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Another Look at p-Adic L-Functions for Totally Real Fields

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Introduction

That p-adic L-functions for totally real fields could be constructed using Shintani's explicit formulas [9] was realized independently by Barsky [1] and by Cassou-Noguès [3]. Although neither of them expresses either his construction or his results in terms of p-adic measures on profinite ray-class groups, Ribet has explained in his Luminy report how their results may be so reformulated.

This paper contains no new results, only a direct conceptual passage from the functions occurring in Shintani's formulas to p-adic measures, by means of "big" Cartier duality, and a trick of Cassou-Noguès. The key point is that the exponentials of linear forms occurring in Shintani's formulas are the analytic expression of algebraic characters of certain algebraic tori attached, albeit not very profoundly, to the number field in question.

From this point of view, we can interpret Shintani's formulas as saying that the values, at the identity, of higher derivatives of certain elementary rational functions on these algebraic tori are equal to the values at negative integers of L-functions attached to the totally real number field in question (a sort of multi-dimensional compatibility between Abel summation and Dirichlet summation). The "trick" of Cassou-Noguès is to pick just the "right" rational function; then "big" Cartier duality interprets this rational function as *being* the required measure! In the case $K=\mathbb{Q}$, this method is none other than the "Eulerian" approach to p-adic L-function for $\mathbb{Q}$ that is explained in [5].

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I. p-Adic Measures and Trivial Tori (Compare [5], [6])

Fix a free, finitely generated abelian group $\mathscr{A}$, a prime number p and an integer $N \geqq 1$. In this section we will recall the relation between p-adic measures on the space $\varprojlim_r \mathscr{A}/Np^r\mathscr{A}$ and functions on a suitable completion of the algebraic torus over $\mathbb{Z}$ defined by

$$T(\mathscr{A}) = \mathrm{Hom}(\mathscr{A}, \mathbb{G}_m).$$

For any ring R, the group of R-valued points of $T(\mathscr{A})$ is given by

$$T(\mathscr{A})(R) = \mathrm{Hom}(\mathscr{A}, R^\times).$$

Each element $\alpha \in \mathscr{A}$ defines a character $T^\alpha \in \mathrm{Hom}(T(\mathscr{A}), \mathbb{G}_m)$, by

$$T^\alpha(\xi) = \xi(\alpha) \quad \text{for} \quad \xi \in T(\mathscr{A})(R) = \mathrm{Hom}(\mathscr{A}, R^\times).$$

The coordinate ring $A(T(\mathscr{A}))$ of $T(\mathscr{A})$ is the free $\mathbb{Z}$-module on the T^α. Each "numerical polynomial" on $\mathscr{A}$

$$P : \mathscr{A} \to \mathbb{Z}$$

defines an invariant differential operator D_P on $A(T(\mathscr{A}))$ by

$$D_P(T^\alpha) = P(\alpha)\, T^\alpha.$$

For each integer $N \geqq 1$, we denote by $T_N(\mathscr{A})$ the sub-groupscheme of points of order N in $T(\mathscr{A})$. Thus we have

$$\begin{cases} T_N(\mathscr{A}) = \mathrm{Hom}(\mathscr{A}, \boldsymbol{\mu}_N) \\ T_N(\mathscr{A})(R) = \mathrm{Hom}(\mathscr{A}, \boldsymbol{\mu}_N(R)). \end{cases}$$

The *formal completion* of $T(\mathscr{A})$ along $T_N(\mathscr{A})$, denoted $\hat{T}_N(\mathscr{A})$, is the sub-group-functor of $T(\mathscr{A})$ defined by

$\hat{T}_N(\mathscr{A})(R) =$ Kernel of the composite map

$T(\mathscr{A})(R) \to T(\mathscr{A})(R^{\mathrm{red}}) \xrightarrow{\times N} T(\mathscr{A})(R^{\mathrm{red}})$

$=$ inverse image of $T_N(\mathscr{A})(R^{\mathrm{red}})$ under the canonical map

$T(\mathscr{A})(R) \to T(\mathscr{A})(R^{\mathrm{red}})$.

When the prime numer p is nilpotent in R, then we have the simpler description

$$\hat{T}_N(\mathscr{A})(R) = \mathrm{Hom}(\mathscr{A}, \boldsymbol{\mu}_{Np^\infty}(R)) = \mathrm{Hom}_{\mathrm{contin}}\left(\varprojlim_r \mathscr{A}/Np^r\mathscr{A}, R^\times\right).$$

In the second description, $R^\times$ is given the discrete topology.

If we "extend" the functor $\hat{T}_N(\mathscr{A})$ to the category of adic rings, by defining

$$\hat{T}_N(\mathscr{A})(R) = \varprojlim_n \hat{T}_N(\mathscr{A})(R/J^nR),$$

J an ideal of definition of R, then it is represented by the adic ring

$$A(\hat{T}_N(\mathscr{A})) = \varprojlim_n A(T(\mathscr{A}))/I(N)^n,$$

where $I(N)$ is the ideal in $A(T(\mathscr{A}))$ defining $T_N(\mathscr{A})$.

For a fixed adic ring R_0, with ideal of definition J_0, we may restrict this "extended" funtor to the category of continuous adic R_0-algebras (i.e. adic R_0-algebras R such that $J_0 R$ lies in an ideal of definition of R). There it is represented by the adic ring

$$A(\hat{T}_N(\mathscr{A}))\hat{\otimes} R_0 \overset{\text{dfn}}{=} \varprojlim_n A(T(\mathscr{A})) \bigotimes_{\mathbb{Z}} R_0/(J_0, I(N))^n.$$

If p is topologically nilpotent in R_0, then it is topologically nilpotent in any continuous adic R_0-algebra R. On such rings we have a simple description of the extended functor:

$$\begin{aligned}
\hat{T}_N(\mathscr{A})(R) &\overset{\text{dfn}}{=} \varprojlim_n \hat{T}_N(\mathscr{A})(R/J^n R) \\
&= \varprojlim_n \operatorname{Hom}(\mathscr{A}, \boldsymbol{\mu}_{Np^\infty}(R/J^n R)) \\
&= \varprojlim_n \operatorname{Hom}_{\text{contin}}\left(\varprojlim_r \mathscr{A}/Np^r\mathscr{A}, (R/J^n R)^\times\right) \\
&= \operatorname{Hom}_{\text{contin}}\left(\varprojlim_r \mathscr{A}/Np^r\mathscr{A}, R^\times\right).
\end{aligned}$$

In this last description, $R^\times$ is topologized by the system of neighborhoods $1+J^n$ of 1.

The following Theorem is just a restatement of Cartier duality for toroidal groups (cf. [6]).

Theorem 1. *Let R be an adic ring in which p is topologically nilpotent, and $N \geqq 1$ an integer. There is a natural isomorphism of continuous adic R-algebras between $A(\hat{T}_N(\mathscr{A}))\hat{\otimes} R$ and the convolution algebra Meas $\left(\varprojlim_r \mathscr{A}/Np^r\mathscr{A}, R\right)$ of all R-valued measures on the space $\varprojlim_r \mathscr{A}/Np^r\mathscr{A}$. Formation of this isomorphism, which we denote $f \leftrightarrow \mu_f$, commutes with arbitrary continuous extensions of scalars $R \to R'$ of adic rings in which p is topologically nilpotent. This isomorphism is uniquely characterized by the following integration formulas*

(1) *for any continuous adic R-algebra R', and any R'-valued point*

$$\xi \in \hat{T}_N(\mathscr{A})(R') = \operatorname{Hom}_{\text{contin}}\left(\varprojlim_r \mathscr{A}/Np^r\mathscr{A}, (R')^\times\right)$$

we have

$$\int_{\varprojlim_r \mathscr{A}/Np^r\mathscr{A}} \xi(\alpha)\, d\mu_f(\alpha) = f(\xi)$$

(2) *For any numerical polynomial $P: \mathscr{A} \to \mathbb{Z}$, extended by continuity to a $\mathbb{Z}_p$-valued function on $\varprojlim_r \mathscr{A}/Np^r\mathscr{A}$, if we denote by D_P the corresponding invariant differential operator (which extends to an invariant differential operator on $A(\hat{T}_N(\mathscr{A}))$), we have, for any point ξ as above*

$$\int P(\alpha)\xi(\alpha)\, d\mu_f(\alpha) = (D_P f)(\xi)$$

(3) *For* ξ, P *as above, and any continuous function* G *on* $\varprojlim_r \mathscr{A}/Np^r\mathscr{A}$, *we have*

$$\int G(\alpha)\,P(\alpha)\,\xi(\alpha)\,d\mu_f(\alpha) = \int G(\alpha)\,d\mu_F(\alpha)$$

with

$$F(x) = (D_P f)(X\xi)$$

II. Functions on Tori Arising from Shintani Decompositions (cf. Shintani [9])

Let K be a totally real number field of finite degree g over $\mathbb{Q}$. We denote by Σ the set $\mathrm{Hom}(K, \mathbb{R})$ of real embeddings of K, and by $\mathbb{R}_+ \subset \mathbb{R}$ the strictly positive reals. We denote by $(K\otimes\mathbb{R})_+$ the subset of $K\otimes\mathbb{R} \xrightarrow{\sim} \mathbb{R}^\Sigma$ which is $(\mathbb{R}_+)^\Sigma$. For any subset $A \subset K \subset K\otimes\mathbb{R}$, we denote by $A_+ \subset A$ the set of its totally positive elements

$$A_+ = A \cap (K\otimes\mathbb{R})_+ \quad \text{inside} \quad K\otimes\mathbb{R}.$$

We denote by E the group $\mathcal{O}(K)^\times$ of units of K.

By a Shintani decomposition $\mathscr{S}$ of K, we mean a finite collection $\mathscr{S}$ of finite subsets $V \subset K_+$ such that

(1) each V consists of elements of K_+ which are linearly independent over $\mathbb{Q}$.

(2) if, for each V, we define $\mathrm{Cone}(V) \subset (K\otimes R)_+$ to be the $\mathbb{R}_+$-span of the elements $v \in V$, then we have a *disjoint union*

$$(K\otimes\mathbb{R})_+ = \bigcup_{e\in E_+} \bigcup_{V\in\mathscr{S}} e\cdot\mathrm{Cone}(V)$$

According to a fundamental Theorem of Shintani, such decompositions exist for any totally real field K. Notice that if $\mathscr{S} = \{V\}$ is one Shintani decomposition, then for any $\lambda \in K_+$, the collection $\{\lambda V\}$ is another; in particular, given any non-zero integral ideal $\mathscr{A}$ of K, there exist Shintani decompositions $\mathscr{S}$ with all $V \subset \mathscr{A}_+$.

Given a non-zero integral ideal $\mathscr{A}$, and *any* Shintani decomposition $\mathscr{S}$, we have a disjoint union decomposition

$$\mathscr{A}_+ = \bigcup_{e\in E_+} \bigcup_{V\in\mathscr{S}} e(\mathscr{A} \cap \mathrm{Cone}(V)).$$

In particular, the set

$$\mathscr{S}\text{-rep}(\mathscr{A}) \overset{\mathrm{dfn}}{=} \bigcup_{V\in\mathscr{S}} \mathscr{A} \cap \mathrm{Cone}(V)$$

provides a set of *representations* for the quotient space

$$\mathscr{A}_+/\text{mult. by } E_+.$$

The construction

$$\alpha \in \mathscr{A}_+ \mapsto \text{the ideal} \quad (\alpha)\mathscr{A}^{-1}$$

provides a bijection

$$\mathscr{A}_+/\text{mult. by } E_+ \xrightarrow{\sim} \left\{\begin{matrix}\text{integral ideals in the strict} \\ \text{ideal class of } \mathscr{A}^{-1}.\end{matrix}\right\}$$

Thus an integral ideal in the strict ideal class of $\mathscr{A}^{-1}$ may be written uniquely as $(\alpha)\mathscr{A}^{-1}$ with $\alpha\in\mathscr{S}$-rep$(\mathscr{A})$.

Consider henceforth a non-zero integral ideal $\mathscr{A}$, and a Shintani decomposition $\mathscr{S}$ with all $V\subset\mathscr{A}$. We will define a rational function $f(\mathscr{A},\mathscr{S})$ on the torus $T(\mathscr{A})$ as follows.

For each $V\in\mathscr{S}$, let $R(V,\mathscr{A})$ be the subset of $\mathscr{A}_+$ defined by

$$R(V,\mathscr{A})=\mathscr{A}_+\cap\left\{\sum_{v\in V}x_v v\in\mathrm{Cone}(V)|\text{all } x_v\in\mathbb{Q}_+, 0<x_v\leqq 1\right\}.$$

Then each element of the set $\mathscr{A}\cap\mathrm{Cone}(V)$ can be written uniquely in the form

$$r+\sum_{v\in V}n_v v$$

for some $r\in R(V,\mathscr{A})$ and integers $n_v\geqq 0$.

The set $R(V,\mathscr{A})$ being *finite*, we may define a rational function $f(\mathscr{A},\mathscr{S})$ on $T(\mathscr{A})$ by the formula

$$f(\mathscr{A},\mathscr{S})=\sum_{V\in\mathscr{S}}\sum_{r\in R(V,\mathscr{A})}\frac{T^r}{\prod_{v\in V}(1-T^v)}.$$

Given any ring R and any point $\xi\in T(\mathscr{A})(R)=\mathrm{Hom}(\mathscr{A},R^\times)$, we denote by $f(\mathscr{A},\mathscr{S};\xi)$ the function obtained by translating $f(\mathscr{A},\mathscr{S})$ by ξ:

$$f(\mathscr{A},\mathscr{S};\xi)=\sum_{V\in\mathscr{S}}\sum_{r\in R(V,\mathscr{A})}\frac{\xi(r)\,T^r}{\prod_{v\in V}(1-\xi(v)\,T^v)}$$

Because $\mathscr{A}$ is an integral ideal, the norm mapping of $K/\mathbb{Q}$ defines a $\mathbb{Z}$-valued polynomial function

$$\mathbb{N}:\mathscr{A}\to\mathbb{Z},$$

and the corresponding invariant differential operator $D_{\mathbb{N}}$ on the torus $T(\mathscr{A})$.

Theorem 2. (compare Shintani [9], Prop. 1). *Let $\xi\in T(\mathscr{A})(\mathbb{C})=\mathrm{Hom}(\mathscr{A},\mathbb{C}^\times)$ satisfy $\forall V\in\mathscr{S}$, and $\forall v\in V$, we have $\xi(v)\neq 1$, $|\xi(v)|\leqq 1$.*

Then the Dirichlet series $\varphi(s)$ defined as

$$\varphi(s)=\sum_{V\in\mathscr{S}}\sum_{r\in R(V,\mathscr{A})}\sum_{\{n_v\}\geqq 0}\xi(r+\Sigma n_v v)\,\mathbb{N}(r+\Sigma n_v v)^{-s}$$

convergent for $\mathrm{Re}(s)\gg 0$, *extends to an entire function of s, whose values at negative integers $-n$, for $n=0, 1, 2, \ldots$, are given by the formula*

$$\varphi(-n)=(D_{\mathbb{N}})^n(f(\mathscr{A},\mathscr{S};\xi))(1).$$

or

$$\varphi(-n)=(D_{\mathbb{N}})^n(f(\mathscr{A},\mathscr{S}))(\xi).$$

Remark. We can also write $\varphi(s)$ as

$$\varphi(s)=\sum_{V\in\mathscr{S}}\sum_{\alpha\in\mathscr{A}\cap\mathrm{Cone}(V)}\xi(\alpha)\,\mathbb{N}(\alpha)^{-s}$$

$$=\sum_{\alpha\in\mathscr{S}-\mathrm{rep}(\mathscr{A})}\xi(\alpha)\,\mathbb{N}(\alpha)^{-s}.$$

III. Construction of a Measure (following Cassou-Noguès [3])

Fix a prime ideal $\mathscr{L}$ of K of residue degree one; thus

$$\mathcal{O}(K)/\mathscr{L} \xrightarrow{\sim} \mathbb{Z}/l\mathbb{Z},$$

where l is the residue characteristic of $\mathscr{L}$. Given a non-zero integral ideal $\mathscr{A}$ of K, the quotient $\mathscr{A}/\mathscr{L}\mathscr{A}$ is a one-dimensional vector space over $\mathcal{O}(K)/\mathscr{L} \simeq \mathbb{Z}/l\mathbb{Z}$, and in particular $\mathscr{A}/\mathscr{L}\mathscr{A}$ is a cyclic group of order l. Each of the l characters ψ of the group $\mathscr{A}/\mathscr{L}\mathscr{A}$ with values in the subgroup $\boldsymbol{\mu}_l$ of l-th roots of unity inside $\mathbb{Z}[\zeta_l]$ defines a point, still denoted ψ, of the torus $T(\mathscr{A})$ with values in $\mathbb{Z}[\zeta_l]$, namely the composite homomorphism

$$\mathscr{A} \to \mathscr{A}/\mathscr{L}\mathscr{A} \xrightarrow{\psi} \boldsymbol{\mu}_l \to (\mathbb{Z}[\zeta_l])^\times,$$

viewed as an element

$$\psi \in T(\mathscr{A})(\mathbb{Z}[\zeta_l]) \stackrel{\text{dfn}}{=} \operatorname{Hom}(\mathscr{A}, (\mathbb{Z}[\zeta_l])^\times).$$

Following [3], we use the $l-1$ *nontrivial* characters of $\mathscr{A}/\mathscr{L}\mathscr{A}$ to define a rational function $f(\mathscr{A}, \mathscr{S}; \mathscr{L})$ on the algebraic torus $T(\mathscr{A})$ by the formula

$$\begin{aligned} f(\mathscr{A}, \mathscr{S}; \mathscr{L}) &= - \sum_{\text{non triv } \psi} f(\mathscr{A}, \mathscr{S}; \psi) \\ &= - \sum_{V \in \mathscr{S}} \sum_{r \in R(V, \mathscr{A})} \sum_{\text{non triv } \psi} \frac{\psi(r) T^r}{\sum_{v \in V} (1 - \psi(v) T^v)}. \end{aligned}$$

Suppose now that $\mathscr{L}$ is so chosen that for all $V \in \mathscr{S}$, all $v \in V$, we have $v \notin \mathscr{A}\mathscr{L}$. Then for each of the non-trivial ψ above, we'll have

$$\psi(v) = \text{a primitive } l\text{-th root of unity.}$$

It is then plain from the explicit formula for $f(\mathscr{A}, \mathscr{S}; \mathscr{L})$ that if $N \geqq 1$ is any integer prime to l, and R is any adic ring in which l is invertible, then $f(\mathscr{A}, \mathscr{S}; \mathscr{L})$ is a well defined element of $A(\hat{T}_N(\mathscr{A})) \hat{\otimes} R$. Interpreting this element as a measure, we obtain from Theorems 1 and 2 the following

Theorem 3. *Let $\mathscr{A}$ be an integral ideal of K, $\mathscr{S}$ a Shintani decomposition of K with each $V \in \mathscr{S}$ contained in $\mathscr{A}$, and $\mathscr{L}$ a prime ideal of K of residue degree one such that for all $V \in \mathscr{S}$, and all $v \in V$, $v \notin \mathscr{A}\mathscr{L}$. Then for any integer $N \geqq 1$ prime to l, and any prime number $p \neq l$, there exists a $\mathbb{Z}_p$-valued measure $\mu_{f(\mathscr{A}, \mathscr{S}; \mathscr{L})}$ on $\varprojlim_r \mathscr{A}/Np^r\mathscr{A}$, such that if*

$$g : \varprojlim_r \mathscr{A}/Np^r\mathscr{A} \to \mathbb{Z}$$

is any locally constant function, and if $k \geqq 0$ is any non-negative integer, then we have the integration formula

$$\int_{\varprojlim_r \mathscr{A}/Np^r\mathscr{A}} g(\alpha)\, \mathbb{N}(\alpha)^k \, d\mu_{f(\mathscr{A}, \mathscr{S}; \mathscr{L})} = \text{the value at } s = -k \text{ of the analytic continuation of the Dirichlet series}$$

$$\sum_{\alpha \in \mathscr{S} - \text{rep}(\mathscr{A})} g(\alpha)\Big(-\sum_{\psi \text{ non-triv}} \psi(\alpha)\Big)\mathbb{N}(\alpha)^{-s}$$

(equality in $\mathbb{Z}[1/l]$*).*

NB:

$$-\sum_{\psi \text{ non-triv}} \psi(\alpha) = \begin{cases} 1 & \text{if } \alpha \notin \mathscr{A}\mathscr{L} \\ 1-l & \text{if } \alpha \in \mathscr{A}\mathscr{L} \end{cases}$$

IV. Measures on Ray-Class Monoids; the Main Theorem

For any integral ideal $\mathscr{N}$ of K, we define the ray-class monoid $M(\mathscr{N})$ of ideals mod $\mathscr{N}$ to be the quotient

$$M(\mathscr{N}) = \{\text{non-zero integral ideals of } K\}/\widetilde{\mathscr{N}}$$

where $\widetilde{\mathscr{N}}$ is the equivalence relation

$$\mathscr{A} \underset{\mathscr{N}}{\sim} \mathscr{B} \quad \text{if} \quad \mathscr{A}\mathscr{B}^{-1} = (\alpha) \quad \text{with} \quad \alpha \in K_+, \quad \alpha \in 1 + \mathscr{N}\mathscr{B}^{-1}.$$

Multiplication of ideals passes over to define a multiplication on $M(\mathscr{N})$, with the trivial ideal (1) as the unit element.

We denote by

$$G(\mathscr{N}) \subset M(\mathscr{N})$$

the *group* of invertible elements of $M(\mathscr{N})$. Thus

$$G(\mathscr{N}) = \{\text{prime-to-}\mathscr{N} \text{ integral ideals}\}/\widetilde{\mathscr{N}} \xrightarrow{\sim}$$
$$\{\text{fractional ideals prime to } \mathscr{N}\}/\{(\alpha) | \alpha \in K_+, \alpha \equiv 1 \bmod^{\times} \mathscr{N}\}$$

is the usual "ray-class group mod $\mathscr{N}$". Notice that

$$M((1)) = G((1)) = \text{the strict ideal class group.}$$

Let $\{\mathscr{A}_i\}$ be a set of integral ideals of K whose *inverses* give a set of representatives for the strict ideal class group. Then any integral ideal $\mathscr{B}$ can be written, for some unique i, in the form

$$\mathscr{B} = (b)\mathscr{A}_i^{-1} \quad \text{with} \quad b \in (\mathscr{A}_i)_+,$$

and b is unique up to multiplication by E_+. Two integral ideals $\mathscr{B}, \mathscr{C}$ are $\widetilde{\mathscr{N}}$ if and only if they are in the same strict ideal class, say that of $\mathscr{A}_i^{-1}$, *and* if we have

$$\mathscr{B} = (b)\mathscr{A}_i^{-1}, \quad \mathscr{C} = (c)\mathscr{A}_i^{-1}$$

with $b \in (\mathscr{A}_i)_+$, $c \in (\mathscr{A}_i)_+$, and $b \equiv c \bmod \mathscr{A}_i\mathscr{N}$.

Given a function

$$g: M(\mathcal{N}) \to \mathbb{Q},$$

the associated L-series $L(s,g)$ is, by definition, the Dirichlet series

$$L(s,g) = \sum_{\mathscr{A} \text{ integral}} g(\mathscr{A}) \mathbb{N}\mathscr{A}^{-s}.$$

In terms of any set of integral ideals $\mathscr{A}_i$ whose inverses give representations for the strict ideal class group, and any Shintani decomposition $\mathscr{S}$ with all $V \subset \bigcap_i \mathscr{A}_i$, we may rewrite $L(s,g)$ as a finite sum

$$L(s,g) = \sum_{\mathscr{A}_i} \sum_{\alpha \in \mathscr{S} - \mathrm{rep}(\mathscr{A}_i)} g((\alpha)\mathscr{A}_i^{-1}) \mathbb{N}((\alpha)\mathscr{A}_i^{-1})^{-s}$$

$$= \sum_i \mathbb{N}\mathscr{A}_i^s \sum_{\alpha \in \mathscr{S} - \mathrm{rep}(\mathscr{A}_i)} g((\alpha)\mathscr{A}_i^{-1}) \mathbb{N}(\alpha)^{-s}.$$

By Shintani's Theorems each of the individual Dirichlet series

$$\sum_{\alpha \in \mathscr{S} - \mathrm{rep}(\mathscr{A}_i)} g((\alpha)\mathscr{A}_i^{-1}) \mathbb{N}(\alpha)^{-s}$$

has a meromorphic continuation to the entire s-plane, which assumes *rational* values at negative integers $s = 0, -1, -2, \ldots$.

For any $\mathscr{C} \in M(\mathcal{N})$, we denote by

$$g_{\mathscr{C}}: M(\mathcal{N}) \to \mathbb{Q}$$

the function defined by

$$g_{\mathscr{C}}(\mathscr{A}) = g(\mathscr{A}\mathscr{C}).$$

Given a prime number p and an integer $N \geqq 1$, we define

$$M(Np^\infty) \overset{\mathrm{dfn}}{=} \varprojlim_r M((Np^r)),$$

$$G(Np^\infty) = \varprojlim_r G((Np^r)).$$

Then $G(Np^\infty)$ is the group of units in the profinite monoid $M(Np^\infty)$. The norm function extends to a continuous function

$$\mathbb{N}: M(Np^\infty) \to \mathbb{Z}_p.$$

(Main) Theorem 4. *Let $\mathscr{C}$ be any element of $G(Np^\infty)$. Then there exists a $\mathbb{Z}_p$-valued measure $\mu_{\mathscr{C}}$ on $M(Np^\infty)$ such that for any locally constant function*

$$g: M(Np^\infty) \to \mathbb{Z}$$

and any non-negative integer $k \geqq 0$, we have the formula

$$\int_{M(Np^\infty)} g(\mathscr{A}) \mathbb{N}\mathscr{A}^k d\mu_{\mathscr{C}} = L(-k,g) - \mathbb{N}\mathscr{C}^{k+1} L(-k, g_{\mathscr{C}}).$$

Proof. Let $\mathscr{A}_i$ be a set of integral ideals of K whose inverses $\mathscr{A}_i^{-1}$ give representations of the strict ideal class group. Let $\mathscr{S}$ be an Shintani decomposition with each $V \subset \bigcap_i \mathscr{A}_i$. Let $\mathscr{L}$ be a prime ideal of K of residue degree one such that

$$\begin{cases} \mathscr{L} \text{ is prime to } pN \\ \text{for all } V \in \mathscr{S}, \text{ and for all } v \in V,\ v \notin \mathscr{L}. \end{cases}$$

By Chebataroff, the images in $G(Np^\infty)$ of such prime ideals $\mathscr{L}$ are everywhere dense, and a routine limiting argument shows that it suffices to construct the measure $\mu_{\mathscr{L}}$ for such $\mathscr{L}$.

Let $G: M(Np^\infty) \to \mathbb{Z}_p$ be a continuous function. For each i, the function on $(\mathscr{A}_i)_+$ defined by

$$\alpha \in (\mathscr{A}_i)_+ \to G((\alpha)\mathscr{A}_i^{-1})$$

extends uniquely to a continuous $\mathbb{Z}_p$-valued function on $\varprojlim_r \mathscr{A}_i/Np^r\mathscr{A}_i$, which we still note $\alpha \mapsto G((\alpha)\mathscr{A}_i^{-1})$. We now *define* the measure $\mu_{\mathscr{L}}$ by decreeing that for any continuous G, we are to have

$$\int_{M(Np^\infty)} G(\mathscr{A})\,d\mu_{\mathscr{L}} \overset{\text{dfn}}{=} \sum_i \int_{\varprojlim_r \mathscr{A}_i/Np^r\mathscr{A}_i} G((\alpha)\mathscr{A}_i^{-1})\,d\mu_{f(\mathscr{A}_i,\mathscr{S};\mathscr{L})}.$$

When we have

$$G(\mathscr{A}) = g(\mathscr{A})\,\mathbb{N}\mathscr{A}^k,$$

with $g(\mathscr{A})$ locally constant and $\mathbb{Z}$-valued, and $k \geqq 0$, then the required integral formulas follow immediately from Theorem 3:

$$\int_{M(Np^\infty)} g(\mathscr{A})\,\mathbb{N}\mathscr{A}^k d\mu_{\mathscr{L}} = \sum_i \int_{\varprojlim_r \mathscr{A}_i/\mathbb{N}p^r\mathscr{A}_i} g((\alpha)\mathscr{A}_i^{-1})\,\mathbb{N}(\alpha)^k\,\mathbb{N}\mathscr{A}_i^{-k} d\mu_{f(\mathscr{A}_i,\mathscr{S};\mathscr{L})}$$

= the value at $s = -k$ of

$$\sum_i \sum_{\alpha \in \mathscr{S}-\mathrm{rep}(\mathscr{A}_i)} g((\alpha)\mathscr{A}_i^{-1})\,\mathbb{N}(\alpha)^{-s}\,\mathbb{N}\mathscr{A}_i^s \bigstar_{\mathscr{A}_i\mathscr{L}}(\alpha)$$

with

$$\bigstar_{\mathscr{A}_i\mathscr{L}}(\alpha) = \begin{cases} 1 & \text{if } \alpha \notin \mathscr{A}_i\mathscr{L} \\ 1 - l = 1 - \mathbb{N}\mathscr{L} & \text{if } \alpha \in \mathscr{A}_i\mathscr{L} \end{cases}$$

= the value at $s = -k$ of

$$\sum_i \sum_{\substack{\text{integral ideals} \\ \mathscr{A} \sim \mathscr{A}_i^{-1}}} g(\mathscr{A})\,\mathbb{N}\mathscr{A}^{-s} \bigstar_{\mathscr{L}}(\mathscr{A})$$

with

$$\bigstar_{\mathscr{L}}(\mathscr{A}) = \begin{cases} 1 & \text{if } \mathscr{A} \not\equiv 0(\mathscr{L}) \\ 1 - \mathbb{N}\mathscr{L} & \text{if } \mathscr{A} \equiv 0(\mathscr{L}) \end{cases}$$

= the value at $s=-k$ of

$$\sum_{\text{integral ideals }\mathscr{A}} g(\mathscr{A})\mathbb{N}\mathscr{A}^{-s} - \sum_{\substack{\text{integral }\mathscr{A}\\ \text{div by }\mathscr{L}}} \mathbb{N}\mathscr{L}\cdot g(\mathscr{A})\cdot\mathbb{N}(\mathscr{A})^{-s}$$

$$= \sum_{\text{integral }\mathscr{A}} g(\mathscr{A})\mathbb{N}\mathscr{A}^{-s} - \sum_{\text{integral }\mathscr{A}} \mathbb{N}\mathscr{L}\cdot g(\mathscr{A}\mathscr{L})\cdot\mathbb{N}(\mathscr{A}\mathscr{L})^{-s}$$

$$= L(-k,g) - \mathbb{N}\mathscr{L}^{k+1}L(-k,g_{\mathscr{L}}).$$ Q.E.D.

V. Comments

Let $\bar{E}_+$ denote the closure of E_+ in $\varprojlim_r (\mathcal{O}(K)/Np^r\mathcal{O}(K))^\times$. Then the space $M(Np^\infty)$ is the disjoint union of the compact open sets

$$\begin{array}{c} M(Np^\infty)_{\mathscr{A}_i} \overset{\text{dfn}}{=} \text{fibre of } M(Np^\infty)\to M((1)) \text{ over } \mathscr{A}_i^{-1} \\ \wr\big\uparrow \times\mathscr{A}_i^{-1} \\ \left(\varprojlim_r \mathscr{A}_i/Np^r\mathscr{A}_i\right)\Big/\text{mult. by } \bar{E}_+ . \end{array}$$

The *restriction* of $\mu_{\mathscr{L}}$ to $M(Np^\infty)_{\mathscr{A}_i}$ is the *direct image* of $\mu_{f(\mathscr{A}_i,\mathscr{S};\mathscr{L})}$ under the canonical projection

$$\begin{array}{ccc} \varprojlim_r \mathscr{A}_i/Np^r\mathscr{A}_i & \twoheadrightarrow & \left(\varprojlim_r \mathscr{A}_i/Np^r\mathscr{A}_i\right)\Big/\text{mult. by } \bar{E}_+ \\ & & \wr\big\downarrow \times\mathscr{A}_i^{-1} \\ & & M(Np^\infty)_{\mathscr{A}_i} \\ & & \cap \\ & & M(Np^\infty) \end{array}$$

There are many possible Shintani decompositions $\mathscr{S}$. *Each* of them which "avoids" $\mathscr{L}$ gives rise to a measure $\mu_{f(\mathscr{A}_i,\mathscr{S};\mathscr{L})}$ on $\varprojlim_r \mathscr{A}_i/Np^r\mathscr{A}_i$. The only apparent link between these measures is that they all project to the *same* measure on the quotient by $\bar{E}_+$; and there is no obvious reason to prefer one to another. Presumably, to obtain the 2-adic divisibilities given by the Deligne-Ribet approach will require a better understanding of the various measures $\mu_{f(\mathscr{A}_i,\mathscr{S};\mathscr{L})}$ for variable $\mathscr{S}$, and their interrelations. How can one bring to bear the "real Frobenii"? It is even possible that one could *exploit* the abundance of measures on $\varprojlim_r \mathscr{A}_i/Np^r\mathscr{A}_i$ with given projection mod $\bar{E}_+$ to obtain *new* results, but this is idle speculation.

Another mystery is the relation between the Shintani approach to L-values, through decompositions of $(K\otimes\mathbb{R})_+$ into open simplicial cones, and the Deligne-Ribet approach (cf. [4]) through modular forms on the associated Hilbert-Blumenthal moduli scheme. One cannot help being struck by the fact that the natural compactifications of this moduli variety are described in terms of just such decompositions of $(K\otimes\mathbb{R})_+$ into open simplicial cones (cf. [7])!

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