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Characters and Jacquet Modules

W. Casselman

Department of Mathematics, University of British Columbia, Vancouver, Canada

Let k be a locally compact $\mathfrak{p}$-adic field with integers $\mathfrak{o}$, G the group of k-rational points of a reductive algebraic group defined over k. In this paper I shall generalize a recent result of Deligne [6] on the support of the character of an absolutely cuspidal representation and relate the character of any finitely generated admissible representation of G to that of its associated Jacquet modules.

1. First I must collect some facts about tori in G for which I have found no simple reference. Let A_ϕ be a maximal split torus in G, P_ϕ a minimal parabolic subgroup containing A_ϕ, Σ the set of roots of G relative to A_ϕ, Δ the simple roots corresponding to the choice of P_ϕ. The sets Σ and Δ may be identified with subsets of the real vector space $X = X(A_\phi) \otimes \boldsymbol{R}$, where $X(A_\phi)$ is the group of rational characters of A_ϕ. Let W be the corresponding Weyl group. For $\theta \subsetneqq \Delta$, set $A_\theta = \cap \ker(\alpha)$ $(\alpha \in \theta)$. I define a *standard* torus of G to be any conjugate of one of these A_θ. The standard tori contained in A_ϕ, for example, correspond bijectively to the faces of the linear dissection of X determined by the root hyperplanes: to the face F corresponds $A_F = \cap \ker(\alpha)$ $(\alpha | F = 0)$.

1.1. Lemma. *If A is any split torus of G and $\bar{A}$ is the smallest standard torus containing A, then the centralizer $Z_G(A)$ of A is G is equal to that of $\bar{A}$.*

The case of a reductive group over an algebraically closed field is dealt with in [10], and the general case follows directly from that since by [1] any standard torus in G is also one in the extension of G to an algebraic closure.

1.2. Corollary. *The maximal split subtorus of any maximal torus in G is a standard torus.*

Proof. Let T be the given maximal torus, A the maximal split subtorus of T. If $\bar{A}$ is the smallest standard torus containing A, then 1.1 implies that T and $\bar{A}$ commute, hence that $A \cdot T$ is a torus of G. Since T is maximal, $\bar{A} \subsetneqq T$. Since A is maximal split in T, $\bar{A} = A$.

In this situation, of course, T will also be a maximal torus of $Z_G(A)$. Incidentally, since A is a standard torus it is conjugate to some A_θ, and $Z_G(A)$ is therefore conjugate to a reductive factor of the standard parabolic P_θ. In fact, it follows from 1.1 that if A is any split torus of G then $Z_G(A)$ is conjugate to a reductive factor of some standard parabolic.

Recall that a semi-simple element $x \in G$ is said to be *regular* if $Z_G(x)$ is a maximal torus. The element x will clearly also be a regular element of any reductive subgroup of G containing $Z_G(x)$.

2. Let $g \in G$ be regular, semi-simple. Let T be $Z_G(g)$, A the maximal split subtorus of T, S the maximal anisotropic subtorus of T. Since T is isogenous to $S \times A$, some positive power of g will factor as $s \cdot a$ with $s \in S$, $a \in A$.

The map $\chi \to |\chi|$ allows one to identify the real vector space X with the linear dual of $\mathscr{A} = A_\phi / A_\phi(\mathfrak{o}) \otimes \mathbf{R}$. In $\mathscr{A}$ the image of $A_\phi^- = \{x \in A_\phi \,|\, |\alpha(x)| \leqq 1 \text{ for all } \alpha \in \Delta\}$ is a fundamental chamber for the Weyl group, and hence there exists $y \in G$ such that $yay^{-1} \in A_\phi^-$. Let $\Omega = \{\alpha \in \Delta \,|\, |\alpha(yay^{-1})| = 1\}$, and define P_g to be the parabolic subgroup $y^{-1} P_\Omega y$. It has the Levi decomposition $P_g = M_g N_g$, where N_g is its unipotent radical and $M_g = y^{-1} Z_G(A_\Omega) y$. It is clear that g is a regular semi-simple element of M_g. Furthermore, it is easy to see from the construction that this P_g and the one constructed by Deligne in [6] are the same.

Let N_g^- be the unipotent radical of the parabolic $P_{g^{-1}}$ opposite to P_g.

2.1. Lemma (Deligne). *There exists a decreasing sequence $\{K_i\}$ of compact open subgroups in G which form a basis for the neighborhoods of the identity and such that, where $N_i = N_g \cap K_i$, $M_i = M_g \cap K_i$, $N_i^- = N_g^- \cap K_i$:*

(a) $K_i = N_i^- M_i N_i$;

(b) $g N_i g^{-1} \subsetneqq N_i$, $g M_i g^{-1} = M_i$, $g^{-1} N_i^- g \subsetneqq N_i^-$;

(c) *If U_1 and U_2 are any two compact open subgroups of N, then there exists $n \geqq 0$ such that $g^n U_1 g^{-n} \subsetneqq U_2$, and similarly for N^- and g^{-1}.*

(d) *In the Hecke algebra $\mathscr{H}(G, K_i)$, for $n \geqq 0$:*

$(K_i g K_i)^n = K_i g^n K_i$.

This is proven in [6].

From now on I shall fix g and set $P = P_g$, $M = M_g$, $N = N_g$.

3. Let (π, V) be a finitely generated admissible representation of G. Recall that the *Jacquet module* associated to V and P is the space V_N defined as the largest quotient of V on which N acts trivially, together with the natural representation π_V of M on this space. For any compact subgroup $H \subseteqq G$, let $\mathscr{P}_H$ be the operator $(\text{meas}\, H)^{-1} \int_H \pi(h)\, dh$.

3.1. Lemma. (a) *If $v \in V$ is fixed by $M_i N_i^-$ then $\mathscr{P}_{N_i}(v) = \mathscr{P}_{K_i}(v)$;*

(b) *The natural map from V^{K_i} to V^{M_i} is surjective;*

(c) *The representation (π_N, V_N) is a finitely generated admissible representation of M.*

This is proven in § 3 of [3]. Of course (a) is trivial. It plays a role in proving (b), which in turn implies (c) almost immediately.

3.2. Corollary. *For any $v \in V^{K_i}$ with image $u \in V_N$, (meas $K_i g K_i)^{-1} \pi(K_i g K_i) v$ has image $\pi_N(g) u$.*

This follows from 2.1 and 3.1, since $\pi(g)v$ is fixed by $M_i N_i^-$ and

$$(\text{meas } K_i g K_i)^{-1} \pi(K_i g K_i) v = \mathscr{P}_{K_i}(\pi(g)v).$$

3.3. Proposition. *For each K_i there exists a space $V_g^{K_i} \subseteq V^{K_i}$ such that*

(a) *The projection from $V_g^{K_i}$ to $V_N^{M_i}$ is a linear isomorphism;*

(b) *For each $n \geqq 0$, $V_g^{K_i}$ is stable with respect to $\pi(K_i g^n K_i)$;*

(c) *There exists n such that $\pi(K_i g^n K_i) V^{K_i} \subseteq V_g^{K_i}$.*

Proof. The argument is much like that used to construct the canonical liftings in § 4 of [3], but I shall repeat it.

Recall from [3] that for any compact subgroup $U \subseteq N$ the space $V(U)$ is that of all $v \in V$ such that

$$\int_U \pi(u) v du = 0,$$

and that V_N is the quotient of V by the union $V(N)$ of all the $V(U)$. Choose a fixed compact open subgroup $U \subseteq N$ such that $V(N) \cap V^{K_i} \subseteq V(U)$ and $N_i \subseteq U$.

3.4. Lemma. *If $g^n U g^{-n} \subseteq N_i$ and $v \in V(N) \cap V^{K_i}$, then $\pi(K_i g^n K_i) v = 0$.*

Proof. The vector $\pi(K_i g^n K_i) v$ differs from $\mathscr{P}_{K_i}(\pi(g^n)v)$ by only a scalar. By 3.1, this latter is equal to $\mathscr{P}_{N_i}(\pi(g^n)v)$. But

$$\begin{aligned}\mathscr{P}_{N_i}(\pi(g^n)v) &= (\text{const}) \int_{N_i} \pi(x)\pi(g^n) v dx \\ &= (\text{const})\, \pi(g^n) \int_{g^{-n} N_i g^n} \pi(x) v dx \\ &= 0.\end{aligned}$$

Choose n to be large enough so that $g^n U g^{-n} \subseteq N_i$, and define $V_g^{K_i}$ to be $\pi(K_i g^n K_i) V^{K_i}$.

Proof of 3.3(a). First, surjectivity. Consider $u \in V_N^{M_i}$. Since g normalizes M_i, $\pi_N(g^{-n}) u \in V_N^{M_i}$. By 3.1 there exists $v \in V^{K_i}$ whose image in V_N is $\pi_N(g^{-n})u$. But then by 3.2, $\mathscr{P}_{K_i}(\pi(g^n)v)$ has image u.

Second, injectivity. Suppose that $v \in V(N) \cap V_n$, say $v = \pi(K_i g^n K_i) v_0$, $v_0 \in V^{K_i}$. By the choice of U, $v \in V(U)$. Now v is also, up to a constant, equal to $\mathscr{P}_{K_i}(\pi(g^n)v_0) = \mathscr{P}_{N_i}(\pi(g^n)v_0)$. Therefore

$$\begin{aligned}\int_U \pi(u) v du &= 0 \\ &= \int_U \pi(u) du \int_{N_i} \pi(n_i) \pi(g^n) v_0 dn_i \\ &= \int_U \pi(u) \pi(g^n) v_0 du \\ &= \pi(g^n) \int_{g^{-n} U g^n} \pi(u) v_0 du\end{aligned}$$

so $v_0 \in V(N)$ also and $v = 0$ by 3.4.

Proof of (b). The above argument is independent of large n, so that all the spaces $\pi(K_i g^n K_i) V^{K_i}$ have the same dimension. But for $m \geqq n$, 2.1(d) implies that $\pi(K_i g^m K_i) V^{K_i} \subseteq \pi(K_i g^n K_i) V^{K_i}$.

Statement (c) is immediate from the definition.

3.5. Corollary. *For* $n \gg 0$,

$$\mathrm{Tr}\,[(\mathrm{meas}\, K_i g^n K_i)^{-1} \pi(K_i g^n K_i)] = \mathrm{Tr}\,[\pi_N(g_n) | V_N^{M_i}].$$

4. Let $\mathscr{C}(\boldsymbol{Z})$ be the space of all complex-valued functions on the integers. Define $\tau : \mathscr{C}(\boldsymbol{Z}) \to \mathscr{C}(\boldsymbol{Z})$ by the formula

$$\tau F(x) = F(x-1).$$

A function $F \in \mathscr{C}(\boldsymbol{Z})$ is said to be $\boldsymbol{Z}$-finite if it is contained in a finite-dimensional subspace of $\mathscr{C}(\boldsymbol{Z})$ stable under τ, or equivalently if the subspace spanned by $\{\tau^n F | n \in \boldsymbol{Z}\}$ is finite-dimensional. This condition is also equivalent to the existence of a polynomial $P(\tau) \neq 0$ such that $P(\tau)F = 0$.

4.1. Lemma. *Let* F_1 *and* F_2 *be two* $\boldsymbol{Z}$*-finite functions. If there exists* $n \in Z$ *such that* $F_1(x) = F_2(x)$ *for all* $x \geqq n$, *then* $F_1 = F_2$.

I leave this as an exercise.

The simplest example of a $\boldsymbol{Z}$-finite function is $F(n) = \lambda^n$, where $\lambda \in \boldsymbol{C}^\times$. Also, of course, any linear combination of $\boldsymbol{Z}$-finite functions is $\boldsymbol{Z}$-finite.

Let X be any endomorphism of a finite-dimensional complex vector space, and suppose that its non-zero eigenvalues are $\lambda_1, \ldots, \lambda_r$. For $n \geqq 1$, $\mathrm{Tr}(X^n) = \lambda_1^n + \ldots + \lambda_r^n$; the function $n \to \mathrm{Tr}(X^n)$ may therefore be extended to a unique $\boldsymbol{Z}$-finite function on all of $\boldsymbol{Z}$.

4.2. Corollary. *If* X *and* Y *are two finite-dimensional endomorphisms such that* $\mathrm{Tr}(X^n) = \mathrm{Tr}(Y^n)$ *for* $n \gg 0$, *then* $\mathrm{Tr}(X^n) = \mathrm{Tr}(Y^n)$ *for all* $n \geqq 1$.

5. The main result is now almost immediate. Adopt the notation of § 3.

5.1. Lemma. *For all* $n \geqq 1$,

$$\mathrm{Tr}[(\mathrm{meas}\, K_i g_n K_i)^{-1} \pi(K_i g^n K_i)] = \mathrm{Tr}[(\mathrm{meas}\, M_i)^{-1} \pi_N(g^n M_i)].$$

This is a corollary of 3.5 and 4.2.

According to a result of Harish-Chandra and Howe there exists a locally constant function $\boldsymbol{ch}_\pi$ defined on the open set of regular semi-simple elements of G such that for any $f \in C_c^\infty(G)$ with support in this set

$$\mathrm{Tr}(\pi(f)) = \int_G f(x) \boldsymbol{ch}_\pi(x) dx.$$

(The case when k has characteristic 0 is discussed in [2], [7], and [8]. In fact, what is proven there under this assumption is the much deeper result that the character of π—i.e. the functional on $C_c^\infty(G)$ which takes f to the trace of $\pi(f)$ – is determined by the function $\boldsymbol{ch}_\pi$ on the regular semi-simple elements. The result needed above is more elementary than this and has been recently established by Harish-Chandra without assumption on the characteristic of k.)

This result applies as well to the Jacquet module π_N. Therefore, setting f equal to the characteristic function of $K_i g K_i$ and letting i increase, one has:

5.2. Theorem. *Let* π *be a finitely generated admissible representation of* G, g *a regular semi-simple element of* G, $P = P_g = MN$. *Then*

$$\boldsymbol{ch}_\pi(g) = \boldsymbol{ch}_{\pi_N}(g).$$

5.3. Remarks. (a) If π is absolutely cuspidal then $\pi_N = 0$ for all non-trivial N and one recovers Deligne's theorem.

6. Let G be a reductive group over $\boldsymbol{R}$, $\mathfrak{g}$ its Lie algebra, K a maximal compact subgroup. There are two conjectures in this case which amount to an analogue of the above theorem.

Conjecture 1. *Let $P = MN$ be a parabolic subgroup of G, $\mathfrak{m}$ and $\mathfrak{n}$ the Lie algebras of M and N, $K_M = K \cap M$. If (π, V) is a finitely generated admissible representation of $(\mathfrak{g}, K)$ then each homology group $H_n(\mathfrak{n}, V)$ is one of $(\mathfrak{m}, K_M)$.*

This is known to be true if P is minimal, or if $n = 0$ (see [5]; for a few results on the homology, see [4]). It is perhaps not too difficult to prove in general.

For any regular semi-simple element $g \in G$ one can define $P = P_g$ as for the $\mathfrak{p}$-adic case. It is a classical theorem of Harish-Chandra that each finitely generated admissible representation of $(\mathfrak{g}, K)$ has a character a smooth function on the set of such elements.

Conjecture 2. *For any regular semi-simple g,*

$$\boldsymbol{ch}_\pi(g) = \frac{\Sigma(-1)^i \boldsymbol{ch}(g|H_i(n, V))}{\Sigma(-1)^i \boldsymbol{ch}(g|\Lambda^i \mathfrak{n})}.$$

For the case of P_g minimal, this conjecture is due to Osborne [9], and has been verified in an *ad hoc* manner for a number of cases. When V is finite-dimensional this is almost trivially true and plays a role in Kostant's proof of the Weyl character formula.

Note added in proof: Several people have noticed that conjecture 1 is easy. Conjecture 2 has been proven by H. Hecht and W. Schmid.

References

1. Borel, A., Tits, J.: Groupes réductifs. Publ. Math. I.H.E.S. 55—151 (1965)
2. Cartier, P.: Les représentations des groupes réductifs p-adiques et leurs caracteres. Seminaire Bourbaki No. 471
3. Casselman, W.: Introduction to the theory of admissible representations of p-adic reductive groups. To appear
4. Casselman, W., Osborne, M.S.: The n-cohomology of representations with an infinitesimal character. Comp. Math. **31**, 219—227 (1975)
5. Casselman, W., Osborne, M.S.: The restriction of an admissible representation to n (to appear)
6. Deligne, P.: Le support du caractere d'une représentation supercuspidale (to appear in the Comptes Rendues de l'Académie des Sciences)
7. Harish-Chandra: The characters of reductive p-adic groups (to appear)
8. Howe, R.: The Fourier transform and germs of characters. Math. Ann. **208**, 305—322 (1974)
9. Osborne, M.S.: Thesis, Yale, 1973
10. Springer, T.A., Steinberg, R.: Conjugacy classes. In Seminar on algebraic groups and related finite groups. Berlin, Heidelberg, New York: Springer 1970

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