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Stable Reflexive Sheaves

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Contents

0. Introduction

Let $\mathscr{F}$ be a coherent sheaf on a scheme X. The *dual* of $\mathscr{F}$ is the sheaf $\mathscr{F}^\vee = \mathscr{H}om(\mathscr{F}, \mathscr{O}_X)$. If the natural map $\mathscr{F} \to \mathscr{F}^{\vee\vee}$ of $\mathscr{F}$ to its double dual is an isomorphism, we say that $\mathscr{F}$ is *reflexive*.

We use the definition of stability given by Mumford and Takemoto [18]. Let X be a normal projective variety with a fixed very ample divisor H. For any coherent sheaf $\mathscr{F}$ on X, define $\mu(\mathscr{F}) = \deg c_1(\mathscr{F})/\operatorname{rank} \mathscr{F}$. Here $c_1(\mathscr{F})$ denotes the first Chern class of $\mathscr{F}$, considered as an element of the Picard group $\operatorname{Pic} X$, and deg denotes its degree with respect to H. A torsion-free coherent sheaf $\mathscr{F}$ is *stable* if for every coherent subsheaf $\mathscr{F}' \subsetneqq \mathscr{F}$ with $0 < \operatorname{rank} \mathscr{F}' < \operatorname{rank} \mathscr{F}$, the inequality $\mu(\mathscr{F}') < \mu(\mathscr{F})$ holds. If the weaker inequality $\mu(\mathscr{F}') \leqq \mu(\mathscr{F})$ holds, we say that $\mathscr{F}$ is *semistable*.

This paper begins the study of stable reflexive sheaves (especially those of rank 2 on $\mathbb{P}^3$) considered as a generalization of stable vector bundles.

There are several reasons for studying reflexive sheaves. The first reason is natural curiosity. Reflexive sheaves are more general than vector bundles (one might think of them as "vector bundles with singularities"), yet behave in many

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ways like vector bundles. So one can try to extend to reflexive sheaves the results already known for vector bundles [1, 9]. This turns out to work well, and without too much difficulty once one has mastered the elementary properties of reflexive sheaves. Then we can understand vector bundles better by regarding them as a special kind of reflexive sheaves.

One of the basic problems in the study of vector bundles is to understand their variety of moduli. How do reflexive sheaves fit in this problem? One knows [12] that stable vector bundles with given Chern classes (let us say of rank 2 on $\mathbb{P}^3$, to fix the ideas) are parametrized by a quasiprojective scheme M, which can be compactified to a projective scheme $\bar{M}$ by adding certain semistable torsion free sheaves. The stable reflexive sheaves do *not* appear in the closure of the moduli M of stable vector bundles. Rather, they form new moduli separate from M. The reason for this, as we will see (2.6), is that for a rank 2 reflexive sheaf $\mathscr{F}$ on $\mathbb{P}^3$, the third Chern class c_3 is equal to the number of points where $\mathscr{F}$ is not locally free. Now the Chern classes are topological invariants, and a rank 2 vector bundle has $c_3=0$. Therefore any stable reflexive sheaf appearing in the closure of the moduli of vector bundles M would have $c_3=0$ and so would be a vector bundle itself.

The second reason for studying reflexive sheaves is that they arise naturally from vector bundles of higher rank. For example, let $\mathscr{E}$ be a rank 3 vector bundle on $\mathbb{P}^3$. If $\mathscr{E}$ is generated by global sections, then a sufficiently general section $s\in H^0(\mathscr{E})$ will vanish at only finitely many points. In that case there is an exact sequence

$$0\to\mathcal{O}\xrightarrow{s}\mathscr{E}\to\mathscr{F}\to 0,$$

where $\mathscr{F}$ is a rank 2 reflexive sheaf. $\mathscr{F}$ will be a vector bundle only in the rare case when $c_3(\mathscr{E})=0$. Now we can study $\mathscr{E}$ in terms of $\mathscr{F}$ and the extension of $\mathscr{F}$ by $\mathcal{O}$.

The third reason for studying reflexive sheaves, in particular those of rank 2 on $\mathbb{P}^3$, relates to the classification of space curves [11]. There is a well known correspondence between rank 2 vector bundles on $\mathbb{P}^3$ and curves Y in $\mathbb{P}^3$ which are locally complete intersections and have the property $\omega_Y\cong\mathcal{O}_Y(l)$ for some $l\in\mathbb{Z}$ [9, Sect. 1]. The condition $\omega_Y\cong\mathcal{O}_Y(l)$ imposes strong restrictions on the curves Y which can occur. Using reflexive sheaves instead of vector bundles, we find a correspondence between rank 2 reflexive sheaves on $\mathbb{P}^3$ and arbitrary curves in $\mathbb{P}^3$. This gives a dictionary between reflexive sheaves and curves, which should provide applications to the classification of curves.

The fourth and crucial reason for studying reflexive sheaves is that they arise naturally in the study of rank 2 vector bundles on $\mathbb{P}^3$, and provide a means for studying the moduli of these vector bundles by an induction on c_2. For example, let $\mathscr{E}$ be a stable rank 2 vector bundle on $\mathbb{P}^3$ with $c_1=0$. We say a plane $H\subsetneq\mathbb{P}^3$ is an *unstable plane* for $\mathscr{E}$ if $H^0(\mathscr{E}_H(-r))\neq 0$ for some $r>0$. The largest such r is the *order* of the unstable plane H. Now suppose that $\mathscr{E}$ has an unstable plane H of order r. Choose a section $s\in H^0(\mathscr{E}_H(-r))$. This defines a map $\mathcal{O}_H\to\mathscr{E}_H(-r)$. Dualizing and twisting we get a map $\mathscr{E}_H\to\mathcal{O}_H(-r)$, whose image we call $\mathscr{I}_{Z,H}(-r)$ for a suitable closed subscheme $Z\subsetneq H$. Now map $\mathscr{E}$ to $\mathscr{I}_{Z,H}(-r)$ by first restricting to H, then using the above map on $\mathscr{E}_H$. Let $\mathscr{E}'$ be the kernel, so there is an exact sequence

$$\mathcal{O}\to\mathscr{E}'\to\mathscr{E}\to\mathscr{I}_{Z,H}(-r)\to 0.$$

Then $\mathscr{E}'$ will be a rank 2 reflexive sheaf on $\mathbb{P}^3$ with $c_1 = -1$. It will be stable if $\mathscr{E}$ is stable. And its second Chern class c_2' is equal to $c_2 - r$, which is strictly less than c_2. We call this construction the *reduction step* [see (5.2) and (9.1)] for $\mathscr{E}$ with respect to the unstable plane H.

Now we can study $\mathscr{E}$ in terms of $\mathscr{E}'$ and the extension of $\mathscr{I}_{Z,H}(-r)$ by $\mathscr{E}'$. The reduction step works similarly for bundles with $c_1 = -1$ and also for reflexive sheaves. Thus we have an inductive procedure on c_2 for studying vector bundles or reflexive sheaves whenever we can establish the existence of an unstable plane. The existence of unstable planes and use of the reduction step are the main new techniques which appear in this paper.

One motivation for this work is to attack the conjecture formulated in my earlier paper [9, 8.2.2]. The conjecture is that if $\mathscr{E}$ is a rank 2 vector bundle on $\mathbb{P}^3$ with $c_1 = 0$ and $c_2 \geqq 0$, and if t is an integer such that $t > \sqrt{3c_2 + 1} - 2$, then $H^0(\mathscr{E}(t)) \neq 0$. The conjecture is still unproven, but in this paper (Sect. 10) we show that it is reasonable to extend the conjecture to reflexive sheaves, and that the numerical evidence for the conjecture is good. We hope that these same methods may settle the conjecture in the future.

The paper is organized as follows. Sections 1–4 contain preliminary material on reflexive sheaves and their Chern classes. Section 5 contains a key result on the cohomology of rank 2 vector bundles on $\mathbb{P}^3$. The main tool is a reduction step on $\mathbb{P}^2$ similar to the one described above on $\mathbb{P}^3$.

Section 7 contains the central result of the paper. It is a generalization of the work of Barth and Elencwajg [3]. For any semistable reflexive sheaf we define a numerical invariant called its *spectrum*, and we establish various properties of the spectrum. As corollaries (in Sect. 8) we obtain 1) a vanishing theorem for $H^1(\mathscr{E}(l))$, for l negative; 2) a vanishing theorem for $H^2(\mathscr{E}(l))$, for l positive; and 3) a bound for c_3 in terms of c_1 and c_2. We give examples to show that all these results are the best possible.

In Sect. 9 we study the extremal bundles and sheaves for which the bounds of the previous section are sharp. In some cases we can give a fairly complete description of their properties and variety of moduli.

The last Sect. 10 gives a bound for t such that $H^0(\mathscr{E}(t)) \neq 0$ which is weaker than the conjectured bound above, but which suffices to prove the conjecture for all values of $c_2 \leqq 25$.

Throughout the paper we work over an algebraically closed ground field k of arbitrary characteristic. In particular, we do not use[1] the theorem of Grauert-Mülich or Barth's theorem on the restriction of stable bundles [1], which fail in characteristic $p > 0$. However we will indicate where some proofs can be shortened by using those results in characteristic 0.

In a future paper we plan to develop these techniques further, to prove for example the nonexistence of bundles with certain spectra (7.6.1), the nonexistence of reflexive sheaves with certain c_3 (4.2.0), (4.2.3), and to obtain better bounds for the least t such that $H^0(\mathscr{E}(t)) \neq 0$.

1 Except in one instance (7.5)

1. Basic Properties of Reflexive Sheaves

In this section we gather together some basic properties of reflexive sheaves. These results should be well known, but for lack of adequate references we will include proofs here. (See [17, II, Sect. 1.1] for analogous results in the analytic case.)

Throughout this section X will denote an integral noetherian scheme. If $\mathscr{F}$ is a coherent sheaf on X, we define the *dual* of $\mathscr{F}$ to be the coherent sheaf $\mathscr{F}^\vee = \mathscr{H}om(\mathscr{F}, \mathcal{O}_X)$. There is a natural map of $\mathscr{F}$ to its double dual $\mathscr{F}^{\vee\vee}$. Since X is an integral scheme, the stalk of $\mathscr{F}$ at the generic point of X is a finite-dimensional vector space over the function field of X. This vector space is isomorphic to its double dual. Therefore the kernel and the cokernel of the natural map $\mathscr{F} \to \mathscr{F}^{\vee\vee}$ are torsion sheaves on X, i.e. they are supported on proper closed subsets of X. In fact it is easy to see that the kernel of this map is equal to the torsion subsheaf of $\mathscr{F}$.

Definition. A coherent sheaf $\mathscr{F}$ on X is *reflexive* if the natural map $\mathscr{F} \to \mathscr{F}^{\vee\vee}$ is an isomorphism.

For example, any locally free sheaf is reflexive. On the other hand, any reflexive sheaf is torsion-free. So the reflexive sheaves form a wider class than the locally free sheaves, but are not as general as all torsion-free sheaves.

Proposition 1.1. *A coherent sheaf $\mathscr{F}$ on a noetherian integral scheme X is reflexive if and only if (at least locally) it can be included in an exact sequence*

$$0 \to \mathscr{F} \to \mathscr{E} \to \mathscr{G} \to 0,$$

where $\mathscr{E}$ is locally free and $\mathscr{G}$ is torsion-free.

Proof. Suppose $\mathscr{F}$ is reflexive. Then locally one can find a resolution of the dual sheaf $\mathscr{F}^\vee$ by locally free sheaves

$$\mathscr{L}_1 \to \mathscr{L}_0 \to \mathscr{F}^\vee \to 0.$$

The dualizing functor $\mathscr{H}om(\cdot, \mathcal{O}_X)$ is left exact, so taking duals gives an exact sequence

$$0 \to \mathscr{F}^{\vee\vee} \to \mathscr{L}_0^\vee \to \mathscr{L}_1^\vee.$$

Now $\mathscr{F}$ is reflexive so $\mathscr{F} \cong \mathscr{F}^{\vee\vee}$. The middle sheaf $\mathscr{L}_0^\vee$ is locally free. Taking $\mathscr{G}$ to be the image of the map $\mathscr{L}_0^\vee \to \mathscr{L}_1^\vee$ gives an exact sequence of the required form with $\mathscr{G}$ torsion-free.

Conversely, suppose there is an exact sequence

$$0 \to \mathscr{F} \to \mathscr{E} \to \mathscr{G} \to 0$$

with $\mathscr{E}$ locally free and $\mathscr{G}$ torsion-free. Then $\mathscr{F}$ is torsion-free, so the natural map $\mathscr{F} \to \mathscr{F}^{\vee\vee}$ is injective. On the other hand, $\mathscr{E}$ is locally free, hence reflexive, so $\mathscr{F}^{\vee\vee} \subseteq \mathscr{E}$. Therefore the quotient $\mathscr{F}^{\vee\vee}/\mathscr{F}$, which is a torsion sheaf, is a subsheaf of $\mathscr{G}$. Since $\mathscr{G}$ is torsion-free, it is zero, so $\mathscr{F}$ is reflexive.

Corollary 1.2. *The dual of any coherent sheaf is reflexive.*

Proof. Given a coherent sheaf $\mathscr{F}$, take a local resolution by locally free sheaves

$$\mathscr{L}_1 \to \mathscr{L}_0 \to \mathscr{F} \to 0.$$

The duals of these sheaves form an exact sequence

$$0 \to \mathscr{F}^{\vee} \to \mathscr{L}_0^{\vee} \to \mathscr{L}_1^{\vee}.$$

Then, as above, taking $\mathscr{G}$ to be the image of the map $\mathscr{L}_0^{\vee} \to \mathscr{L}_1^{\vee}$, there is an exact sequence

$$0 \to \mathscr{F}^{\vee} \to \mathscr{L}_0^{\vee} \to \mathscr{G} \to 0$$

with $\mathscr{L}_0^{\vee}$ locally free and $\mathscr{G}$ torsion-free. So by (1.1), $\mathscr{F}^{\vee}$ is reflexive.

Proposition 1.3. *Assume that X is normal, i.e. all its local rings are integrally closed domains. Then a coherent sheaf $\mathscr{F}$ is reflexive if and only if*

1) *$\mathscr{F}$ is torsion-free, and*
2) *for every $x \in X$ such that $\dim \mathcal{O}_x \geqq 2$, depth $\mathscr{F}_x \geqq 2$.*

Proof. Here x ranges over all scheme points of X, $\mathcal{O}_x$ is the local ring of x on X, $\mathscr{F}_x$ is the stalk of $\mathscr{F}$ at x, and the depth is measured over the local ring $\mathcal{O}_x$.

First suppose $\mathscr{F}$ is reflexive. Then $\mathscr{F}$ is torsion free. Further by (1.1) we can find a local exact sequence

$$0 \to \mathscr{F} \to \mathscr{E} \to \mathscr{G} \to 0$$

with $\mathscr{E}$ locally free and $\mathscr{G}$ torsion-free. Let x be a point with $\dim \mathcal{O}_x \geqq 2$, and consider the exact sequence of $\mathcal{O}_x$-modules

$$0 \to \mathscr{F}_x \to \mathscr{E}_x \to \mathscr{G}_x \to 0.$$

X normal and $\dim \mathcal{O}_x \geqq 2$ imply that depth $\mathcal{O}_x \geqq 2$ by Serre's criterion "normal $\Leftrightarrow R_1 + S_2$" [14, Theorem 39, p. 125]. Then since $\mathscr{E}$ is locally free, depth $\mathscr{E}_x \geqq 2$. On the other hand, since $\mathscr{G}$ is torsion-free, depth $\mathscr{G}_x \geqq 1$. It follows that depth $\mathscr{F}_x \geqq 2$. For example, one can use the local cohomology vanishing criterion for depth [7, Corollary 3.10, p. 47] together with the long exact sequence of local cohomology [7, 1.1] for H_x^i over the local ring.

Conversely, suppose that $\mathscr{F}$ is a coherent sheaf on X satisfying the conditions 1) and 2) of the proposition. Since $\mathscr{F}$ is torsion-free, the natural map $\mathscr{F} \to \mathscr{F}^{\vee\vee}$ is injective. Let $\mathscr{R}$ be the quotient:

$$0 \to \mathscr{F} \to \mathscr{F}^{\vee\vee} \to \mathscr{R} \to 0.$$

If $x \in X$ is a point for which $\dim \mathcal{O}_x = 1$, then $\mathcal{O}_x$ is a discrete valuation ring, by the R_1 part of Serre's criterion quoted above. Therefore $\mathscr{F}_x$, being a torsion-free $\mathcal{O}_x$-module, is actually free, so $\mathscr{F}_x \cong \mathscr{F}_x^{\vee\vee}$, and $\mathscr{R}_x = 0$. Thus $\mathscr{R}$ has support of codimension $\geqq 2$. Suppose $\mathscr{R} \neq 0$. Let $x \in \operatorname{Ass} \mathscr{R}$ be an associated point of $\mathscr{R}$, for example, the generic point of an irreducible component of the support of $\mathscr{R}$. Then depth $\mathscr{R}_x = 0$. In the exact sequence

$$0 \to \mathscr{F}_x \to \mathscr{F}_x^{\vee\vee} \to \mathscr{R}_x \to 0$$

since $\mathscr{F}^{\vee\vee}$ is reflexive and $\dim \mathcal{O}_x \geqq 2$, it follows that depth $\mathscr{F}_x^{\vee\vee} \geqq 2$ by the first part of the proof above. Then using the exact sequence of local cohomology and the depth criterion as above, depth $\mathscr{F}_x = 1$. This contradicts condition 2). So we conclude that $\mathscr{R} = 0$, hence $\mathscr{F} \cong \mathscr{F}^{\vee\vee}$ is reflexive.

Corollary 1.4. *Assume that X is regular, i.e. all its local rings are regular local rings. Then a reflexive sheaf $\mathscr{F}$ on X is locally free except along a closed subset Y of codimension $\geqq 3$. In particular, a reflexive sheaf on a regular scheme of dimension 2 is locally free.*

Proof. The set of points where a coherent sheaf is not locally free is a proper closed subset of X [8, II, Exercise 5.8], so it suffices to show for any point $x \in X$ with $\dim \mathcal{O}_x = 2$ that $\mathscr{F}_x$ is a free $\mathcal{O}_x$-module. Indeed, by (1.3) depth $\mathscr{F}_x = 2$. Since $\mathcal{O}_x$ is a regular local ring, this implies that the homological dimension of $\mathscr{F}_x$ is zero, i.e. $\mathscr{F}_x$ is free [14, Exercise 4, p. 113].

Corollary 1.5. *Assume X is normal, and let*

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{R} \to 0$$

be an exact sequence of coherent sheaves, with $\mathscr{F}$ reflexive. Then $\mathscr{F}'$ is reflexive if and only if $\operatorname{Ass}\mathscr{R}$ *consists of points of codimensions* 0 *and* 1 *only.*

Proof. In any case $\mathscr{F}'$ is torsion-free, so $\mathscr{F}'$ is reflexive if and only if it satisfies condition 2) of (1.3). For any point $x \in X$ with $\dim \mathcal{O}_x \geqq 2$, depth $\mathscr{F}_x \geqq 2$, so as above, depth $\mathscr{F}'_x \geqq 2$ if and only if depth $\mathscr{R}_x \geqq 1$, i.e. $x \notin \operatorname{Ass}\mathscr{R}$. In other words, $\mathscr{R}$ must have no associated points of codimension $\geqq 2$. This means that $\operatorname{Ass}\mathscr{R}$ must be of pure codimension 1.

Next we will show that reflexive sheaves are determined by their behavior off of subsets of codimension $\geqq 2$. At the same time we show that reflexive sheaves are the same as torsion-free sheaves which are "normal" in the sense of Barth.

Definition (Barth [1, p. 128]). A coherent sheaf $\mathscr{F}$ on X is *normal* if for every open set $U \subseteq X$ and every closed subset $Y \subseteq U$ of codimension $\geqq 2$, the restriction map $\mathscr{F}(U) \to \mathscr{F}(U-Y)$ is bijective.

Proposition 1.6. *Let $\mathscr{F}$ be a coherent sheaf on a normal integral scheme X. The following conditions are equivalent:*

(i) *$\mathscr{F}$ is reflexive;*

(ii) *$\mathscr{F}$ is torsion-free and normal;*

(iii) *$\mathscr{F}$ is torsion-free, and for each open $U \subseteq X$ and each closed subset $Y \subseteq U$ of codimension $\geqq 2$, $\mathscr{F}_U \cong j_* \mathscr{F}_{U-Y}$, where $j: U-Y \to U$ is the inclusion map.*

Proof. Since reflexive implies torsion-free, we may assume that $\mathscr{F}$ is torsion-free. We will show that each of the above conditions is equivalent to

(iv) *for each closed subset $Y \subseteq X$ of codimension $\geqq 2$, $\mathscr{H}^1_Y(\mathscr{F}) = 0$, where $\mathscr{H}^1_Y$ denotes the local cohomology sheaf.*

(iii)⇔(iv). This is a standard property of local cohomology sheaves [7, 1.9].

(iv)⇒(ii). The restriction map $\mathscr{F}(U) \to \mathscr{F}(U-Y)$ is injective since $\mathscr{F}$ is torsion-free. Thus there is an exact sequence

$$0 \to \mathscr{F}(U) \to \mathscr{F}(U-Y) \to H^1_Y(U, \mathscr{F}).$$

Since $\mathscr{F}$ is torsion free, $\mathscr{H}^0_Y(\mathscr{F}) = 0$ also, so the spectral sequence [7, 1.4] of local cohomology groups and sheaves implies that $H^1_Y(U, \mathscr{F}) = 0$. Here we apply (iv) to the closure $\bar{Y}$ of Y in X.

(ii)⇒(iv). The sheaf $\mathscr{H}_Y^1(\mathscr{F})$ is associated to the presheaf $U \to H^1_{Y \cap U}(U, \mathscr{F})$. Letting U range over an affine base for the topology of X, the next term in the exact sequence above is $H^1(U, \mathscr{F}) = 0$, so condition (ii) implies that $H^1_{Y \cap U}(U, \mathscr{F}) = 0$. Therefore $\mathscr{H}_Y^1(\mathscr{F}) = 0$.

(iv)⇒(i). To show that $\mathscr{F}$ is reflexive we use the criterion of (1.3). For any point $x \in X$ with $\dim \mathcal{O}_x \geqq 2$, let Y be the closure of x. Then $\mathscr{H}_Y^1(\mathscr{F}) = 0$ implies $H^1_x(\mathscr{F}_x) = 0$, so depth $\mathscr{F}_x \geqq 2$.

(i)⇒(iv). Let $Y \subseteq X$ be a closed subset of codimension $\geqq 2$. For any closed subset $Z \subseteq Y$ there is a spectral sequence associated to the composite functor $\Gamma_Z = \Gamma_Z \circ \Gamma_Y$. Since $\mathscr{F}$ is torsion-free, $\mathscr{H}_Y^0(\mathscr{F}) = 0$, so we find $\mathscr{H}_Z^1(\mathscr{F}) = \mathscr{H}_Z^0(\mathscr{H}_Y^1(\mathscr{F}))$. Now assume $\mathscr{F}$ is reflexive, and suppose $\mathscr{H}_Y^1(\mathscr{F}) \neq 0$. Let $x \in \operatorname{Ass} \mathscr{H}_Y^1(\mathscr{F})$. Then taking Z to be the closure of x, the above formula shows $\mathscr{H}_Z^1(\mathscr{F})_x \neq 0$. Therefore depth $\mathscr{F}_x = 1$, which contradicts (1.3). We conclude that $\mathscr{H}_Y^1(\mathscr{F}) = 0$.

Corollary 1.7. *Let $f: X \to Y$ be a proper dominant morphism of normal integral schemes, with all fibers of the same dimension. If $\mathscr{F}$ is a coherent reflexive sheaf on X, then $f_* \mathscr{F}$ is a coherent reflexive sheaf on Y.*

Proof. The coherence of $f_* \mathscr{F}$ follows of course from the hypothesis f proper. Since $\mathscr{F}$ is torsion-free and f dominant, $f_* \mathscr{F}$ is also torsion-free. To show that $f_* \mathscr{F}$ is reflexive, we use (1.6) (ii). Then $\mathscr{F}$ is normal and we must show $f_* \mathscr{F}$ normal. If U is any open subset of Y and $Z \subseteq Y$ a closed subset of codimension $\geqq 2$, then $f^{-1}(U)$ is open in X and $f^{-1}(Z)$ closed of codimension $\geqq 2$ in $f^{-1}(U)$ because of the hypothesis on the dimensions of fibres. Now $f_* \mathscr{F}(U) = \mathscr{F}(f^{-1}(U))$ and $f_* \mathscr{F}(U - Z) = \mathscr{F}(f^{-1}(U) - f^{-1}(Z))$ so $f_* \mathscr{F}$ is normal.

Remarks 1.7.1. The implication $\mathscr{F}$ normal implies $f_* \mathscr{F}$ normal is due to Barth [1, Sect. 4.2].

1.7.2. If f is a smooth proper morphism of nonsingular varieties of relative dimension n, and $\mathscr{F}$ is locally free on X, then one can also deduce this result from the relative duality theorem. Indeed, $f_* \mathscr{F}$ is the dual of $R^n f_*(\mathscr{F}^{\vee} \otimes \omega_{X/Y})$, hence is reflexive by (1.2).

Example 1.7.3. The following example, due to Strømme, shows that $\mathscr{F}$ locally free does not imply $f_* \mathscr{F}$ locally free, even if f is a smooth morphism of nonsingular varieties. On $\mathbb{P}^1$, consider extensions

$$0 \to \mathcal{O}(-2) \to \mathscr{E} \to \mathcal{O}(2) \to 0.$$

They are classified by $H^1(\mathbb{P}^1, \mathcal{O}(-4))$, which is a 3-dimensional vector space. Let T be this vector space, considered as an affine 3-space, and let $\mathscr{F}$ be the universal extension of $\mathcal{O}(2)$ by $\mathcal{O}(-2)$ on $X = \mathbb{P}^1 \times T$. Then $\mathscr{F}$ is locally free of rank 2 on $\mathbb{P}^1 \times T$. Let $f: X \to T$ be the projection. For any $t \in T$ let $\mathscr{F}_t$ be the fibre on $\mathbb{P}^1$. Then $\mathscr{F}_t \cong \mathcal{O}(2) \oplus \mathcal{O}(-2)$ for $t = 0$, but $\mathscr{F}_t \cong \mathcal{O}(1) \oplus \mathcal{O}(-1)$ or $\mathcal{O} \oplus \mathcal{O}$ for $t \neq 0$. Thus

$$h^0(\mathbb{P}^1, \mathscr{F}_t) = \begin{cases} 3 & \text{for } t = 0 \\ 2 & \text{for } t \neq 0 \end{cases}$$

and

$$h^1(\mathbb{P}^1, \mathscr{F}_t) = \begin{cases} 1 & \text{for } t = 0 \\ 0 & \text{for } t \neq 0. \end{cases}$$

We will show that $f_*\mathscr{F}$ is not locally free. One might think this was obvious from the calculation of $h^0(\mathscr{F}_t)$ above. To be sure, there is a natural map

$$f_*\mathscr{F}\otimes k(t)\to H^0(\mathscr{F}_t)$$

for any t, but it need not be an isomorphism [8, III, Sect. 12]. Indeed, one can show that for $t=0$ it is the zero map! So this does not help to see if $f_*\mathscr{F}$ is locally free.

Instead, we use a different method. Let $P\in T$ be the point $t=0$, and let $Z=f^{-1}(P)\subsetneq X$. The functor Γ_Z, which takes sections of a sheaf on X with support in Z, can be written as a composite functor $\Gamma_Z=\Gamma_P\circ f_*$. This gives rise to a spectral sequence

$$E_2^{pq}=H_P^p(T,R^qf_*(\mathscr{F}))\Rightarrow E^n=H_Z^n(X,\mathscr{F}),$$

for any coherent sheaf $\mathscr{F}$ on X. In our case, since $\mathscr{F}$ is locally free on X and Z has codimension 3, the abutment $H_Z^n(X,\mathscr{F})=0$ for $n=0,1,2$. Thus the d_2 map gives an isomorphism of initial terms

$$H_P^0(R^1f_*\mathscr{F})\xrightarrow{\sim}H_P^2(f_*\mathscr{F}).$$

Now R^1f_* commutes with base extension [8, III, 12.11] since $R^2f_*=0$, so the calculations of $h^1(\mathscr{F}_t)$ above show that $R^1f_*\mathscr{F}$ is a torsion sheaf concentrated at P. Therefore $H_P^0(R^1f_*\mathscr{F})\neq 0$, so $H_P^2(f_*\mathscr{F})\neq 0$, so $f_*\mathscr{F}$ is a reflexive sheaf, by (1.7), on T whose depth at P is 2. Thus $f_*\mathscr{F}$ is not locally free.

Proposition 1.8. *Let $f:X\to Y$ be a flat morphism of noetherian integral schemes, and let $\mathscr{F}$ be a reflexive coherent sheaf on Y. Then $f^*\mathscr{F}$ is reflexive on X.*

Proof. We will show for any coherent sheaf $\mathscr{F}$ on Y, that $f^*(\mathscr{F}^\vee)=(f^*\mathscr{F})^\vee$. From this and the definition of reflexive sheaves, the proposition follows immediately.

The question is local on X and Y, so we reduce to the following easy result about flat ring extensions. Let A be a noetherian ring, let M,N be A-modules, with M finitely generated, and let $A\to B$ be a flat ring homomorphism. Then the natural map

$$\mathrm{Hom}_A(M,N)\bigotimes_A B\to\mathrm{Hom}_B\left(M\bigotimes_A B,N\bigotimes_A B\right)$$

is an isomorphism. To prove this it suffices to note that both sides are contravariant left-exact functors in M, and that the result is trivially true for $M=A$.

Proposition 1.9. *Assume X is integral and locally factorial (i.e. all its local rings are unique factorization domains). Then any reflexive* rank 1 *sheaf is invertible.*

Proof (cf. Barth [1, Lemma 1, p. 128]). Since X is normal and $\mathscr{F}$ torsion-free, there is a closed subset Y of codimension $\geqq 2$ such that $\mathscr{F}$ is locally free (hence invertible) on $X-Y$. On the other hand, since X is locally factorial, $\mathrm{Pic}X\to\mathrm{Pic}(X-Y)$ is bijective [8, II, 6.5 and II, 6.16]. Therefore there is an invertible sheaf $\mathscr{L}$ on X with $\mathscr{L}_{X-Y}\cong\mathscr{F}_{X-Y}$. Then since $\mathscr{F}$ and $\mathscr{L}$ are both reflexive, using (1.6)(iii), $\mathscr{F}\cong j_*\mathscr{F}_{X-Y}$ and $\mathscr{L}\cong j_*\mathscr{L}_{X-Y}$, so $\mathscr{F}\cong\mathscr{L}$. Thus $\mathscr{F}$ is invertible.

Example 1.9.1. To show that (1.4) and (1.9) are best possible, we give some examples of non-locally-free reflexive rank 2 sheaves on a regular scheme of dimension 3. Let $\mathscr{E}$ be a rank 2 vector bundle on $\mathbb{P}^2$, and let $M = \bigoplus_{n \in \mathbb{Z}} H^0(\mathbb{P}^2, \mathscr{E}(n))$ be the corresponding graded module over the homogeneous polynomial ring $S = k[x_0, x_1, x_2]$. Then $\mathscr{F} = \tilde{M}$ is a reflexive sheaf on Spec S. It is locally free if and only if $\mathscr{E}$ is a direct sum of line bundles. Since one knows many indecomposable rank 2 vector bundles on $\mathbb{P}^2$, this gives many examples of non-locally-free reflexive sheaves on $\mathbb{A}^3$.

It is instructive to study these examples further. The sheaf $\mathscr{F}$ on Spec S is locally free everywhere except possibly at the origin P. There, its local cohomology groups $H^i_P(\mathscr{F})$ are 0 for $i=0,1$ by construction, and $H^2_P(\mathscr{F}) \cong \bigoplus_{n \in \mathbb{Z}} H^1(\mathbb{P}^2, \mathscr{E}(n))$. Thus depth $\mathscr{F}_P \geqq 2$, which proves $\mathscr{F}$ is reflexive. And $\mathscr{F}$ will be locally free if and only if depth $\mathscr{F}_P = 3$, which is equivalent to $H^2_P(\mathscr{F}) = 0$. Thus the non-vanishing cohomology groups $H^1(\mathbb{P}^2, \mathscr{E}(n))$ are the obstructions to $\mathscr{F}$ being locally free.

We close this section with a special property of rank 2 reflexive sheaves. If $\mathscr{E}$ is a rank 2 vector bundle, then the natural map $\mathscr{E} \otimes \mathscr{E} \to \wedge^2 \mathscr{E}$ gives the well-known isomorphism $\mathscr{E}^\vee \cong \mathscr{E} \otimes (\wedge^2 \mathscr{E})^{-1}$, where $\wedge^2 \mathscr{E}$ is an invertible sheaf which we may call the determinant, or first Chern class of $\mathscr{E}$. How can we generalize this to reflexive sheaves? In general, tensor operations on non-locally-free sheaves are poorly behaved. Therefore to obtain reasonable definitions and results, we will usually restrict to an open set where a sheaf is locally free, do tensor operations there, and then extend back to the whole space.

Definition. Let $\mathscr{F}$ be a torsion-free coherent sheaf on an integral locally factorial scheme X. We define an invertible sheaf det $\mathscr{F}$, the *determinant* of $\mathscr{F}$, as follows. Let Y be a closed subset of codimension $\geqq 2$ of X such that $\mathscr{F}$ is locally free on $X - Y$ of rank r. Then $\wedge^r(\mathscr{F}_{X-Y})$ is an invertible sheaf on $X - Y$. As in the proof of (1.9) there is a unique invertible sheaf on X whose restriction to $X - Y$ is $\wedge^r(\mathscr{F}_{X-Y})$. This is det $\mathscr{F}$.

Proposition 1.10. *Let $\mathscr{F}$ be a* rank 2 *reflexive sheaf on an integral locally factorial scheme X. Then $\mathscr{F}^\vee \cong \mathscr{F} \otimes (\det \mathscr{F})^{-1}$.*

Proof. Again choose Y of codimension $\geqq 2$ such that $\mathscr{F}_{X-Y}$ is locally free. Then on $X - Y$ the formula for a rank 2 vector bundle holds:

$$\mathscr{F}^\vee_{X-Y} \cong \mathscr{F}_{X-Y} \otimes (\wedge^2 \mathscr{F}_{X-Y})^{-1}.$$

Since $\mathscr{F}^\vee$ and $\mathscr{F} \otimes (\det \mathscr{F})^{-1}$ are reflexive sheaves on X, whose restrictions to $X - Y$ are isomorphic, by (1.6) (iii) they are isomorphic on X.

2. Numerical Invariants

In this section we review results we will need such as Chern classes, Riemann-Roch theorem, and Serre duality, giving special attention to the case of rank 2 reflexive sheaves on $\mathbb{P}^3$.

We begin with Chern classes. They are usually defined for vector bundles $\mathscr{E}$ on a nonsingular variety X, and take values in the Chow ring [8, Appendix A]. In case $X=\mathbb{P}^n$, the Chow ring is $\mathbb{Z}[t]/t^{n+1}$, where t is the class of a hyperplane, so we may consider the *Chern classes* $c_i(\mathscr{E})$ as integers, and the *Chern polynomial*

$$c_t(\mathscr{E})=1+c_1(\mathscr{E})t+\ldots+c_r(\mathscr{E})t^r$$

as a polynomial with integer coefficients.

One knowns that c_t is multiplicative in the following sense: if

$$0\to\mathscr{E}'\to\mathscr{E}\to\mathscr{E}''\to 0$$

is a short exact sequence of vector bundles, then

$$c_t(\mathscr{E})=c_t(\mathscr{E}')\,c_t(\mathscr{E}'').$$

Therefore the Chern polynomial c_t can be defined on the Grothendieck group $K(X)$ of vector bundles on X [8, p. 435]. For a nonsingular variety X, the Grothendieck groups of all vector bundles and of all coherent sheaves coincide [8, III, Exercise 6.9]. Therefore c_t is defined for all coherent sheaves on X. In particular, this gives a definition of Chern classes $c_i(\mathscr{F})\in\mathbb{Z}$ for all coherent sheaves $\mathscr{F}$ on $\mathbb{P}^n$ and all $i=1,2,\ldots,n$. For a vector bundle $\mathscr{E}$ of rank r, one knows that $c_i(\mathscr{E})=0$ for $i>r$, but of course this need not hold for coherent sheaves.

Lemma 2.1. *Let $\mathscr{F}$ be a coherent sheaf of* rank $r\geqq 0$ *on $\mathbb{P}^n$, and let $l\in\mathbb{Z}$. Then the Chern classes of $\mathscr{F}(l)$ are given by*

$$c_i(\mathscr{F}(l))=c_i(\mathscr{F})+(r-i+1)lc_{i-1}(\mathscr{F})+\binom{r-i+2}{2}l^2c_{i-2}(\mathscr{F})+\ldots+\binom{r}{i}l^i.$$

Proof. Because of the universal nature of such formulas, it is enough to check the case $\mathscr{F}$ locally free and $i\leqq r$. Furthermore, by the splitting principle, one may assume $\mathscr{F}\cong\sum_{i=1}^{r}\mathcal{O}(a_i)$. Then the calculation is tedious but straightforward.

Corollary 2.2. *Let $\mathscr{F}$ be a coherent sheaf of* rank 2 *on $\mathbb{P}^3$ with Chern classes c_1, c_2, c_3. Then the Chern classes of $\mathscr{F}(l)$ are*

$$c_1'=c_1+2l$$
$$c_2'=c_2+c_1l+l^2$$
$$c_3'=c_3.$$

Note in particular that c_3 does not change with twisting.

Theorem 2.3 (Riemann-Roch). *Let $\mathscr{F}$ be a coherent sheaf of* rank r *on $\mathbb{P}^3$, with Chern classes c_1, c_2, c_3, and let $\chi(\mathscr{F})=\sum(-1)^ih^i(\mathscr{F})$ be its Euler characteristic. Then*

$$\chi(\mathscr{F})=r+\binom{c_1+3}{3}-2c_2+\tfrac{1}{2}(c_3-c_1c_2)-1.$$

Proof. This can be deduced from the generalized Grothendieck-Hirzebruch-Riemann-Roch theorem [8, Appendix A, 4.1]. Or it can be proved independently on $\mathbb{P}^n$ by showing that $K(X)\cong\mathbb{Z}[h]/(1-h)^{n+1}$ where h is the class of $\mathcal{O}(1)$, and then reducing to sheaves of the form $\sum\mathcal{O}(a_i)$. In either case, once one knows that $\chi(\mathscr{F})$ can be expressed as a polynomial with rational coefficients in the Chern classes, the coefficients themselves can be recovered by computing with sheaves of the form $\sum \mathcal{O}(a_i)$, which is easy.

Corollary 2.4. *If $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^3$, then $c_1c_2 \equiv c_3 (\mathrm{mod}\, 2)$.*

Proof. The Riemann-Roch formula above shows that $\frac{1}{2}(c_3 - c_1c_2)$ is an integer.

Theorem 2.5. (Serre duality). *Let $\mathcal{F}$ be a reflexive coherent sheaf on $\mathbb{P}^3$. Then there are isomorphisms*

$$H^0(\mathcal{F}^\vee \otimes \omega) \to H^3(\mathcal{F})'$$

$$H^3(\mathcal{F}^\vee \otimes \omega) \to H^0(\mathcal{F})'$$

and an exact sequence

$$0 \to H^1(\mathcal{F}^\vee \otimes \omega) \to H^2(\mathcal{F})' \to H^0(\mathcal{E}xt^1(\mathcal{F}, \omega)) \to H^2(\mathcal{F}^\vee \otimes \omega) \to H^1(\mathcal{F})' \to 0 .$$

Proof. The usual Serre duality theorem says that $H^i(\mathcal{F})$ and $\mathrm{Ext}^{3-i}(\mathcal{F}, \omega)$ are dual vector spaces [8, III, 7.1]. We combine this with the spectral sequence of local and global Ext functors:

$$E_2^{pq} = H^p(\mathcal{E}xt^q(\mathcal{F}, \omega)) \Rightarrow E^n = \mathrm{Ext}^n(\mathcal{F}, \omega) .$$

Since $\mathcal{F}$ is reflexive, the E_2^{pq} terms are zero except for $q=0,1$. Indeed, $\mathcal{E}xt^0(\mathcal{F}, \omega) = \mathcal{H}om(\mathcal{F}, \omega) = \mathcal{F}^\vee \otimes \omega$. Furthermore, $\mathcal{F}$ is locally free except at a finite number of points P_i by (1.4). At those points $\mathcal{F}$ has depth 2, hence homological dimension 1. Thus $\mathcal{E}xt^i(\mathcal{F}, \omega) = 0$ for $i \geqq 2$, and $\mathcal{E}xt^1(\mathcal{F}, \omega)$ is a coherent sheaf supported at the points P_i. It follows then that $E_2^{p1} = 0$ except for $p = 0$. Now the spectral sequence degenerates into two isomorphisms and one 5-term exact sequence. Substituting $H^{3-i}(\mathcal{F})'$ for $\mathrm{Ext}^i(\mathcal{F}, \omega)$ gives the result.

Remark 2.5.1. Serre's vanishing theorem [8, III, 5.2] says that $H^i(\mathcal{F}(l)) = 0$ for $i > 0$ and $l \gg 0$. In the case of a locally free sheaf $\mathcal{E}$ on $\mathbb{P}^3$, this implies that $H^i(\mathcal{E}(l)) = 0$ for $i < 3$ and $l \ll 0$. If $\mathcal{F}$ is reflexive, this version of Serre duality shows that $H^i(\mathcal{F}(l)) = 0$ for $i = 0, 1$, and $l \ll 0$, and that $H^2(\mathcal{F}(l))$ is of constant dimension $h^0(\mathcal{E}xt^1(\mathcal{F}, \omega))$ for $l \ll 0$.

Proposition 2.6. *Let $\mathcal{F}$ be a* rank 2 *reflexive sheaf on $\mathbb{P}^3$. Then $c_3(\mathcal{F}) = h^0(\mathcal{E}xt^1(\mathcal{F}, \omega))$. In particular, $c_3 \geqq 0$, and $c_3 = 0$ if and only if $\mathcal{F}$ is locally free.*

Proof[2]. We will calculate the Chern classes of $\mathcal{F}^\vee$ in two different ways. Let $\mathcal{F}$ have Chern classes c_1, c_2, c_3. Since the determinant sheaf of $\mathcal{F}$ defined in Sect. 1 is $\mathcal{O}(c_1)$, we see that $\mathcal{F}^\vee \cong \mathcal{F}(-c_1)$ by (1.10). Therefore by (2.2)

$$c_t(\mathcal{F}^\vee) = 1 - c_1 t + c_2 t^2 + c_3 t^3 .$$

On the other hand, since $\mathcal{F}$ is reflexive, it has homological dimension $\leqq 1$ at every point, so we can find an exact sequence

$$0 \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0$$

2 The idea for this proof, simpler than my original proof, is due to W. E. Lang

with $\mathscr{E}_1$ and $\mathscr{E}_0$ locally free. Therefore

$$c_t(\mathscr{F}) = c_t(\mathscr{E}_0)c_t(\mathscr{E}_1)^{-1}.$$

Taking duals gives an exact sequence

$$0 \to \mathscr{F}^\vee \to \mathscr{E}_0^\vee \to \mathscr{E}_1^\vee \to \mathscr{E}xt^1(\mathscr{F}, \mathcal{O}) \to 0.$$

Therefore

$$c_t(\mathscr{F}^\vee) = c_t(\mathscr{E}_0^\vee)\, c_t(\mathscr{E}_1^\vee)^{-1}\, c_t(\mathscr{E}xt^1(\mathscr{F}, \mathcal{O})).$$

For a locally free sheaf $\mathscr{E}$,

$$c_t(\mathscr{E}^\vee) = c_{-t}(\mathscr{E}).$$

On the other hand, $\mathscr{E}xt^1(\mathscr{F}, \mathcal{O})$ is a sheaf supported at the non-locally-free points of $\mathscr{F}$, whose length is equal to $h^0(\mathscr{E}xt^1(\mathscr{F}, \omega))$, since the twist ω has no effect. Call that length n. Then by (2.7) below,

$$c_t(\mathscr{E}xt^1(\mathscr{F}, \mathcal{O})) = 1 + 2nt^3.$$

So we find

$$\begin{aligned} c_t(\mathscr{F}^\vee) &= c_{-t}(\mathscr{E}_0)\, c_{-t}(\mathscr{E}_1)^{-1}(1 + 2nt^3) \\ &= c_{-t}(\mathscr{F})(1 + 2nt^3) \\ &= (1 - c_1 t + c_2 t^2 - c_3 t^3)(1 + 2nt^3) \\ &= 1 - c_1 t + c_2 t^2 + (2n - c_3)t^3. \end{aligned}$$

Comparing with the other expression for $c_t(\mathscr{F}^\vee)$ we see that $c_3 = 2n - c_3$, so $c_3 = n$, as required.

Lemma 2.7. *Let $\mathscr{G}$ be a sheaf on $\mathbb{P}^3$ concentrated at a finite number of points, of length n. Then*

$$c_t(\mathscr{G}) = 1 + 2nt^3.$$

Proof. Using induction on n and the multiplicativity of c_t we reduce to the case $n = 1$. Then $\mathscr{G} = \mathcal{O}_P$, where P is a point in $\mathbb{P}^3$. The sheaf $\mathcal{O}_P$ has a resolution on $\mathbb{P}^3$ of the form

$$0 \to \mathcal{O}(-3) \to \mathcal{O}(-2)^3 \to \mathcal{O}(-1)^3 \to \mathcal{O} \to \mathcal{O}_P \to 0.$$

Therefore

$$c_t(\mathcal{O}_P) = 1 \cdot (1 - t)^{-3}(1 - 2t)^3(1 - 3t)^{-1}.$$

A short calculation then gives $c_t(\mathcal{O}_P) = 1 + 2t^3$, as required.

Remark 2.7.1. Since the sheaf $\mathscr{E}xt^1(\mathscr{F}, \omega)$ has support at the points where $\mathscr{F}$ is not locally free, and its length is a measure of the nonfreeness of $\mathscr{F}$ at those points, we will refer to (2.6) by the catch phrase "c_3 is the number of points where $\mathscr{F}$ is not free". This result gives more insight into the fact (2.2) that c_3 is invariant under twisting. Note also, by the local duality theorem [7, 6.3] that at each point P, $\mathscr{E}xt^1(\mathscr{F}, \omega)_P$ is dual to the local cohomology group $H_P^2(\mathscr{F})$ which measures the extent to which $\mathscr{F}$ does not have depth 3. So c_3 is also equal to the sum of the lengths of the modules $H_P^2(\mathscr{F})$ taken over the non-free points of $\mathscr{F}$. Compare (1.9.1) for examples of these local cohomology modules.

3. Stable and Semistable Reflexive Sheaves

We continue to use the definition of Mumford and Takemoto [18].

Definition. A reflexive coherent sheaf $\mathscr{F}$ on $\mathbb{P}^n$ is *stable* (resp. *semistable*) if for every coherent subsheaf $\mathscr{F}'$ of $\mathscr{F}$, with $0<\operatorname{rank}\mathscr{F}'<\operatorname{rank}\mathscr{F}$,

$$\mu(\mathscr{F}')<\mu(\mathscr{F})$$

(resp. $\leqq$), where $\mu=c_1/\operatorname{rank}$.

Lemma 3.1. *Let $\mathscr{F}$ be a* rank 2 *reflexive sheaf on $\mathbb{P}^n$, which is normalized so that $c_1(\mathscr{F})=0$ or -1. Then $\mathscr{F}$ is stable if and only if $H^0(\mathscr{F})=0$. If $c_1=0$, then $\mathscr{F}$ is semistable if and only if $H^0(\mathscr{F}(-1))=0$.*

Proof. To test a rank 2 sheaf for stability we need only consider rank 1 coherent subsheaves. Since $\mathscr{F}$ is reflexive, we can replace any rank 1 subsheaf by its double dual without decreasing μ, so it is enough to consider rank 1 reflexive subsheaves, which are invertible (1.9). On $\mathbb{P}^n$, an invertible sheaf is of the form $\mathcal{O}(l)$ for some $l\in\mathbb{Z}$. Thus $\mathscr{F}$ is stable if and only if for any subsheaf $\mathcal{O}(l)\subsetneqq\mathscr{F}$, $l<0$. This is equivalent to $H^0(\mathscr{F})=0$. In case $c_1=0$, $\mathscr{F}$ is semistable if for any such subsheaf, $l\leqq 0$. This is equivalent to $H^0(\mathscr{F}(-1))=0$.

Remark 3.1.1. The definition of stability or semistability due to Gieseker and Maruyama (GM-stable) replaces μ in the definition above by P/rank, where P is the Hilbert polynomial. Since the leading term of the Hilbert polynomial is determined by c_1, it is clear that stable $\Rightarrow$ GM-stable $\Rightarrow$ GM-semistable $\Rightarrow$ semistable. In the case of rank 2 reflexive sheaves on $\mathbb{P}^3$, one can show in fact that stable $\Leftrightarrow$ GM-stable, and that the only GM-semistable sheaf which is not stable is $\mathcal{O}\oplus\mathcal{O}$. For c_1 odd there is nothing to prove, so we may assume $c_1=0$. Then the Hilbert polynomial of $\mathscr{F}$ is [9, 8.1]

$$P_{\mathscr{F}}(l)=\tfrac{1}{3}(l+1)(l+2)(l+3)-c_2(l+2)+\tfrac{1}{2}c_3.$$

If $\mathscr{F}$ is GM-semistable but not stable, then $H^0(\mathscr{F})\neq 0$, so there is an exact sequence

$$0\to\mathcal{O}\to\mathscr{F}\to\mathscr{I}_Y\to 0,$$

and

$$P_{\mathcal{O}}(l)\leqq\tfrac{1}{2}P_{\mathscr{F}}(l).$$

Since $P_{\mathcal{O}}=\frac{1}{6}(l+1)(l+2)(l+3)$, we find

$$0\leqq -c_2(l+2)+\tfrac{1}{2}c_3,$$

the inequality being understood as polynomials in l. In particular, $c_2\leqq 0$. On the other hand, $c_2\geqq 0$ by (3.3) below. Therefore $c_2=0$, so $Y=\emptyset$, and $\mathscr{F}=\mathcal{O}\oplus\mathcal{O}$.

This shows that the only GM-semistable sheaf which is not stable is $\mathcal{O}\oplus\mathcal{O}$, and in particular, stable⇔GM-stable.

Theorem 3.2. *Let $\mathcal{F}$ be a semistable* rank 2 *reflexive sheaf on* $\mathbb{P}^3$. *Then for a general plane $H\subsetneq\mathbb{P}^3$, the restriction $\mathcal{F}_H$ is a semistable vector bundle on H.*

Proof. This generalizes the corresponding result for vector bundles [9, 3.3]. The proof is slightly different, because of the non-locally-free points.

Observe first that since $\mathcal{F}$ is locally free except at a finite number of points, the restriction to a general plane $\mathcal{F}_H$ is locally free. On the other hand, if H is a special plane, containing some points where $\mathcal{F}$ is not locally free, there is in any case an exact sequence

$$0\to\mathcal{F}(-1)\to\mathcal{F}\to\mathcal{F}_H\to 0.$$

Since $\mathcal{F}$ is reflexive, by (1.5) the associated points of $\mathcal{F}_H$ all have codimension 1. In other words, $\mathcal{F}_H$ is a torsion-free coherent sheaf on H. Furthermore, this sequence shows that the Chern classes of $\mathcal{F}_H$ are independent of H, special or not. Therefore, as H varies, the sheaves $\mathcal{F}_H$ form a flat family [8, III, 9.9], and we can apply the theorems of semicontinuity of cohomology [8, III, Sect. 12].

Now suppose that $\mathcal{F}$ is normalized with $c_1=0$ or -1. Then by the lemma, $H^0(\mathcal{F}(-1))=0$, and if $c_1=-1$, $H^0(\mathcal{F})=0$. If $\mathcal{F}_H$ is not semistable for a general plane H, then there is an integer m with $m+c_1<0$ such that $H^0(\mathcal{F}_H(m))\neq 0$. By semicontinuity this is in fact true for every plane H.

For each plane H, let m_H be the least integer for which $H^0(\mathcal{F}_H(m_H))\neq 0$. Let $m_0=\max\{m_H\}$. Then for most planes H, $m_H=m_0$. Fix a plane H, not containing any of the non-locally-free points of $\mathcal{F}$, for which $m_H=m_0$. Let $s\in H^0(\mathcal{F}_H(m_0))$, and write the corresponding exact sequence of sheaves on H,

$$0\to\mathcal{O}_H\to\mathcal{F}_H(m_0)\to\mathcal{I}_{Z,H}(c_1+2m_0)\to 0,$$

where Z is a 0-dimensional subscheme of H. Since $c_1+2m_0<0$, this shows that $h^0(\mathcal{F}_H(m_0))=1$.

Now take a line L in H, not meeting Z. Then $\mathcal{F}_L\cong\mathcal{O}_L(-m_0)\oplus\mathcal{O}_L(c_1+m_0)$, so $h^0(\mathcal{F}_L(m_0))=1$ also, and $h^0(\mathcal{F}_L(m'))=0$ for $m'<m_0$.

Next, we consider the pencil of planes containing L. I claim that for every plane H containing L, $h^0(\mathcal{F}_H(m_0))=1$. First note for any m, the exact sequence

$$0\to\mathcal{F}_H(m-1)\to\mathcal{F}_H(m)\to\mathcal{F}_L(m)\to 0.$$

Since $h^0(\mathcal{F}_L(m'))=0$ for $m'<m_0$, it follows that $h^0(\mathcal{F}_H(m'))=0$ for $m'<m_0$. On the other hand, $h^0(\mathcal{F}_H(m_0))\neq 0$ by choice of m_0. So the exact sequence

$$0\to H^0(\mathcal{F}_H(m_0))\to H^0(\mathcal{F}_L(m_0))$$

shows that $h^0(\mathcal{F}_H(m_0))=1$.

Now we can complete the proof as in [9, 3.3] by blowing up the line L. Let $p:X\to\mathbb{P}^3$

$$\begin{array}{ccc} Y\subseteq X & \xrightarrow{q} & \mathbb{P}^1 \\ {\scriptstyle p}\downarrow & & \\ L\subseteq\mathbb{P}^3 & & \end{array}$$

be the blowing-up map, let $Y=p^{-1}(L)$, and let $q:X\to\mathbb{P}^1$ be the map given by the pencil of planes through L. Then $p^*\mathcal{F}$ is reflexive on X (since $\mathcal{F}$ is locally free along L), and $q_*(p^*\mathcal{F}(m_0))$ is an invertible sheaf on $\mathbb{P}^1$ [8, III, 12.9]. But we have seen that for every plane H containing L, the map $H^0(\mathcal{F}_H(m_0))\to H^0(\mathcal{F}_L(m_0))$ is an isomorphism. Therefore $q_*(p^*\mathcal{F}(m_0))\to q_*(p^*\mathcal{F})_Y(m_0)$ is an isomorphism. And this latter sheaf is just $\mathcal{O}_{\mathbb{P}^1}$, since

$$(p^*\mathcal{F})_Y=p^*(\mathcal{F}_L) \quad\text{and}\quad \mathcal{F}_L\cong\mathcal{O}_L(-m_0)\oplus\mathcal{O}_L(c_1+m_0).$$

We conclude that $q_*(p^*\mathcal{F}(m_0))\cong\mathcal{O}_{\mathbb{P}^1}$, which implies that $H^0(\mathcal{F}(m_0))\neq 0$ on $\mathbb{P}^3$, contradicting the semistability of $\mathcal{F}$.

Corollary 3.3. *If $\mathcal{F}$ is a semistable* rank 2 *reflexive sheaf on $\mathbb{P}^3$, then $c_1^2-4c_2\leqq 0$.*

Proof. The same is true for a semistable rank 2 vector bundle on $\mathbb{P}^2$ [9, 3.2].

Remark 3.3.1. Over a field of characteristic zero, one can generalize Barth's restriction theorem [1] for stable vector bundles: if $\mathcal{F}$ is a stable reflexive rank 2 sheaf on $\mathbb{P}^3$, then the restriction $\mathcal{F}_H$ of $\mathcal{F}$ to a general plane is stable, unless $\mathcal{F}$ is isomorphic to a null-correlation bundle. There are proofs (unpublished) by Gruson and Peskine and by Wever. An analogous restriction theorem in characteristic $p>0$ has recently been proved by Ein [5]: the restriction $\mathcal{F}_H$ of a stable rank 2 reflexive sheaf on $\mathbb{P}^3$ to a general plane is stable, unless $\mathcal{F}$ is either a null-correlation bundle or a Frobenius pullback of a null-correlation bundle.

Now we turn to the question of moduli. Maruyama [12, 13] has shown that there is a coarse moduli scheme for stable torsion-free sheaves with given Chern classes. One knows from deformation theory that the Zariski tangent space to the moduli scheme at the point corresponding to a stable sheaf $\mathcal{F}$ is $\mathrm{Ext}^1(\mathcal{F},\mathcal{F})$, and that the obstructions to extending an infinitesimal deformation lie in $\mathrm{Ext}^2(\mathcal{F},\mathcal{F})$. The next result will give the "expected" dimension for the moduli space.

Proposition 3.4. *Let $\mathcal{F}$ be a* rank 2 *reflexive sheaf on $\mathbb{P}^3$ with Chern classes c_1, c_2, c_3. Then*

$$\sum_{i=0}^{3}(-1)^i\dim\mathrm{Ext}^i(\mathcal{F},\mathcal{F})=2c_1^2-8c_2+4.$$

Proof. Compare [9, 4.2] where the same result is proved for a locally free rank 2 sheaf on $\mathbb{P}^3$. In our case, since $\mathcal{F}$ is reflexive, we can find a resolution

$$0\to\mathcal{E}_1\to\mathcal{E}_0\to\mathcal{F}\to 0,$$

where $\mathcal{E}_0$, $\mathcal{E}_1$ are locally free sheaves. Let $\mathcal{E}_1$ have rank k. Then $\mathcal{E}_0$ has rank $k+2$. For the purposes of this calculation, we may assume that $\mathcal{E}_0$ and $\mathcal{E}_1$ are sums of line bundles. So let

$$\mathcal{E}_0=\sum_{i=1}^{k+2}\mathcal{O}(a_i)$$

and

$$\mathcal{E}_1=\sum_{j=1}^{k}\mathcal{O}(b_j).$$

Replacing $\mathscr{F}$ by the complex $\mathscr{E}_1 \to \mathscr{E}_0$, we can calculate the $\mathrm{Ext}^i(\mathscr{F}, \mathscr{F})$ as the hypercohomology of the complex

$$\mathscr{E}_0^\vee \otimes \mathscr{E}_1 \to \mathscr{E}_0^\vee \otimes \mathscr{E}_0 \oplus \mathscr{E}_1^\vee \otimes \mathscr{E}_1 \to \mathscr{E}_1^\vee \otimes \mathscr{E}_0 .$$

So the alternating sum of the dimensions of the Ext^i will be equal to the alternating sum of the Euler characteristics χ of these sheaves. We know $\chi(\mathcal{O}(l)) = \frac{1}{6}(l+1)(l+2)(l+3)$. Substituting the expressions for $\mathscr{E}_0$ and $\mathscr{E}_1$ above, we obtain our answer in terms of the a_i and b_j. On the other hand, these are related to the Chern classes c_i of $\mathscr{F}$. After a somewhat messy calculation in which the linear and cubic terms all cancel, we obtain the formula of the proposition.

Remark 3.4.1 If $\mathscr{F}$ is a stable rank 2 reflexive sheaf on $\mathbb{P}^3$, then one sees easily that $\mathrm{Hom}(\mathscr{F}, \mathscr{F}) = k$ and $\mathrm{Ext}^3(\mathscr{F}, \mathscr{F}) = 0$. So the proposition gives

$$\dim \mathrm{Ext}^1(\mathscr{F}, \mathscr{F}) - \dim \mathrm{Ext}^2(\mathscr{F}, \mathscr{F}) = \begin{cases} 8c_2 - 3 & \text{if } c_1 = 0 \\ 8c_2 - 5 & \text{if } c_1 = -1 . \end{cases}$$

This is the same result as for rank 2 vector bundles [9, 4.2]. Note in particular that it is independent of c_3. We conclude [15] that the dimension of each irreducible component of the moduli space is $\geqq 8c_2 - 3$ (resp. $8c_2 - 5$). Furthermore, if $\mathrm{Ext}^2(\mathscr{F}, \mathscr{F}) = 0$, then it is nonsingular [13, 6.7] of dimension equal to $8c_2 - 3$ (resp. $8c_2 - 5$).

4. Correspondence with Curves in $\mathbb{P}^3$

In this section we explain the correspondence between rank 2 reflexive sheaves on $\mathbb{P}^3$ and curves in $\mathbb{P}^3$. This generalizes the known correspondence for vector bundles [9, 1.1]. To simplify our statement, we will identify line bundles on $\mathbb{P}^3$ with the standard line bundles $\mathcal{O}(l)$. Then we discuss the problem of determining the possible Chern classes c_1, c_2, c_3 of stable rank 2 reflexive sheaves on $\mathbb{P}^3$, and give some examples.

Theorem 4.1. *Fix an integer c_1. Then there is a one-to-one correspondence between*

(i) *pairs $(\mathscr{F}, s)$ where $\mathscr{F}$ is a* rank 2 *reflexive sheaf on $\mathbb{P}^3$ with $c_1(\mathscr{F}) = c_1$, and $s \in H^0(\mathscr{F})$ is a global section whose zero-set has codimension* 2, *and*

(ii) *pairs (Y, ξ), where Y is a Cohen-Macaulay curve in $\mathbb{P}^3$, generically locally complete intersection, and $\xi \in H^0(\omega_Y(4 - c_1))$ is a global section which generates the sheaf $\omega_Y(4 - c_1)$ except at finitely many points.*

Furthermore under this correspondence

$$c_2 = d$$
$$c_3 = 2p_a - 2 + d(4 - c_1),$$

where c_2, c_3 are the Chern classes of $\mathscr{F}$, and d, p_a are the degree and arithmetic genus of Y.

Proof. We follow the idea of the proof of [9, 1.1] noting necessary modifications. Given $(\mathscr{F}, s)$ as in (i), the section s defines a map $\mathcal{O} \xrightarrow{s} \mathscr{F}$. Taking duals gives a map $\mathscr{F}^\vee \to \mathcal{O}$. Its image is an ideal sheaf we call $\mathscr{I}_Y$. This defines the curve Y. By (1.5), the kernel of the map $\mathscr{F}^{\vee} \to \mathscr{I}_Y$ is a reflexive, rank 1 sheaf $\mathscr{L}$, which by (1.9) is invertible. Since Y has codimension 2 by hypothesis, $\mathscr{L}$ must be $\mathcal{O}(-c_1)$. So there is an exact sequence

$$0 \to \mathcal{O}(-c_1) \to \mathscr{F}^{\vee} \to \mathscr{I}_Y \to 0. \qquad (1)$$

Since $\mathscr{F}$ is locally free except at a finite number of points, we see already that $\mathscr{I}_Y$ is locally generated by two elements, except at those points, so Y is generically locally complete intersection. This sequence also shows that $\operatorname{depth}_P \mathscr{I}_P = 2$ at the points where $\mathscr{F}$ is not locally free. This implies that $\operatorname{depth}_P \mathcal{O}_Y = 1$ at those points, so Y has no embedded or isolated points. In other words, Y is a Cohen-Macaulay curve.

Now the exact sequence (1) defines an element $\xi \in \operatorname{Ext}^1(\mathscr{I}_Y, \mathcal{O}(-c_1))$. Using the exact sequence $0 \to \mathscr{I}_Y \to \mathcal{O} \to \mathcal{O}_Y \to 0$, this group is isomorphic to $\operatorname{Ext}^2(\mathcal{O}_Y, \mathcal{O}(-c_1))$, which in turn is isomorphic to $H^0(Y, \omega_Y(4-c_1))$, exactly as in the vector bundle case [9, proof of 1.1]. Thus we obtain $\xi \in H^0(\omega_Y(4-c_1))$.

Applying the functor $\mathscr{H}om(\cdot, \mathcal{O}(-c_1))$ to the sequence (1) gives an exact sequence

$$\begin{aligned} 0 \to \mathcal{O}(-c_1) \to \mathscr{F}(-c_1) \to \mathcal{O} \xrightarrow{\delta} \mathscr{E}xt^1(\mathscr{I}_Y, \mathcal{O}(-c_1)) \\ \to \mathscr{E}xt^1(\mathscr{F}^{\vee}, \mathcal{O}(-c_1)) \to 0. \end{aligned} \qquad (2)$$

Here the sheaf $\mathscr{E}xt^1(\mathscr{I}_Y, \mathcal{O}(-c_1))$ can be identified with $\omega_Y(4-c_1)$ as above, in which case δ is the map which sends 1 to ξ. Since the last sheaf $\mathscr{E}xt^1(\mathscr{F}^{\vee}, \mathcal{O}(-c_1))$ is supported at the points where $\mathscr{F}$ is not locally free, ξ must generate $\omega_Y(4-c_1)$ except at those points.

This shows how the data (i) determines the data (ii).

Conversely, suppose given (Y, ξ) as in (ii). Think of ξ as an element of $\operatorname{Ext}^1(\mathscr{I}_Y, \mathcal{O}(-c_1))$ and let it determine an extension (1) as above. This defines a coherent sheaf $\mathscr{F}^{\vee}$. Since Y is Cohen-Macaulay, $\mathscr{F}^{\vee}$ has depth $\geqq 2$ at every point. Therefore from the sequence (2) it will be locally free except at the points where ξ fails to generate $\omega_Y(4-c_1)$, and at those points at least it will be reflexive (1.3). Then we can define $\mathscr{F} = (\mathscr{F}^{\vee})^{\vee}$, and the section s is obtained as the dual of the map $\mathscr{F}^{\vee} \to \mathscr{I}_Y$ in (1).

This shows how the data (ii) determines the data (i), and establishes the desired correspondence between reflexive sheaves and curves.

The fact that $c_2 = d$ is the same as in the vector bundle case [9, 2.1]: one need only apply it to the vector bundle obtained by removing the points where $\mathscr{F}$ is not locally free. To compute c_3, write the last part of the sequence (2) as

$$\mathcal{O} \xrightarrow{\xi} \omega_Y(4-c_1) \to \mathscr{E}xt^1(\mathscr{F}^{\vee}, \mathcal{O}(-c_1)) \to 0.$$

Recall also $\mathscr{F}^{\vee} \cong \mathscr{F}(-c_1)$ by (1.10), so the sheaf on the right, up to a twist, which is irrelevant, is the sheaf whose length we know to be c_3 (2.6). So $c_3 = \deg \omega_Y(4-c_1) = 2p_a - 2 + d(4-c_1)$.

Remark 4.1.1. If $\mathscr{E}$ is a rank 2 vector bundle on $\mathbb{P}^3$, then we know [9, 1.4] that for $n \gg 0$, there is a section $s \in H^0(\mathscr{E}(n))$ corresponding to a *nonsingular* curve Y. We may ask if this is also true for reflexive sheaves. The answer is *no* in general. Indeed, a necessary condition that a reflexive sheaf $\mathscr{F}$ correspond to a nonsingular curve is that at each point P where $\mathscr{F}$ is not locally free, the module $\mathrm{Ext}^1(\mathscr{F}_P, \mathcal{O}_P)$ over the local ring $\mathcal{O}_P$ be isomorphic to $\mathcal{O}_P/(x, y, z^m)$ for some choice of regular parameters x, y, z, and some $m \geqq 1$. This is apparent from the exact sequence (2). For if Y is nonsingular, then ω_Y is an invertible sheaf. Take x, y local parameters at P defining the curve. Then the module $\omega_Y(4-c_1)/\xi$ will be of the form given.

We do not know if this condition is sufficient for some twist of $\mathscr{F}$ to correspond to a nonsingular curve.

Proposition 4.2. *Let $\mathscr{F}$ be a* rank 2 *reflexive sheaf on* $\mathbb{P}^3$ *corresponding to a curve Y in $\mathbb{P}^3$ as in* (4.1). *Then $\mathscr{F}$ is stable (respectively, semistable), if and only if*

(1) $c_1 > 0$ *(respectively,* $c_1 \geqq 0$*), and*

(2) *Y is not contained in any surface of degree* $\leqq \frac{1}{2}c_1$ *(respectively* $< \frac{1}{2}c_1$*).*

Proof. In view of (3.1), the proof is the same as the proof for vector bundles [9, 3.1].

Remark 4.2.0. Now let us consider the problem of determining the possible values of the Chern classes c_1, c_2, c_3 of stable rank 2 reflexive sheaves on $\mathbb{P}^3$. We can *normalize* any rank 2 reflexive sheaf by a suitable twist (2.2) so that $c_1 = 0$ or -1. The possible values of c_2, c_3 for stable rank 2 reflexive sheaves are then limited by the following results.

a) $c_2 > 0$. For $c_1 = -1$ this follows from (3.3). For $c_1 = 0$, the same result (3.3) shows that $c_2 \geqq 0$. The fact that $c_2 > 0$ follows from (9.7).

b) $c_3 \geqq 0$ (2.6) and $c_3 \equiv c_1 c_2 (\mathrm{mod}\, 2)$ (2.4).

c) c_3 is bounded above by a quadratic polynomial in c_2 (8.2).

d) For a given c_2, not all values of c_3 satisfying b) and c) actually occur. Examples of impossible values of c_3 will be given in a later paper.

We will now give some examples of stable reflexive sheaves to illustrate these properties. Of course for $c_3 = 0$ we are simply talking about rank 2 vector bundles, so we refer to the earlier paper [9] for examples.

Example 4.2.1. Using the correspondence between curves and reflexive sheaves (4.1) we can construct stable rank 2 reflexive sheaves $\mathscr{F}$ on $\mathbb{P}^3$ with $c_1 = 0$ as extensions

$$0 \to \mathcal{O} \to \mathscr{F}(1) \to \mathscr{I}_Y(2) \to 0,$$

where Y is a curve (Cohen-Macaulay and generically local complete intersection) of degree $d = c_2 + 1$, such that there exists a section $\xi \in H^0(\omega_Y(2))$ which generates that sheaf except at finitely many points, and such that Y is not contained in a plane (4.2). If Y is a reduced curve, the existence of such a ξ is automatic. So we have only to investigate the possible arithmetic genera p_a of such curves Y, to obtain reflexive sheaves with $c_3 = 2d + 2p_a - 2$.

First let $c_2 = 1$. Then $d = 2$, and the only reduced curve of degree 2 not contained in a plane is the union of two skew lines. This gives the nullcorrelation bundle [9, 3.1.1]. In fact, this is the only stable reflexive sheaf with $c_2 = 0$, $c_2 = 1$. Indeed, if $\mathscr{F}$ is such a sheaf, then by (10.3), $h^0(\mathscr{F}(1)) \neq 0$, so $\mathscr{F}(1)$ has a section corresponding to a degree 2 curve not contained in a plane. This must be two skew lines, or a multiplicity 2 structure on a line. In the latter case one can show $c_3 = 0$, so it is a vector bundle, hence a nullcorrelation bundle [9, 8.4.1].

Next let $c_2=2$. Then $d=3$ and there are three possibilities:

c_3	p_a	Y
0	-2	3 skew lines
2	-1	conic + skew line
4	0	twisted cubic curve

One can verify that each of these forms a family of dimension 13, which is equal to the expected dimension $8c_2-3$ (3.4.1).

Now let $c_2=3$. Then $d=4$ and there are several possibilities for Y.

c_3	p_a	Y
0	-3	4 skew lines
2	-2	conic + 2 skew lines
4	-1	2 skew conics
4	-1	twisted cubic + skew line
6	0	plane cubic + skew line
6	0	rational quartic curve
8	1	elliptic quartic curve

Note that for $c_3=4$, 6 there are two different ways to construct these sheaves. Is one family a specialization of the other, or do they belong to separate irreducible components of the moduli space?

Let $c_2=4$. Then $d=5$. We do not list all possible Y, only enough to show that all even values of c_3 between 0 and 14 are possible.

c_3	p_a	Y
0	-4	5 skew lines
2	-3	conic + 3 skew lines
4	-2	2 conics + skew line
6	-1	twisted cubic + conic
8	0	rational quintic
10	1	elliptic quintic
12	2	quintic of genus 2
14	3	plane quartic with line attached

For $c_3=14$ we must take Y to be a singular curve, namely the union of a plane quartic curve (say nonsingular) with a line, not in the plane of the quartic curve, which meets the quartic curve in one point.

Let $c_2=5$. Then $d=6$. Using nonsingular curves Y similar to those above, one can construct sheaves having all even c_3 satisfying $0 \leqq c_3 \leqq 20$. On the other hand, if Y is a plane quintic curve with a line attached as above, then $c_3=22$.

Example 4.2.2. We can guess the maximum c_3 for $c_1=0$ and given $c_2>0$ by generalizing the examples above. Let Y be a plane curve of degree c_2 with a line attached. Then the arithmetic genus of Y is the same as the genus of the plane curve, namely $\frac{1}{2}(c_2-1)(c_2-2)$. So we obtain a stable reflexive sheaf $\mathscr{F}$ with $c_1=0$, $c_2>0$ and

$$c_3=2d+2p_a-2=c_2^2-c_2+2.$$

We will see later (8.2) that this is indeed the maximum c_3 possible.

Example 4.2.3. If $c_1=-1$ we can make an analogous construction using extensions

$$0\to\mathcal{O}\to\mathscr{F}(1)\to\mathscr{I}_Y(1)\to 0.$$

In this case Y has degree $d=c_2$, the existence of an appropriate $\xi\in H^0(\omega_Y(3))$ imposes no condition on a reduced Y, and stability imposes no condition on Y (except Y nonempty). The third Chern class is $c_3=3d+2p_a-2$.

If $c_2=1$, Y is a line, and we get $c_3=1$.

If $c_2=2$, Y can be two skew lines or a conic, giving $c_3=2, 4$. Of course $c_3=0$ is possible, but cannot be obtained by a *reduced* curve using this construction. Instead, we must construct $\mathscr{F}$ by a section of $\mathscr{F}(2)$: see (4.2.4) below.

If $c_2=3$, this construction using nonsingular curves Y gives $c_3=3, 5, 7, 9$.

Let us do the case $c_2=4$ in more detail. Here are the possible c_3 obtained using reduced curves Y:

c_3	p_a	Y
4	-3	4 skew lines
6	-2	conic + 2 skew lines
8	-1	2 conics
10	0	rational quartic
12	1	elliptic quartic
16	3	plane quartic

The values $c_3=0$, 2 can be obtained using sections of $\mathscr{F}(2)$ (4.2.4). Note that $c_3=14$ is absent. In fact, we will see in a later paper that the triple $c_1=-1$, $c_2=4$, $c_3=14$ is impossible.

Example 4.2.4. We can construct further examples of stable $\mathscr{F}$ with $c_1=-1$ as extensions

$$0\to\mathcal{O}\to\mathscr{F}(2)\to\mathscr{I}_Y(3)\to 0.$$

Then Y is a curve of degree $d=c_2+2$, and $c_3=d+2p_a-2$. The existence of an appropriate $\xi\in H^0(\omega_Y(1))$ requires that Y should have no unattached lines, and the stability of $\mathscr{F}$ requires that Y should not lie in a plane. We obtain the same possible values of c_3 as in (4.2.3), including the low values which were missing there. For example in case $c_2=4$ the possibilities are

c_3	p_a	Y
0	−2	3 conics
2	−1	2 twisted cubics
4	0	rational sextic
6	1	elliptic sextic
8	2	sextic of genus 2
10	3	sextic of genus 3
12	4	sextic of genus 4
16	6	plane sextic.

Example 4.2.5. For $c_1=-1$ and $c_2>0$ we can guess that the maximum c_3 is given by the construction of (4.2.3) using a plane curve of degree c_2. This gives $c_3=c_2^2$. We will see later (8.2) that this is indeed the maximum.

5. A Technical Result on $\mathbb{P}^2$

Let $\mathscr{E}$ be a rank 2 vector bundle on $\mathbb{P}^2$, and let $S=k[x_0,x_1,x_2]$ be the homogeneous coordinate ring of $\mathbb{P}^2$. We consider the graded S-module $M=\bigoplus_{l\in\mathbb{Z}} H^1(\mathbb{P}^2, \mathscr{E}(l))$. In this section we will prove a result about the dimensions of the components of an arbitrary graded submodule N of M.

The motivation for this is that when studying a rank 2 reflexive sheaf $\mathscr{F}$ on $\mathbb{P}^3$, we will consider its restriction $\mathscr{E}$ to a general plane H. To get information about the cohomology groups $H^1(\mathbb{P}^3, \mathscr{F}(l))$, we consider the natural restriction maps $H^1(\mathscr{F}(l))\to H^1(\mathscr{E}(l))$. The images of these maps, summed over all l, give a graded submodule N of M. If we understand the module structure of M and its possible submodules, then we can deduce useful information about the cohomology of $\mathscr{F}$. This will be done in Sect. 7.

Our main tool in analyzing the module structure of M is to consider the jumping lines of $\mathscr{E}$.

Definition. Let $\mathscr{E}$ be a rank 2 vector bundle on $\mathbb{P}^2$ with $c_1=0$ or -1. A *jumping line* for $\mathscr{E}$ is a line L such that $H^0(\mathscr{E}_L(-r))\neq 0$ for some $r>0$. The largest such integer r is called the *order* of the jumping line L.

Remark 5.0.1. If $\mathscr{E}$ is a semistable bundle in characteristic 0, the theorem of Grauert-Mülich [1, Theorem 1] says that for a general line L, the restriction of $\mathscr{E}$ has splitting type $(0,0)$ or $(0,-1)$. Thus the jumping lines are those for which the splitting type "jumps" to (r, c_1-r) with $r>0$. The theorem of Grauert-Mülich fails if $\mathscr{E}$ is not semistable, or if the characteristic is not 0, but we use the terminology of jumping lines anyway, keeping in mind that it may happen that every line is a jumping line.

Suppose that L is a jumping line of order r of a semistable bundle $\mathscr{E}$. The exact sequence

$$0\to\mathscr{E}(-1)\to\mathscr{E}\to\mathscr{E}_L\to 0$$

gives an exact cohomology sequence

$$H^0(\mathscr{E}(l)) \to H^0(\mathscr{E}_L(l)) \to H^1(\mathscr{E}(l-1)).$$

Since $H^0(\mathscr{E}(l))=0$ for $l<0$, but $H^0(\mathscr{E}_L(l))\neq 0$ for $-r\leqq l$, the jumping line contributes a submodule to M. So in studying the module structure of M, we will be concerned with the existence of jumping lines.

We begin with some preliminaries.

Lemma 5.1 (Bilinear map lemma). *Let $\phi: V_1 \times V_2 \to W$ be a map of nonzero finite-dimensional vector spaces which is bilinear and nondegenerate, i.e. for each $v_1 \neq 0$ in V_1 and each $v_2 \neq 0$ in V_2, $\phi(v_1, v_2)\neq 0$. Then*

$$\dim W \geqq \dim V_1 + \dim V_2 - 1.$$

Proof. This result may be well-known but for lack of a suitable reference we include the proof. It is closely related to the lemma [8, IV, 5.5] used in the proof of Clifford's theorem.

Let $X \subseteq V_1 \otimes V_2$ be the image of $V_1 \times V_2$. Then X is an algebraic variety in the affine space $V_1 \otimes V_2$, of dimension equal to $\dim V_1 + \dim V_2 - 1$. Indeed, it is the cone over the Segre embedding of $\mathbb{P}(V_1) \times \mathbb{P}(V_2)$ in $\mathbb{P}(V_1 \otimes V_2)$ [8, I, Exercise 2.14]. The given map ϕ extends to a linear map $\bar{\phi}: V_1 \otimes V_2 \to W$, whose kernel meets X only at zero. Therefore $\dim W \geqq \dim X$ by the affine dimension theorem [8, I, 7.1].

Proposition 5.2 (Reduction step). *Let $\mathscr{E}$ be a* rank 2 *vector bundle on $\mathbb{P}^2$ with $c_1=0$ or -1, and let L be a jumping line of order r for $\mathscr{E}$.*

(a) *If $c_1=0$, there is an exact sequence*

$$0 \to \mathscr{E}' \to \mathscr{E} \to \mathcal{O}_L(-r) \to 0,$$

where $\mathscr{E}'$ is a rank 2 *vector bundle with $c_1'=-1$ and $c_2'=c_2-r$. Furthermore $t'\geqq t$, where t (resp. t') is the least integer for which $H^0(\mathscr{E}(t))\neq 0$ (resp. $H^0(\mathscr{E}'(t'))\neq 0$), and if $r>t$, then $t'=t$. There is also a dual exact sequence*

$$0 \to \mathscr{E} \to \mathscr{E}'(1) \to \mathcal{O}_L(r+1) \to 0.$$

(b) *If $c_1=-1$, there is an exact sequence*

$$0 \to \mathscr{E}'(-1) \to \mathscr{E} \to \mathcal{O}_L(-r-1) \to 0.$$

where $\mathscr{E}'$ is a rank 2 *vector bundle with $c_1'=0$ and $c_2'=c_2-r-1$. Furthermore $t'\geqq t-1$, and if $r>t-1$, then $t'=t-1$. There is also a dual exact sequence*

$$0 \to \mathscr{E} \to \mathscr{E}' \to \mathcal{O}_L(r+1) \to 0.$$

Proof. We do only part (a) since (b) is similar. By definition of r, the restriction $\mathscr{E}_L$ of $\mathscr{E}$ to L must be isomorphic to $\mathcal{O}_L(r)\oplus\mathcal{O}_L(-r)$. We define the map $\mathscr{E}\to\mathcal{O}_L(-r)$ by first restricting to L and then projecting on the second factor. Let $\mathscr{E}'$ be the kernel of this map. By (1.5) $\mathscr{E}'$ is reflexive, and in fact locally free (1.4). Therefore $\mathscr{E}'$ is a vector bundle.

To compute the Chern classes of $\mathscr{E}'$ we first use the sequence

$$0 \to \mathcal{O}(-r-1) \to \mathcal{O}(-r) \to \mathcal{O}_L(-r) \to 0$$

from which we find

$$c_t(\mathcal{O}_L(-r)) = 1 + t + (r+1)t^2 .$$

Then we use the defining sequence for $\mathcal{E}'$ to find

$$c_t(\mathcal{E}') = 1 - t + (c_2 - r)t^2 .$$

The statements about the integers t and t' are clear by considering the exact sequence of H^0 associated to the defining sequence for $\mathcal{E}'$.

Finally, to dualize, we apply the functor $\mathcal{H}om(\cdot, \mathcal{O})$ and get

$$0 \to \mathcal{H}om(\mathcal{O}_L(-r), \mathcal{O}) \to \mathcal{E}^\vee \to \mathcal{E}'^\vee \to \mathcal{E}xt^1(\mathcal{O}_L(-r), \mathcal{O}) \to 0 .$$

The first term is 0; $\mathcal{E}^\vee \cong \mathcal{E}$ since $c_1 = 0$; $\mathcal{E}'^\vee \cong \mathcal{E}'(1)$ since $c_1(\mathcal{E}') = -1$; and $\mathcal{E}xt^1(\mathcal{O}_L(-r), \mathcal{O}) \cong \mathcal{O}_L(r+1)$ by a simple computation with a resolution of $\mathcal{O}_L$.

Theorem 5.3. *Let $\mathcal{E}$ be a* rank 2 *vector bundle on $\mathbb{P}^2$ with $c_1 = 0$ or -1 and such that $H^0(\mathcal{E}(-1)) = 0$. Let $M = \bigoplus_{l \in \mathbb{Z}} H^1(\mathcal{E}(l))$, and let N be a graded submodule. Let $n_l = \dim N_l$ where N_l is the graded component of N in degree l. Then*

(a) $n_{-2} \leqq n_{-1}$.

(b) $n_l < n_{l+1}$ *if* $l \leqq -3$ *and* $n_l \neq 0$.

(c) *if* $n_l + 1 = n_{l+1}$ *for some* $l \leqq -4$, *then there is a linear form* $x \in H^0(\mathcal{O}(1))$ *such that* x *annihilates* $N_{l'}$ *for all* $l' \leqq l$.

Furthermore,

(1) *if* $\mathcal{E}$ *is stable, then* (c) *holds also for* $l = -3$,

(2) *if* $\mathcal{E}$ *is stable and* $0 < n_{-2} < c_2$, *then* $n_{-2} < n_{-1}$;

(3) *if* $\mathcal{E}$ *is stable with* $c_1 = -1$, *and* $n_{-1} < c_2$, *then* $n_{-1} \leqq n_0$.

Proof. Let t be the least integer for which $H^0(\mathcal{E}(t)) \neq 0$. Then $t \geqq 0$ by hypothesis.

Case 1. $t = 0$. Then $\mathcal{E}$ itself has a section so there is an exact sequence

$$0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{I}_Z(c_1) \to 0$$

for some codimension 2 subscheme Z of $\mathbb{P}^2$. Let L be a line not meeting Z. Restricting to L we find $\mathcal{E}_L \cong \mathcal{O}_L \oplus \mathcal{O}_L(c_1)$. Therefore L is not a jumping line, and $H^0(\mathcal{E}_L(l)) = 0$ for any $l < 0$.

Let x be the equation of the line L. The exact sequence

$$0 \to \mathcal{E}(-1) \xrightarrow{x} \mathcal{E} \to \mathcal{E}_L \to 0$$

gives an exact cohomology sequence

$$H^0(\mathcal{E}_L(l+1)) \to H^1(\mathcal{E}(l)) \xrightarrow{x} H^1(\mathcal{E}(l+1)) .$$

So we see that the map x on $H^1(\mathcal{E}(l))$ is injective whenever $l \leqq -2$. Since x maps N_l to N_{l+1}, this shows that $n_l \leqq n_{l+1}$ for $l \leqq -2$. In particular, this proves (a).

Next, consider any l for which $N_l \neq 0$, and look at the natural map

$$N_l \times H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \to N_{l+1} .$$

If for every $x \in H^0(\mathcal{O}(1))$ the induced map of N_l to N_{l+1} is injective, then by the bilinear map lemma (5.1) we find $n_{l+1} \geqq n_l + 2$, which proves (b). If not, let x be a linear form for which the induced map N_l to N_{l+1} is not injective, and let L be the line $x=0$. As above, we consider the exact sequence

$$H^0(\mathscr{E}_L(l+1)) \xrightarrow{\delta} H^1(\mathscr{E}(l)) \xrightarrow{x} H^1(\mathscr{E}(l+1)).$$

Summing over all $l \in \mathbb{Z}$ gives an exact sequence of graded S-modules. We consider N as a submodule of the middle term. Let $N'' = xN$ be the image of N by x, and let $N' = \delta^{-1}N$ be the inverse image on the left. This gives an exact sequence of graded S-modules

$$N' \xrightarrow{\delta} N \xrightarrow{x} N'' \to 0.$$

We take the grading inherited from the original sequence, so δ is a map of degree -1, and x is a map of degree $+1$.

Now $H^0(\mathscr{E}(l+1)) = 0$ for $l \leqq -2$ by hypothesis, so in that range the map δ is injective. Thus

$$n_l = n'_{l+1} + n''_{l+1} \quad \text{if} \quad l \leqq -2$$

and

$$n_{l+1} = n'_{l+2} + n''_{l+2} \quad \text{if} \quad l \leqq -3.$$

Since N'' is another graded submodule of M, by the inequality already proved, $n''_{l+1} \leqq n''_{l+2}$ for $l \leqq -3$. On the other hand, since $\mathscr{E}_L$ is a vector bundle on the line L, the module $\bigoplus_{l \in \mathbb{Z}} H^0(\mathscr{E}_L(l))$ is a torsion-free graded $k[y,z]$-module. Therefore N' is also a torsion-free $k[y,z]$-module, and it follows that $n'_{l+1} < n'_{l+2}$ [use for example (5.1) again, applied to the map $N'_{l+1} \times H^0(\mathcal{O}_L(1)) \to N'_{l+2}$]. Note also $n'_{l+1} \neq 0$ by choice of x above.

Combining these results, we conclude that $n_l < n_{l+1}$ if $l \leqq -3$ and $n_l \neq 0$, which proves (b).

To prove (c), let $l \leqq -4$, and suppose in the above proof that $n''_{l+1} \neq 0$. Then by the result (b) just proved, $n''_{l+1} < n''_{l+2}$, so the proof above actually gives $n_{l+1} \leqq n_l + 2$. Therefore the equality $n_l + 1 = n_{l+1}$ implies $N''_{l+1} = 0$. It follows from our first inequality that $N''_{l'+1} = 0$ for all $l' \leqq l$. In other words, the linear form x annihilates $N_{l'}$ for all $l' \leqq l$.

Since $t=0$, $\mathscr{E}$ cannot be stable, so the statements (1), (2), (3) do not apply. This completes the proof of the case $t=0$.

Case 2. $t>0$. Then $\mathscr{E}$ is stable (3.1) and $c_2 > 0$ [9, 3.2]. We will use induction on c_2, beginning with the case $c_2 = 0$ which is vacuous. We also divide the proof into two subcases.

Case 2.1. $c_1 = 0$. Take an $l \leqq -2$ for which $N_l \neq 0$, and as in Case 1 consider the map

$$N_l \times H^0(\mathcal{O}_{\mathbf{P}^2}(1)) \to N_{l+1}.$$

If the induced map $x: N_l \to N_{l+1}$ is injective for each $x \in H^0(\mathcal{O}(1))$, then as before $n_{l+1} \geqq n_l + 2$, which proves (a), (b), and (2), and (c), (1), (3) are vacuous.

So suppose to the contrary for some linear form x, the map $x: N_l \to N_{l+1}$ is not injective, and let L be the line $x=0$. By one of the exact sequences used earlier, this implies $H^0(\mathscr{E}_L(l+1)) \neq 0$, so L is a jumping line of order $r \geqq -l-1>0$. We now apply the reduction step (5.2) to $\mathscr{E}$ and the line L. This gives a new rank 2 bundle $\mathscr{E}'$ with $c_1'=-1$, $c_2'=c_2-r$, and $t' \geqq t>0$. Since $c_2'<c_2$ we can apply the induction hypothesis to $\mathscr{E}'$. The dual exact sequence of (5.2) twisted by l gives an exact sequence of cohomology

$$H^0(\mathscr{E}'(l+1)) \to H^0(\mathcal{O}_L(l+r+1)) \xrightarrow{\delta} H^1(\mathscr{E}(l)) \xrightarrow{\alpha} H^1(\mathscr{E}'(l+1)).$$

Summing over all $l \in \mathbb{Z}$ gives an exact sequence of graded S-modules. Our given module N is a submodule of the third term. Let $N''=\alpha(N)$, and let $N'=\delta^{-1}(N)$. Then there is an exact sequence

$$N' \xrightarrow{\delta} N \xrightarrow{\alpha} N'' \to 0,$$

with gradings inherited from the above sequence. Furthermore, since $t'>0$, $H^0(\mathscr{E}'(l+1))=0$ for $l \leqq -1$, so δ is injective in that range. We conclude that for $l \leqq -2$,

$$n_l = n'_{l+r+1} + n''_{l+1}$$
$$n_{l+1} = n'_{l+r+2} + n''_{l+2}.$$

Now we proceed as in the previous case. Since N' is a torsion-free $k[y,z]$-module, $n'_{l+r+1}<n'_{l+r+2}$. If $l \leqq -3$, the induction hypothesis (b) and (2) applied to $\mathscr{E}'$ shows that $n''_{l+1} \leqq n''_{l+2}$, with strict inequality if $n''_{l+1} \neq 0$. This implies that $n_l<n_{l+1}$, which proves (b). Furthermore, if $n''_{l+1} \neq 0$, then $n_{l+1} \geqq n_l+2$, which proves (c) and (1).

Statement (3) does not apply in this case, so it remains to show that if $0<n_{-2}<c_2$, then $n_{-2}<n_{-1}$. This will prove (2). It also proves (a), because if $n_{-2}=0$, (a) is trivial, and if $n_{-2}=c_2$, then $N_{-2}=H^1(\mathscr{E}(-2))$, so $N_{-1}=H^1(\mathscr{E}(-1))$ and both have dimension c_2.

So let $l=-2$. If $n''_{-1}<c_2(\mathscr{E}')$, then by the induction hypothesis (3), $n''_{-1} \leqq n''_0$, so the same argument as above shows that $n_{-2}<n_{-1}$. So suppose to the contrary $n''_{-1}=c_2(\mathscr{E}')$, i.e. $N''_{-1}=H^1(\mathscr{E}'(-1))$. Then we consider the defining sequence

$$0 \to \mathscr{E}' \to \mathscr{E} \to \mathcal{O}_L(-r) \to 0$$

of $\mathscr{E}'$ (5.2) and the associated cohomology sequence

$$H^1(\mathscr{E}'(l)) \xrightarrow{\beta} H^1(\mathscr{E}(l)) \xrightarrow{\gamma} H^1(\mathcal{O}_L(l-r)).$$

Summing over l gives another exact sequence of graded S-modules. Our module N is a submodule of the middle term. Let $N'''=\beta^{-1}(N)$ and $N^{iv}=\gamma(N)$. Then there is an exact sequence

$$N''' \xrightarrow{\beta} N \xrightarrow{\gamma} N^{iv} \to 0.$$

Furthermore, β is injective for $l \leqq 0$, so

$$n_{-2} = n'''_{-2} + n^{iv}_{-2-r}$$
$$n_{-1} = n'''_{-1} + n^{iv}_{-1-r}.$$

The induction hypothesis for $\mathscr{E}'$ implies $n'''_{-2} < n'''_{-1}$ unless $n'''_{-1} = 0$. On the other hand, N^{iv} is a submodule of $\oplus H^1(\mathcal{O}_L(l-r))$, and it is easy to see that $n_l^{iv} \leqq n_{l+1}^{iv}$ for any l unless $N_l^{iv} = H^1(\mathcal{O}_L(l-r))$. [To prove this let R be the quotient of $\oplus H^1(\mathcal{O}_L(l-r))$ by N^{iv}. Then the dual of R, R', is a submodule of $\oplus H^0(\mathcal{O}_L(r-l-2))$. Therefore R' is a torsion-free $k[y,z]$-module, so $r'_l < r'_{l+1}$ if $r'_l \neq 0$. Translating this back to R proves the result.] Combining these inequalities gives $n_{-2} < n_{-1}$, which is what we want, unless $n'''_{-1} = 0$ or $N^{iv}_{-2} = H^1(\mathcal{O}_L(-2-r))$.

Now let x be the equation of L, and consider the multiplication $x: N_{-2} \to N_{-1}$. Since x annihilates N^{iv}, we obtain a factorization of x by a map $\theta: N_{-2} \to N'''_{-1}$. This is induced from a similar map $\theta: H^1(\mathscr{E}(-2)) \to H^1(\mathscr{E}'(-1))$, and a small calculation with local coordinates shows that θ is none other than the map α used earlier in the proof. We have assumed that $N''_{-1} = H^1(\mathscr{E}'(-1))$, so we conclude also $N'''_{-1} = H^1(\mathscr{E}'(-1))$. In particular, $n'''_{-1} \neq 0$, which takes care of one of the exceptions above.

To complete the proof, suppose $N^{iv}_{-2} = H^1(\mathcal{O}_L(-2-r))$. Then also $N^{iv}_{-1} = H^1(\mathcal{O}_L(-1-r))$, so we find $N_{-1} = H^1(\mathscr{E}(-1))$, and therefore $n_{-1} = c_2$. Since we assumed $n_{-2} < c_2$, this proves (2) in this case also.

Case 2.2. $c_1 = -1$. We begin as before. Take $l \leqq -2$ for which $N_l \neq 0$. If $x: N_l \to N_{l+1}$ is injective for all x, we are done [except for statement (3) which will be proved below]. In the contrary case, we take x for which the map $x: N_l \to N_{l+1}$ is not injective. Then the line L defined by $x=0$ is a jumping line of order $r \geqq -l-1 > 0$, and we apply the reduction step (5.2) to $\mathscr{E}$ and L. This gives a bundle $\mathscr{E}'$ with $c'_1 = 0$, $c'_2 = c_2 - r - 1$, and $t' \geqq t - 1 \geqq 0$. If $t' = 0$ we can apply Case 1 to $\mathscr{E}'$. If $t' > 0$, then we are in Case 2, and can apply the induction hypothesis to $\mathscr{E}'$ since $c'_2 < c_2$. The dual sequence of (5.2) twisted by l gives an exact cohomology sequence

$$H^0(\mathscr{E}'(l)) \to H^0(\mathcal{O}_L(l+r+1)) \xrightarrow{\delta} H^1(\mathscr{E}(l)) \xrightarrow{x} H^1(\mathscr{E}'(l))$$

similar to the one in Case 2.1 above. Since $t' \geqq 0$, $H^0(\mathscr{E}'(l)) = 0$ for $l \leqq -1$, so δ is injective in that range. Now an argument exactly like the one used in Case 2.1 for $l \leqq -3$ proves (a)–(c), (1), and (2).

It remains to prove (3). For this we use Serre duality on $\mathbb{P}^2$ and the fact that $\mathscr{E}^\vee \cong \mathscr{E}(1)$. This shows that the module M is self-dual, up to shift in degrees. To be precise, M_{-1} is self-dual, and M_0 is dual to M_{-2}. We apply the results already proved to the dual of M/N, which is another graded submodule of M. In particular,

$$\dim(M/N)'_{-2} < \dim(M/N)'_{-1}$$

unless $(M/N)'_{-2} = 0$. Note $\dim(M/N)'_{-2} = c_2$ is impossible because $\dim M_{-2} = c_2 - 1$. This translates by duality as

$$m_0 - n_0 < m_{-1} - n_{-1}$$

unless $n_0 = m_0$. Since $m_{-1} = c_2$ and $m_0 = c_2 - 1$ by Riemann-Roch, we find $n_{-1} \leqq n_0$, as required, unless $n_0 = m_0$. But in the case $n_0 = m_0$, we have $n_0 = c_2 - 1$, so the inequality still holds because of the hypothesis $n_{-1} < c_2$. This completes the proof.

Remark 5.3.1. In the case $c_1=0$, $\mathscr{E}$ stable, and char. $k=0$, Barth and Elencwajg [3] have proved (2) by a different method. It was an attempt to understand their result which led to this theorem.

6. An Application: Bounds for $H^1(\mathscr{E}(l))$ on $\mathbb{P}^2$

Let $\mathscr{E}$ be a stable rank 2 bundle on $\mathbb{P}^2$ with $c_1=0$ or -1 and let t be the least integer for which $H^0(\mathscr{E}(t))\neq 0$. Applying the result (5.3) of the last section to the module $M=\bigoplus_{l\in\mathbb{Z}} H^1(\mathscr{E}(l))$ itself, we find that the dimension $n_l=h^1(\mathscr{E}(l))$ is strictly decreasing for $l\leqq -2$ as l decreases, and so by Serre duality it is also strictly decreasing for $l\geqq 0$ as l increases. This fact, together with the Riemann-Roch theorem, gives a new proof of [9, 7.4], which gave bounds on $h^1(\mathscr{E}(l))$ for all l. in terms of c_1, c_2, and t.

But now we can prove a stronger result. Using part (c) of (5.3) we can show that the existence of large values of l for which $h^1(\mathscr{E}(l))\neq 0$ is equivalent to the existence of jumping lines of high order, and we can give sharper bounds on $h^1(\mathscr{E}(l))$. Here is a precise statement.

Theorem 6.1. *Let $\mathscr{E}$ be a stable* rank 2 *vector bundle on $\mathbb{P}^2$ with $c_1=0$. Let $r>0$ be the largest order of a jumping line, or let $r=0$ if there are no jumping lines. Let m be the largest integer such that $H^1(\mathscr{E}(m))\neq 0$. Then*

(a) *$m\geqq\frac{1}{2}(c_2-t^2+t-2)$ if and only if $r\geqq\frac{1}{2}(c_2-t^2+t+2)$. In that case $m=r-2$, the jumping line of order r is unique, and*

$$h^1(\mathscr{E}(l))\begin{cases}\leqq c_2-t^2+t-2l-2 & \text{for } t\leqq l\leqq c_2-t^2+t-r-2\\ =r-l-1 & \text{for } c_2-t^2+t-r-1\leqq l\leqq r-1\\ =0 & \text{for } l\geqq r-1.\end{cases}$$

(b) *$m<\frac{1}{2}(c_2-t^2+t-2)$ if and only if $r<\frac{1}{2}(c_2-t^2+t+2)$. In that case $m\geqq r-2$, and*

$$h^1(\mathscr{E}(l))\begin{cases}\leqq c_2-t^2+t-2l-2 & \text{for } t\leqq l\leqq\frac{1}{2}(c_2-t^2+t-2)\\ =0 & \text{for } l\geqq\frac{1}{2}(c_2-t^2+t-2).\end{cases}$$

Similar bounds for $l<0$ can be obtained by duality.

Proof. For $0\leqq l<t$, the value of $h^1(\mathscr{E}(l))$ is given by the Riemann-Roch theorem [9, 7.4]. In particular, $h^1(\mathscr{E}(t-1))=c_2-t^2-t$. If there are no jumping lines, the case (c) of (5.3) cannot occur, so we conclude, taking $N=M$, that $n_l\leqq n_{l+1}-2$ if $n_l\neq 0$ and $l\leqq -3$. By Serre duality, this implies that the function $h^1(\mathscr{E}(l))$ drops by at least 2 as l increases, $l\geqq 0$. This gives the bounds of (b) above.

Now suppose that $r>0$, and let L be a jumping line of maximal order r. From the exact sequence

$$0\to\mathscr{E}(-1)\to\mathscr{E}\to\mathscr{E}_L\to 0$$

and the stability of $\mathscr{E}$, we have an injective map

$$0\to H^0(\mathscr{E}_L(l+1))\to H^1(\mathscr{E}(l))$$

for each $l \leqq -1$. On the other hand, $\mathscr{E}_L \cong \mathcal{O}_L(r) \oplus \mathcal{O}_L(-r)$, so

$$h^0(\mathscr{E}_L(l+1)) = h^0(\mathcal{O}_L(r+l+1)) = r+l+2.$$

We conclude that

$$n_l \geqq r+l+2$$

for $l \geqq -1$.

On the other hand, suppose $n_l + 1 = n_{l+1}$ for some $l \leqq -3$. Then by (5.3), there is a linear form x annihilating N_l, so $n_l = h^0(\mathscr{E}_L(l+1))$, where L is the line $x=0$. Since r is the maximum order of a jumping line, $h^0(\mathscr{E}_L(l+1)) \leqq r+l+2$. Combining with the above inequality, $n_l = r+l+2$.

Thus we see that as $l \leqq -2$ decreases, the function $n_l = h^1(\mathscr{E}(l))$ drops by at least 2 each time, until it becomes equal to 0, 1, or $r+l+2$, whichever occurs first. After that it drops by exactly 1 each time, so is either 0 or equal to $r+l+2$.

By duality, as $l \geqq -1$ increases, n_l drops by at least 2 each time, until it becomes equal to 0, 1, or $r-l-1$, whichever occurs first. After that it is equal to 0 or $r-l-1$.

The function $\psi(l)$ which begins with $\psi(t-1) = c_2 - t^2 - t$, and drops by 2 for each increase in l is

$$\psi(l) = c_2 - t^2 + t - 2l - 2 \quad \text{for} \quad l \geqq t-1.$$

Let us solve the equation $\psi(l) = r-l-1$ for l. It gives

$$l_0 = c_2 - t^2 + t - r - 1.$$

The corresponding value of ψ is

$$\psi(l_0) = 2r - c_2 + t^2 - t.$$

If $\psi(l_0) \geqq 2$, i.e. $r \geqq \frac{1}{2}(c_2 - t^2 + t + 2)$, then $h^1(\mathscr{E}(l))$ is bounded by $\psi(l)$ for $l \leqq l_0$, and is equal to $r-l-1$ thereafter. This gives the bounds in (a).

If $\psi(l_0) \leqq 1$, i.e. $r < \frac{1}{2}(c_2 - t^2 + t + 2)$, then $h^1(\mathscr{E}(l))$ is bounded by $\psi(l)$ until it becomes 0. This gives the bounds of (b).

Now consider the invariant m. In case (a), clearly $m = r-2 \geqq \frac{1}{2}(c_2 - t^2 + t - 2)$. In case (b), $m < \frac{1}{2}(c_2 - t^2 + t - 2)$. This proves the if and only if statements.

It remains to prove the uniqueness of the jumping line of order r in case (a). Indeed, in that case M coincides with the submodule obtained as the image of $\oplus H^0(\mathscr{E}_L(l+1))$ at least in its most negative degree $l = -r-1$. So the equation of L can be recovered as the annihilator of a generator of M in degree $-r-1$.

Theorem 6.2. *Let $\mathscr{E}$ be a stable* rank 2 *bundle on* $\mathbb{P}^2$ *with* $c_1 = -1$. *Let r and m be as in* (6.1). *Then*

(a) $m \geqq \frac{1}{2}(c_2 - t^2 + 2t - 2)$ *if and only if* $r \geqq \frac{1}{2}(c_2 - t^2 + 2t)$. *In that case* $m = r-1$, *the jumping line of order r is unique, and*

$$h^1(\mathscr{E}(l)) \begin{cases} \leqq c_2 - t^2 + 2t - 2l - 2 & \textit{for} \quad t \leqq l \leqq c_2 - t^2 + 2t - r - 3 \\ = r - l & \textit{for} \quad c_2 - t^2 + 2t - r - 2 \leqq l \leqq r \\ = 0 & \textit{for} \quad l \geqq r. \end{cases}$$

(b) *If* $m<\frac{1}{2}(c_2-t^2+2t-2)$ *then*

$$h^1(\mathscr{E}(l))\begin{cases}\leqq c_2-t^2+2t-2l-2 & \text{for} \quad t\leqq l\leqq\frac{1}{2}(c_2-t^2+2t-2)\\ =0 & \text{for} \quad l\geqq\frac{1}{2}(c_2-t^2+2t-2).\end{cases}$$

Proof. Same as (6.1).

Example 6.2.1. One can show by examples that the bounds in these two theorems are the best possible for all values of c_2, t, r for which there exist stable bundles. We will illustrate this in a special case.

Take $c_1=0$, $c_2>0$, $t=1$, and any integer r satisfying $\frac{1}{2}(c_2+2)\leqq r<c_2$. Then we construct a bundle $\mathscr{E}$ as an extension

$$0\to\mathcal{O}\to\mathscr{E}(1)\to\mathscr{I}_Z(2)\to 0,$$

where Z is a set of c_2+1 points, consisting of $Z_1=$ a set of $r+1$ points on a line L and $Z_2=$ a set of c_2-r points on another line L'. Then Z is not contained in a single line, so $\mathscr{E}$ is stable. The line L is a jumping line of order r, so we are in case (a) of (6.1). It is easy to verify that the inequalities of (6.1a) are all equalities in this case. One way to see this is to use the reduction step for $\mathscr{E}$ and L. The new bundle $\mathscr{E}'$ is then an extension

$$0\to\mathcal{O}\to\mathscr{E}'(1)\to\mathscr{I}_{Z_2}(1)\to 0$$

whose cohomology is easy to find.

Example 6.2.2. It is possible to have all equalities in part (b) of the theorems, and yet have no jumping lines of order $r\geqq 2$ at all. Take $c_1=0$, c_2 odd $\geqq 3$. We construct $\mathscr{E}$ by

$$0\to\mathcal{O}\to\mathscr{E}(1)\to\mathscr{I}_Z(2)\to 0,$$

where Z consists of c_2+1 points on an irreducible conic C. Clearly Z is not contained in a line, so $\mathscr{E}$ is stable and $t=1$. To find the jumping lines of $\mathscr{E}$, we reason as follows. If L is a line which does not meet Z, then restricting to L gives

$$0\to\mathcal{O}_L\to\mathscr{E}_L(1)\to\mathcal{O}_L(2)\to 0.$$

Therefore $\mathscr{E}_L$ is either $\mathcal{O}_L\oplus\mathcal{O}_L$ or $\mathcal{O}_L(1)\oplus\mathcal{O}_L(-1)$. If L is a line which does meet Z, then the number of points of Z on L is either 1 or 2, since Z lies on the conic C. Then the restriction $\mathscr{E}_L$ is again one of the same two types, and the second type definitely exists. Thus $r=1$, so $r<\frac{1}{2}(c_2+2)$ and we find ourselves in case (b) of (6.1).

Next we compute $H^1(\mathscr{E}(l))=H^1(\mathscr{I}_Z(l+1))$. It is the cokernel of the map $H^0(\mathcal{O}(l+1))\to H^0(\mathcal{O}_Z(l+1))$, which factors through $H^0(\mathcal{O}_C(l+1))$, which has dimension $2l+3$. Thus for $2l+3<c_2+1$, i.e. $2l+3\leqq c_2$, $h^1(\mathscr{E}(l))\neq 0$. In particular, if $l=\frac{1}{2}(c_2-3)$, then $h^1(\mathscr{E}(l))\neq 0$. Therefore $m=\frac{1}{2}(c_2-3)$, and all the inequalities of (b) are forced to be equalities.

Remark 6.2.3. These theorems and the examples given show that one can "explain" the nonvanishing cohomology groups $H^1(\mathscr{E}(l))$ in the range $l>\sim\frac{1}{2}c_2$ by the existence of high order jumping lines. On the other hand, Brun [4] has shown that for the generic stable rank 2 bundle on $\mathbb{P}^2$ with $c_1=0$ or -1 and given c_2, the dimension of $H^1(\mathscr{E}(l))$ is given by the Riemann-Roch theorem: $h^1(\mathscr{E}(l)) = -\chi(\mathscr{E}(l))$ whenever $\chi(\mathscr{E}(l))<0$, and $h^1(\mathscr{E}(l))=0$ otherwise. In particular $h^1(\mathscr{E}(l))=0$ for $l > \sim \sqrt{c_2}$.

To explain the nonvanishing $h^1(\mathscr{E}(l))$ in the intermediate range $\sqrt{c_2} \leqq m \leqq \frac{1}{2}c_2$ something else is needed. Clearly in example (6.2.2) the conic C plays a special role. More generally, one can define the notion of an *unstable curve* C for a stable bundle $\mathscr{E}$, as a curve for which $H^0(\mathscr{E}_C(-r)) \neq 0$ for some $r>0$, and one can explain the nonvanishing cohomology groups $H^1(\mathscr{E}(l))$ in this intermediate range by the existence of suitable unstable curves.

7. The Spectrum of a Reflexive Sheaf

The spectrum of a vector bundle was introduced in the paper of Barth and Elencwajg [3]. We recall their definition. Let $\mathscr{E}$ be a stable rank 2 vector bundle with $c_1=0$ on $\mathbb{P}^3$, over a field of characteristic 0. Let L be a general line of $\mathbb{P}^3$, let $p: X \to \mathbb{P}^3$ be the blowing-up of L, and let $q: X \to \mathbb{P}^1$ be the morphism determined by the pencil of planes through L. Then the sheaf $\mathscr{H} = R^1 q_* p^* \mathscr{E}(-1)$ is locally free of rank c_2 on $\mathbb{P}^1$, so it can be written

$$\mathscr{H} \cong \bigoplus_{i=1}^{c_2} \mathcal{O}_{\mathbb{P}^1}(k_i)$$

for suitable integers $k_1, \ldots, k_{c_2}$. This set of integers $\{k_i\}$ is called the *spectrum* of $\mathscr{E}$. The principal properties of the spectrum are

(1) $\{k_i\}$ is symmetric around 0,

(2) $\{k_i\}$ is a connected set of integers,

(3) for any $l \leqq -1$, $H^1(\mathbb{P}^3, \mathscr{E}(l)) \cong H^0(\mathbb{P}^1, \mathscr{H}(l+1))$.

From these properties it is easy to deduce a vanishing theorem

(4) $H^1(\mathscr{E}(l))=0$ for $l \leqq -\frac{1}{2}c_2 - 1$.

Furthermore, the spectrum provides a stratification of the moduli space which is useful in the classification problem.

Our purpose in this section is to generalize these results to stable or semistable rank 2 reflexive sheaves $\mathscr{F}$ on $\mathbb{P}^3$ with $c_1=0$ or -1, and over a field of arbitrary characteristic. At the same time, we provide a new proof of the original results of Barth and Elencwajg.

In characteristic 0 it is possible to use the above definition of the spectrum under these broader hypotheses. However in characteristic $p>0$ we do not know if that definition works, so we will take a different approach. We will show (7.1) that there exists a unique set of integers $\{k_i\}$ with the required properties, but we do not know if they give the decomposition of $R^1 q_* p^* \mathscr{F}(-1)$.

Our main results are similar to the ones above. For reflexive sheaves which are not locally free, the symmetry property is replaced by a formula (7.3) relating the spectrum to c_3. In the case of semistable sheaves the connectedness (7.6) must be weakened slightly, by allowing possible gaps at 0 or -1. Since Serre duality is more complicated for reflexive sheaves, we relate the k_i to $H^1(\mathscr{F}(l))$ and $H^2(\mathscr{F}(l))$. As consequences of these results, we will obtain in Sect. 8 vanishing theorems for $H^1(\mathscr{F}(l))$ for l negative, and for $H^2(\mathscr{F}(l))$ for l positive, and we get bounds on c_3 in terms of c_1 and c_2. Finally we will give examples to show that all these results are the best possible.

Theorem 7.1. *Let $\mathscr{F}$ be a* rank 2 *reflexive sheaf with $c_1=0$ or -1 on $\mathbb{P}^3$, and assume $H^0(\mathscr{F}(-1))=0$. Then there exists a unique set of integers $\{k_i\}_{i=1,\ldots,c_2}$ called the spectrum of $\mathscr{F}$, with the following properties (where $\mathscr{H}$ denotes the sheaf $\oplus\mathcal{O}(k_i)$ on $\mathbb{P}^1$):*

(a) $h^1(\mathbb{P}^3,\mathscr{F}(l))=h^0(\mathbb{P}^1,\mathscr{H}(l+1))$ *for* $l\leqq -1$.

(b) $h^2(\mathbb{P}^3,\mathscr{F}(l))=h^1(\mathbb{P}^1,\mathscr{H}(l+1))$ *for* $l\geqq -3$ *if* $c_1=0$, *and* $l\geqq -2$ *if* $c_1=-1$.

Proof. Let H be a general plane in $\mathbb{P}^3$, and let $\mathscr{E}$ be the restriction $\mathscr{F}_H$. Then $\mathscr{E}$ is a rank 2 vector bundle on H, and $H^0(\mathscr{E}(-1))=0$. If $\mathscr{F}$ is semistable, this follows from the fact (3.2) that $\mathscr{E}$ is also semistable. In the remaining case $c_1=-1$ and $H^0(\mathscr{F})\neq 0$, there is an exact sequence

$$0\to\mathcal{O}\to\mathscr{F}\to\mathscr{I}_Y(-1)\to 0$$

for a suitable curve Y in $\mathbb{P}^3$. Then any H which does not contain any irreducible component of Y will give an $\mathscr{E}=\mathscr{F}_H$ with $H^0(\mathscr{E}(-1))=0$.

We will now apply the results of Sect. 5 to $\mathscr{E}$. Let $M=\oplus H^1(H,\mathscr{E}(l))$, and let N be the submodule defined by the images of the natural maps $H^1(\mathbb{P}^3,\mathscr{F}(l))\to H^1(H,\mathscr{E}(l))$. Then there is an exact sequence

$$H^0(\mathscr{E}(l))\to H^1(\mathscr{F}(l-1))\to H^1(\mathscr{F}(l))\to N_l\to 0.$$

Since $H^0(\mathscr{E}(l))=0$ for $l\leqq -1$, we find

$$n_l=h^1(\mathscr{F}(l))-h^1(\mathscr{F}(l-1))$$

for $l\leqq -1$.

We are looking for integers $\{k_i\}$ satisfying (a) and (b) above. The condition (a) can be expressed as

$$h^1(\mathscr{F}(l))=\sum_{k_i\geqq -l-1}(k_i+l+2).$$

Substituting above,

$$n_l=\sum_{k_i\geqq -l-1}1.$$

In other words, for each $l\leqq -1$

$$\#\{k_i=-l-1\}=n_l-n_{l-1}.$$

According to (5.3), the quantities n_l-n_{l-1} are nonnegative for all $l\leqq -1$, so it is possible to find integers k_i satisfying these conditions, and the $k_i\geqq 0$ are uniquely determined by (a).

Now we turn to (b). Let R be the quotient module of M defined by the kernels of the natural maps $H^2(\mathscr{F}(l))\to H^2(\mathscr{F}(l+1))$. Then there are exact sequences

$$0\to R_{l+1}\to H^2(\mathscr{F}(l))\to H^2(\mathscr{F}(l+1))\to H^2(\mathscr{E}(l+1)).$$

Since $H^0(\mathscr{E}(-1))=0$, by duality $H^2(\mathscr{E}(-2))=0$ if $c_1=0$, and $H^2(\mathscr{E}(-1))=0$ if $c_1=-1$. Therefore, letting $r_l=\dim R_l$,

$$r_{l+1}=h^2(\mathscr{F}(l))-h^2(\mathscr{F}(l+1))$$

for $l\geqq -3$ if $c_1=0$ and $l\geqq -2$ if $c_1=-1$. The condition (b) can be expressed as

$$h^2(\mathscr{F}(l))=\sum_{k_i\leqq -l-3}(-k_i-l-2).$$

So

$$r_{l+1}=\sum_{k_i\leqq -l-3}1,$$

and therefore

$$\#\{k_i=-l-3\}=r_{l+1}-r_{l+2}$$

for all l in the range given. We apply (5.3) to the dual of R, which is a submodule of M. This shows that the quantities $r_{l+1}-r_{l+2}$ are all nonnegative in the range given. Therefore such integers k_i exist, and the $k_i\leqq 0$ if $c_1=0$ and $k_i\leqq -1$ if $c_1=-1$ are uniquely determined by condition (b).

In case $c_1=0$ we now have two determinations of $\#\{k_i=0\}$. We must check that they agree. In other words, we must show that $n_{-1}-n_{-2}=r_{-2}-r_{-1}$. Indeed, this follows from the exact sequences for all l

$$0\to N_l\to M_l\to R_l\to 0$$

and the fact that $m_{-1}=m_{-2}=c_2$.

Thus we have proved that there exists a unique set of integers $\{k_i\}$ satisfying (a) and (b). It remains to show that the number of k_i is equal to c_2. Indeed, summing the expressions above for the number of k_i equal to each given integer shows

$$\#\{k_i\geqq 0\}=n_{-1}$$
$$\#\{k_i<0\}=r_{-1}.$$

But $n_{-1}+r_{-1}=m_{-1}=h^1(\mathscr{E}(-1))$ which is equal to c_2 by Riemann-Roch.

Remark 7.1.1. In the statements (a) and (b) of the theorem, we assert only that the dimensions of the cohomology groups in question are equal. We do not claim there is any natural isomorphism of vector spaces, such as results from the identification of $R^1q_*p^*\mathscr{E}(-1)$ with $\mathscr{H}$ in the paper of Barth and Elencwajg.

Remark 7.1.2. If $\operatorname{char}k=0$, and if L is a general line in $\mathbb{P}^3$, then one can show, using a method similar to the method of Barth and Elencwajg, that $R^1q_*p^*\mathscr{F}(-1)$ (using the notation at the beginning of this section) is locally free of rank c_2 on $\mathbb{P}^1$, and that it is isomorphic to $\mathscr{H}$. Thus our definition of the spectrum agrees with theirs. The characteristic 0 hypothesis is used to show that $\mathscr{F}_L$ is isomorphic to $\mathcal{O}_L\oplus\mathcal{O}_L$ or $\mathcal{O}_L\oplus\mathcal{O}_L(-1)$.

Proposition 7.2 (Symmetry). *With the hypotheses of* (7.1), *assume furthermore that* $\mathscr{F}$ *is locally free. Then*

$$\{-k_i\}=\{k_i\} \qquad \textit{if} \quad c_1=0,$$

and

$$\{-k_i\}=\{k_i+1\} \quad \textit{if} \quad c_1=-1.$$

Proof. This is a consequence of Serre duality. Using the notation of the proof of (7.1),

$$n_l=h^1(\mathscr{F}(l))-h^1(\mathscr{F}(l-1)).$$

By duality,

$$n_l=h^2(\mathscr{F}(-l-4-c_1))-h^2(\mathscr{F}(-l-3-c_1)),$$

which is equal to r_{-l-3-c_1}. So for any $l\leqq -1$,

$$\begin{aligned}\#\{k_i=-l-1\}&=n_l-n_{l-1}\\&=r_{-l-3-c_1}-r_{-l-2-c_1}\\&=\#\{k_i=l+1+c_1\}.\end{aligned}$$

This proves the stated symmetry of the spectrum.

Proposition 7.3. *Let $\mathscr{F}$ be as in* (7.1). *Then*

$$c_3=-2\sum k_i \qquad \textit{if} \quad c_1=0$$

$$c_3=-2\sum k_i-c_2 \quad \textit{if} \quad c_1=-1.$$

Proof. For this we use Serre duality for the reflexive sheaf $\mathscr{F}$ (2.5), and the fact (2.6) that $c_3=h^0(\mathscr{E}xt^1(\mathscr{F},\omega))$. First we need a lemma.

Lemma 7.4. *Let $\mathscr{F}$ be a* rank 2 *reflexive sheaf on $\mathbb{P}^3$, let H be a plane not containing any of the non-locally-free points of $\mathscr{F}$, let $M=\oplus H^1(H,\mathscr{F}_H(l))$, let N be the submodule of M obtained as the image of $\oplus H^1(\mathscr{F}(l))$, and let R be the quotient module of M obtained as the kernel of the natural map $\oplus H^2(\mathscr{F}(l-1))\to\oplus H^2(\mathscr{F}(l))$, so that there is an exact sequence*

$$0\to N\to M\to R\to 0.$$

For each l, the duality morphism $H^1(\mathscr{F}(l))\to H^2(\mathscr{F}(-l-4-c_1))'$ of (2.5) *induces an inclusion*

$$0\to N_l\to R'_{-l-3-c_1}.$$

Summing over l gives a module homomorphism $N\to R'$ whose quotient we call V:

$$0\to N\to R'\to V\to 0.$$

Then V is a module of total length c_3, and the self-duality isomorphisms $M_l\cong M'_{-l-3-c_1}$ of M induce a self-duality $V_l\cong V'_{-l-3-c_1}$ of V.

Proof. Let $\mathscr{G}=\mathscr{E}xt^1(\mathscr{F},\omega)$. Then the exact sequence of (2.5) gives an exact sequence of graded $S=k[x_0,x_1,x_2,x_3]$-modules

$$0\to\oplus H^1(\mathscr{F}(l))\to\oplus H^2(\mathscr{F}(-l-4-c_1))'\to\oplus H^0(\mathscr{G}).$$

Let the image of this last map be $A \subseteq \oplus H^0(\mathscr{G})$. Then $A_l = 0$ for $l \ll 0$, by Serre's vanishing theorem for H^2, and $A_l = H^0(\mathscr{G})$ for $l \gg 0$ again by Serre's vanishing theorem, because the next term in the sequence (2.5) is $H^2(\mathscr{F}(l))$.

Let $h \in H^0(\mathcal{O}_{\mathbb{P}^3}(1))$ be the equation of H, and regard A as a graded $k[h]$-module. Since H does not meet the support of $\mathscr{G}$, the map $h: H^0(\mathscr{G}) \to H^0(\mathscr{G})$ is bijective. Hence A is a torsion-free $k[h]$-module. The properties $A_l = 0$ for $l \ll 0$ and $A_l = H^0(\mathscr{G})$ for $l \gg 0$ show that A is finitely generated and of rank $= h^0(\mathscr{G}) = c_3$. By the structure theorem for modules over a principal ideal domain, it is a free graded $k[h]$-module of rank c_3.

Tensoring the above sequence of S-modules with S/hS gives

$$0 \to \oplus N_l \to \oplus R'_{-l-3-c_1} \to A/hA \to 0,$$

and the structure of A just described implies that $V = A/hA$ is a k-vector space of dimension c_3.

The dual of this sequence is

$$0 \to V' \to R \to N' \to 0.$$

Since $R \cong M/N$ and $N' \cong M'/R'$ and $M \cong M'$, we conclude $V' \cong V$.

Proof of (7.3), *continued.* Let us compute $\sum k_i$. Using the notation of the proof of (7.1) and the formula

$$\#\{k_i = -l-1\} = n_l - n_{l-1}$$

for each $l \leqq -1$, we find

$$\sum_{k_i > 0} k_i = \sum_{l \leqq -2} n_l.$$

Then using the formula

$$\#\{k_i = -l-3\} = r_{l+1} - r_{l+2}$$

for each $l \geqq -2$ we find

$$\sum_{k_i < 0} k_i = -\sum_{l \geqq -1} r_l.$$

Now suppose $c_1 = 0$. Then we sum these two expressions and change variables in the second, so as to obtain

$$\sum k_i = \sum_{l \leqq -2} (n_l - r_{-l-3}).$$

By the lemma, this gives

$$\sum k_i = -\sum_{l \leqq -2} v_l,$$

where $v_l = \dim V_l$. The self-duality of V shows that

$$\sum_{l \leqq -2} v_l = \sum_{l \geqq -1} v_l,$$

so each of these sums is equal to $\frac{1}{2}\dim V=\frac{1}{2}c_3$. We conclude that $c_3=-2\sum k_i$ as required.

In the case $c_1=-1$ we sum the above expressions in the following way:

$$\sum k_i=\sum_{l\leqq -2}(n_l-r_{-l-2})-r_{-1}=-\sum_{l\leqq -2}v_l-r_{-1}.$$

In this case the self-duality of V shows that

$$\sum_{l\leqq -2}v_l=\sum_{l\geqq 0}v_l.$$

Therefore

$$\sum_{l\leqq -2}v_l=\tfrac{1}{2}c_3-\tfrac{1}{2}v_{-1}.$$

So we get

$$c_3=-2\sum k_i+v_{-1}-2r_{-1}.$$

Then using $v_{-1}=r_{-1}-n_{-1}$ and $n_{-1}+r_{-1}=m_{-1}=c_2$ gives the required result.

Theorem 7.5. *Let $\mathscr{F}$ be as in* (7.1), *and let $\{k_i\}$ be the spectrum.*

(a) *Assume $H^0(\mathscr{F}(-1))=0$.*

1) *If there is a $k>0$ in the spectrum, then $1,2,\ldots,k$ also occur in the spectrum.*

2) *If there is a $k<-1$, then $-1,-2,\ldots,k$ also occur if $c_1=0$, and $-2,-3,\ldots,k$ also occur if $c_1=-1$.*

(b) *Assume $\mathscr{F}$ is stable.*

1) *If there is a $k>0$, then $0,1,\ldots,k$ also occur*[3].

2) *If there is a $k<0$, then $-1,-2,\ldots,k$ also occur. Furthermore, if $c_1=0$, then either 0 also occurs, or -1 occurs at least twice*[3].

Proof. Let H be a general plane, and let $\mathscr{E}=\mathscr{F}_H$. We apply the strict inequalities of (5.3) to $\mathscr{E}$. In case (a), the hypothesis $H^0(\mathscr{F}(-1))=0$ implies $H^0(\mathscr{E}(-1))=0$, as in the proof of (7.1). Then (5.3) tells us that $n_{l-1}<n_l$ for all $l\leqq -2$, unless $n_l=0$. Using the formula from the proof of (7.1)

$$\#\{k_i=-l-1\}=n_l-n_{l-1}$$

for all $l\leqq -1$, we see that if there is a $k>0$ in the spectrum, then the integers $1,2,\ldots,k$ also occur in the spectrum. This is 1).

Similarly, using the formula

$$\#\{k_i=-l-3\}=r_{l+1}-r_{l+2},$$

and applying (5.3) to R', we get conclusion 2).

Now suppose $\mathscr{F}$ is stable, and assume for the moment that $\mathscr{E}=\mathscr{F}_H$ is also stable. Then (5.3) gives the additional conclusion that $n_{-2}<n_{-1}$, unless $n_{-2}=c_2$. Applying (5.3) to the module $N=$ image of $\oplus H^1(\mathscr{F}(l))$, the inclusion $N\subsetneqq R'$ shows

3 The case $c_1=0$, $\mathscr{F}$ stable requires the restriction theorems of Barth (char0) and Ein (char$p$). We will signal with * later results which depend on this one

that $n_{-2} \leqq \frac{1}{2}c_2$, so the inequality $n_{-2} < n_{-1}$ holds without exception, and this proves 1).

For statement 2), we apply (5.3) to $N = R'$. If $c_1 = -1$, then $\dim M_{-2} = c_2 - 1$, so again $n_{-2} < n_{-1}$ without exception, which shows that $-1, -2, \ldots, k$ occur. If $c_1 = 0$, then either $n_{-2} < n_{-1}$, which shows that $0, -1, \ldots, k$ occur, or else $n_{-2} = n_{-1} = c_2$. In the latter case, $\dim M_{-3} = c_2 - 2$, so $n_{-2} - n_{-3} \geqq 2$, which shows that -1 must occur at least twice.

It remains to consider the case $\mathscr{F}$ stable, and $\mathscr{F}_H$ not stable for a general plane H, which can happen only when $c_1 = 0$. If $\operatorname{char} k = 0$, the restriction theorem of Barth [1], which holds also for reflexive sheaves (3.3.1), implies that $\mathscr{F}$ is the nullcorrelation bundle. The spectrum of the nullcorrelation bundle is $\{0\}$, so 1) and 2) hold vacuously.

If $\operatorname{char} k = p > 0$, the restriction theorem of Ein [5] implies that $\mathscr{F} \cong F^{r*}\mathscr{E}_0$, where $\mathscr{E}_0$ is the nullcorrelation bundle, F is the Frobenius morphism of $\mathbb{P}^3$ to itself, and $r \geqq 0$ is an integer. In this case we will show by explicit computation that 0 occurs in the spectrum. There is an exact sequence

$$0 \to \mathcal{O} \to \mathscr{E}_0(1) \to \mathscr{I}_Y(2) \to 0,$$

where Y is a disjoint union of two lines L_1 and L_2 in $\mathbb{P}^3$ [9, 8.4.1]. Applying F^{r*} gives an exact sequence

$$0 \to \mathcal{O} \to \mathscr{F}(q) \to \mathscr{I}_{Y^{(q)}}(2q) \to 0,$$

where $q = p^r$ and $Y^{(q)}$ is the scheme defined by the q^{th} powers of the equations defining Y. In fact, $Y^{(q)} = L_1^{(q)} \cup L_2^{(q)}$, where $L_1^{(q)}$ is a complete intersection (x^q, y^q), if x, y are the linear forms defining L_1, and similarly for L_2.

The number of times 0 occurs in the spectrum of $\mathscr{F}$ is $n_{-1} - n_{-2} = h^1(\mathscr{F}(-1)) + h^1(\mathscr{F}(-3)) - 2h^1(\mathscr{F}(-2))$. Using the isomorphisms $H^1(\mathscr{F}(l)) \cong H^1(\mathscr{I}_{Y^{(q)}}(l+q))$ and the fact that $L_1^{(q)}$ is not contained in any surface of degree $< q$, it is easy to compute these numbers. We find that 0 occurs exactly q times in the spectrum. Since $q \geqq 1$, this proves the result.

Corollary 7.6. *If $\mathscr{F}$ is semistable, the spectrum is connected, except possibly for a gap at 0. If $\mathscr{F}$ is stable, the spectrum is connected*[4].

Remark 7.6.1. For given c_1, c_2, c_3, one can ask what are the possible spectra of stable or semistable reflexive sheaves with the given Chern classes. The results of this section provide various necessary conditions for a set of integers $\{k_i\}$ to be a spectrum, but not all sets of integers satisfying these conditions actually occur. For example, consider stable vector bundles with $c_1 = 0$ and $c_2 = 8$. There are eight sets of $\{k_i\}$ which satisfy the symmetry and connectedness conditions, but only seven of these correspond to bundles (see Table 1).

The notation used in the table is as follows. Y_d denotes an elliptic curve of degree d. Unions are supposed to be disjoint. $F_{a,b}$ denotes a complete intersection of surfaces of degrees a and b. Z corresponds to a section of $\mathscr{E}'(3)$ where $\mathscr{E}'$ is an instanton bundle [i.e. a stable bundle with $c_1 = 0$ and $H^1(\mathscr{E}'(-2)) = 0$] with $c_2 = 3$. Z' corresponds to a section of $\mathscr{E}''(4)$ where $\mathscr{E}''$ is a nullcorrelation bundle.

4 See footnote to (7.5)

Table 1

Spectrum	Existence
0^8	$\mathscr{E}(1)\leftrightarrow 9$ skew lines (instanton bundle)
$-1, 0^6, 1$	$\mathscr{E}(2)\leftrightarrow Y_6\cup Y_6$
$-1, -1, 0^4, 1, 1$	$\mathscr{E}(2)\leftrightarrow Y_3\cup Y_3\cup Y_6$
$-1, -1, -1, 0^2, 1, 1, 1$	$\mathscr{E}(2)\leftrightarrow Y_3\cup Y_3\cup Y_3\cup Y_3$
$-2, -1, 0^4, 1, 2$	$\mathscr{E}(3)\leftrightarrow F_{1,5}\cup Z$
$-2, -1, -1, 0^2, 1, 1, 2$	$\mathscr{E}(3)\leftrightarrow F_{2,4}\cup F_{3,3}$
$-2, -2, -1, 0^2, 1, 2, 2$	does not exist
$-3, -2, -1, 0^2, 1, 2, 3$	$\mathscr{E}(4)\leftrightarrow F_{1,7}\cup Z'$

We will show in a later article that there is no bundle with spectrum $-2, -2, -1, 0, 0, 1, 2, 2$.

8. Vanishing Theorems and Bounds on c_3

In this section we will use the properties of the spectrum developed in the last section to prove vanishing theorems for $H^1(\mathscr{F}(l))$, $l<0$, and $H^2(\mathscr{F}(l))$, $l>0$, and give bounds for c_3 in terms of c_1 and c_2. We also give examples to show these are the best possible.

The vanishing theorem for $H^1(\mathscr{F}(l))$ was proved by Barth and Elencwajg [3] for a stable vector bundle with $c_1=0$ in characteristic 0. Our result generalizes and gives a new proof of theirs.

Theorem 8.1. *Let $\mathscr{F}$ be a rank 2 reflexive sheaf on $\mathbb{P}^3$ with $c_1=0$ or -1.*

(a) *Assume $H^0(\mathscr{F}(-1))=0$. Then $H^1(\mathscr{F}(l))=0$ for $l\leqq -\frac{1}{2}c_2-\frac{3}{2}$.*

(b) *Assume $\mathscr{F}$ stable and $c_1=-1$. Then $H^1(\mathscr{F}(l))=0$ for $l\leqq -\frac{1}{2}c_2-\frac{1}{2}$.*

(c) *Assume $\mathscr{F}$ stable and $c_1=0$. Then $H^1(\mathscr{F}(l))=0$ for $l\leqq -\frac{1}{2}c_2-1$.*

Proof. Let $\{k_i\}$ be the spectrum of $\mathscr{F}$, and let $k=\max\{k_i\}$. Then by (7.1a), $H^1(\mathscr{F}(l))$ will be 0 for $l+k+1<0$, i.e. $l<-k-1$. So our technique is to use the properties of the spectrum to bound k.

Let us begin with the case $\mathscr{F}$ semistable and $c_1=0$. Since $c_3\geqq 0$, by (7.3) we see $\sum k_i\leqq 0$. On the other hand, by (7.6), the spectrum is connected, except possibly for a gap at 0. So if c_2 is even, the largest possible k is $\frac{1}{2}c_2$, corresponding to the spectrum $\{-\frac{1}{2}c_2,\ldots, -1, 1,\ldots, \frac{1}{2}c_2\}$. If c_2 is odd, the largest $k=\frac{1}{2}(c_2-1)$, corresponding for example to the spectrum $\{-\frac{1}{2}(c_2+1),\ldots,-1,1,\ldots,\frac{1}{2}(c_2-1)\}$ or $\{-\frac{1}{2}(c_2-1),\ldots,-1,0,1,\ldots,\frac{1}{2}(c_2-1)\}$. Therefore $H^1(\mathscr{F}(l))=0$ for $l<-\frac{1}{2}c_2-1$ if c_2 even, and $l<-\frac{1}{2}c_2-\frac{1}{2}$ if c_2 odd. Since l is an integer, both of these are equivalent to the condition $l\leqq -\frac{1}{2}c_2-\frac{3}{2}$ of (a).

In case $H^0(\mathscr{F}(-1))=0$ and $c_1=-1$, then by (7.3), $\sum k_i\leqq -\frac{1}{2}c_2$. And by (7.5), the spectrum is connected, except for possible gaps at $0, -1$. Looking at the possible spectra, we find again that $k=\frac{1}{2}c_2$ if c_2 even and $k=\frac{1}{2}(c_2-1)$ if c_2 odd are the largest possible values of k. We get the same result (a) in this case also.

In case (b), again $\sum k_i\leqq -\frac{1}{2}c_2$, and by (7.6) the spectrum is connected. If c_2 is even, the largest possible k is $\frac{1}{2}(c_2-2)$, corresponding to the spectrum $\{-\frac{1}{2}c_2,\ldots,-1,0,1,\ldots,\frac{1}{2}(c_2-2)\}$. If c_2 is odd, $k=\frac{1}{2}(c_2-3)$ is the largest possible. We find that $H^1(\mathscr{F}(l))=0$ for $l\leqq -\frac{1}{2}c_2-\frac{1}{2}$, as required.

To prove (c), we could use an entirely similar argument, based on (7.6)* which says the spectrum is connected. But that depends on the restriction theorems of Barth and Ein, so it may be of interest to give another more elementary proof not depending on those results.

So suppose $\mathscr{F}$ is stable and $c_1=0$. From part (a) we know already that $H^1(\mathscr{F}(l))=0$ for $l\leqq -\frac{1}{2}c_2-\frac{3}{2}$. If c_2 is odd, this is equivalent to $l\leqq -\frac{1}{2}c_2-1$, so there is nothing to prove. If c_2 is even, we must show that $H^1(\mathscr{F}(-\frac{1}{2}c_2-1))$ is 0.

Suppose on the contrary $H^1(\mathscr{F}(-\frac{1}{2}c_1-1))\neq 0$. Let H be a general plane, let $\mathscr{E}=\mathscr{F}_H$, which in any case is semistable, and use the notations of the proof of (7.3). Then $n_{-\frac{1}{2}c_2-1}\geqq 1$, and because of the inequalities of (5.3) we have

$$
\begin{aligned}
&n_{-\frac{1}{2}c_2-1}\geqq 1\\
&\quad n_{-\frac{1}{2}c_2}\geqq 2\\
&\dots\\
&\quad n_{-2}\geqq \tfrac{1}{2}c_2\\
&\quad n_{-1}\geqq \tfrac{1}{2}c_2\,.
\end{aligned}
$$

On the other hand, because $N\subseteq R'$ and $R=M/N$, we have $n_{-2}+n_{-1}\leqq \dim M_{-2}=c_2$. This forces all the inequalities above to be equalities.

Now for each l, let $a_l=h^1(\mathscr{F}(l))$. From the equation $n_l=a_l-a_{l-1}$ for all $l<0$ we find

$$
\begin{aligned}
&a_{-\frac{1}{2}c_2-1}=1\\
&\quad a_{-\frac{1}{2}c_2}=3\\
&\dots\\
&\quad a_{-2}=\tfrac{1}{8}c_2(c_2+2)\\
&\quad a_{-1}=\tfrac{1}{8}c_2(c_2+6)\,.
\end{aligned}
$$

Consider the natural map

$$H^1(\mathscr{F}(-\tfrac{1}{2}c_2-1))\times H^0(\mathcal{O}_{\mathbf{P}^3}(1))\to H^1(\mathscr{F}(-\tfrac{1}{2}c_2))\,.$$

By a trivial case of the bilinear map lemma (5.1), there is a linear form x which annihilates $H^1(\mathscr{F}(-\frac{1}{2}c_2-1))$. Let H_0 be the corresponding plane. Then $H^0(\mathscr{F}_{H_0}(-\frac{1}{2}c_2))\neq 0$, so there is an injective map $\mathcal{O}_{H_0}\to\mathscr{F}_{H_0}(-\frac{1}{2}c_2)$.

Next look at the exact sequence

$$H^0(\mathscr{F}(l+1))\to H^0(\mathscr{F}_{H_0}(l+1))\to H^1(\mathscr{F}(l))\,.$$

Because $\mathscr{F}$ is stable (here we use the hypothesis $\mathscr{F}$ stable for the first time), $H^0(\mathscr{F}(l+1))=0$ for $l\leqq -1$, so the next map is injective. Combining with the inclusion $\mathcal{O}_{H_0}\to\mathscr{F}_{H_0}(-\frac{1}{2}c_2)$ we find

$$h^1(\mathscr{F}(-1)\geqq h^0(\mathscr{F}_{H_0})\geqq h^0(\mathcal{O}_{H_0}(\tfrac{1}{2}c_2))=\tfrac{1}{8}(c_2+2)(c_2+4)\,.$$

This contradicts the result $a_{-1}=\frac{1}{8}c_2(c_2+6)$ above, and completes the proof.

Example 8.1.1. For $c_1=0$ and each c_2 odd $\geqq 1$, an example of a stable vector bundle $\mathscr{E}$ with $H^1(\mathscr{E}(-\frac{1}{2}c_2-\frac{1}{2}))\neq 0$. Take an integer $m\geqq 1$. Let Y_1 be a plane curve of degree $2m-1$, and let Y_2 be a complete intersection of two surfaces of degree m. Let Y be the disjoint union of Y_1 and Y_2. Then

$$d = \deg Y = m^2 + 2m - 1$$
$$\omega_Y \cong \mathcal{O}_Y(2m-4).$$

So we can use Y to define a bundle $\mathcal{E}(m)$ by the exact sequence

$$0 \to \mathcal{O} \to \mathcal{E}(m) \to \mathcal{I}_Y(2m) \to 0.$$

By construction $c_1(\mathcal{E})=0$, $c_2(\mathcal{E})=2m-1$, and $\mathcal{E}$ is stable, because Y is not contained in any surface of degree m. Indeed, Y_2 is contained in a pencil of surfaces of degree m, but none of them can contain Y_1. Finally, since Y has two connected components, $H^1(\mathcal{I}_Y)=1$, so $h^1(\mathcal{E}(-m))=1$. But $m=\frac{1}{2}(c_2+1)$, so this gives the required example.

Since $H^1(\mathcal{E}(-\frac{1}{2}c_2-\frac{1}{2}))\neq 0$, the spectrum must contain $\frac{1}{2}(c_2-1)$. So by symmetry (7.2) and the fact that $\mathcal{E}$ semistable implies the spectrum connected except possibly for 0 (7.6), we see that the spectrum must be $\{-\frac{1}{2}(c_2-1), \ldots, -1, 0, 1, \ldots, \frac{1}{2}(c_2-1)\}$. In particular, it is connected.

For $c_2=1$ we get the null correlation bundle. For $c_2=3$ we get bundles with α-invariant 1 [9, 3.1.3]. For $c_2 \geqq 5$ we get the oversized family discussed by Barth and Hulek [2]. See (9.9) for further discussion of these bundles.

Example 8.1.2. For $c_1=0$ and each c_2 even $\geqq 2$, a semistable vector bundle $\mathcal{E}$ with $H^1(\mathcal{E}(-\frac{1}{2}c_2-1))\neq 0$. This is an analogous construction. Take $m \geqq 1$, let Y_1 be a plane curve of degree $2m-1$, let Y_2 be a complete intersection of surfaces of degrees $m-1, m+1$, and let $Y = Y_1 \coprod Y_2$. Then

$$d = m^2 + 2m - 2$$
$$\omega_Y = \mathcal{O}_Y(2m-4),$$

so we can construct a bundle $\mathcal{E}$ by

$$0 \to \mathcal{O} \to \mathcal{E}(m) \to \mathcal{I}_Y(2m) \to 0.$$

It has $c_1=0$, $c_2=2m-2$, and is semistable, because Y is not contained in any surface of degree $m-1$. Since Y has two components, as above $h^1(\mathcal{E}(-m))=1$. Since $m=\frac{1}{2}c_2+1$, this gives $H^1(\mathcal{E}(-\frac{1}{2}c_2-1))\neq 0$.

The construction works for any $m \geqq 2$, so gives examples for all even $c_2 \geqq 2$. The spectrum contains $\frac{1}{2}c_2$, so by symmetry and connectedness (except at 0) it must be $\{-\frac{1}{2}c_2, \ldots, -1, 1, \ldots, \frac{1}{2}c_2\}$.

Example 8.1.3. For $c_1=0$ and each c_2 even $\geqq 2$, a stable vector bundle $\mathcal{E}$ with $H^1(\mathcal{E}(-\frac{1}{2}c_2))\neq 0$. This time the construction is a little different regarding the curve Y_2. Take $m \geqq 1$, take Y_1 as above a plane curve of degree $2m-1$. For Y_2 we need a curve of degree m^2-1 with $\omega_{Y_2} \cong \mathcal{O}_{Y_2}(2m-4)$. To prove the existence of such a curve, let $\mathcal{E}_0$ be the null correlation bundle, and let Y_2 be the zero-set of a general section of $\mathcal{E}_0(m)$. For $m=1$, Y_2 will be a disjoint union of 2 lines. For $m \geqq 2$ we can take Y_2 nonsingular and irreducible [9, 1.4], and it will have degree m^2+1 and $\omega = \mathcal{O}(2m-4)$ as required.

Now take $Y = Y_1 \coprod Y_2$. Then

$$d = m^2 + 2m$$

$$\omega_Y = \mathcal{O}_Y(2m-4)$$

so we construct $\mathcal{E}$ as before. It will have $c_1 = 0$, $c_2 = 2m$, and will be stable. In fact, since the null correlation bundle $\mathcal{E}_0$ is stable, Y_2 is not contained in any surface of degree m, so a fortiori Y is not contained in any surface of degree m. In fact, if $m \geqq 2$, then $\deg Y_1 \geqq m+1$, so Y is not even contained in any surface of degree $m+1$. In that case $H^0(\mathcal{E}(1)) = 0$.

If $m = 1$, Y consists of 3 skew lines, $c_2 = 2$, and these are the bundles studied in [9, Sect. 9]. They have $h^1(\mathcal{E}(-1)) = 2$, and spectrum $\{0, 0\}$. For $m \geqq 2$, Y has two components so $h^1(\mathcal{E}(-m)) = 1$. Since $m = \frac{1}{2}c_2$, we get $H^1(\mathcal{E}(-\frac{1}{2}c_2)) \neq 0$. The spectrum is $\{-\frac{1}{2}(c_2-2), \ldots, -1, 0, 0, 1, \ldots, \frac{1}{2}(c_2-2)\}$. One can show that the dimension of the family of these bundles is $3m^2+4m+8$ [see Sect. 9 for a discussion of the analogous situation for (8.1.1)]. For $m \geqq 3$ this number is $> 8c_2 - 3$, which shows the existence of oversized families for all even $c_2 \geqq 6$.

Example 8.1.4. For $c_1 = -1$ and each c_2 even $\geqq 2$, a vector bundle $\mathcal{E}$ with $h^0(\mathcal{E}(-1)) = 0$ and $H^1(\mathcal{E}(-\frac{1}{2}c_2 - 1)) \neq 0$. The construction is similar. Take $m \geqq 3$, let Y_1 be a plane curve of degree $2m-2$, and let Y_2 be a complete intersection of surfaces of degrees $m-2$ and $m+1$, and let $Y = Y_1 \coprod Y_2$. Then

$$d = m^2 + m - 4$$

$$\omega_Y \cong \mathcal{O}_Y(2m-5).$$

So we can construct a bundle $\mathcal{E}$ by

$$0 \to \mathcal{O} \to \mathcal{E}(m) \to \mathcal{I}_Y(2m-1) \to 0$$

which will have $c_1 = -1$, $c_2 = 2m-4$. Since Y is not contained in any surface of degree $m-2$, we see that $H^0(\mathcal{E}(-1)) = 0$. Since Y has two components, $h^1(\mathcal{I}_Y) = 1$, so $h^1(\mathcal{E}(-m+1)) = 1$. But $m = \frac{1}{2}c_2 + 2$, so $H^1(\mathcal{E}(-\frac{1}{2}c_2 - 1)) \neq 0$. The spectrum is $\{-\frac{1}{2}c_2 - 1, \ldots, -2, 1, 2, \ldots, \frac{1}{2}c_2\}$.

Example 8.1.5. For $c_1 = -1$ and each c_2 even $\geqq 2$, a stable vector bundle $\mathcal{E}$ with $H^1(\mathcal{E}(-\frac{1}{2}c_2)) \neq 0$. Take $m \geqq 2$, let Y_1 be a plane curve of degree $2m-2$, let Y_2 be a complete intersection of surfaces of degrees m, $m-1$, and let $Y = Y_1 \coprod Y_2$. Then

$$d = m^2 + m - 2$$

$$\omega_Y \cong \mathcal{O}_Y(2m-5)$$

so we construct $\mathcal{E}$ by

$$0 \to \mathcal{O} \to \mathcal{E}(m) \to \mathcal{I}_Y(2m-1) \to 0.$$

It has $c_1 = -1$, $c_2 = 2m-2$, and is stable because Y is not contained in any surface of degree $m-1$. For $m \geqq 2$, Y has two components, so as before, $H^1(\mathcal{E}(-m+1)) \neq 0$. Since $m = \frac{1}{2}c_2 + 1$, $H^1(\mathcal{E}(-\frac{1}{2}c_2)) \neq 0$. The spectrum is $\{-\frac{1}{2}c_2, \ldots, -1, 0, 1, \ldots, \frac{1}{2}(c_2-2)\}$.

Remark 8.1.6. These examples show that the bounds in (8.1) are best possible. Note that the theorem applies to all reflexive sheaves, while the examples just given are all vector bundles. One might ask whether a reflexive sheaf for which the bounds of (8.1) are sharp is necessarily a vector bundle. The answer is yes and no. Take the case $c_1=0$, c_2 odd, and $\mathscr{F}$ stable as in (8.1.1) for example. If $\mathscr{F}$ is such a reflexive sheaf with $H^1(\mathscr{F}(-\frac{1}{2}c_2-\frac{1}{2}))\neq 0$, then the spectrum contains $\frac{1}{2}(c_2-1)$. The conditions $\sum k_i \leqq 0$ and $\{k_i\}$ connected (7.6)* force the spectrum to be $\{-\frac{1}{2}(c_2-1), \ldots, -1, 0, 1, \ldots, \frac{1}{2}(c_2-1)\}$. Then by (7.3) $c_3=0$ so $\mathscr{F}$ is a vector bundle.

A similar argument shows that a reflexive sheaf $\mathscr{F}$ which is semistable with $c_1=0$ and c_2 even, or stable with $c_1=-1$ and c_2 even and for which the bounds of (8.1) are sharp must be a vector bundle. These are illustrated in (8.1.2) and (8.1.5) above.

However, if $c_1=0$, c_2 even, $\mathscr{F}$ stable or if $c_1=0$, c_2 odd, $\mathscr{F}$ semistable, then there are non-locally-free reflexive sheaves for which the bounds of (8.1) are sharp. Here are some examples.

Example 8.1.7. For $c_1=0$ and each c_2 even $\geqq 2$, a stable reflexive sheaf $\mathscr{F}$ with $H^1(\mathscr{F}(-\frac{1}{2}c_2))\neq 0$, and $c_3=c_2>0$. Take $m \geqq 1$, let Y_1 be a plane curve of degree $2m$, and let Y_2 be a complete intersection of two surfaces of degree m. Then

$$d=m^2+2m$$

$$\omega_{Y_1}=\mathcal{O}_{Y_1}(2m-3)$$

$$\omega_{Y_2}=\mathcal{O}_{Y_2}(2m-4).$$

We construct $\mathscr{F}$ by an exact sequence

$$0 \to \mathcal{O} \to \mathscr{F}(m) \to \mathscr{I}_Y(2m) \to 0.$$

The extension is determined by an element $\xi \in H^0(\omega_Y(4-2m)) = H^0(\mathcal{O}_{Y_1}(1)) \oplus H^0(\mathcal{O}_{Y_2})$. So ξ vanishes on a set of $2m$ collinear points in the plane, and $c_3=2m$. We find $c_1=0$, $c_2=2m$, and $\mathscr{F}$ is stable because Y is contained in no surface of degree m. Since Y has two components, $H^1(\mathscr{I}_Y)=1$, so $H^1(\mathscr{F}(-m))=1$. This shows $H^1(\mathscr{F}(-\frac{1}{2}c_2))\neq 0$ as required. The spectrum is $\{-\frac{1}{2}c_2, \ldots, -1, 0, 1, \frac{1}{2}(c_2-2)\}$.

It is interesting to compare this example with (8.1.3). The construction is simpler, and the spectrum has no repeated terms. One could say this reflexive sheaf is the natural example of sharpness of (8.1) in this case, and that explains why the construction of (8.1.3) was more subtle than the other vector bundle examples.

Example 8.1.8. For $c_1=0$ and each c_2 odd $\geqq 3$, a semistable reflexive sheaf $\mathscr{F}$ with $H^1(\mathscr{F}(-\frac{1}{2}c_2-\frac{1}{2}))\neq 0$ and $c_3=c_2+1$. The construction is similar. Take $m \geqq 2$, let Y_1 be a plane curve of degree $2m$, and let Y_2 be a complete intersection of surfaces of degrees $m-1$, $m+1$. Construct $\mathscr{F}$ as in (8.1.7). Then $c_1=0$, $c_2=2m-1$, $c_3=2m$, $\mathscr{F}$ is semistable, and $H^1(\mathscr{F}(-m))\neq 0$, so $H^1(\mathscr{F}(-\frac{1}{2}c_2-\frac{1}{2}))\neq 0$. The spectrum is $\{-\frac{1}{2}(c_2+1), \ldots, -1, 1, \ldots, \frac{1}{2}(c_2-1)\}$. Compare with (8.1.1).

Theorem 8.2. *Let $\mathscr{F}$ be a* rank 2 *reflexive sheaf on* $\mathbb{P}^3$.

(a) *Assume $c_1=0$ and $\mathscr{F}$ semistable. Then $H^2(\mathscr{F}(l))=0$ for $l \geqq c_2-2$. Also $c_3 \leqq c_2^2+c_2$.*

(b)[5] *Assume* $c_1=0$ *and* $\mathscr{F}$ *stable. Then* $H^2(\mathscr{F}(l))=0$ *for* $l\geqq c_2-3$. *Also* $c_3\leqq c_2^2-c_2+2$.

(c) *Assume* $c_1=-1$ *and* $H^0(\mathscr{F}(-1))=0$. *Then* $H^2(\mathscr{F}(l))=0$ *for* $l\geqq c_2-1$. *Also* $c_3\leqq c_2^2+2c_2$.

(d) *Assume* $c_1=-1$ *and* $\mathscr{F}$ *stable. Then* $H^2(\mathscr{F}(l))=0$ *for* $l\geqq c_2-2$. *Also* $c_3\leqq c_2^2$.

Proof. Let $\{k_i\}$ be the spectrum of $\mathscr{F}$. According to (7.1b), $H^2(\mathscr{F}(l))=0$ for $l\geqq -k-2$, where $k=\min\{k_i\}$. Also, by (7.3), $c_3=-2\sum k_i$ if $c_1=0$, and $c_3=-2\sum k_i-c_2$ if $c_1=-1$. So our strategy to bound l and to bound c_3 is the same: we see what is the most negative the spectrum can be. For this we use the results of (7.5).

(a) If $c_1=0$ and $\mathscr{F}$ is semistable, the most negative spectrum is $-1, -2, \ldots, -c_2$. So $k=-c_2$, and $H^2(\mathscr{F}(l))=0$ for $l\geqq c_2-2$. From (7.3), $c_3=2\cdot\frac{1}{2}c_2(c_2+1)=c_2^2+c_2$.

(b) If $c_1=0$ and $\mathscr{F}$ is stable, the most negative spectrum is $-1, -1, -2, \ldots, -c_2+1$. This gives the result similarly.

(c) If $c_1=-1$ and $H^0(\mathscr{F}(-1))=0$, the most negative spectrum is $-2, \ldots, -c_2-1$, which gives the result.

(d) Similarly, if $c_1=-1$ and $\mathscr{F}$ is stable, the most negative spectrum is $-1, -2, \ldots, -c_2$ which gives the result in this case.

Example 8.2.1. For $c_1=0$ and each $c_2>0$, a semistable reflexive sheaf $\mathscr{F}$ with $H^2(\mathscr{F}(c_2-3))\neq 0$, and $c_3=c_2^2+c_2$. Take a plane curve Y of degree c_2, and construct $\mathscr{F}$ by an extension

$$0\to\mathscr{O}\to\mathscr{F}\to\mathscr{I}_Y\to 0.$$

Then $\mathscr{F}$ is semistable with $c_1=0$ and the given c_2. The extension is determined by $\xi\in H^0(\omega_Y(4))=H^0(\mathscr{O}_Y(c_2+1))$. Therefore $c_3=c_2^2+c_2$, and the set of non-locally-free points of $\mathscr{F}$ is a complete intersection of Y with a curve of degree c_2+1 in the plane. For $l\geqq -3$, $H^2(\mathscr{F}(l))\cong H^2(\mathscr{I}_Y(l)\cong H^1(\mathscr{O}_Y(l))$. Since $\omega_Y\cong\mathscr{O}_Y(c_2-3)$, and $H^1(Y,\omega_Y)\neq 0$, we see $H^2(\mathscr{F}(c_2-3))\neq 0$.

Example 8.2.2. For $c_1=0$ and each $c_2\geqq 2$, a stable reflexive sheaf $\mathscr{F}$ with $H^2(\mathscr{F}(c_2-4))\neq 0$ and $c_3=c_2^2-c_2+2$, cf. (4.2.2). Let Y be the union of a plane curve Y_1 of degree c_2 with a line L, not in the plane Y_1, but meeting Y_1 at one point. Construct $\mathscr{F}$ by an extension

$$0\to\mathscr{O}\to\mathscr{F}(1)\to\mathscr{I}_Y(2)\to 0.$$

Then $c_1=0$, c_2 is as given, and we compute c_3 by the formula $c_3=2p_a-2+2d$ of (4.1). The arithmetic genus p_a of Y is the same as for Y_1, so $c_3=c_2(c_2-3)+2(c_2+1)$ $=c_2^2-c_2+2$. $\mathscr{F}$ is stable because Y is not contained in a plane (here we need $c_2\geqq 2$). For $l\geqq -2$, $H^2(\mathscr{F}(l))\cong H^2(\mathscr{I}_Y(l+1))\cong H^1(\mathscr{O}_Y(l+1))$. This last group maps surjectively to $H^1(\mathscr{O}_{Y_1}(l+1))$, which is nonzero for $l+1=c_2-3$ as above. Therefore $H^2(\mathscr{F}(c_2-4))\neq 0$.

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5 See footnote to (7.5)

Example 8.2.3. For $c_1=-1$ and each $c_2>0$, a reflexive sheaf $\mathscr{F}$ with $H^0(\mathscr{F}(-1))=0$, $H^2(\mathscr{F}(c_2-2))\neq 0$, and $c_3=c_2^2+2c_2$. Let Y be a plane curve of degree c_2, and construct $\mathscr{F}$ as an extension

$$0\to\mathcal{O}\to\mathscr{F}\to\mathscr{I}_Y(-1)\to 0.$$

The verification of its properties is similar to the previous examples.

Example 8.2.4. For $c_1=-1$ and each $c_2>0$, a stable reflexive sheaf $\mathscr{F}$ with $H^2(\mathscr{F}(c_2-3))\neq 0$ and $c_3=c_2^2$, cf. (4.2.5). Again take Y a plane curve of degree c_2, and construct $\mathscr{F}$ as an extension

$$0\to\mathcal{O}\to\mathscr{F}(1)\to\mathscr{I}_Y(1)\to 0.$$

The properties are verified as before.

Remark 8.2.5. These examples show that the statements of (8.2) are best possible, both as regards the vanishing of $H^2(\mathscr{F}(l))$ and as regards c_3. It is clear that any sheaf for which c_3 is equal to the bound given must have the most negative spectrum, and so for that sheaf the bounds on $H^2(\mathscr{F}(l))$ vanishing are sharp. We see from the proof of (8.2) that the converse (sharp H^2 – vanishing⇒maximum c_3) is true also except in case (b), where the are two possible spectra which would give the same k, namely $-1, -1, -2, \ldots, -c_2+1$ and $0, -1, -2, \ldots, -c_2+1$. The next example shows the latter can actually occur.

Example 8.2.6. For $c_1=0$ and each $c_2\geqq 1$, a stable reflexive sheaf with $H^2(\mathscr{F}(c_2-4))\neq 0$ and $c_3=c_2^2-c_2$. This time take Y to be the disjoint union of a plane curve Y_1 of degree c_2 and a line L, and then construct $\mathscr{F}$ as in (8.2.2). The only difference is that p_a is smaller by 1, so c_3 is smaller by 2.

9. Classification of Some Extremal Sheaves

In this section we begin the work of classifying stable vector bundles and reflexive sheaves with given Chern classes and given spectrum, using the method of unstable planes. The idea is to make a reduction step similar to the one on $\mathbb{P}^2$ described in Sect. 5, reducing to the study of sheaves with smaller c_2. We illustrate this method in two cases of sheaves or bundles which are extremal for the vanishing theorems in Sect. 8. The method works well in these cases. Other cases are more complicated, and will be deferred to a subsequent paper.

First of all we must make precise the concept of an unstable plane. Let $\mathscr{E}$ be a stable or semistable rank 2 reflexive sheaf on $\mathbb{P}^3$ with $c_1=0$ or -1. An unstable plane for $\mathscr{E}$ should be a plane H for which the restriction $\mathscr{E}_H$ is unstable. If $\mathscr{E}$ is a vector bundle, then $\mathscr{E}_H$ is also, and $\mathscr{E}_H$ will be unstable if and only if $H^0(\mathscr{E}_H(-r))\neq 0$ for some $r>0$ if $c_1=0$ or $r\geqq 0$ if $c_1=-1$. However, if $\mathscr{E}$ is reflexive but not locally free, then $\mathscr{E}_H$ is only torsion-free. In this case the stability of $\mathscr{E}_H$ cannot be measured by sections of $\mathscr{E}_H$. Rather we must use the dual $\mathscr{E}_H^\vee$ or double dual $\mathscr{E}_H^{\vee\vee}$ which are vector bundles on H. It seems most convenient to use the dual $\mathscr{E}_H^\vee$, which has $c_1=0$ or 1, and gives a unified definition in both cases, as follows.

Definition. Let $\mathcal{E}$ be a rank 2 semistable reflexive sheaf on $\mathbb{P}^3$ with $c_1=0$ or -1. A plane H in $\mathbb{P}^3$ is an *unstable plane* for $\mathcal{E}$ if there is an integer $r>0$ for which $H^0(\mathcal{E}_H^\vee(-r))\neq 0$. The largest such integer r is the *order* of the unstable plane.

Remark 9.0.1. The notation $\mathcal{E}_H^\vee$ means restrict $\mathcal{E}$ to H first, then take the dual. By Serre duality on H, the condition $H^0(\mathcal{E}_H^\vee(-r))\neq 0$ is equivalent to $H^2(\mathcal{E}_H(r-3))\neq 0$, because $H^2(\mathcal{E}_H(r-3))$ is dual to $\operatorname{Ext}^0(\mathcal{E}_H(r-3),\mathcal{O}_H(-3))=H^0(\mathcal{E}_H^\vee(-r))$.

Proposition 9.1 (Reduction Step). *Let $\mathcal{E}$ be a* rank 2 *reflexive sheaf on $\mathbb{P}^3$ with $c_1=0$ or -1, c_2, and c_3 given, and let H be an unstable plane for $\mathcal{E}$ of order r. Then there is an exact sequence*

$$0\to\mathcal{E}'\to\mathcal{E}\to\mathcal{I}_{Z,H}(-r)\to 0,$$

where Z is a zero-dimensional subscheme of H. Let $s=\text{length}\ \mathcal{O}_Z$. Then $\mathcal{E}'$ has Chern classes

$$c_1'=c_1-1$$

$$c_2'=c_2-r-c_1$$

$$c_3'=c_3-c_2-c_1r-r^2+2s.$$

There is also a dual exact sequence

$$0\to\mathcal{E}^\vee\to\mathcal{E}'^\vee\to\mathcal{I}_{W,H}(r+1)\to 0,$$

where W is a zero-dimensional subscheme of H of length $c_2+c_1r+r^2-s$.

Furthermore, if $\mathcal{E}$ is a vector bundle, then $s=c_2+c_1r+r^2$, $c_3'=s$, and W is empty.

Proof. [Compare with (5.2) and note change in role of r and $\mathcal{E}'$ in case $c_1=-1$.] Since H is a unstable plane of order r, by definition $H^0(\mathcal{E}_H^\vee(-r))\neq 0$, and this is the least twist of $\mathcal{E}_H^\vee$ having a section. A section of this sheaf gives a map $\mathcal{O}_H\to\mathcal{E}_H^\vee(-r)$. Dualizing and twisting by $-r$ gives a map $\mathcal{E}_H^{\vee\vee}\to\mathcal{O}_H(-r)$ which is surjective except at a finite number of points. Composing with the natural inclusion $\mathcal{E}_H\to\mathcal{E}_H^{\vee\vee}$ gives a map to $\mathcal{O}_H(-r)$ whose image defines the subscheme Z of H:

$$\mathcal{E}_H\to\mathcal{I}_{Z,H}(-r)\to 0.$$

Then we compose with the restriction map $\mathcal{E}\to\mathcal{E}_H$ and let $\mathcal{E}'$ be the kernel, giving the exact sequence

$$0\to\mathcal{E}'\to\mathcal{E}\to\mathcal{I}_{Z,H}(-r)\to 0$$

of the proposition. To compute Chern classes, we use the exact sequence

$$0\to\mathcal{O}(-r-1)\to\mathcal{O}(-r)\to\mathcal{O}_H(-r)\to 0$$

to find

$$c_t(\mathcal{O}_H(-r))=1+t+(r+1)t^2+(r^2+2r+1)t^3.$$

Then from the exact sequence

$$0\to\mathcal{I}_{Z,H}(-r)\to\mathcal{O}_H(-r)\to\mathcal{O}_Z\to 0$$

and the fact that $c_t(\mathcal{O}_Z)=1+2st^3$ by (2.7), we find

$$c_t(\mathcal{I}_{Z,H}(-r))=1+t+(r+1)t^2+(r^2+2r+1-2s)t^3.$$

Finally, the exact sequence of the proposition allows us to compute the Chern classes c_1', c_2', c_3' of $\mathcal{E}'$ as stated.

For the dual exact sequence, we apply the functor $\mathcal{H}om(\cdot,\mathcal{O})$ to the original sequence. We get

$$\begin{aligned}&0\to\mathcal{E}^{\vee}\to\mathcal{E}'^{\vee}\to\mathcal{E}xt^1(\mathcal{I}_{Z,H}(-r),\mathcal{O})\\ &\to\mathcal{E}xt^1(\mathcal{E},\mathcal{O})\to\mathcal{E}xt^1(\mathcal{E}',\mathcal{O})\to\mathcal{E}xt^2(\mathcal{I}_{Z,H}(-r),\mathcal{O})\to 0.\end{aligned}$$

An easy calculation shows that $\mathcal{E}xt^1(\mathcal{I}_{Z,H}(-r),\mathcal{O})\cong\mathcal{O}_H(r+1)$. The map from $\mathcal{E}'^{\vee}$ to this sheaf is surjective except possibly at the points where $\mathcal{E}xt^1(\mathcal{E},\mathcal{O})\neq 0$, which are the non-locally-free points of $\mathcal{E}$. So the image defines a zero-dimensional subscheme W of H:

$$0\to\mathcal{E}^{\vee}\to\mathcal{E}'^{\vee}\to\mathcal{I}_{W,H}(r+1)\to 0.$$

The length of W can be computed now using the fact that $\mathcal{E}xt^1(\mathcal{E},\mathcal{O})$ has length c_3 by (2.6), similarly $\mathcal{E}xt^1(\mathcal{E}',\mathcal{O})$ has length c_3', and $\mathcal{E}xt^2(\mathcal{I}_{Z,H}(-r),\mathcal{O})\cong\omega_Z$, which has length s.

Finally, if $\mathcal{E}$ is locally free, then $c_3=0$, $\mathcal{E}_H$ is a vector bundle, Z is the zero set of a section of $\mathcal{E}_H^{\vee}(-r)$ so its length is $c_2(\mathcal{E}_H^{\vee}(-r))=c_2+c_1r+r^2$, and $\mathcal{E}xt^1(\mathcal{E},\mathcal{O})$ is 0, so W is empty.

Remark 9.1.1. If $c_1=-1$, then $c_1'=-2$, so $\mathcal{E}'$ is not normalized. The corresponding normalized sheaf is $\mathcal{E}'(1)$ which has Chern classes $c_1''=0$, $c_2''=c_2-r$, and $c_3''=c_3'$.

Remark 9.1.2. If $c_3\neq 0$, then s is not uniquely determined by c_1,c_2,c_3,r. It may depend on the position of H with respect to the non-locally-free points of $\mathcal{E}$. Thus c_3' also is not uniquely determined by c_1,c_2,c_3, and r.

Remark 9.1.3. If t is the least integer for which $H^0(\mathcal{E}(t))\neq 0$, and t' the corresponding integer for $\mathcal{E}'$, then as in (5.2) $t'\geqq t$ with equality if $r>t$.

Now we come to our first main result of this section, which is the classification of stable reflexive sheaves with c_1 odd and maximum possible c_3, namely $c_3=c_2^2$ (8.2).

Theorem 9.2. *For any $m\geqq 1$, the moduli of stable* rank 2 *reflexive sheaves on $\mathbb{P}^3$ with Chern classes $c_1=-1$, $c_2=m$, $c_3=m^2$ is irreducible, nonsingular, and rational of dimension* 3 *if $m=1$, and m^2+3m+1 if $m\geqq 2$.*

Remark 9.2.1. For $m=1,2,3$ this is the expected dimension (3.4.1) given by Riemann-Roch, namely $8c_2-5$. For $m\geqq 4$, the dimension is $>8c_2-5$, so we have an oversized moduli space.

The proof of the theorem will follow after several lemmas.

Lemma 9.3. *For each $m\geqq 1$, the construction of* (8.2.4) *gives a family of stable reflexive sheaves with Chern classes $(-1,m,m^2)$ which is irreducible, nonsingular, and rational of dimension* 3 *if $m=1$ and m^2+3m+1 if $m\geqq 2$.*

Proof. The sheaves of (8.2.4) are constructed by taking a plane curve Y of degree m and then forming $\mathcal{F}(1)$ as an extension

$$0\to\mathcal{O}\to\mathcal{F}(1)\to\mathcal{I}_Y(1)\to 0.$$

The extension is given by an element $\xi\in H^0(\omega_Y(3))$, which must generate that sheaf except at a finite number of points. In this case $\omega_Y\cong\mathcal{O}_Y(m-3)$, so $\xi\in H^0(\mathcal{O}_Y(m))$.

If $m=1$, Y is a line, so $h^0(\mathcal{I}_Y(1))=2$, and two global sections generate the sheaf $\mathcal{I}_Y(1)$. It follows that $h^0(\mathcal{F}(1))=3$, and $\mathcal{F}(1)$ is generated by global sections. So there is an exact sequence

$$0\to\mathcal{L}\to\mathcal{O}^3\to\mathcal{F}(1)\to 0,$$

where $\mathcal{L}$ is defined as the kernel of the natural map $\mathcal{O}^3\to\mathcal{F}(1)$. Now $\mathcal{L}$ has rank 1, and is invertible, since $\mathcal{F}$ has homological dimension 1. Its first Chern class is -1, so $\mathcal{L}\cong\mathcal{O}(-1)$. The dual exact sequence is

$$0\to\mathcal{F}\to\mathcal{O}^3\to\mathcal{I}_{\mathbb{P}}(1)\to 0,$$

where $\mathbb{P}$ is a point. The map $\mathcal{O}^3\to\mathcal{I}_{\mathbb{P}}(1)$ is essentially unique, so $\mathcal{F}$ is determined by the point $\mathbb{P}$. Thus the family in this case is parametrized by $\mathbb{P}^3$, the point being the unique non-locally-free point of $\mathcal{F}$.

Now let $m\geqq 2$. Then Y lies in a unique plane H, so $h^0(\mathcal{I}_Y(1))=1$, and $h^0(\mathcal{F}(1))=2$. Taking two sections of $\mathcal{F}(1)$ gives an exact sequence

$$0\to\mathcal{O}^2\to\mathcal{F}(1)\to\mathcal{I}_{Y,H}(1)\to 0.$$

Note that $\mathcal{I}_{Y,H}(1)\cong\mathcal{O}_H(1-m)$. The dual exact sequence is

$$0\to\mathcal{F}\to\mathcal{O}^2\to\mathcal{I}_{Z,H}(m)\to 0,$$

where Z is the zero-dimensional subscheme of H defined by the zeros of $\xi\in H^0(\mathcal{O}_Y(m))$.

Now we see easily that giving $\mathcal{F}$ is equivalent to giving H and a 2-dimensional subspace of $H^0(\mathcal{O}_H(m))$ corresponding to a pencil of curves of degree m without fixed components. To give H is to give a point $h\in\mathbb{P}^{3*}$. So the family of possible sheaves $\mathcal{F}$ obtained by this construction is an open subset of the Grassmann variety of rank 2 subsheaves of $q_*p^*\mathcal{O}(m)$, where $p:X\to\mathbb{P}^3$ and $q:X\to\mathbb{P}^{3*}$ are the maps associated with the incidence correspondence between $\mathbb{P}^3$ and $\mathbb{P}^{3*}$. So the family constructed is irreducible, nonsingular, and rational. Its dimension is 3 for the choice of H plus $2\left(\binom{m+2}{2}-2\right)$ for the choice of a 2-dimensional subspace of $H^0(\mathcal{O}_H(m))$, which gives m^2+3m+1.

Lemma 9.4. *Every stable reflexive sheaf with Chern classes* $(-1,1,1)$ *is among those described in* (9.3) *for* $m=1$.

Proof. Let $\mathcal{F}$ be such a sheaf. Since $\mathcal{F}$ is stable, $H^0(\mathcal{F}(l))=0$ for $l\leqq 0$. By Serre duality and the isomorphism $\mathcal{F}^\vee\cong\mathcal{F}(1)$, $H^3(\mathcal{F}(l))=0$ for $l\geqq -3$. Next consider the spectrum of $\mathcal{F}$. It consists of a single integer, since $c_2=1$, and by (7.3) it must be -1. So by (7.1)(b) we have $H^2(\mathcal{F}(l))=0$ for $l\geqq -1$. Now the Riemann-Roch theorem gives $\chi(\mathcal{F})=0$, so from the above we conclude $H^1(\mathcal{F})=0$.

Now we are in a position to apply Castelnuovo's theorem [16, p. 99]. Since $H^1(\mathscr{F})=0$, $H^2(\mathscr{F}(-1))=0$, and $H^3(\mathscr{F}(-2))=0$, we conclude that $\mathscr{F}(1)$ is generated by global sections, and $H^i(\mathscr{F}(l-i))=0$ for $i>0$ and $l\geqq 1$. Then Riemann-Roch for $\mathscr{F}(1)$ gives $h^0(\mathscr{F}(1))=3$. So $\mathscr{F}(1)$ is generated by 3 global sections, and the argument used in the proof of (9.3) applies to show it is one of those.

Lemma 9.5. *Let $m\geqq 2$ and let $\mathscr{F}$ be a stable reflexive sheaf with Chern classes $(-1,m,m^2)$. Then $\mathscr{F}$ has an unstable plane of order m.*

Proof. (This result is true also for $m=1$, as one can see from the above description of $\mathscr{F}$ in that case, but the proof we will give now does not work in that case.) Consider the spectrum of $\mathscr{F}$. It consists of $m=c_2$ integers, and because $c_3=m^2$, the formula of (7.3) for c_3 and the connectedness of the spectrum (7.5) force the spectrum to be $-1,-2,\ldots,-m$ [see also (8.2.5)]. Now from (7.1) and the hypothesis $m\geqq 2$ we find that $h^2(\mathscr{F}(m-3))=1$ and $h^2(\mathscr{F}(m-4))=3$. Therefore there exists a linear form $x\in H^0(\mathcal{O}(1))$ such that the map multiplication by x from $H^2(\mathscr{F}(m-4))$ to $H^2(\mathscr{F}(m-3))$ is zero. Let H be the plane $x=0$. Then the exact sequence of cohomology shows that $H^2(\mathscr{F}_H(m-3))\neq 0$. So H is an unstable plane of order m (9.0.1).

Lemma 9.6. *Any stable reflexive sheaf $\mathscr{F}$ with Chern classes $(-1,m,m^2)$ and $m\geqq 2$ is among those described in* (9.3).

Proof. By (9.5) there is an unstable plane H of order m. We perform a reduction step for H (9.1) which gives an exact sequence

$$0\to\mathscr{E}'\to\mathscr{F}\to\mathscr{I}_{Z,H}(-m)\to 0.$$

The new sheaf $\mathscr{E}'$ has $c_1=-2$ and $h^0(\mathscr{E}')=0$, so it is semistable. The normalized sheaf $\mathscr{E}'(1)$ has $c_1''=0$ and $c_2''=0$ since $m=c_2(\mathscr{E})$. Therefore, by (9.7) below, $\mathscr{E}'(1)\cong\mathcal{O}\oplus\mathcal{O}$. The dual exact sequence, untwisted once, is therefore

$$0\to\mathscr{F}\to\mathcal{O}^2\to\mathscr{I}_{W,H}(m)\to 0.$$

Thus $\mathscr{F}$ is of the form described in the proof of (9.3).

Lemma 9.7. *A semistable* rank 2 *reflexive sheaf with $c_1=c_2=0$ must be $\mathcal{O}\oplus\mathcal{O}$. In particular, $c_3=0$.*

Proof. Let $\mathscr{F}$ be such a sheaf. According to (8.2), $c_3\leqq c_2^2+c_2=0$. Therefore $c_3=0$, $\mathscr{F}$ is a vector bundle, and in that case it is easy to see $\mathscr{F}=\mathcal{O}\oplus\mathcal{O}$. For example, by [9, 8.2], $\mathscr{F}$ has a global section, so $\mathscr{F}$ is an extension

$$0\to\mathcal{O}\to\mathscr{F}\to\mathscr{I}_Y\to 0$$

where Y has degree $c_2=0$, i.e. Y is empty. Then the extension splits, so $\mathscr{F}=\mathcal{O}\oplus\mathcal{O}$.

Lemma 9.8. *Let $\mathscr{F}$ be one of the sheaves constructed in* (9.3), *for any $m\geqq 1$. Then* $\dim\operatorname{Ext}^1(\mathscr{F},\mathscr{F})$ *is* 3 *if $m=1$ and m^2+3m+1 if $m\geqq 2$.*

Proof. We use the exact sequence (untwisted once) from (9.3)

$$0\to\mathcal{O}(-1)^2\to\mathscr{F}\to\mathcal{O}_H(-m)\to 0.$$

Applying the functor $\mathrm{Hom}(\cdot,\mathscr{F})$ we get an exact sequence

$$0\to\mathrm{Hom}(\mathscr{F},\mathscr{F})\to\mathrm{Hom}(\mathcal{O}(-1)^2,\mathscr{F})$$
$$\to\mathrm{Ext}^1(\mathcal{O}_H(-m),\mathscr{F})\to\mathrm{Ext}^1(\mathscr{F},\mathscr{F})\to\mathrm{Ext}^1(\mathcal{O}(-1)^2,\mathscr{F}).$$

Now $\mathrm{Hom}(\mathscr{F},\mathscr{F})$ has dimension 1 (3.4.1), and $\mathrm{Hom}(\mathcal{O}(-1)^2,\mathscr{F})=2H^0(\mathscr{F}(1))$ has dimension 6 if $m=1$ and 4 if $m\geqq 2$. At the end of the sequence, $\mathrm{Ext}^1(\mathcal{O}(-1)^2,\mathscr{F})=2H^1(\mathscr{F}(1))$, which is 0, as we see directly from the exact sequence for $\mathscr{F}$ above.

It remains to calculate $\mathrm{Ext}^1(\mathcal{O}_H(-m),\mathscr{F})$. For this we use the exact sequence

$$0\to\mathcal{O}(-m-1)\to\mathcal{O}(-m)\to\mathcal{O}_H(-m)\to 0,$$

which gives

$$0\to\mathrm{Hom}(\mathcal{O}(-m),\mathscr{F})\to\mathrm{Hom}(\mathcal{O}(-m-1),\mathscr{F})\to\mathrm{Ext}^1(\mathcal{O}_H(-m),\mathscr{F})$$
$$\to\mathrm{Ext}^1(\mathcal{O}(-m),\mathscr{F}).$$

This last term is $H^1(\mathscr{F}(m))=0$ as above, and the first two are $H^0(\mathscr{F}(m))$ and $H^0(\mathscr{F}(m+1))$, which are easily calculated from the above sequence for $\mathscr{F}$.

Putting all these together gives $\dim\mathrm{Ext}^1(\mathscr{F},\mathscr{F})=3$ if $m=1$ and m^2+3m+1 if $m\geqq 2$ as required.

Proof of Theorem 9.2. We have constructed families in (9.3) of the required dimension which are irreducible, nonsingular, and rational. We have seen (9.4) and (9.6) that any stable reflexive sheaf with the given Chern classes is among those constructed. Finally (9.8) we have seen that for any of those sheaves, the dimension of $\mathrm{Ext}^1(\mathscr{F},\mathscr{F})$ is equal to the dimension of the total family. This implies by deformation theory that the moduli space is everywhere reduced and nonsingular [13, 6.7]. So the families constructed give the whole moduli space, which completes the proof.

Theorem 9.9. *For any $m\geqq 1$, the stable* rank 2 *vector bundles on $\mathbb{P}^3$ with $c_1=0$, $c_2=2m-1$, and the maximum spectrum $-m+1,\ldots,0,\ldots,m-1$, form an irreducible, nonsingular, rational family of dimension* 5 *if $m=1$ and $3m^2+4m+1$ if $m\geqq 2$.*

Remark 9.9.1. In the case $m=1$ these are the null correlation bundles [9, 8.4.1] for which the result is known. In case $m=2$, having the spectrum $-1, 0, 1$ is equivalent to having α-invariant 1. So in this case the stable bundles with $c_1=0$, $c_2=3$, and $\alpha=1$ form an irreducible, nonsingular, rational family of dimension 21. This was first proved by Ellingsrud and Strømme [6]. In case $m\geqq 3$, the dimension of the family is $>8c_2-3$, so it is an oversized family, first discovered by Barth and Hulek [2]. Until now, we have been unable to compute $h^1(\mathscr{E}nd\,\mathscr{E})$ for these bundles, so we do not know if they form a connected component of the moduli space or if the moduli space is reduced there. However, Strømme [6, 4.7] has been able to show by another method that this family is (set-theoretically) an irreducible component of the moduli space.

The proof of the theorem will follow after several lemmas.

Lemma 9.10. *For each $m\geqq 2$ the construction of* (8.1.1) *gives an irreducible family of stable bundles with $c_1=0$, $c_2=2m-1$, and spectrum $-m+1,\ldots,0,\ldots,m-1$, which has dimension $3m^2+4m+1$.*

Proof. The bundles in example (8.1.1) are constructed by extensions

$$0 \to \mathcal{O} \to \mathcal{E}(m) \to \mathcal{I}_Y(2m) \to 0,$$

where Y is a disjoint union of a plane curve Y_1 of degree $2m-1$ and a complete intersection Y_2 of two surfaces of degree m. The numerical invariants were calculated in (8.1.1).

It remains to show the family is irreducible and to find its dimension. The bundle $\mathcal{E}$ is determined by the choices of Y_1, Y_2, and $\xi \in H^0(\mathcal{O}_Y)$. These are all irreducible choices, so the family is irreducible.

The choice of the plane H containing Y_1 is 3 parameters. The choice of Y_1 in H is $\frac{1}{2}(2m+1)(2m)-1$. The choice of Y_2 is equivalent to the choice of a 2-dimensional subvector space of $H^0(\mathcal{O}(m))$. That is $2\left(\binom{m+3}{3}-2\right)$ parameters. The choice of ξ is 2. Then we must subtract $h^0(\mathcal{E}(m)) = 1 + h^0(\mathcal{I}_Y(2m))$. Any surface of degree $2m$ containing Y must have H as a component. Therefore $h^0(\mathcal{I}_Y(2m)) = h^0(\mathcal{I}_{Y_2}(2m-1))$. This can be calculated from the resolution

$$0 \to \mathcal{O}(-2m) \to \mathcal{O}(-m)^2 \to \mathcal{I}_{Y_2} \to 0,$$

giving $h^0(\mathcal{I}_{Y_2}(2m-1)) = 2\binom{m+2}{3}$. Combining all these shows that the family of bundles $\mathcal{E}$ constructed depends on $3m^2+4m+1$ parameters.

Lemma 9.11. *Let $m \geq 2$ and let $\mathcal{E}$ be any stable bundle with Chern classes and spectrum as in (9.9). Then $\mathcal{E}$ has a unique unstable plane H of order $m-2$. The reduction step with respect to H gives exact sequences*

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{I}_{Z,H}(-m+1) \to 0, \tag{1}$$

$$0 \to \mathcal{E} \to \mathcal{E}'(1) \to \mathcal{O}_H(m) \to 0, \tag{2}$$

where $\mathcal{E}'$ is a stable reflexive sheaf with Chern classes $(-1, m, m^2)$. The plane H is also the unique unstable plane of order m for $\mathcal{E}'$.

Proof. Knowing the spectrum of $\mathcal{E}$ and using (7.1) we find $h^1(\mathcal{E}(-m)) = 1$ and $h^1(\mathcal{E}(-m+1)) = 3$. Therefore there is a linear form $h \in H^0(\mathcal{O}(1))$ such that the induced map $h: H^1(\mathcal{E}(-m)) \to H^1(\mathcal{E}(-m+1))$ is zero. Let H be the plane $h=0$. Then from the exact sequence $0 \to \mathcal{E}(-1) \xrightarrow{h} \mathcal{E} \to \mathcal{E}_H \to 0$ it follows that $H^0(\mathcal{E}_H(-m+1)) = 1$, so H is an unstable plane of order $m-1$. The reduction step (9.1) then gives the exact sequences (1) and (2) and shows that $\mathcal{E}'$ is stable with Chern classes $(-1, m, m^2)$.

Such reflexive sheaves $\mathcal{E}'$ were studied in (9.2)–(9.8). In particular since the plane H contains the non-locally-free points of $\mathcal{E}'$, it follows that the same H is the unstable plane of order m for $\mathcal{E}'$ found in (9.5). By the proof of (9.3) the reduction step for $\mathcal{E}'$ and H gives exact sequences

$$0 \to \mathcal{O}^2 \to \mathcal{E}'(1) \to \mathcal{O}_H(-m+1) \to 0, \tag{3}$$

$$0 \to \mathcal{E}' \to \mathcal{O}^2 \to \mathcal{I}_{Z,H}(m) \to 0 \tag{4}$$

with the same Z as above in (1).

Now if H' is any plane different from H, it follows from (1) and (3) that for any $r \geqq 0$

$$H^0(\mathscr{E}_{H'}(-r) = H^0(\mathscr{E}'_{H'}(-r)) = H^0(\mathcal{O}^2_{H'}(-r-1)) = 0.$$

Therefore H is the unique unstable plane for $\mathscr{E}$. Similarly, restricting (3) to H' gives $0 \to \mathcal{O}^2_{H'} \to \mathscr{E}'_{H'}(1)$, so dualizing, $0 \to \mathscr{E}'^{\vee}_{H'}(-1) \to \mathcal{O}^2_{H'}$, so H is the unique unstable plane for $\mathscr{E}'$ of order $\geqq 2$.

Lemma 9.12. *The restrictions of $\mathscr{E}$ and $\mathscr{E}'$ to H are described as follows: there is an exact sequence*

$$0 \to \mathcal{O}_H(m-1) \to \mathscr{E}_H \to \mathscr{I}_{Z,H}(-m+1) \to 0 \tag{5}$$

and an isomorphism

$$\mathscr{E}'_H \cong \mathcal{O}_H(-m) \oplus \mathscr{I}_{Z,H}(m-1). \tag{6}$$

Proof. The exact sequence (5) comes from the definition of Z in the reduction step. The kernel of the map $\mathscr{E}_H \to \mathscr{I}_{Z,H}(-m+1)$ is determined by its Chern class.

To determine $\mathscr{E}'_H$, we tensor the sequence (4) with $\mathcal{O}_H$, giving

$$0 \to \mathscr{T}or_1(\mathscr{I}_{Z,H}(m), \mathcal{O}_H) \to \mathscr{E}'_H \to \mathcal{O}^2_H \to \mathscr{I}_{Z,H}(m) \to 0.$$

To compute $\mathscr{T}or_1$, use the resolution

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_H \to 0$$

of $\mathcal{O}_H$. This gives

$$\mathscr{T}or_1(\mathscr{I}_{Z,H}(m), \mathcal{O}_H) \cong \mathscr{I}_{Z,H}(m-1).$$

The map $\mathcal{O}^2_H \to \mathscr{I}_{Z,H}(m) \to 0$ is given by two forms of degree m on H having no common factor. So the kernel of this map is $\mathcal{O}_H(-m)$. Thus there is an exact sequence

$$0 \to \mathscr{I}_{Z,H}(m-1) \to \mathscr{E}'_H \to \mathcal{O}_H(-m) \to 0. \tag{7}$$

This sequence gives an element

$$\xi \in \operatorname{Ext}^1_H(\mathcal{O}_H(-m), \mathscr{I}_{Z,H}(m-1)) = H^1(\mathscr{I}_{Z,H}(2m-1)).$$

But it is easy to see from the exact sequence

$$0 \to \mathcal{O}_H(-2m) \to \mathcal{O}_H(-m)^2 \to \mathscr{I}_{Z,H} \to 0$$

that this H^1 is 0. Hence the sequence (7) splits, giving the required isomorphism (6).

Lemma 9.13. *For each $m \geqq 2$, to give a bundle $\mathscr{E}$ with Chern classes and spectrum as in (9.9) it is equivalent to give the following data.*

(a) *A plane H in $\mathbb{P}^3$.*

(b) *A 2-dimensional subspace V of $H^0(\mathcal{O}_H(m))$, corresponding to a linear system without fixed components. The base points of this linear system determine Z.*

(c) *A form $f \in H^0(\mathcal{O}_H(2m-1))$, not vanishing at any point of Z.*

Proof. Given $\mathscr{E}$, let H be the unique unstable plane for $\mathscr{E}$ and make reduction steps using notation as in (9.11). Take V to be the image of $H^0(\mathcal{O}^2)$ by the second map of (4). Now consider the second map of (2). It factors through $\mathscr{E}'_H(1)$, so gives a map, untwisted once,

$$\mathscr{E}'_H \to \mathcal{O}_H(m-1) \to 0.$$

Using the isomorphism (6) of (9.12) this is a map

$$\mathcal{O}_H(-m) \oplus \mathscr{I}_{Z,H}(m-1) \to \mathcal{O}_H(m-1) \to 0.$$

This is given by a form f of degree $2m-1$ on the first factor, and a scalar on the second factor. For the map to be surjective, f must not vanish at any point of Z.

Conversely, suppose given the data (a)–(c). Then (a) and (b) determine a stable reflexive sheaf $\mathscr{E}'$, as in the proof of (9.3). The form f allows us to define the surjective map $\mathscr{E}'_H \to \mathcal{O}_H(m-1) \to 0$ as above. Composing with the restriction map $\mathscr{E}' \to \mathscr{E}'_H$ and twisting gives a surjective map $\mathscr{E}'(1) \to \mathcal{O}_H(m) \to 0$, and we take $\mathscr{E}$ to be the kernel, as in (2). It is easy to verify that $\mathscr{E}$ is a stable vector bundle with the required Chern classes and spectrum.

Remark 9.13.1. Strictly speaking, the form f in (c) is not determined until we choose a basis for V and a splitting of the sequence (7). For simplicity we will slur over this subtlety.

Corollary 9.14. *For each $m \geqq 2$, the stable bundles with Chern classes and spectra as in* (9.9) *form an irreducible, non-singular, rational family of dimension* $3m^2+4m+1$.

Proof. The choice of H is a point in $\mathbb{P}^{3*}$, a rational variety. The choices of V and f, which are essentially independent of each other, are points in a certain Grassmann variety and a geometric vector bundle over $\mathbb{P}^{3*}$. So the family is irreducible, nonsingular, and rational.

The choice of H is 3 parameters. The choice of V is $2(\frac{1}{2}(m+2)(m+1)-2)$ parameters. The choice of f is $\frac{1}{2}(2m+1)(2m)$ parameters. Adding gives $3m^2+4m+1$ for the dimension of the family.

Lemma 9.15. *For each $m \geqq 2$, if $\mathscr{E}$ is a bundle as in* (9.9) *then $h^0(\mathscr{E}(1))=2$, and for any nonzero section $s \in H^0(\mathscr{E}(1))$, the zero set of s is a curve Y of degree $2m$ with $\omega_Y \cong \mathcal{O}_Y(-2)$, which is a multiplicity 2 structure on a plane curve Y_0 of degree m. Conversely any such curve Y gives bundles of the required type.*

Proof. Referring to the exact sequences of (9.11) it follows from (1) and (3) that $H^0(\mathscr{E}(1)) \cong H^0(\mathscr{E}'(1)) \cong H^0(\mathcal{O}^2)$, so $h^0(\mathscr{E}(1))=2$. Since $\mathscr{E}$ is stable, $\mathscr{E}(1)$ is the first twist of $\mathscr{E}$ having sections, so any nonzero section $s \in H^0(\mathscr{E}(1))$ will vanish in codimension 2. Its zero set Y will be a locally complete intersection curve with degree $2m$ and $\omega_Y \cong \mathcal{O}_Y(-2)$, and fits in an exact sequence

$$0 \to \mathcal{O} \xrightarrow{s} \mathscr{E}(1) \to \mathscr{I}_Y(2) \to 0.$$

The sheaves $\mathscr{E}(1)$, $\mathscr{E}'(1)$, and $\mathcal{O}^2$ are isomorphic outside of H by the sequences (1) and (3) so it is clear that Y has support in H. To study Y more closely, consider the restriction of s to H. This gives a section of $\mathscr{E}_H(1)$. From (5) we see that

$H^0(\mathscr{E}_H(1)) \cong H^0(\mathcal{O}_H(m))$. In other words, every section of $\mathscr{E}_H(1)$ is obtained by multiplying the unique (up to scalar) section of $\mathscr{E}_H(-m+1)$ by a form of degree m. Therefore s_H vanishes along a curve Y_0 of degree m in H, and at the points of Z. So Y has support equal to the curve Y_0, and $Z \subsetneq Y_0$. No irreducible component of Y can be equal to the corresponding component of Y_0, because of $\omega_Y \cong \mathcal{O}_Y(-2)$, so from the fact that $\deg Y = 2 \deg Y_0$ we conclude that Y is a multiplicity 2 structure on Y_0.

Conversely it is easy to see that such a curve Y gives a bundle of the required type.

Remark 9.15.1. Since the multiplicity 2 structures on curves have been classified by Ferrand [9, 1.5], this gives another way of constructing bundles of the type considered in (9.9).

Lemma 9.16. *Let $m \geqq 2$ and let $\mathscr{E}$ be a bundle as in* (9.9). *Then $\mathscr{E}(m)$ has sections vanishing in codimension* 2 *only, and if s is a sufficiently general section, its zero set Y is a disjoint union of a plane curve Y_1 of degree $2m-1$ and a curve Y_2 which is a complete intersection of two surfaces of degree m. In particular, every bundle of the type considered in* (9.9) *is among those constructed in* (8.1.1).

Proof. We have seen (9.15) that $h^0(\mathscr{E}(1)) = 2$. Similarly, from (1) and (3) it follows for every $l < m$ that $H^0(\mathscr{E}(l)) \cong H^0(\mathscr{E}'(l)) \cong H^0(\mathcal{O}^2(l-1))$, and that $H^0(\mathscr{E}(m)) \cong H^0(\mathscr{E}'(m))$ is strictly greater than $H^0(\mathcal{O}^2(m-1))$. Therefore $\mathscr{E}(m)$ has sections which are not in the image of $H^0(\mathscr{E}(m-1)) \otimes H^0(\mathcal{O}(1)) \to H^0(\mathscr{E}(m))$, and such sections vanish only in codimension 2.

Let s be such a section, and let Y be its zero set. Then we know from general principles that Y has degree $m^2 + 2m - 1$ and that $\omega_Y \cong \mathcal{O}_Y(2m-4)$. Consider the restriction of s to H. It is a section of $\mathscr{E}_H(m)$, and from (5) we see that s_H is obtained by multiplying the unique (up to scalar) section of $\mathscr{E}_H(-m+1)$ by a form of degree $2m-1$. Therefore s_H vanishes along a curve Y_1 of degree $2m-1$ and at the m^2 points Z where the section of $\mathscr{E}_H(-m+1)$ vanishes. I claim the curve Y_1 does not meet Z. Indeed, by choice of s, it corresponds to a section of $\mathscr{E}'(m)$ whose image in $\mathcal{O}_H$ by (3) is nonzero. Now the inclusion $\mathscr{E}'(m) \to \mathscr{E}(m)$ from (1) induces a map $\mathscr{E}'_H(m) \to \mathscr{E}_H(m)$ by restriction to H. In fact, using (5) and (6) this map can be factored

$$\mathscr{E}'_H(m) \cong \mathcal{O}_H \oplus \mathscr{I}_{Z,H}(2m-1) \to \mathcal{O}_H(2m-1) \to \mathscr{E}_H(m),$$

where the first map is given by the form f of (9.13c) plus inclusion, and the second map comes from (5). Since f does not vanish at any of the points of Z, and since s_H is in the image of this map and comes from a section of $\mathscr{E}'(m)$ whose image in $\mathcal{O}_H$ is nonzero, we see that $Y_1 \cap Z = \emptyset$.

Now consider s outside of H. Using (1) and (4), s gives a section of $\mathcal{O}^2(m)$ which vanishes along a complete intersection curve Y_2 of two surfaces of degree m. So outside of H, $Y = Y_2$. We conclude, considering the degrees of Y_1, Y_2, and Y, that $Y = Y_1 \cup Y_2$, and that $Y_2 \cap H = Z$, so Y_1 and Y_2 are disjoint.

Proof of 9.9. If $m = 1$, the result is known [9, 8.4.1], so we may assume $m \geqq 2$. Then (9.14) shows that these bundles form a family of the type required, and (9.16) shows these are the same bundles constructed in (8.1.1).

We end this section with some further properties of these bundles.

Proposition 9.17. *Let $m \geqq 1$ and let $\mathscr{E}$ be one of the bundles described in* (9.9). *Then*
(a) $H^1(\mathscr{E}(l))=0$ *for* $l \geqq \frac{3}{2}(c_2-1)$,
(b) $\mathscr{E}(l)$ *is generated by global sections for* $l \geqq c_2$, *and both bounds are sharp.*

Proof. If $m=1$ the result is easy, so we assume $m \geqq 2$. To prove (a) we use the sequence (2) of (9.11). It follows from (3) that $H^1(\mathscr{E}'(l))=0$ for all l, so from (2) there is an exact sequence

$$H^0(\mathscr{E}'(l+1)) \to H^0(\mathcal{O}_H(l+m)) \to H^1(\mathscr{E}(l)) \to 0 .$$

The first map factors through a surjective map to $H^0(\mathscr{E}'_H(l+1))$, which can be interpreted using (6) of (9.12), so we get

$$H^0(\mathcal{O}_H(l-m+1)) \oplus H^0(\mathscr{I}_{Z,H}(l+m)) \to H^0(\mathcal{O}_H(l+m)) \to H^1(\mathscr{E}(l))$$

Now let f_1 and f_2 be forms of degree m generating $\mathscr{I}_{Z,H}$, and let f_3 be the form of degree $2m-1$ defining the above map [called f in (9.13c)]. Then we see that the module $M=\bigoplus_{l \in \mathbb{Z}} H^1(\mathscr{E}(l))$ is just the coordinate ring $k[x_0, x_1, x_2]$ of H, divided by the ideal generated by f_1, f_2, f_3 and shifted in degree by $-m$. These forms f_i have no common zero in H, so they define a complete intersection ideal in the polynomial ring. The largest degree of a nonzero graded component of $k[x_0, x_1, x_2]/(f_1, f_2, f_3)$ is $\sum(\deg f_i - 1)$ which is $4m-4$. Therefore, shifting by $-m$, we find $h^1(\mathscr{E}(l))=0$ for $l \geqq 3m-3$ precisely. Since $c_1=2m-1$, this says $l \geqq \frac{3}{2}(c_2-1)$.

To prove (b) we use the sequences (1) and (3). From (3) it follows that $\mathscr{E}'(l)$ is generated by global sections for $l \geqq m$. Then from (1) it follows [using the fact that $H^1(\mathscr{E}'(l))=0$ for all l] that $\mathscr{E}(l)$ is generated by global sections as soon as $l \geqq m$ and $\mathscr{I}_{Z,H}(l-m+1)$ is generated by global sections. Since Z is the complete intersection of two curves of degree m, we need $l-m+1 \geqq m$ for this to hold. In other words $l \geqq 2m-1=c_2$.

Remark 9.17.1. Since these bundles have the maximum spectrum, we might expect that they exhibit the worst possible behavior with respect to vanishing of H^1 and generation by global sections. So we pose the *question*: do the statements (a) and (b) hold for all stable rank 2 bundles on $\mathbb{P}^3$ with $c_1=0$ and given c_2?

10. Nonvanishing of $H^0(\mathscr{E}(t))$ on $\mathbb{P}^3$

In this section we return to the question of finding the least integer t such that $H^0(\mathscr{E}(t)) \neq 0$ for a stable rank 2 vector bundle on $\mathbb{P}^3$. This question was discussed in [9, Sect. 8] and also in [10] and [11], where it is seen to be related to finding the maximum genus of a space curve not contained in a surface of a given degree. At that time we stated a conjecture [9, 8.2.2] which we now restate more generally for reflexive sheaves.

Conjecture 10.1. *Let $\mathscr{F}$ be a* rank 2 *reflexive sheaf on* $\mathbb{P}^3$ *with* $c_1=0$ *and* $c_2 \geqq 0$, *and let* $t > \sqrt{3c_2+1}-2$. *Then* $H^0(\mathscr{F}(t)) \neq 0$.

While we do not yet have a proof of this conjecture, we will show that it holds for all values of $c_2 \leqq 25$. The method is the same as the one used in [9, Sect. 8], with two improvements. The first is that by taking into account the spectrum, we can get a better estimate on $h^2(\mathscr{F}(t))$. The second is that this method works for reflexive sheaves as well as for vector bundles.

For simplicity we will discuss only the case $c_1=0$.

Theorem 10.2. *Let $\mathscr{F}$ be a* rank 2 *reflexive sheaf on $\mathbb{P}^3$ with $c_1=0$ and $c_2\geqq 0$. Assume either*

(a) *c_2 is odd, $0\leqq t\leqq \frac{1}{2}(c_2-3)$, and $8t^3+36t^2+40t+3>3c_2^2+12c_2t+24c_2$, or*

(b) *c_2 is even, $0\leqq t\leqq \frac{1}{2}(c_2-2)$, and $8t^3+36t^2+52t+24>3c_2^2+12c_2t+30c_2$.*

Then $H^0(\mathscr{F}(t))\neq 0$.

Proof. Let us first treat the case c_2 odd and $\mathscr{F}$ a vector bundle. If $\mathscr{F}$ is not stable, then $H^0(\mathscr{F})\neq 0$ so $H^0(\mathscr{F}(t))\neq 0$ for any $t\geqq 0$. Thus we may assume $\mathscr{F}$ is stable. We will use the Riemann-Roch theorem [9, 8.1] which says

$$\chi(\mathscr{F}(t))=\tfrac{1}{3}(t+1)(t+2)(t+3)-c_2(t+2).$$

Since $\mathscr{F}$ is stable, $h^3(\mathscr{F}(t))=0$ for $t\geqq 0$. Suppose now that $h^0(\mathscr{F}(t))=0$ for some $t\geqq 0$. Then

$$\chi(\mathscr{F}(t))=-h^1(\mathscr{F}(t))+h^2(\mathscr{F}(t))\leqq h^2(\mathscr{F}(t)).$$

We will bound $h^2(\mathscr{F}(t))$ using the spectrum (7.1). Let $c_2=2k+1$. Then the maximum possible spectrum, by (7.2) and (7.5) is $-k,\ldots,0,\ldots,k$. Therefore by (7.1) $h^2(\mathscr{F}(t))=0$ for $t\geqq k-2$, and for $t\leqq k-1$,

$$h^2(\mathscr{F}(t))\leqq \tfrac{1}{2}(k-t-2)(k-t-1).$$

Now substituting $k=\frac{1}{2}(c_2-1)$ and using the inequality $\chi(\mathscr{F}(t))\leqq h^2(\mathscr{F}(t))$, we find

$$8t^3+36t^2+40t+3\leqq 3c_2^2+12c_2t+24c_2.$$

Thus if $t\leqq k-1=\frac{1}{2}(c_2-3)$, and the opposite inequality is satisfied, we conclude that $h^0(\mathscr{F}(t))\neq 0$. By the way, if $t>k-1$, then $h^2(\mathscr{F}(t))=0$, and that gives the conjectured inequality (10.1) in that case.

The argument for a vector bundle with c_2 even is entirely analogous, using the fact that if $\mathscr{F}$ is semistable with $c_2=2k$, the maximum spectrum is $-k,\ldots,-1,1,\ldots,k$. We omit the details.

Now suppose that $\mathscr{F}$ is a reflexive sheaf with $c_2>0$. There are two changes. The Riemann-Roch theorem (2.3) is given by the above formula $+\frac{1}{2}c_3$. On the other hand, the spectrum can be more negative, so the bound on $h^2(\mathscr{F}(t))$ is weaker. However, these two effects cancel each other, because of (7.3), so that the same conclusion holds. To be precise, the bound used in the vector bundle case can be written

$$h^2(\mathscr{F}(t))\leqq 1+2+\ldots+(k-t-1).$$

In our case, suppose the maximum spectrum is $-k',\ldots,0,\ldots,k''$ with $k''\leqq k\leqq k'$. Then

$$h^2(\mathscr{F}(t))\leqq 1+2+\ldots+(k'-t-1).$$

Table 2

c_2	$A(c_2)$	$B(c_2)$	c_2	$A(c_2)$	$B(c_2)$
11	4	4	26	7	8
12	5	5	27	8	8
13	5	5	28	8	8
14	5	5	29	8	8
15	5	5	30	8	8
16	6	6	31	8	8
17	6	6	32	8	9
18	6	6	33	9	9
19	6	6	34	9	9
20	6	6	35	9	9
21	7	7	36	9	9
22	7	7	37	9	9
23	7	7	38	9	10
24	7	7	39	9	10
25	7	7	40	10	10

On the other hand, by (7.3), $c_3 = -2\sum k_i$, so

$$\tfrac{1}{2}c_3 = (k''+1) + \ldots + k'.$$

Therefore

$$h^2(\mathscr{F}(t) - \tfrac{1}{2}c_3 \leqq 1 + 2 + \ldots + (k'' - t - 1),$$

which is better than the original bound. So the same result holds.

Corollary 10.3. *The conjecture* (10.1) *is true for all odd* $c_2 \leqq 37$ *and all even* $c_2 \leqq 24$.

Proof. This is simply a consequence of numerical computations expressed in Table 2. Here $A(c_2)$ is the conjectured bound and $B(c_2)$ is the least t satisfying the hypotheses of (10.2). For $c_2 \leqq 10$, $h^2(\mathscr{F}(t)) = 0$ in the proof of (10.2) so the conjecture holds. For $c_2 \geqq 11$, the quantity $t = B(c_2)$ satisfies $t \leqq \frac{1}{2}(c_2 - 3)$ [resp. $t \leqq \frac{1}{2}(c_2 - 2)$] so we can conclude $h^0(\mathscr{F}(t)) \neq 0$ from (10.2).

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Note added in proof. The conjecture (10.1) is now proven to be true for all c_2 (at least in characteristic 0). Furthermore, examples found in joint work with A. Hirschowitz show this result is the best possible for each c_2. Proofs will appear shortly.