_{Ku_{tt}})(s) = 0 \quad \text{whenever} \quad F(s) = 0. \tag{4.5}$$

In the case of water waves (4.1) we are in this situation.

Of the integral operator we ask these properties. Form the symmetric part of $K\dfrac{\partial}{\partial x}$;

(i) $\left(K\dfrac{\partial}{\partial x} - \dfrac{\partial}{\partial x}K^*\right)$ positive, of order 1

so that its square root can contribute to an energy norm. Example (4.3) gives $2|D|$. Secondly

(ii) $\left(K\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial x}K^*\right)$ of at most order 1/2

(iii) $(1+K^2)$ smoothing of at least 1/2 order.

That is we ask that the estimates hold

$$\left\|\left(K\frac{\partial}{\partial x} + \frac{\partial}{\partial x}K^*\right)f\right\|_{L^2}^2 + \left\|(1+K^2)\frac{\partial}{\partial x}f\right\|_{L^2}^2 \leqq \text{const} \int f\left(K\frac{\partial}{\partial x} - \frac{\partial}{\partial x}K^*\right) f\,dx.$$

Finally it is required that commutators with K are smoothing.

(iv) $$\left\|\left[\frac{\partial}{\partial t}, K\right]f\right\|_{L^2} \leqq \text{const}\, \|f\|_{L^2}$$

(v) $$\left\|[b(x), K]\frac{\partial}{\partial x}f\right\|_{L^2} \leqq \text{const}(b)\, \||D|^{1/2}f\|_{L^2}.$$

The constant depends upon Sobolev or L^∞ estimates of $b(x)$ and its derivatives, for instance (4.3) admits the commutator estimate Lemma 2.1 [3], with $\text{const}(b) \leqq |b(x)|_{C,1}$.

The main step is the reduction of (4.2) to an appropriate quasilinear form. The idea originates in the study of (4.1) in a paper of Nalimov, and is more involved than the process of Sects. 2 and 3. Take one time derivative of (4.2), and denote $u_{tt}=v$

$$F_{u_{tt}}v_t+F_{u_x}u_{xt}+F_{Ku_{tt}}\frac{\partial}{\partial t}Kv+F_{Ku_x}Ku_{xt}+F_{Ku_x}\left[\frac{\partial}{\partial t},K\right]u_x=0. \tag{4.6}$$

This is treated as an equation for $u_x=w$;

$$(F_{Ku_x}K+F_{u_x})w_t+F_{u_{tt}}v_t+F_{Ku_{tt}}\frac{\partial}{\partial t}Kv+F_{Ku_x}\left[\frac{\partial}{\partial t},K\right]u_x=0.$$

Controlling $|(v,Kv,w,Kw)|_{L^\infty}<R$ we may invert $(F_{Ku_x}K+F_{u_x})$,

$$w_t=e(v,Kv,v_t,Kv_t,w,Kw). \tag{4.7}$$

Now take a second time derivative of (4.2),

$$F_{u_{tt}}v_{tt}+F_{u_x}v_x+F_{Ku_{tt}}Kv_{tt}+F_{Ku_x}Kv_x+g_1=0. \tag{4.8}$$

Here $g_1(v,Kv,v_t,Kv_t,w,Kw)=F_{Ku_x}\left[\left(\frac{\partial}{\partial t}\right)^2,K\right]w+F_{Ku_{tt}}\left[\left(\frac{\partial}{\partial t}\right)^2,K\right]v+F_{ij}$, where F_{ij} are terms involving second derivatives of F, which are lower order in v, w. Working with (4.8) we have the identities;

Lemma 4.1. (i)

$$\begin{aligned}&(F_{u_{tt}}-F_{Ku_{tt}}K)(F_{u_{tt}}+F_{Ku_{tt}}K)\\&\quad=(F^2_{u_{tt}}+F^2_{Ku_{tt}})+F_{Ku_{tt}}([F_{u_{tt}},K]+[F_{Ku_{tt}},K]K+F_{Ku_{tt}}(1+K^2))\end{aligned}$$

Thus

(ii) $$(F_{u_{tt}}+F_{Ku_{tt}}K)^{-1}=(F^2_{u_{tt}}+F^2_{Ku_{tt}})^{-1}(F_{u_{tt}}-F_{Ku_{tt}}K)+H_1,$$

where H_1 is smoothing of order at least 1/2.

For $|(v,Kv,w,Kw)|_{L^\infty}<R$, $(F^2_{u_{tt}}+F^2_{Ku_{tt}})^{-1}$ is positive and finite, giving

$$v_{tt}+(F^2_{u_{tt}}+F^2_{Ku_{tt}})^{-1}(F_{u_{tt}}-F_{Ku_{tt}}K)(F_{Ku_x}-F_{u_x}K)Kv_x+g_2=0. \tag{4.9}$$

In this expression

$$g_2=H_1(F_{Ku_x}-F_{u_x}K)Kv_x+(F_{u_{tt}}+F_{Ku_{tt}}K)^{-1}(g_1+F_{u_x}(1+K^2)v_x).$$

Finally we compute that

Lemma 4.2

$$\begin{aligned}&(F_{u_{tt}}-F_{Ku_{tt}})(F_{Ku_x}-F_{u_x}K)Kv_x\\&\quad=(F_{u_{tt}}F_{Ku_x}-F_{Ku_{tt}}F_{u_x})Kv_x+(F_{u_{tt}}F_{u_x}+F_{Ku_{tt}}F_{Ku_x})v_x+H_2v_x,\end{aligned}$$

where $H_2=\{F_{Ku_{tt}}([K,F_{u_x}]+[K,F_{Ku_x}]K+F_{Ku_x}(1+K^2))+(F_{Ku_{tt}}KF_{u_x}-F_{u_{tt}}F_{u_x})\times(1+K^2)\}$, *a smoothing term.*

The remarkable fact is that for solutions $F=0$, the coefficient of v_x in (4.9) is zero, $(F_{u_{tt}}F_{u_x}+F_{Ku_{tt}}F_{Ku_x})=0$ from (4.5). Thus one may drop terms containing v_x alone, keeping terms containing Kv_x and lower order terms, to obtain a quasilinear system for (v, w).

$$\begin{aligned} v_{tt}+aKv_x+c&=0, \\ w_t-e&=0. \end{aligned} \tag{4.10}$$

The coefficient $a=(F^2_{u_{tt}}+F^2_{Ku_{tt}})^{-1}(F_{u_{tt}}F_{Ku_x}-F_{Ku_{tt}}F_{u_x})\geqq\gamma>0$ for $|(v, Kv, w, Kw)|_{L^\infty}<R$. Furthermore (a, c, e) are functions of $(v, Kv, v_t, Kv_t, w, Kw, w_t)$, with e independent of w_t.

System (4.10) contains no pure v_x terms, and is amenable to energy estimates of the character of Sect. 3. In particular the natural energy norm for the linearized problem becomes

$$\|v\|_E^2=\int v_t^2+\frac{1}{2}a\left(\left(K\frac{\partial}{\partial x}-\frac{\partial}{\partial x}K^*\right)^{1/2}v\right)^2 dx$$

and a similar procedure of bounding higher Sobolev estimates gives a result of local existence in time for Sobolev initial data.

Theorem 4.3. *The equation* (4.2) *possesses solutions locally in time for initial data such that*

$$\|f(x)\|_{H,r+1/2}+\|g\|_{H,r}<R, \qquad r\geqq 3.$$

Solutions obey an energy estimate

$$\|u(t)\|_{E,r}\leqq\exp\left(\mathrm{const}(r)\int_0^t|(u, Ku)|_{C,2}d\tau\right)\|u(0)\|_{E,r}.$$

Furthermore C^∞ initial data generate C^∞ solutions if indeed F, $K(x, t, D)$ are smooth.

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Received August 28, 1986