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The Restriction of Admissible Representations to $\mathfrak{n}$

William Casselman and M. Scott Osborne

Department of Mathematics, University of British Columbia, Vancouver, Canada, and
Department of Mathematics, University of Washington, Seattle, Wash. 98195, USA

Let G be the group of real-valued points on a reductive algebraic group defined over $\boldsymbol{R}$. We offer here, among other things, a completely algebraic proof of the fact that every irreducible admissible representation of G may be embedded in a principal series representation, under the assumption that the unipotent radical of a minimal parabolic is abelian.

In this direction, Harish-Chandra [4] has proven, without assumption on $\mathfrak{n}$, that every irreducible admissible representation of G appears as a subquotient of a principal series representation. Analysis of the asymptotic behavior of matrix coefficients may be used to improve this to obtain an imbedding. The techniques we use here, however, seem to us to have an independent interest.

1. Let

P = a minimal parabolic with Levi decomposition $P=MN$;
A = the maximal split torus in the center of M;
K = a maximal compact subgroup of G;
$K_M = K \cap M$ = a maximal compact in M.

Let $\mathfrak{g}, \mathfrak{p}$, etc. be their complexified Lie algebras, and let $\mathfrak{Z}(\mathfrak{g})$, $\mathfrak{Z}(\mathfrak{m})$ be the centers of the universal enveloping algebras of $\mathfrak{g}, \mathfrak{m}$. The graded rings associated to these by their canonical filtrations are isomorphic to the rings $I(\mathfrak{g})$, $I(\mathfrak{m})$ of $\mathfrak{g}$-, $\mathfrak{m}$-invariants in the symmetric algebras $S(\mathfrak{g})$, $S(\mathfrak{m})$ respectively.

Since $\mathfrak{g}$ is the direct sum of $\mathfrak{n}$, $\mathfrak{a}$, and $\mathfrak{k}$, $\mathfrak{U}(\mathfrak{g})$ is the (vector space) tensor product of $\mathfrak{U}(\mathfrak{n})$, $\mathfrak{U}(\mathfrak{a})$, and $\mathfrak{U}(\mathfrak{k})$ by Poincare-Birkhoff-Witt. There exists therefore a canonical linear map from $\mathfrak{U}(\mathfrak{g})$ to $\mathfrak{U}(\mathfrak{a})$ which annihilates $\mathfrak{n}\mathfrak{U}(\mathfrak{g})+\mathfrak{U}(\mathfrak{g})\mathfrak{k}$ and is the identity on $\mathfrak{U}(\mathfrak{a})$. Since $[\mathfrak{a},\mathfrak{n}]\subset\mathfrak{n}$, it is in fact a $(\mathfrak{U}(\mathfrak{a}+\mathfrak{n}), \mathfrak{U}(\mathfrak{k}))$-bimodule map, where $\mathfrak{k}$ and $\mathfrak{n}$ act trivially on $\mathfrak{U}(\mathfrak{a})$. It preserves canonical filtrations as well, so it induces an algebra homomorphism of the associated graded algebras, $\sigma: S(\mathfrak{g})\rightarrow S(\mathfrak{a})$. The ring $S(\mathfrak{a})$ is a finitely generated module over $\sigma(I(\mathfrak{g}))$ [5, Lemma 2.1.5.4].

Let i be the involution of $\mathfrak{U}(\mathfrak{k})$ which takes $X\in\mathfrak{k}$ to $-X$. Left multiplication by elements of $\mathfrak{U}(\mathfrak{n})$ and $\mathfrak{Z}(\mathfrak{g})$ and right multiplication by the involute of an element of $\mathfrak{U}(\mathfrak{k})$ makes $\mathfrak{U}(\mathfrak{g})$ into a left $\mathfrak{U}(\mathfrak{n})\otimes\mathfrak{Z}(\mathfrak{g})\otimes\mathfrak{U}(\mathfrak{k})$-module. This structure is compatible with canonical filtrations, and $S(\mathfrak{g})$ thus becomes a module over $S(\mathfrak{n})\otimes I(\mathfrak{g})\otimes S(\mathfrak{k})$.

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1.1. Proposition. *Any homogeneous elements of* $S(\mathfrak{a})$ *which generate* $S(\mathfrak{a})$ *as a module over* $\sigma(I(\mathfrak{g}))$ *also generate* $S(\mathfrak{g})$ *as a module over* $S(\mathfrak{n})\otimes I(\mathfrak{g})\otimes S(\mathfrak{k})$.

Proof. Suppose $\alpha_1, \ldots, \alpha_m$ are homogeneous and generate $S(\mathfrak{a})$ over $\sigma(I(\mathfrak{g}))$. The proof proceeds by induction on degree. It therefore suffices to show that given a homogeneous $\alpha \in S(\mathfrak{a})$, there exist $X_1, \ldots, X_m \in I(\mathfrak{g})$ such that

$$\alpha - \sum X_i \alpha_i \in \mathfrak{n}S(\mathfrak{g}) + S(\mathfrak{g})\mathfrak{k}.$$

There certainly exist homogeneous $X_1, \ldots, X_m \in I(\mathfrak{g})$ such that $\alpha = \sum \sigma(X_i)\alpha_i$, and $\deg(\alpha) = \deg(X_i) + \deg(\alpha_i)$, all i. But σ is a ring homomorphism:

$$\sigma(\alpha - \sum X_i \alpha_i) = \alpha - \sum \sigma(X_i)\alpha_i = 0,$$

so that

$$\alpha - \sum X_i \alpha_i \in \ker(\sigma) = \mathfrak{n}S(\mathfrak{g}) + S(\mathfrak{g})\mathfrak{k}.$$

1.2. Corollary. *The algebra* $\mathfrak{U}(\mathfrak{g})$ *is finitely generated as a module over* $\mathfrak{U}(\mathfrak{n})\otimes\mathfrak{Z}(\mathfrak{g})\otimes\mathfrak{U}(\mathfrak{k})$.

2. A representation (π, V) of $\mathfrak{g}$ and K on the same space V is said to be *admissible* if

1. The restriction to K is an algebraic direct sum of irreducible, finite-dimensional, continuous representations of K, each isomorphism class occurring finitely often;
2. The representation of $\mathfrak{k}$ is the differential of the representation of K;
3. For any $k \in K$, $X \in \mathfrak{g}$, one has

$$\pi(k)\pi(X)\pi(k^{-1}) = \pi(\mathrm{Ad}(k)X).$$

2.1. Proposition. *A finite-dimensional admissible representation of* $\mathfrak{g}$ *and* K *may be extended uniquely to a continuous representation of* G.

This is because $G/K = AN$ is connected and simply connected.

Any endomorphism of the space of an admissible representation which commutes with $\mathfrak{g}$ also commutes with the identity component of K, and therefore any vector in the space is contained in a finite-dimensional subspace stable with respect to this endomorphism. In particular, the representation is locally $\mathfrak{Z}(\mathfrak{g})$-finite. Therefore:

2.2. Proposition. *If* (π, V) *is a finitely generated admissible representation of* $\mathfrak{g}$ *and* K, *then there exists an ideal of* $\mathfrak{Z}(\mathfrak{g})$ *of finite codimension which annihilates it.*

The main result of this section is:

2.3. Theorem. *Any finitely generated admissible representation of* $\mathfrak{g}$ *and* K *is also a finitely generated* $\mathfrak{U}(\mathfrak{n})$*-module.*

Proof. Let $W \subsetneq V$ be a finite-dimensional K-stable subspace which generates V as a $\mathfrak{g}$-module and which is a sum of full $\mathfrak{k}$-isotypes—W is then $\mathfrak{Z}(\mathfrak{g})$-stable, too. Let $\alpha_1, \ldots, \alpha_n$ be elements of $\mathfrak{U}(\mathfrak{g})$ generating $\mathfrak{U}(\mathfrak{g})$ over $\mathfrak{U}(\mathfrak{n})\otimes\mathfrak{Z}(\mathfrak{g})\otimes\mathfrak{U}(\mathfrak{k})$. Then any basis of $\sum \pi(\alpha_i)W$ generates V as a $\mathfrak{U}(\mathfrak{n})$-module.

2.4. Corollary. *If (π, V) is a finitely generated admissible representation of $\mathfrak{g}$ and K, then the homology spaces $H_q(\mathfrak{n}, V)$ are finite-dimensional.*

This follows directly from the fact that

$$H_q(\mathfrak{n}, V) = \mathrm{Tor}^q_{\mathfrak{U}(\mathfrak{n})}(C, V).$$

For more on $\mathfrak{n}$-homology of $\mathfrak{g}$-representations, refer to [3]. (Since [3] deals mostly, and perhaps unfortunately, with cohomology instead of homology, we would like to make here a few observations relating the two. If $r = \dim \mathfrak{n}$, then the wedge map:

$$\Lambda^q \mathfrak{n} \otimes [(\Lambda^{r-q} \mathfrak{n}) \otimes V] \to \Lambda^r \mathfrak{n} \otimes V$$

induces a natural isomorphism of $\Lambda^{r-q} \mathfrak{n} \otimes V$ with $\mathrm{Hom}(\Lambda^q \mathfrak{n}, \Lambda^r \mathfrak{n} \otimes V)$. Up to sign, this commutes with the standard differentials, so that $H_{r-q}(\mathfrak{n}, V)$ is naturally isomorphic to $H^q(\mathfrak{n}, \Lambda^r \mathfrak{n} \otimes V)$, which in turn is naturally isomorphic to $\Lambda^r \mathfrak{n} \otimes H^q(\mathfrak{n}, V)$ since $\mathfrak{n}$ acts trivially on $\Lambda^r \mathfrak{n}$.)

2.5. Proposition. *If (π, V) is an admissible representation of $\mathfrak{g}$ and K, then any $\mathfrak{n}$-invariant vector in V generates (over $\mathfrak{g}$) a finite-dimensional subspace of V.*

Proof. Writing backwards, $\mathfrak{U}(\mathfrak{g})$ is finitely generated over $\mathfrak{U}(\mathfrak{k}) \otimes \mathfrak{Z}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{n})$. The proposition now follows from local $\mathfrak{U}(\mathfrak{k})$- and $\mathfrak{Z}(\mathfrak{g})$-finiteness.

Combining with highest-weight theory, we have:

2.6. Corollary. *If V is an irreducible admissible representation of $\mathfrak{g}$ and K, then V is finite-dimensional if and only if V contains a nonzero $\mathfrak{n}$-invariant vector.*

3. Let δ denote the modular function of P, $\delta(mn) = \det(\mathrm{Ad}_{\mathfrak{n}}(m))$.

If (ω, U) is a continuous finite-dimensional representation of M, define $\mathrm{Ind}(\omega|P, G)$ to be the right regular representation of $\mathfrak{g}$ and K on the space of K-finite functions $f: G \to U$ such that $f(nmg) = [\omega\delta^{1/2}](m) f(g)$ for all $n \in N$, $m \in M$, $g \in G$. Let $\Lambda: \mathrm{Ind}(\omega|P, G) \to \omega\delta^{1/2}$ be the P-morphism $f \to f(1)$.

3.1. Lemma (Frobenius reciprocity). *If (π, V) is any admissible representation of $\mathfrak{g}$ and K, and (ω, U) is a continuous finite-dimensional representation of M, then composition with Λ induces an isomorphism*

$$\Lambda^*: \mathrm{Hom}_{\mathfrak{g}-K}(V, \mathrm{Ind}(\omega|P, G)) \cong \mathrm{Hom}_{\mathfrak{p}-K_M}(V, \omega\delta^{1/2}).$$

Proof. It suffices to exhibit the inverse of Λ^*. Given a $\mathfrak{p} - K_M$ map $F: V \to \omega\delta^{1/2}$, associate to it the $\mathfrak{g} - K$-map $\Phi: V \to \mathrm{Ind}(\omega|P, G)$ which takes $v \in V$ to Φ_v, where

$$\Phi_v(nmk) = \omega\delta^{1/2}(m) F(\pi(k)v)\,; \qquad n \in N,\ m \in M,\ k \in K.$$

If V is any module over $\mathfrak{U}(\mathfrak{n})$, define $V(\mathfrak{n})$ to be the linear span of $\{\pi(X)v | X \in \mathfrak{n}, v \in V\}$, and $V_{\mathfrak{n}}$ to be $V/V(\mathfrak{n})$. [Of course, $V_{\mathfrak{n}} = H_0(\mathfrak{n}, V)$.] If (π, V) is a representation of $\mathfrak{p}$ and K_M, then $V_{\mathfrak{n}}$ is naturally a representation of $\mathfrak{m}$ and K_M. The space $V_{\mathfrak{n}}$ is the universal $\mathfrak{n}$-trivial quotient of V, so that 3.1 implies immediately:

3.2. Proposition. *If (π, V) is any admissible representation of $\mathfrak{g}$ and K, and if (ω, U) is a continuous finite-dimensional representation of M, then*

$$\mathrm{Hom}_{\mathfrak{g}-K}(V, \mathrm{Ind}(\omega|P, G)) \cong \mathrm{Hom}_{\mathfrak{m}-K_M}(V_{\mathfrak{n}}, \omega\delta^{1/2}).$$

3.3. Remark. If (π, V) is finitely generated and admissible, then $V_{\mathfrak{n}}$ is finite-dimensional. The group M is also the group of rational points on a real reductive group, so that 2.1 implies that $V_{\mathfrak{n}}$ is in a unique and natural way the space of a continuous representation of M. In these circumstances, then, the above says that

$$\mathrm{Hom}_{\mathfrak{g}-K}(V, \mathrm{Ind}(\omega|P, G)) \cong \mathrm{Hom}_M(V_{\mathfrak{n}}, \omega\delta^{1/2}).$$

4. Let R denote an arbitrary commutative Noetherian ring, and let ∂ be a derivation of R.

4.1. Lemma. *If I is an ideal of R, then $\partial(I^n) \subset I^{n-1}$.*

This is straightforward.

4.2. Lemma. *If I is an ideal of R and $x \in I$, then*

$$\partial^m x^m \equiv m!(\partial x)^m \,(\mathrm{mod}\, I).$$

Proof. This is trivial for $m=1$. Proceed by induction. Leibniz' formula gives

$$\begin{aligned}\partial^m x^m &= \partial^m x \cdot x^{m-1} + m\partial^{m-1} x \cdot \partial x^{m-1} + \ldots + m\partial x \cdot \partial^{m-1} x^{m-1} + x \cdot \partial^m x^{m-1} \\ &\equiv m\partial x \partial^{m-1} x^{m-1} (\mathrm{mod}\, I)\end{aligned}$$

since $\partial^p x^n \in I^{n-p}$ $(p<n)$ by 4.1. Use induction.

4.3. Proposition. *Assume that $\mathbb{Q} \subset R$. Let P be a prime ideal of R, I an ideal of R with $P^m \subset I \subset P$ for some $m \geqq 1$. If $\partial I \subset I$, then $\partial P \subset P$.*

Proof. Suppose $x \in P$. Then $x^m \in I$, so every $\partial^p x^m \in I$ too. But

$$\partial^m x^m \equiv m!(\partial x)^m (\mathrm{mod}\, P)$$

by 4.2. Since $I \subset P$ and $m!$ is invertible in R, $(\partial x)^m \in P$, so that $\partial x \in P$.

If V is an R-module, recall that a derivation of V over ∂ is a map $\delta: V \to V$ such that $\delta(ru) = \partial r \cdot u + r \cdot \delta u$ for all $r \in R$, $u \in V$.

4.4. Proposition. *If there exists on V a derivation over ∂, then $\mathrm{Ann}_R(V)$ is ∂-stable.*

Straightforward.

4.5. Theorem. *If V is a finitely generated R-module on which there exists a derivation over ∂, then every associated prime ideal of V is ∂-stable.*

Proof. If $P \in \mathrm{Ass}(V)$, $P = \mathrm{Ann}(x)$, then $x \in V\{P\} = \{v \in V | P^n v = 0 \text{ for some } n \geqq 1\}$, so $V\{P\} \neq 0$, and $P \in \mathrm{Ass}(V\{P\})$. By 4.1, $V\{P\}$ is stable with respect to any derivation over ∂, so we may as well assume that $V = V\{P\}$. By the ACC on $V_N\{P\} = \{v \in V | P^N v = 0\}$, there exists an m such that $P^m V = 0$. Setting $I = \mathrm{Ann}_R(V)$ the result follows from 4.4 and 4.3.

5. Return to Lie Algebras. Assume $\mathfrak{g}$ to be simple (not an important restriction) and $\mathfrak{n}$—or equivalently N—to be abelian. This means that G is isogenous (over $\boldsymbol{R}$) to some orthogonal group $\mathrm{SO}(1, m+1)$, $\mathfrak{m}$ is the Lie algebra of the multiplicator group $\mathrm{GO}(m)$ of the quadratic form $x_1^2+\ldots+x_m^2$, $\mathfrak{n}$ is isomorphic to $\boldsymbol{C}^m$, and $\mathfrak{U}(\mathfrak{n})$ is the polynomial algebra in m variables.

5.1. Theorem. *If (π, V) is a finitely-generated admissible representation of $\mathfrak{g}$ and K, then every prime ideal in $\mathfrak{U}(\mathfrak{n})$ associated to V as a $\mathfrak{U}(\mathfrak{n})$-module is stable with respect to $\mathfrak{m}$.*

This follows immediately from Theorems 2.3 and 4.5, since the elements of $\mathfrak{m}$ are derivations of $\mathfrak{U}(\mathfrak{n})$.

For any prime ideal P let $\mathscr{V}(P)$ be the variety of points on which elements of P vanish. The ideal P is stable with respect to $\mathfrak{m}$ if and only if $\mathscr{V}(P)$ is stable with respect to the connected component of $\mathrm{GO}(m)$. Therefore, the varieties attached to the possible primes associated to admissible representations of $\mathfrak{g}$ are exactly the closures of the orbits of $\mathrm{GO}(m)$. More precisely, the possible $\mathscr{V}(P)$ are (1) when $m=1$: the origin and all of $\boldsymbol{C}^1$; (2) when $m=2$: the origin, the two lines $x \pm iy=0$, and all of $\boldsymbol{C}^2$; (3) when $m \geqq 3$: the origin, the 0-sphere, and all of $\boldsymbol{C}^m$ (this by Witt's theorem). Furthermore, in the case $m=2$ one can show by considering K_M as well that the prime $(x+iy)$ is an associated prime if and only if $(x-iy)$ is.

Each of these prime ideals is contained in the maximal ideal corresponding to the origin—in other words, the origin lies in the closure of every orbit of the connected component of $\mathrm{GO}(m)$— which is the ideal of $\mathfrak{U}(\mathfrak{n})$ generated by $\mathfrak{n}$ itself (cf. below). According to [1], Chapter IV, § 1.2, Proposition 6, V embeds in its localization at $\mathfrak{U}(\mathfrak{n})\mathfrak{n}$; thus by [1], Chapter II, § 2.2, Corollary 3, $V_{\mathfrak{n}} \neq 0$. By 2.3, 2.4, and 3.3:

5.2. Corollary. *If π is an irreducible admissible representation of $\mathfrak{g}$ and K, then there exists a continuous finite-dimensional representation (ω, U) of M such that π may be embedded into* $\mathrm{Ind}(\omega|P, G)$.

Dixmier has remarked that one can assemble a shorter proof of this fact by considering $\mathrm{Ann}_{\mathfrak{U}(\mathfrak{n})}(V)$ alone (applying 4.4).

Note that by Krull's theorem, the localization of V at $\mathfrak{U}(\mathfrak{n})\mathfrak{n}$ embeds in the completion. This implies a property which is stronger than "$V_{\mathfrak{n}} \neq 0$".

5.3. Corollary. *If (ω, V) is a finitely generated admissible representation of $\mathfrak{g}$ and K, then the canonical map from V into its completion at the ideal $\mathfrak{U}(\mathfrak{n})\mathfrak{n}$ is an injection.*

This result, unlike 5.2, uses the full force of Theorem 5.1, and in fact, is equivalent to it.

Remarks. (a) When $G=\mathrm{SL}_2(\boldsymbol{R})$, $\mathfrak{U}(\mathfrak{n})$ is a polynomial algebra in one variable. Because every finitely generated module is then the sum of its torsion and free submodules, our proofs are especially simple in this case.

(b) Even when $\mathfrak{n}$ is not necessarily abelian, Corollary 5.3 makes sense, and in fact the analysis of the asymptotic behavior of matrix coefficients mentioned earlier shows it to be true. We expect the completions at $\mathfrak{U}(\mathfrak{n})\mathfrak{n}$ to play an important role in the theory of admissible representations. For example, one may apply 5.3 to show quickly that any finitely generated admissible representation has finite length, a sort of result hitherto accessible only through deep analytical results of Harish-Chandra.

(c) To what extent can our techniques be adapted to the case of non-abelian $\mathfrak{n}$? A great deal of commutative algebra may be carried over to the enveloping algebras of nilpotent Lie algebras, but unfortunately not what we need, as an example of Dixmier shows.

Let $\mathfrak{n} = \boldsymbol{C}X + \boldsymbol{C}Y + \boldsymbol{C}Z$

$$[X, Y] = Z, \quad [X, Z] = [Y, Z] = 0.$$

Let $\mathfrak{U} = \mathfrak{U}(\mathfrak{n})$, $V =$ the $\mathfrak{U}$-module $\mathfrak{U}/\mathfrak{U}(1+X)$. The annihilator of every non-zero submodule of V is 0, so 0 is the only associated prime of V in the non-commutative sense. However, $V(\mathfrak{n}) = V$ and $V_{\mathfrak{n}} = 0$.

Our conclusion is that there is likely to exist some category of special $\mathfrak{U}(\mathfrak{n})$-modules for which the theory goes through, and that this category contains the restriction of admissible representations of $\mathfrak{g} - K$ to $\mathfrak{n}$. But we have no substantial result in this direction.

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